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New Results on Truncated Elliptical Distributions

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Abstract
Truncated elliptical distributions occur naturally in theoretical and applied statistics and are essential for the study of other classes of multivariate distributions. Two members of this class are the multivariate truncated normal and multivariate truncated t distributions. We derive statistical properties of the truncated elliptical distributions. Applications of our results establish new properties of the multivariate truncated slash and multivariate truncated power exponential distributions.

Keywords Elliptical distribution · Log-concavity · Multivariate power exponential distribution · Multivariate slash distribution · Truncated distribution

Mathematics Subject Classification 60E05 · 62E15 · 65C05

1 Introduction

The class of elliptical distributions (see Fang et al. [5]) is of central relevance in applied and theoretical statistics, in particular in multivariate regression analysis. This class comprises a wide variety of distributions for modeling multivariate data, including those with heavier-than-normal or slighter-than-normal tail behavior. Some elliptical distributions are multivariate normal, t, power exponential, slash, scale mixture of normal distributions, among others. A generalization of this class is constructed by truncation: it is derived as the conditional distribution of a random vector, with dimension p say, having an elliptical distribution given that it belongs to a subset of

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$\mathbb{R}^p$. Some works dealing with truncated elliptical distributions are found in Morán-Vásquez and Ferrari [16], Kim [15] and Arellano-Valle et al. [2]. The class of truncated elliptical distributions has as special members the multivariate truncated normal and multivariate truncated t distributions. Some studies regarding to these two distributions can be found in Kan and Robotti [14], Arismendi [1], Ho et al. [11], Nadarajah [17], Horrace [12,13], Geweke [7], Tallis [18–20] and Birnbaum and Meyer [3]. However, other distributions, such as the multivariate truncated slash and multivariate truncated power exponential distributions, have not been studied.

We derive several properties of the truncated elliptical distributions, thus generalizing some results previously shown for the multivariate truncated normal and multivariate truncated t distributions only. Some of our results consider arbitrary truncation subsets of $\mathbb{R}^p$; others are valid only for p-dimensional rectangles. We present a characterization of the truncated elliptical distributions and give some distributional properties obtained through transformations, as well as properties involving the moment generating function (MGF), log-concavity of the joint probability density function (PDF), the MTP_2 property, marginal and conditional distributions, and independence. Additionally, we describe a procedure to evaluate probabilities over rectangles for the multivariate slash distribution, which is useful to compute the PDF of the multivariate truncated slash distribution. Applications of the results derived in this paper allow us to derive new properties of the multivariate truncated slash and multivariate truncated power exponential distributions.

2 The Class of the Truncated Elliptical Distributions

The notation follows that of Morán-Vásquez and Ferrari [16]. Vectors are represented with lowercase Greek letters in bold, and their components with lowercase Greek letters in normal font. For example, if $\boldsymbol{\xi} \in \mathbb{R}^p$, then $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_p)'$. Also, $\boldsymbol{\xi}_{-k} \in \mathbb{R}^{p-1}$, $k = 1, \ldots, p$, is the sub-vector of $\boldsymbol{\xi}$ without its kth component. In the case of random vectors, we use similar notation, but with capital Roman letters. Matrices are denoted by capital Greek letters in boldface and their components in lowercase normal font Greek letters. For instance, if $\boldsymbol{\Delta}(p \times q)$ is a matrix whose elements are real numbers, then $\boldsymbol{\Delta} = (\delta_{jk})_{p\times q}$. If $\boldsymbol{\Delta}$ is a symmetric matrix, $\boldsymbol{\Delta} > 0$ means that $\boldsymbol{\Delta}$ is positive definite. The sub-vector $\boldsymbol{\Delta}_{-k,k} \in \mathbb{R}^{p-1}$ is obtained by deleting the kth component of the kth column of $\boldsymbol{\Delta}(p \times p) > 0$; $\boldsymbol{\Delta}_{k,-k} = \boldsymbol{\Delta}'_{-k,k}$; and $\boldsymbol{\Delta}_{-k,-k} > 0$ is the sub-matrix obtained by deleting the kth row and the kth column of $\boldsymbol{\Delta}(p \times p) > 0$. We denote the rectangles in $\mathbb{R}^p$ with the letter R, which we consider as the Cartesian product of intervals $I_1, \ldots, I_p$ in $\mathbb{R}$, where every I_k can be a finite or infinite interval, that is, $R = I_1 \times \ldots \times I_p$.

The spherical and elliptical distributions play an important role in statistics and its applications; see Fang et al. [5] and Gupta et al. [10]. In what follows, whenever a random vector is said to have a spherical or elliptical distribution, it is assumed that its PDF exists.

Definition 2.1 The random vector $\boldsymbol{S} \in \mathbb{R}^p$ has a spherical (standard elliptical) distribution if its PDF is

$$f_{\boldsymbol{S}}(\boldsymbol{s}) = c_p g(\boldsymbol{s}'\boldsymbol{s}), \quad \boldsymbol{s} \in \mathbb{R}^p. \tag{2.1}$$

The function g is called density generating function (DGF) and satisfies $g(u) \geq 0$, for all $u \geq 0$, and $\int_0^\infty r^{p-1} g(r^2)\,\mathrm{d}r < \infty$. The normalizing constant c_p is

$$c_p = \frac{\Gamma(p/2)}{2\pi^{p/2}} \left(\int_0^\infty r^{p-1} g(r^2)\,\mathrm{d}r \right)^{-1}.$$

We write $\boldsymbol{S} \sim \mathrm{S}_p(g)$.

A generalization of the spherical distributions is constructed from the transformation $\boldsymbol{W} = \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2}\boldsymbol{S}$, where $\boldsymbol{S} \sim \mathrm{S}_p(g)$, $\boldsymbol{\mu} \in \mathbb{R}^p$ and $\boldsymbol{\Sigma}(p \times p) > 0$. The Jacobian of this transformation is $J(\boldsymbol{s} \to \boldsymbol{w}) = \det(\boldsymbol{\Sigma})^{-1/2}$ and, hence, the PDF of $\boldsymbol{W}$ is

$$f_{\boldsymbol{W}}(\boldsymbol{w}) = c_p \det(\boldsymbol{\Sigma})^{-1/2} g((\boldsymbol{w} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{w} - \boldsymbol{\mu})), \quad \boldsymbol{w} \in \mathbb{R}^p. \tag{2.2}$$

