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LOCAL BANACH-SPACE DICHOTOMIES AND ERGODIC SPACES

W. CUELLAR CARRERA.[1], N. DE RANCOURT[2] & V. FERENCZI[1]

Instituto de Matemática e Estatìstica, USP, SP, Brasil[1], U. Lille, França[2]

cuellar@ime.usp.br† noe.derancourtdemimerandvernyuniv-lille.fr‡ ferenczi@ime.usp.br§

Abstract

We prove a local version of Gowers' Ramsey-type theorem [2], as well as local versions both of the Banach space first dichotomy (the "unconditional/HI" dichotomy) of Gowers [2] and of the third dichotomy (the "minimal/tight" dichotomy) due to Ferenczi–Rosendal [1]. As a consequence we obtain new information on the number of subspaces of non-Hilbertian Banach spaces, making some progress towards the "ergodic" conjecture of Ferenczi–Rosendal and towards a question of Johnson.

1 Introduction

A Banach space is said to be *homogeneous* if it is isomorphic to all of its (closed, infinite-dimensional) subspaces. A famous problem due to Banach, and known as the *homogeneous space problem*, asked whether, up to isomorphism, ℓ_2 is the only homogeneous Banach space. The answer turned out to be positive; this problem was eventually solved in the 1990's by a combination of results by Gowers–Maurey [3], Komorowski–Tomczak-Jaegermann [4], and Gowers [2].

The homogeneous space characterization of the Hilbert space shows that, as soon as a separable Banach space X is non-Hilbertian, it should have at least two non-isomorphic subspaces. Thus, the following general question was asked by Godefroy:

Question. (Godefroy) How many different subspaces, up to isomorphism, can a separable, non-Hilbertian Banach space have?

This question seems to be very difficult in general, although good lower bounds for several particular classes of spaces are now known. The seemingly simplest particular case of Godefroy's question is the following question by Johnson:

Question. (Johnson) Does there exist a separable Banach space having exactly two different subspaces, up to isomorphism?

Even this question is still open. More generally, it is not known whether there exist a separable, non-Hilbertian Banach space with at most countably many different subspaces, up to isomorphism.

It seems to be believed that such a space does not exist. In the rest of this paper, a separable Banach space having exactly two different subspaces, up to isomorphism, will be called a *Johnson space*.

2 Main Results

We study dichotomies associated to the *Hilbertian degree*, that is, the local degree defined by the Banach Mazur distance $d_{BM}(F, \ell_2^{\dim(F)})$, for which small spaces are exactly Hilbertian spaces. We shall denote this degree d_2:

$$d_2(F) = d_{BM}(F, \ell_2^{\dim(F)}).$$

To save notation, we say that an FDD $(F_n)_{n\in\mathbb{N}}$ of a Banach space X is *d-better* if $d(X, F_n) \xrightarrow[n\to\infty]{} \infty$. A Banach space X is a d_2-HI space if it contains no direct sum of two non-Hilbertian subspaces, and d_2-minimal if it embeds into all of its non-Hilbertian subspaces ("minimal among non-Hilbertian spaces"). An FDD is d_2-tight if all non-Hilbertian spaces are tight in it. In the case of the Hilbertian degree, our two dichotomies can be summarized as follows:

Theorem 2.1. *Let X be a non-Hilbertian Banach space. Then X has a non-Hilbertian subspace Y satisfying one of the following properties:*

(1) *Y is d_2-minimal and has a d_2-better UFDD;*

(2) *Y has a d_2-better d_2-tight UFDD;*

(3) *Y is d_2-minimal and d_2-hereditarily indecomposable;*

(4) *Y is d_2-tight and d_2-hereditarily indecomposable.*

It is clear from the definitions that if a Banach space X does not contain any isomorphic copy of ℓ_2, then the $d_2 - HI$ property is just the HI property and the d_2-minimality is just classical minimality. It is also easy to check that if X is not ℓ_2-saturated, then our two local dichotomies do not provide more information than the original ones.

For the ergodicity question we have the following consequence

Theorem 2.2. *Every non-ergodic, non-Hilbertian separable Banach space contains a d_2-minimal subspace.*

The results are part of the work: Wilson Cuellar Carrera, Noé de Rancourt, Valentin Ferenczi, *Local Banach-space dichotomies and ergodic spaces*. J. Eur. Math. Soc. (2022), https://doi.org/10.4171/jems/1257.

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