Unitary units in finite dimensional algebras with involution

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1 Introduction

Let R be a ring (with unity). We recall that a map $^* : R \to R$ is called an *involution* in R if for all $a, b \in R$ we have that:

(i) $(a+b)^* = a^* + b^*$.

(ii) $(ab)^* = b^* a^*$.

(iii) $(a^*)^* = a$.

If R is an algebra over a field F and * is an involution of R such that $a^* = a$ for all $a \in F$, then we say that * is an involution *of the first kind*.

For example, if F is field and $M_k(F)$ the algebra of $k \times k$ matrices over F, then map $(a_{ij}) \mapsto (a_{ij})^t = (a_{ji})$ is an involution of $M_k(F)$, which is called the *transpose involution* of $M_k(F)$.

We define the *group of units* of a ring R by:

$$\mathcal{U}(R) = \{x \in R \mid x \text{ is invertible }\}.$$

If R has an involution *, we define the subgroup of *unitary units* by

$$\mathcal{U}_n(R) = \{x \in \mathcal{U}(R) \mid xx^* = 1\},$$

i.e., an element $x \in R$ is a unitary unit if and only if $u^* = u^{-1}$.

We also consider the following sets

$$R^+ = \{x \in R \mid x^* = x\}, \quad \text{symmetric elements}$$

$$R^- = \{x \in R \mid x^* = -x\}, \quad \text{antisymmetric elements}$$

Our aim is to give some information about $\mathcal{U}_n(R)$ in the case when R is a finite dimensional algebra over a field. The results mentioned below were obtained in a joint research with Prof. Antonio Giambruno, of the University of Palermo.

Research partially supported by MURST (Italy) and FAPESP (Brazil)
1991 Mathematics Subject Classification: Primary 16U60; Secondary 16H05, 20F24, 20C05.
Key words and phrases: group ring, unit group, skew elements, unitary units.

2 Matrix algebras

In what follows, F will always denote a field of characteristic different from 2. As mentioned above, an obvious way to consider a full matrix algebra as a ring with involution is to endow it with the transpose involution. Also, if $k = 2m$ is even, every matrix $X \in M_{2m}(F)$ can be written in the form

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where A, B, C and D denote blocks in $M_m(F)$. Then, the map $^s : M_{2m}(F) \to M_{2m}(F)$ given by

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto X^s = \begin{bmatrix} D^t & -B^t \\ -C^t & A^t \end{bmatrix}$$

is an involution of $M_{2m}(F)$ called the *symplectic involution*.

We can loosely say that these are the only involutions of a matrix algebra, in a sense that we now make precise.

Given two involutions * and $^\#$ of a ring R we say that * is equivalent to $^\#$ if there exists an element $a \in R$, which is symmetric under *, such that $x^* = a^{-1}x^\# a$, for all $x \in R$.

Let * be an involution on $M_k(F)$ of the first kind: i.e., which fixes F elementwise. It is well known that * is either equivalent to the transpose involution or the symplectic involution. When F is algebraically closed and * is an involution on $M_k(F)$ then $(M_k(F),^*)$ is actually isomorphic to either $(M_k(F),^t)$ or $(M_k(F),^s)$ (see [3, Theorem 3.1.61]).

When $^* = {}^t$, then $\mathcal{U}_n(M_k(F) = \mathcal{O}_k(F)$ is the orthogonal group and when $^* = {}^s$ then k is always even and we shall denote $\mathcal{U}_n(M_k(F))$ briefly as $\mathcal{S}_k(F)$. Notice that $\mathcal{S}_2(F) = SL_2(F)$, the special linear group of 2×2 matrices. It is well known that for all $u \in \mathcal{O}_k(F)$, we have that $det\ u = \pm 1$ and for all $u \in \mathcal{S}_k(F)$, we have $det\ u = 1$.

Let $X = \{x_1, x_2, \ldots\}$ be a countable set and let $\mathcal{F}$ be the free group on X. If G is a group and $w = w(x_1, \ldots, x_n)$ is a nonempty reduced word on $\mathcal{F}$ in a finite number of variables $x_1, \ldots x_n$, we say that $w = 1$ is a *group identity* for G if $w(g_1, \cdots g_n) = 1$ for all $g_1, \ldots, g_n \in G$.. Without restriction, we may assume that w is a word in at least two variables for, if $x_1^n = 1$ is a group identity for a group G, then $(x_1x_2)^n = 1$ is also a group identity for G. Furthermore, since a free group of finite rank n can be considered as a subgroup of a free group of rank 2 (see, for example [2, Theorem 6.1.1]), we may always assume that w is a word in precisely two variables.

Notice that $SO_2(F) = \{u \in \mathcal{O}_2(F) \mid det\ u = 1\}$ is an abelian group of index 2 and also $(\mathcal{O}_2(F) \setminus SO_2(F))^2 = 1$. Thus $\mathcal{O}_2(F)$ satisfies the group identity $(x_1^2, x_2^2) = 1$. It turns out that, when F is infinite and $k > 2$ then $\mathcal{O}_k(F)$ does not satisfy a group identity. In fact, we have the following.

Lemma 2.1 *Let F be an infinite field. If $k > 2$ then $\mathcal{O}_k(F)$ does not satisfy a group identity and if $k > 1$ then $\mathcal{S}_k(F)$ does not satisfy a group identity.*

We recall that a field F is called *nonabsolute* if either $char(F) = 0$ or $\text{char}(F) = p > 0$ and F is not algebraic over its prime field. For this type of fields, there is a stronger version of the preceeding lemma.

Lemma 2.2 (J.Z. Gonçalves and D.S Passman [1, Lemma 2.5]) *Let F be an absolute field. If $k > 2$ then $\mathcal{O}_k(F)$ does not contain non abelian free groups and if $k > 1$ then $\mathcal{S}_k(F)$ does not contain non abelian free groups.*

3 Finite dimensional algebrs

After the results in the preceeding section, using Wedderburn's Theorem and standard arguments we obtain the following.

Theorem 3.1 *Let R be a semisimple finite dimensional algebra with involution over an algebraically closed field F with $char(F) \neq 2$. Then*

(i) $\mathcal{U}_n(R)$ satisfies a group identity if and only if R^- is commutative.

(ii) If, furthermore, F is nonabsolute, then $\mathcal{U}_n(R)$ contains no free group of rank 2 if and only if R^- is commutative.

Theorem 3.2 *Let F be an algebraically closed field which is nonabsolute, with $char(F) \neq 2$ and let R be a finite dimensional algebra with involution. Then $\mathcal{U}_n(R)$ does not contain a free group of rank 2 if and only if $\mathcal{U}_n(R)$ satisfies the group identity $(x_1^2, x_2^2)^m \equiv 1$, for some positive integer m.*

In then special case when R is the group algebra of a finite group G, one can give a stronger result. We recall the definition of Lie commutators in a ring R, which is inductive:

$$[x_1, x_2] = x_1x_2 - x_2x_1.$$

$$[x_1, x_2, \ldots, x_n] = [[x_1, x_2 \ldots, x_{n-1}], x_n].$$

A ring R is *Lie nilpotent* (of index n) if there exists a positive integer n such that $[x_1, \ldots, x_n] = 0$, for all $x_1, \ldots x_n \in R$.

Corollary 3.3 *Let G be a finite group and F any field with $char(F) \neq 2$. If FG^- is Lie nilpotent, then $\mathcal{U}_n(FG)$ contains no free group of rank 2.*

Proofs will be published elsewhere.

References

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