The random vector $\boldsymbol{W} \in \mathbb{R}^p$ is said to have an elliptical distribution with location vector $\boldsymbol{\mu} \in \mathbb{R}^p$, dispersion matrix $\boldsymbol{\Sigma}(p \times p) > 0$, and DGF g, and we write $\boldsymbol{W} \sim \mathrm{E}\ell_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; g)$. Evidently, if $\boldsymbol{\mu} = \boldsymbol{0}$ and $\boldsymbol{\Sigma} = \boldsymbol{I}_p$, then the PDF (2.2) reduces to PDF (2.1). Any distribution within the elliptical class of distributions is determined by the DGF g. If $g(u) \propto \exp(-u/2)$, $u \geq 0$, we say that $\boldsymbol{W}$ has a multivariate normal distribution with parameters $\boldsymbol{\mu} \in \mathbb{R}^p$ and $\boldsymbol{\Sigma}(p \times p) > 0$, we write $\boldsymbol{W} \sim \mathrm{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. If $g(u) \propto \int_0^1 t^{q+p-1} \exp(-ut^2/2)\,\mathrm{d}t$, $q > 0$, $u \geq 0$, then $\boldsymbol{W}$ has a multivariate slash distribution with parameters $\boldsymbol{\mu} \in \mathbb{R}^p$, $\boldsymbol{\Sigma}(p \times p) > 0$ and $q > 0$, we write $\boldsymbol{W} \sim \mathrm{SL}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, q)$. If $g(u) \propto \exp(-u^\beta/2)$, $\beta > 0$, $u \geq 0$, we say that $\boldsymbol{W}$ has a multivariate power exponential distribution with parameters $\boldsymbol{\mu} \in \mathbb{R}^p$, $\boldsymbol{\Sigma}(p \times p) > 0$ and $\beta > 0$, we write $\boldsymbol{W} \sim \mathrm{PE}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \beta)$. The DGF $g(u) \propto (1 + u/\tau)^{-(\tau+p)/2}$, $\tau > 0$, $u \geq 0$, corresponds to the multivariate t distribution. Extra parameters may appear in PDF (2.2) through the DGF g. For example, the multivariate slash distribution involves the tail parameter q and the multivariate t distribution has the degrees of freedom parameter τ. In these two cases, the extra parameter models the tail behavior. Also, the multivariate power exponential distribution has the kurtosis parameter β. The multivariate normal distribution can be obtained as a limiting case of the multivariate slash and t distributions when $q \to \infty$ and $\tau \to \infty$, respectively, and as a special case of the multivariate power exponential distribution when $\beta = 1$. Further details on elliptical distributions are found in Fang et al. [5]. Detailed studies about the multivariate slash and multivariate power exponential distributions appear in Wang and Genton [22] and Gómez et al. [9], respectively.

Let $\boldsymbol{W} \sim \mathrm{E}\ell_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; g)$ and let $B \subseteq \mathbb{R}^p$ be a measurable set. The distribution obtained by conditioning $\boldsymbol{W}$ on $\{\boldsymbol{W} \in B\}$ is called a truncated elliptical distribution. However, this definition may be given in terms of the PDF as in Definition 2.2.

Definition 2.2 Let $B \subseteq \mathbb{R}^p$ be a measurable set. The random vector $\boldsymbol{X}$ has a truncated elliptical distribution with support B and parameters $\boldsymbol{\mu} \in \mathbb{R}^p$, $\boldsymbol{\Sigma}(p \times p) > 0$ and DGF g, if its PDF is

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{g((\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}))}{\int_B g((\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}))\,\mathrm{d}\boldsymbol{x}}, \quad \boldsymbol{x} \in B. \tag{2.3}$$

The DGF g is such that $g(u) \geq 0$, for all $u \geq 0$, and $\int_0^\infty r^{p-1} g(r^2)\,\mathrm{d}r < \infty$. We write $\boldsymbol{X} \sim \mathrm{TE}\ell_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; B; g)$.

The PDF of $\boldsymbol{X} \sim \mathrm{TE}\ell_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; B; g)$ can be expressed as

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{f_{\boldsymbol{W}}(\boldsymbol{x})}{\mathrm{P}(\boldsymbol{W} \in B)}, \quad \boldsymbol{x} \in B, \tag{2.4}$$

where $f_{\boldsymbol{W}}$ is the PDF of a random vector $\boldsymbol{W} \sim \mathrm{E}\ell_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; g)$.

Note that the PDF of $\boldsymbol{X} \sim \mathrm{TE}\ell_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; B; g)$ exists if the PDF of $\boldsymbol{W} \sim \mathrm{E}\ell_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; g)$ exists, which occurs if $\boldsymbol{\Sigma} > 0$; see Fang et al. [5, Sec. 2.5].

If $B = \mathbb{R}^p$ in (2.4), then $\mathrm{P}(\boldsymbol{W} \in B) = 1$, and we recover PDF (2.2). Hence, the class of truncated elliptical distributions is a generalization of the elliptical class of distributions.

When $\boldsymbol{\mu} = \boldsymbol{0}$ and $\boldsymbol{\Sigma} = \boldsymbol{I}_p$ in Definition 2.2, we say that $\boldsymbol{X}$ has a truncated spherical distribution over B with DGF g, and we write $\boldsymbol{X} \sim \mathrm{TS}_p(B; g)$. The corresponding PDF is given by

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{g(\boldsymbol{x}'\boldsymbol{x})}{\int_B g(\boldsymbol{x}'\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}, \quad \boldsymbol{x} \in B. \tag{2.5}$$

If $B = \mathbb{R}^p$ in (2.5), we arrive at PDF (2.1).

The study of the class of Box–Cox elliptical distributions proposed by Morán-Vásquez and Ferrari [16] is based on the class of the truncated elliptical distributions. These authors showed that, for any p-dimensional rectangle R, the conditional distribution of any component X_k of $\boldsymbol{X} \sim \mathrm{TE}\ell_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; R; g)$, given all the other components, is univariate truncated elliptical with the same support of X_k. This property allowed to propose an algorithm to generate random samples of $\boldsymbol{X}$, which in turn is the basis for generating random samples of the Box–Cox elliptical distributions.

The DGF g provides a correspondence of the members in the class of the truncated elliptical distributions with those in the class of the elliptical distributions. By substituting $g(u) \propto \exp(-u/2)$, $u \geq 0$, in (2.3) we obtain the PDF of a random vector $\boldsymbol{X}$ with a multivariate truncated normal distribution with support B and parameters $\boldsymbol{\mu} \in \mathbb{R}^p$ and $\boldsymbol{\Sigma}(p \times p) > 0$, denoted by $\boldsymbol{X} \sim \mathrm{TN}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; B)$. When $g(u) \propto \int_0^1 t^{q+p-1} \exp(-ut^2/2)\,\mathrm{d}t$, $q > 0$, $u \geq 0$, in (2.3) we obtain the PDF of a random vector $\boldsymbol{X}$ with multivariate truncated slash distribution with support B and parameters $\boldsymbol{\mu} \in \mathbb{R}^p$, $\boldsymbol{\Sigma}(p \times p) > 0$ and $q > 0$, denoted by $\boldsymbol{X} \sim \mathrm{TSL}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, q; B)$. Also, when $g(u) \propto \exp(-u^\beta/2)$, $\beta > 0$, $u \geq 0$, we obtain the PDF of a random vector $\boldsymbol{X}$ with a multivariate truncated power exponential distribution with support B and parameters $\boldsymbol{\mu} \in \mathbb{R}^p$, $\boldsymbol{\Sigma}(p \times p) > 0$ and $\beta > 0$, denoted by $\boldsymbol{X} \sim \mathrm{TPE}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \beta; B)$. Other special case is the multivariate truncated t distribution, which corresponds to the DGF $g(u) \propto (1 + u/\tau)^{-(\tau+p)/2}$, $\tau > 0$, $u \geq 0$. The multivariate truncated normal distribution can be obtained as a limiting case of the multivariate truncated slash and multivariate truncated t distributions when $q \to \infty$ and $\tau \to \infty$, respectively, and as a particular case of the multivariate truncated power exponential distribution when $\beta = 1$.

Computations related to truncated elliptical distributions involve the evaluation of the integral $\int_B g((\boldsymbol{x} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu}))\, \mathrm{d}\boldsymbol{x}$; see (2.3). Genz and Bretz [6] present algorithms to compute this integral when $B = R$ is a rectangle in $\mathbb{R}^p$ and g is the DGF of the multivariate normal and multivariate t distributions. We now propose a procedure to evaluate the integral for the multivariate slash distribution. Let $\boldsymbol{X} \sim \mathrm{TSL}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, q; R)$, where R is a rectangle in $\mathbb{R}^p$. We propose the use of Monte Carlo integration mixed with the algorithm proposed by Genz and Bretz [6] to evaluate the PDF of $\boldsymbol{X}$. In fact, if $\boldsymbol{W} \sim \mathrm{SL}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, q)$, then the PDF of $\boldsymbol{W}$ is $f_{\boldsymbol{W}}(\boldsymbol{w}) = q \int_0^1 u^{q-1} \phi_p(\boldsymbol{w}; \boldsymbol{\mu}, u^{-2}\boldsymbol{\Sigma})\, \mathrm{d}u$, where $\phi_p(\boldsymbol{y}; \boldsymbol{\xi}, \boldsymbol{\Delta})$ is the PDF of $\boldsymbol{Y} \sim \mathrm{N}_p(\boldsymbol{\xi}, \boldsymbol{\Delta})$, and $\mathrm{P}(\boldsymbol{W} \in R) = q \int_0^1 u^{q-1} \Phi_p(R; \boldsymbol{\mu}, u^{-2}\boldsymbol{\Sigma})\, \mathrm{d}u$, where $\Phi_p(R; \boldsymbol{\xi}, \boldsymbol{\Delta})$ is the probability of $\{\boldsymbol{Y} \in R\}$ under $\boldsymbol{Y} \sim \mathrm{N}_p(\boldsymbol{\xi}, \boldsymbol{\Delta})$. This expression for $\mathrm{P}(\boldsymbol{W} \in R)$ is obtained by the Fubini theorem. Then, the PDF of $\boldsymbol{X} \sim \mathrm{TSL}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, q; R)$ can be expressed as

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{\mathrm{E}_q(\phi_p(\boldsymbol{x}; \boldsymbol{\mu}, U^{-2}\boldsymbol{\Sigma}))}{\mathrm{E}_q(\Phi_p(R; \boldsymbol{\mu}, U^{-2}\boldsymbol{\Sigma}))},$$

where the expected values are calculated under the random variable $U \sim \mathrm{beta}(q, 1)$. Therefore, by drawing a random sample of large size n from $U \sim \mathrm{beta}(q, 1)$, say $u_1, \ldots, u_n$, the PDF of $\boldsymbol{X} \sim \mathrm{TSL}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, q; R)$ can be approximated by

$$f_{\boldsymbol{X}}(\boldsymbol{x}) \approx \frac{\sum_{i=1}^n \phi_p(\boldsymbol{x}; \boldsymbol{\mu}, u_i^{-2}\boldsymbol{\Sigma})}{\sum_{i=1}^n \Phi_p(R; \boldsymbol{\mu}, u_i^{-2}\boldsymbol{\Sigma})},$$

where $\Phi_p(R; \boldsymbol{\mu}, u_i^{-2}\boldsymbol{\Sigma})$, $i = 1, \ldots, n$, is evaluated using the algorithm proposed by Genz and Bretz [6].

Figure 1 shows contour plots and PDF plots of bivariate truncated slash distributions. The contours are ellipses projected on the support set. The parameter σ_{12} controls the association between the marginal distributions of X_1 and X_2 (see Fig. 1b, c, and d for null, negative, and positive association, respectively). As the tail parameter q grows, the contours tend to the corresponding contours of the bivariate truncated normal distribution (Fig. 1e, f). The tails of the truncated slash distributions are heavier for smaller values of q.

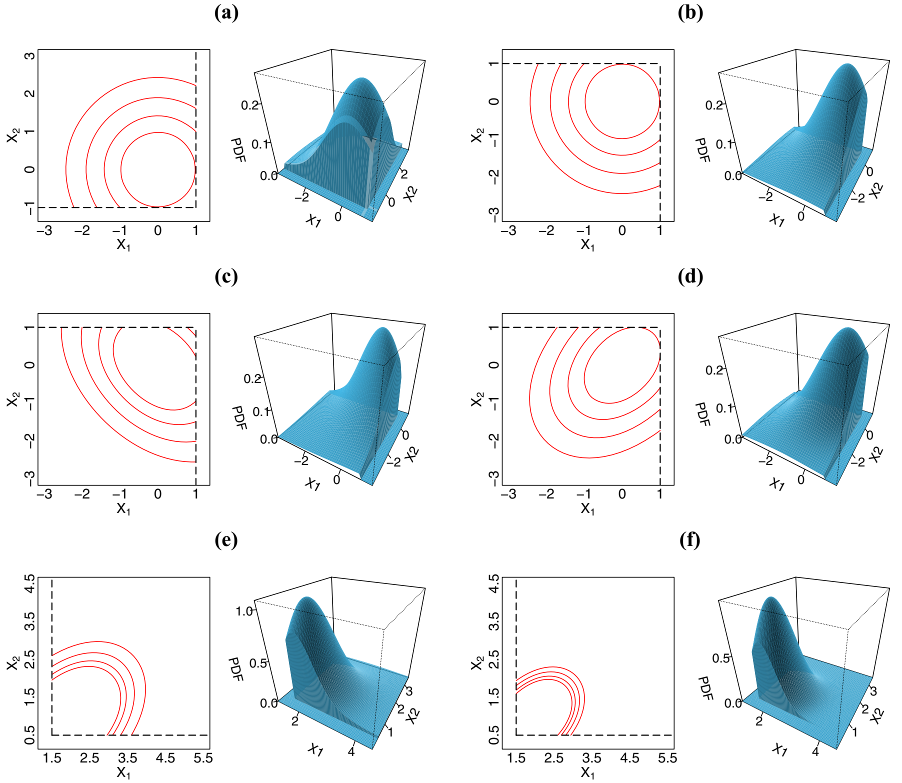

Fig. 1 Contour plots at levels 0.2, 0.15, 0.1, 0.06 and PDF of $\boldsymbol{X} \sim \mathrm{TSL}_2(\boldsymbol{\mu}, \boldsymbol{\Sigma}, q; R)$, where **a** $\mu_1 = \mu_2 = 0$, $\sigma_{11} = \sigma_{22} = 1$, $\sigma_{12} = 0$, $q = 3$, $R = (-\infty, 1) \times (-1, \infty)$, **b** $\mu_1 = \mu_2 = 0$, $\sigma_{11} = \sigma_{22} = 1$, $\sigma_{12} = 0$, $q = 3$, $R = (-\infty, 1) \times (-\infty, 1)$, **c** $\mu_1 = \mu_2 = 0$, $\sigma_{11} = \sigma_{22} = 1$, $\sigma_{12} = -0.36$, $q = 3$, $R = (-\infty, 1) \times (-\infty, 1)$, **d** $\mu_1 = \mu_2 = 0$, $\sigma_{11} = \sigma_{22} = 1$, $\sigma_{12} = 0.36$, $q = 3$, $R = (-\infty, 1) \times (-\infty, 1)$, **e** $\mu_1 = 2$, $\mu_2 = 1$, $\sigma_{11} = \sigma_{22} = 0.25$, $\sigma_{12} = 0.09$, $q = 3$, $R = (1.5, \infty) \times (0.5, \infty)$, **f** $\mu_1 = 2$, $\mu_2 = 1$, $\sigma_{11} = \sigma_{22} = 0.25$, $\sigma_{12} = 0.09$, $q = 12$, $R = (1.5, \infty) \times (0.5, \infty)$

3 Main Results

Let $\mathcal{O}(p)$ be the orthogonal group of $p \times p$ matrices with entries in $\mathbb{R}$, i.e., $\mathcal{O}(p) = \left\{\boldsymbol{H}(p \times p) : \boldsymbol{H}\boldsymbol{H}' = \boldsymbol{H}'\boldsymbol{H} = I_p\right\}$. In Theorem 3.1 we state a property of closedness of the truncated spherical distributions under orthogonal transformations.

Theorem 3.1 *Let $\boldsymbol{S} \sim TS_p(B; g)$ and $T : \mathbb{R}^p \to \mathbb{R}^p$ be the orthogonal transformation $T(\boldsymbol{x}) = \boldsymbol{H}\boldsymbol{x}$, where $\boldsymbol{H} \in \mathcal{O}(p)$. Then $T(\boldsymbol{S}) \sim TS_p(T(B); g)$. The PDF of $\boldsymbol{S}$ satisfies the relation $f_{\boldsymbol{S}}(\boldsymbol{s}) = f_{T(\boldsymbol{S})}(T(\boldsymbol{s}))$, $\boldsymbol{s} \in B$.*

Proof See “Appendix A”. □

If $B = \mathbb{R}^p$ in Theorem 3.1 we have that $\boldsymbol{S}$ and $T(\boldsymbol{S})$ have exactly the same spherical distribution. In other words, the spherical distributions are invariant under orthogonal transformations; Fang et al. [5, Sec. 2.1]. On the other hand, if B is a proper subset of $\mathbb{R}^p$, the distributions of $\boldsymbol{S}$ and $T(\boldsymbol{S})$ are truncated spherical with the same DGF, but the support of $T(\boldsymbol{S})$ is $T(B)$. Figure 1b shows a 90° counterclockwise rotation of the truncated bivariate standard slash distribution shown in Fig. 1a. This orthogonal rotation is obtained using the orthogonal transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $T((x_1, x_2)') = (-x_2, x_1)'$, with associated rotation matrix

$$\boldsymbol{H} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Theorem 3.2 gives a characterization of truncated elliptical distributions through truncated spherical distributions.

Theorem 3.2 *Let $B \subseteq \mathbb{R}^p$ be a measurable set, $\boldsymbol{\mu} \in \mathbb{R}^p$, $\boldsymbol{\Sigma}(p \times p) > 0$ and $T : \mathbb{R}^p \rightarrow \mathbb{R}^p$ be the affine transformation $T(\boldsymbol{x}) = \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2}\boldsymbol{x}$. Then, $\boldsymbol{S} \sim TS_p(B; g)$ if, and only if, $T(\boldsymbol{S}) \sim TE\ell_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; T(B); g)$.*

Proof See "Appendix B". □

Theorem 3.2 with $B = \mathbb{R}^p$ corresponds to the characterization of elliptical distributions from spherical distributions as described in Sect. 2.

Theorem 3.3 gives a closedness property of truncated elliptical distributions under affine transformations.

Theorem 3.3 *Let $B \subseteq \mathbb{R}^p$ be a measurable set and let $T : \mathbb{R}^p \rightarrow \mathbb{R}^p$ be the affine transformation $T(\boldsymbol{x}) = \boldsymbol{\alpha} + \boldsymbol{\Delta}\boldsymbol{x}$, where $\boldsymbol{\alpha} \in \mathbb{R}^p$ and $\boldsymbol{\Delta}(p \times p)$ is a matrix such that $\det(\boldsymbol{\Delta}) \neq 0$. If $\boldsymbol{X} \sim TE\ell_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; B; g)$, then $T(\boldsymbol{X}) \sim TE\ell_p(\boldsymbol{\alpha} + \boldsymbol{\Delta}\boldsymbol{\mu}, \boldsymbol{\Delta}\boldsymbol{\Sigma}\boldsymbol{\Delta}'; T(B); g)$.*

Proof See "Appendix C". □

Figure 1e shows the affine transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of the truncated bivariate slash distribution shown in Fig. 1d, with $T(\boldsymbol{x}) = \boldsymbol{\alpha} + \boldsymbol{\Delta}\boldsymbol{x}$, where

$$\boldsymbol{\alpha} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{\Delta} = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix}.$$

In Theorem 3.4 we state a condition for a truncated elliptical distribution to have MGF.

Theorem 3.4 *Let $B \subseteq \mathbb{R}^p$ be a measurable set, $\boldsymbol{\mu} \in \mathbb{R}^p$ and $\boldsymbol{\Sigma}(p \times p) > 0$. If the MGF of $\boldsymbol{W} \sim E\ell_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; g)$ exists, so does the MGF of $\boldsymbol{X} \sim TE\ell_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; B; g)$.*

Proof See "Appendix D". □

Log-concave PDFs have desirable properties and play an important role in statistics. A PDF $f : \mathbb{R}^p \rightarrow [0, \infty)$ is log-concave if

$$f(\alpha\boldsymbol{x} + (1 - \alpha)\boldsymbol{y}) \geq [f(\boldsymbol{x})]^{\alpha}[f(\boldsymbol{y})]^{1-\alpha} \tag{3.1}$$

is satisfied for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^p$ and for all $\alpha \in [0, 1]$. Some properties of log-concave PDFs are as follows: (i) if $\boldsymbol{X} \in \mathbb{R}^p$ is a random vector having a log-concave PDF $f_{\boldsymbol{X}}$, then the contours of f_X are convex sets and the marginal PDFs are log-concave; (ii) the product of log-concave PDFs is also log-concave; (iii) f_X is A-unimodal, i.e., the set $\{\boldsymbol{x} \in \mathbb{R}^p : f_X(\boldsymbol{x}) \geq \lambda\}$ is convex for all $\lambda > 0$; (iv) if $A_1, \ldots, A_m$ are subsets of $\mathbb{R}^p$ and $\alpha_1, \ldots, \alpha_m$ are real numbers such that $\alpha_i \geq 0$, $i = 1, \ldots, m$ and $\sum_{i=1}^m \alpha_i = 1$, then

$$\mathrm{P}\left[\boldsymbol{X} \in \sum_{i=1}^{m} \alpha_i A_i\right] \geq \prod_{i=1}^{m} [\mathrm{P}(\boldsymbol{X} \in A_i)]^{\alpha_i}, \tag{3.2}$$

where

$$\sum_{i=1}^{m} \alpha_i A_i = \left\{\boldsymbol{z} \in \mathbb{R}^p : \boldsymbol{z} = \sum_{i=1}^{m} \alpha_i \boldsymbol{x}_i,\ \boldsymbol{x}_i \in A_i,\ i = 1, \ldots, m\right\}.$$

Bounds for probabilities involving random vectors having log-concave PDFs may be found from inequality (3.2); see Tong [21, Sec. 4.2].

Theorem 3.5 presents conditions for a truncated elliptical distribution to have a log-concave PDF.

Theorem 3.5 *Let $\boldsymbol{\mu} \in \mathbb{R}^p$, $\boldsymbol{\Sigma}(p \times p) > 0$ and $B \subseteq \mathbb{R}^p$ measurable convex set. If the PDF f_W of $\boldsymbol{W} \sim E\ell_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; g)$ is log-concave, so does the PDF f_X of $\boldsymbol{X} \sim TE\ell_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; B; g)$.*

Proof See "Appendix E". □

The PDF f_X of a random vector having multivariate truncated power exponential distribution, $\boldsymbol{X} \sim \mathrm{TPE}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \beta; B)$, with $B \subseteq \mathbb{R}^p$ being convex, is log-concave for $\beta \geq 1$. This comes from the log-concavity of the PDF f_W of $\boldsymbol{W} \sim \mathrm{PE}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \beta)$ when $\beta \geq 1$ (see details in Appendix F). In particular, if $\beta = 1$, then the PDF f_X of $\boldsymbol{X} \sim \mathrm{TN}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; B)$ is log-concave. This generalizes Theorem 9 of Horrace [13], who shows the log-concavity of the PDF of the multivariate truncated normal distribution with support in the one-sided rectangle $\mathbb{R}^p_{\geq \boldsymbol{\alpha}} = [\alpha_1, \infty) \times \ldots \times [\alpha_p, \infty)$, where $\boldsymbol{\alpha} \in \mathbb{R}^p$.

A PDF $f : \mathbb{R}^p \to [0, \infty)$ is said to be multivariate totally positive of order 2 (MTP_2) if the inequality

$$f(\boldsymbol{y}) f(\boldsymbol{y}^*) \leq f(\boldsymbol{x}) f(\boldsymbol{x}^*) \tag{3.3}$$

holds for all $\boldsymbol{y}$, $\boldsymbol{y}^*$ in the domain of f, where $x_i = \max\{y_i, y_i^*\}$, $x_i^* = \min\{y_i, y_i^*\}$, $i = 1, \ldots, p$; see Tong [21, Sec. 4.3].

In Theorem 3.6 we state conditions for a random vector with a truncated elliptical distribution to have a PDF with the MTP_2 property.

Theorem 3.6 *Let $\boldsymbol{\mu} \in \mathbb{R}^p$, $\boldsymbol{\Sigma}(p \times p) > 0$ and R be a rectangle in $\mathbb{R}^p$. If the PDF f_W of $\boldsymbol{W} \sim E\ell_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; g)$ is MTP_2, so does the PDF f_X of $\boldsymbol{X} \sim TE\ell_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; R; g)$.*

Proof See "Appendix G". □

If the PDF of $\boldsymbol{W} \sim \mathrm{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is MTP_2 (see conditions in Tong [21, Sec. 4.3]), then the PDF of $\boldsymbol{X} \sim \mathrm{TN}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; R)$ is MTP_2, where R is a rectangle in $\mathbb{R}^p$. This generalizes Theorem 16 of Horrace [13], who shows the MTP_2 property of the PDF of the multivariate truncated normal distribution with support in the one-side rectangle $\mathbb{R}^p_{\geq \boldsymbol{\alpha}}$. Theorem 3.6 does not necessarily hold for arbitrary support sets in $\mathbb{R}^p$. For example, if $B \subseteq \mathbb{R}^2$ is the unit ball centered at the origin and $\boldsymbol{W} \sim \mathrm{N}_2(\mathbf{0}, \boldsymbol{\Sigma})$ has PDF with MTP_2 property, then the PDF $f_{\boldsymbol{X}}$ of $\boldsymbol{X} \sim \mathrm{TN}_2(\mathbf{0}, \boldsymbol{\Sigma}; B)$ is such that $f_{\boldsymbol{X}}(\boldsymbol{y}) f_{\boldsymbol{X}}(\boldsymbol{y}^*) > 0$ for $\boldsymbol{y} = (0.8, 0.5)'$ and $\boldsymbol{y}^* = (-0.5, 0.8)'$, but $f_{\boldsymbol{X}}(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x}^*) = 0$, since $\boldsymbol{x} = (0.8, 0.8)' \notin B$ and $\boldsymbol{x}^* = (-0.5, 0.5)' \in B$.

In order to present results on independence and marginal and conditional distributions, we introduce the notation for partitions of $\boldsymbol{X} \in \mathbb{R}^p$, $\boldsymbol{\mu} \in \mathbb{R}^p$ and $\boldsymbol{\Sigma}(p \times p) > 0$ as follows:

$$\boldsymbol{X} = (\boldsymbol{X}_1', \boldsymbol{X}_2')', \quad \boldsymbol{\mu} = (\boldsymbol{\mu}_1', \boldsymbol{\mu}_2')', \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}, \tag{3.4}$$

where $\boldsymbol{X}_1 \in \mathbb{R}^r$, $\boldsymbol{X}_2 \in \mathbb{R}^{p-r}$, $\boldsymbol{\mu}_1 \in \mathbb{R}^r$, $\boldsymbol{\mu}_2 \in \mathbb{R}^{p-r}$, $\boldsymbol{\Sigma}_{11}(r \times r) > 0$, $\boldsymbol{\Sigma}_{22}((p - r) \times (p - r)) > 0$ and $\boldsymbol{\Sigma}_{12}(r \times (p - r))$ is such that $\boldsymbol{\Sigma}_{21} = \boldsymbol{\Sigma}_{12}'$. The support set is assumed to be the rectangle $R \subseteq \mathbb{R}^p$, that may be expressed as the Cartesian product of rectangles $R_1 = I_1 \times \cdots \times I_r \subseteq \mathbb{R}^r$ and $R_2 = I_{r+1} \times \cdots \times I_p \subseteq \mathbb{R}^{p-r}$, i.e.,

$$R = R_1 \times R_2. \tag{3.5}$$

Let $\boldsymbol{X} \in \mathbb{R}^p$, $\boldsymbol{\mu} \in \mathbb{R}^p$, $\boldsymbol{\Sigma}(p \times p) > 0$ be partitioned as in (3.4) and be such that $\boldsymbol{X} \sim \mathrm{TE}\ell_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; R; g)$, with R given in (3.5). The marginal PDF of $\boldsymbol{X}_1$ is given by

$$f_{\boldsymbol{X}_1}(\boldsymbol{x}_1) = \frac{\int_{R_2} g((\boldsymbol{x} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu}))\, \mathrm{d}\boldsymbol{x}_2}{\int_R g((\boldsymbol{x} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu}))\, \mathrm{d}\boldsymbol{x}}, \quad \boldsymbol{x}_1 \in R_1. \tag{3.6}$$

Note that the marginal PDF does not necessarily have the structure of the PDF of a truncated elliptical distribution given in (2.3). Theorem 3.7 gives a condition for marginal distributions to be truncated elliptical.

Theorem 3.7 *Let $\boldsymbol{X} \in R$, $\boldsymbol{\mu} \in \mathbb{R}^p$, $\boldsymbol{\Sigma}(p \times p) > 0$ be partitioned as in (3.4) and such that $\boldsymbol{X} \sim TE\ell_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; R; g)$, with R given in (3.5). If $\boldsymbol{\Sigma}_{12} = \mathbf{0}$, then $\boldsymbol{X}_1 \sim TE\ell_r(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11}; R_1; g_1)$, where*

$$g_1(u) = \int_{T(R_2)} g(u + \boldsymbol{s}'\boldsymbol{s})\, \mathrm{d}\boldsymbol{s}, \quad u \geq 0,$$

with $T : \mathbb{R}^{p-r} \to \mathbb{R}^{p-r}$ being the transformation $T(\boldsymbol{x}) = \boldsymbol{\Sigma}_{22}^{-1/2}(\boldsymbol{x} - \boldsymbol{\mu}_2)$ and g_1 satisfies $\int_0^\infty t^{r-1} g_1(t^2)\, \mathrm{d}t < \infty$.

Proof See "Appendix H". □

In Theorem 3.8 we present the conditional distribution of $\boldsymbol{X}_1|\boldsymbol{X}_2$ for a random vector $\boldsymbol{X} = (\boldsymbol{X}_1', \boldsymbol{X}_2')'$ with truncated elliptical distribution. This conditional distribution is a truncated elliptical distribution.

Theorem 3.8 *Let* $\boldsymbol{X} \in R$, $\boldsymbol{\mu} \in \mathbb{R}^p$, $\boldsymbol{\Sigma}(p \times p) > 0$ *be partitioned as in* (3.4) *and such that* $\boldsymbol{X} \sim TE\ell_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; R; g)$, *with* R *given in* (3.5). *Then,* $\boldsymbol{X}_1|\boldsymbol{X}_2 = \boldsymbol{x}_2 \sim TE\ell_r(\boldsymbol{\mu}_1(\boldsymbol{x}_2), \boldsymbol{\Sigma}_{11.2}; R_1; g_{q(\boldsymbol{x}_2)})$, *where* $\boldsymbol{\mu}_1(\boldsymbol{x}_2) = \boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{x}_2 - \boldsymbol{\mu}_2)$, $\boldsymbol{\Sigma}_{11.2} = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}$ *and* $g_{q(\boldsymbol{x}_2)}(u) = g(u + q(\boldsymbol{x}_2))$, $u \geq 0$, *with* $q(\boldsymbol{x}_2) = (\boldsymbol{x}_2 - \boldsymbol{\mu}_2)'\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{x}_2 - \boldsymbol{\mu}_2)$.

Proof See "Appendix I". □

Theorem 3.8 is a generalization of Theorem 1 of Morán-Vásquez and Ferrari [16], which states that if $\boldsymbol{X} \sim \mathrm{TE}\ell_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; R; g)$, then $X_k|\boldsymbol{X}_{-k} \sim \mathrm{TE}\ell_1(\mu_{k.-k}, \sigma_{k.-k}^2; I_k; g_{k.-k})$, $k = 1, \ldots, p$, where $\mu_{k.-k} = \mu_k + \boldsymbol{\Sigma}_{k,-k}\boldsymbol{\Sigma}_{-k,-k}^{-1}(\boldsymbol{x}_{-k} - \boldsymbol{\mu}_{-k})$, $\sigma_{k.-k}^2 = \sigma_{kk} - \boldsymbol{\Sigma}_{k,-k}\boldsymbol{\Sigma}_{-k,-k}^{-1}\boldsymbol{\Sigma}_{-k,k}$ and $g_{k.-k}(u) = g(u + q(\boldsymbol{x}_{-k}))$, with $q(\boldsymbol{x}_{-k}) = (\boldsymbol{x}_{-k} - \boldsymbol{\mu}_{-k})'\boldsymbol{\Sigma}_{-k,-k}^{-1}(\boldsymbol{x}_{-k} - \boldsymbol{\mu}_{-k})$. This result allows to obtain the complete conditional distributions necessary to use Gibbs sampling for generating random samples of truncated elliptical distributions; see Morán-Vásquez and Ferrari [16]. For example, random samples of $\boldsymbol{X} \sim \mathrm{TSL}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, q; R)$ and $\boldsymbol{X} \sim \mathrm{TPE}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \beta; R)$ may be generated following Algorithm 1 proposed by Morán-Vásquez and Ferrari [16] with $g_{k.-k}(u) \propto \int_0^1 t^{q+p-1} \exp(-(u + q(\boldsymbol{x}_{-k}))t^2/2)\,\mathrm{d}t$ and $g_{k.-k}(u) \propto \exp(-(u + q(\boldsymbol{x}_{-k}))^{\beta}/2)$ as the DGF of $X_k|\boldsymbol{X}_{-k}$, $k = 1, \ldots, p$, respectively.

When the precision matrix $\boldsymbol{\Lambda} = \boldsymbol{\Sigma}^{-1}$ is readily available, Algorithm 1 in Morán-Vásquez and Ferrari [16] may be simplified by writing $\mu_{k.-k} = \mu_k - \boldsymbol{\Lambda}_{k,-k}(\boldsymbol{x}_{-k} - \boldsymbol{\mu}_{-k})/\lambda_{kk}$, $\sigma_{k.-k}^2 = 1/\lambda_{kk}$ and $q(\boldsymbol{x}_{-k}) = (\boldsymbol{x}_{-k} - \boldsymbol{\mu}_{-k})'\boldsymbol{\Lambda}_{-k,-k}(\boldsymbol{x}_{-k} - \boldsymbol{\mu}_{-k})$; Geweke [8, Th. 5.3.1]. This approach avoids multiple matrix inversions and is particularly advantageous when $\boldsymbol{\Sigma}$ is close to a singular matrix.

If $\boldsymbol{\Sigma}_{12} = \boldsymbol{0}$ in Theorem 3.8, then $\boldsymbol{X}_1|\boldsymbol{X}_2 = \boldsymbol{x}_2 \sim \mathrm{TE}\ell_r(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11}; R_1; g_{q(\boldsymbol{x}_2)})$. A comparison of this conditional distribution with the marginal distribution of $\boldsymbol{X}_1$ given in Theorem 3.7 reveals that if $\boldsymbol{\Sigma}_{12} = \boldsymbol{0}$ and the DGFs $g_{q(\boldsymbol{x}_2)}$ and g_1 coincide, then $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are independent. This gives a characterization of the independence of $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$, as stated in Theorem 3.9.

Theorem 3.9 *Let* $\boldsymbol{X} \in \mathbb{R}^p$, $\boldsymbol{\mu} \in \mathbb{R}^p$, $\boldsymbol{\Sigma}(p \times p) > 0$ *be partitioned as in* (3.4) *and such that* $\boldsymbol{X} \sim TE\ell_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; R; g)$, *with* R *given in* (3.5). *Then,* $\boldsymbol{X}_1$ *and* $\boldsymbol{X}_2$ *are independent if, and only if,* $\boldsymbol{X} \sim TN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; R)$ *and* $\boldsymbol{\Sigma}_{12} = \boldsymbol{0}$.

Proof See "Appendix J". □

4 Final Remarks

In this paper we stated several properties of truncated elliptical distributions, which is a generalization of the elliptical distributions. The class of truncated elliptical distributions has as special cases the multivariate truncated normal and multivariate truncated t distributions, among others. Our results provide new properties for the multivariate truncated slash distribution and the multivariate truncated power exponential distribution. The stated properties regard a characterization of the truncated elliptical distributions through the truncated spherical distributions, distributional properties obtained through transformations, MGF, log-concavity, MTP_2 property, marginal and conditional distributions, and independence.

We proposed a procedure to compute the PDF of the multivariate truncated slash distribution by mixing Monte Carlo integration and the algorithm proposed by Genz and Bretz [6] to evaluate probabilities on rectangles through the multivariate normal distribution. This procedure will allow the implementation of maximum likelihood estimation for the multivariate Box–Cox slash distribution, which is a member of the class of Box–Cox elliptical distributions proposed by Morán-Vásquez and Ferrari [16]. The multivariate Box–Cox slash distribution is suitable for modeling correlated multivariate positive data that are skewed and possibly heavy-tailed. Implementation and applications of multivariate Box–Cox slash models will be dealt with in a future paper.

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Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interests regarding the publication of this paper.

Appendix

A Proof of Theorem 3.1

Let $\boldsymbol{U} = T(\boldsymbol{s}) = \boldsymbol{H}\boldsymbol{S}$, with Jacobian $J(\boldsymbol{s} \to \boldsymbol{u}) = \pm 1$. From (2.5) and $\boldsymbol{H}'\boldsymbol{H} = \boldsymbol{I}_p$, we have

$$f_{\boldsymbol{S}}(\boldsymbol{s}) = \frac{g(\boldsymbol{u}'\boldsymbol{u})}{\int_{T(B)} g(\boldsymbol{u}'\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}}, \quad \boldsymbol{u} = T(\boldsymbol{s}), \quad \boldsymbol{s} \in B.$$

This proves the result.

B Proof of Theorem 3.2

Let $\boldsymbol{X} = T(\boldsymbol{S}) = \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2}\boldsymbol{S}$, with Jacobian $J(\boldsymbol{s} \to \boldsymbol{x}) = \det(\boldsymbol{\Sigma})^{-1/2}$. From (2.5) and by noting that $\boldsymbol{s}'\boldsymbol{s} = (\boldsymbol{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu})$, we have that the PDF of $\boldsymbol{X}$ is

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{g((\boldsymbol{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu}))}{\int_{T(B)} g((\boldsymbol{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu}))\,\mathrm{d}\boldsymbol{x}}, \quad \boldsymbol{x} \in T(B). \tag{B.1}$$

Hence, $\boldsymbol{X} \sim \mathrm{TE}\ell_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; T(B); g)$. The result follows by considering the transformation $\boldsymbol{S} = T^{-1}(\boldsymbol{X}) = \boldsymbol{\Sigma}^{-1/2}(\boldsymbol{X} - \boldsymbol{\mu})$ in (B.1), with Jacobian $J(\boldsymbol{x} \to \boldsymbol{s}) = \det(\boldsymbol{\Sigma})^{1/2}$, and by noting that $(\boldsymbol{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu}) = \boldsymbol{s}'\boldsymbol{s}$.

C Proof of Theorem 3.3

Let $\boldsymbol{Y} = T(\boldsymbol{X}) = \boldsymbol{\alpha} + \boldsymbol{\Delta X}$, with Jacobian $J(\boldsymbol{x} \to \boldsymbol{y}) = \det(\Delta)^{-1}$. From (2.3), and by noting that

$$(\boldsymbol{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu}) = (\boldsymbol{y} - (\boldsymbol{\alpha} + \boldsymbol{\Delta\mu}))'(\boldsymbol{\Delta\Sigma\Delta}')^{-1}(\boldsymbol{y} - (\boldsymbol{\alpha} + \boldsymbol{\Delta\mu})),$$

the result follows.

D Proof of Theorem 3.4

Let $\boldsymbol{t} = (t_1, \ldots, t_p)' \in \mathbb{R}^p$ be such that $|t_i| < h$, $i = 1, \ldots, p$, for some $h > 0$. Note that

$$\begin{aligned} M_{\boldsymbol{X}}(\boldsymbol{t}) &= [\mathrm{P}(\boldsymbol{W} \in B)]^{-1} \int_B \exp(\boldsymbol{t}'\boldsymbol{x}) f_{\boldsymbol{W}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &\leq [\mathrm{P}(\boldsymbol{W} \in B)]^{-1} \int_{\mathbb{R}^p} \exp(\boldsymbol{t}'\boldsymbol{x}) f_{\boldsymbol{W}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &= [\mathrm{P}(\boldsymbol{W} \in B)]^{-1} M_{\boldsymbol{W}}(\boldsymbol{t}). \end{aligned}$$

Since $M_{\boldsymbol{W}}(\boldsymbol{t}) < \infty$, we have that $M_{\boldsymbol{X}}(\boldsymbol{t}) < \infty$.

E Proof of Theorem 3.5

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^p$. We will prove that inequality (3.1) holds for $f_{\boldsymbol{X}}$ given in (2.4). If $\boldsymbol{x} \notin B$ or $\boldsymbol{y} \notin B$, we have that $[f_{\boldsymbol{X}}(\boldsymbol{x})]^{\alpha}[f_{\boldsymbol{X}}(\boldsymbol{y})]^{1-\alpha} = 0$, for all $\alpha \in [0, 1]$. Hence, (3.1) holds, because $f_{\boldsymbol{X}}(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \geq 0$. If $\boldsymbol{x}, \boldsymbol{y} \in B$, then, from the convexity of B, it follows that $\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} \in B$, for all $\alpha \in [0, 1]$. Hence, $f_{\boldsymbol{X}}(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) > 0$ and, therefore, $f_{\boldsymbol{W}}(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) > 0$. Since $f_{\boldsymbol{W}}$ is log-concave, then

$$\frac{f_{\boldsymbol{W}}(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y})}{\mathrm{P}(\boldsymbol{W} \in B)} \geq \frac{[f_{\boldsymbol{W}}(\boldsymbol{x})]^{\alpha}[f_{\boldsymbol{W}}(\boldsymbol{y})]^{1-\alpha}}{\mathrm{P}(\boldsymbol{W} \in B)}, \quad \alpha \in [0, 1].$$

Equation (3.1) is seen to be satisfied by noting that $\mathrm{P}(\boldsymbol{W} \in B) = [\mathrm{P}(\boldsymbol{W} \in B)]^{\alpha}[\mathrm{P}(\boldsymbol{W} \in B)]^{1-\alpha}$, for all $\alpha \in [0, 1]$.

F Log-Concavity of the Power Exponential Distribution

The PDF of $\boldsymbol{W} \sim \mathrm{PE}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \beta)$ is given by

$$f_{\boldsymbol{W}}(\boldsymbol{w}) = \frac{p\Gamma(p/2)\det(\boldsymbol{\Sigma})^{-1/2}}{2^{1+p/2\beta}\pi^{p/2}\Gamma(1+p/2\beta)} \exp\left\{-\frac{1}{2}[(\boldsymbol{w}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{w}-\boldsymbol{\mu})]^{\beta}\right\}, \quad \boldsymbol{w} \in \mathbb{R}^p. \tag{F.1}$$

Since the function $u(t) = -t^{\beta}/2$ is concave and non-increasing on $(0, \infty)$ for $\beta \geq 1$, and the function $v(\boldsymbol{w}) = (\boldsymbol{w}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{w}-\boldsymbol{\mu})$ is convex on $\mathbb{R}^p$, then the composition function $h(\boldsymbol{w}) = u(v(\boldsymbol{w})) = -[(\boldsymbol{w}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{w}-\boldsymbol{\mu})]^{\beta}/2$ is concave on $\mathbb{R}^p$ for $\beta \geq 1$; Boyd and Vandenberghe [4, Sec. 3.2.4]. This implies that the PDF (F.1) is log-concave on $\mathbb{R}^p$ for $\beta \geq 1$.

G Proof of Theorem 3.6

Let $\boldsymbol{y}, \boldsymbol{y}^* \in \mathbb{R}^p$. We will prove that the inequality (3.3) holds for $f_{\boldsymbol{X}}$ given in (2.4). If $\boldsymbol{y} \notin R$ or $\boldsymbol{y}^* \notin R$, then $f_{\boldsymbol{X}}(\boldsymbol{y})f_{\boldsymbol{X}}(\boldsymbol{y}^*) = 0$. Also, $f_{\boldsymbol{X}}(\boldsymbol{x})f_{\boldsymbol{X}}(\boldsymbol{x}^*) = 0$, with $x_i = \max\{y_i, y_i^*\}$, $x_i^* = \min\{y_i, y_i^*\}$, $i = 1, \ldots, p$. Hence, (3.3) is satisfied. If $\boldsymbol{y}, \boldsymbol{y}^* \in R$, then $f_{\boldsymbol{X}}(\boldsymbol{y})f_{\boldsymbol{X}}(\boldsymbol{y}^*) > 0$ and, hence, $f_{\boldsymbol{W}}(\boldsymbol{y})f_{\boldsymbol{W}}(\boldsymbol{y}^*) > 0$. In this case, $\boldsymbol{x}, \boldsymbol{x}^* \in R$. Hence, $f_{\boldsymbol{X}}(\boldsymbol{x})f_{\boldsymbol{X}}(\boldsymbol{x}^*) > 0$ and, consequently, $f_{\boldsymbol{W}}(\boldsymbol{x})f_{\boldsymbol{W}}(\boldsymbol{x}^*) > 0$. Since $f_{\boldsymbol{W}}$ is MTP_2, then

$$f_{\boldsymbol{W}}(\boldsymbol{y})f_{\boldsymbol{W}}(\boldsymbol{y}^*) \leq f_{\boldsymbol{W}}(\boldsymbol{x})f_{\boldsymbol{W}}(\boldsymbol{x}^*).$$

By dividing each side of the inequality above by $[\mathrm{P}(\boldsymbol{W} \in R)]^2$, we have that $f_{\boldsymbol{X}}(\boldsymbol{y})f_{\boldsymbol{X}}(\boldsymbol{y}^*) \leq f_{\boldsymbol{X}}(\boldsymbol{x})f_{\boldsymbol{X}}(\boldsymbol{x}^*)$.

H Proof of Theorem 3.7

From (3.6) with $\boldsymbol{\Sigma}_{12} = \boldsymbol{0}$, we have that the marginal PDF of $\boldsymbol{X}_1$ is

$$f_{\boldsymbol{X}_1}(\boldsymbol{x}_1) = \frac{\int_{R_2} g((\boldsymbol{x}_1-\boldsymbol{\mu}_1)'\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{x}_1-\boldsymbol{\mu}_1) + (\boldsymbol{x}_2-\boldsymbol{\mu}_2)'\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{x}_2-\boldsymbol{\mu}_2))\,\mathrm{d}\boldsymbol{x}_2}{\int_{R_1}\int_{R_2} g((\boldsymbol{x}_1-\boldsymbol{\mu}_1)'\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{x}_1-\boldsymbol{\mu}_1) + (\boldsymbol{x}_2-\boldsymbol{\mu}_2)'\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{x}_2-\boldsymbol{\mu}_2))\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}\boldsymbol{x}_1},$$

where $\boldsymbol{x}_1 \in R_1$. Let $\boldsymbol{s} = T(\boldsymbol{x}_2) = \boldsymbol{\Sigma}_{22}^{-1/2}(\boldsymbol{x}_2-\boldsymbol{\mu}_2)$, with Jacobian $J(\boldsymbol{x}_2 \to \boldsymbol{s}) = \det(\boldsymbol{\Sigma}_{22})^{1/2}$. It follows that

$$f_{\boldsymbol{X}_1}(\boldsymbol{x}_1) = \frac{g_1((\boldsymbol{x}_1-\boldsymbol{\mu}_1)'\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{x}_1-\boldsymbol{\mu}_1))}{\int_{R_1} g_1((\boldsymbol{x}_1-\boldsymbol{\mu}_1)'\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{x}_1-\boldsymbol{\mu}_1))\,\mathrm{d}\boldsymbol{x}_1}, \quad \boldsymbol{x}_1 \in R_1.$$

On the other hand, note that

$$g_1(u) = \int_{T(R_2)} g(u + \boldsymbol{s}'\boldsymbol{s})\,\mathrm{d}\boldsymbol{s} \le \int_{\mathbb{R}^{p-r}} g(u + \boldsymbol{s}'\boldsymbol{s})\,\mathrm{d}\boldsymbol{s} = h_1(u), \quad u \ge 0.$$

The function h_1 is such that $\int_0^\infty t^{r-1} h_1(t^2)\,\mathrm{d}t < \infty$; Fang et al. [5, Sec. 2.2].

I Proof of Theorem 3.8

The conditional PDF of $\boldsymbol{X}_1|\boldsymbol{X}_2$ is

$$f_{X_1|X_2}(\boldsymbol{x}_1) = \frac{g((\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}))}{\int_{R_1} g((\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}))\,\mathrm{d}\boldsymbol{x}_1}, \quad \boldsymbol{x}_1 \in R_1.$$

From the identity $(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}) = (\boldsymbol{x}_1 - \boldsymbol{\mu}_1(\boldsymbol{x}_2))'\boldsymbol{\Sigma}_{11.2}^{-1}(\boldsymbol{x}_1 - \boldsymbol{\mu}_1(\boldsymbol{x}_2)) + q(\boldsymbol{x}_2)$, the result follows.

J Proof of Theorem 3.9

$\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are independent if, and only if, the PDF of $\boldsymbol{X} \sim \mathrm{TE}\ell_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; R; g)$ is factored as $f_X(\boldsymbol{x}) = f_{X_1}(\boldsymbol{x}_1) f_{X_2}(\boldsymbol{x}_2)$. This equality holds if, and only if, $\boldsymbol{\Sigma}_{12} = \boldsymbol{0}$ and $g(u+v) = g(u)g(v)$, with $u \ge 0$ and $v \ge 0$. This functional equation has as solution the DGF $g(u) = \exp(-ku)$, for some $k \ge 0$; Gupta et al. [10, Sec. 1.3]. From $\int_0^\infty t^{p-1}\exp(-kt^2)\,\mathrm{d}t = 2^{p/2-1}\Gamma(p/2)$, we find that $k = 1/2$. Therefore, $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are independent if, and only if, $\boldsymbol{X} \sim \mathrm{TN}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; R)$, with $\boldsymbol{\Sigma}_{12} = \boldsymbol{0}$.

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