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Some aspects of Reidemeister fixed point theory, equivariant fixed point theory and coincidence theory

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Abstract

In this survey the authors present a brief description of their contribution to Nielsen fixed point theory. Aspects of Reidemeister theory, equivariant fixed point theory and coincidence theory are discussed.

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1 Introduction

This survey compiles the major part of the work developed by the authors (not as a group) on Nielsen fixed point theory in a broad sense. It is divided into four sections which may be read independently of one another. The main results are explicitly stated and, even though not much details are provided, we present all necessary references. The appendix, on the other hand, contains detailed alternative proofs of certain results on equivariant fixed point theory. For an extensive survey on the subject we refer to the Handbook of Topological Fixed Point Theory, see [10].

The well known Lefschetz Fixed Point Theorem states that for a finite simplicial complex K and a continous function $f : K \to K$ with $L(f)$, the Lefschetz number of f, different from zero, the existence of a fixed point of f is guaranteed.

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By a fixed point of f we mean a point $x \in K$ such that $f(x) = x$. Since the Lefschetz number is a homotopy invariant, we conclude that every map g homotopic to f will also have a fixed point.

In this context, it is natural to ask wheather a map $f : K \to K$ with $L(f) = 0$ can be deformed to a map with no fixed points. This result is not true in general and Nielsen in [44], introduced another homotopy invariant, known as the Nielsen number of f, $N(f)$, which counts the number of essencial fixed point classes.

In Nielsen Fixed Point Theory, a notion of a fixed point class is defined by saying that two fixed points x_0 and x_1 are in the same Nielsen class of f if there exists a path λ, from x_0 to x_1, such that $f(\lambda)$ is homotopic, as paths, to λ. An index of a Nielsen class is also defined and the Nielsen number is the number of classes with non zero index, the essencial ones. The Lefschetz number then represents the global index of the fixed point set and $N(f)$ is a lower bound to the minimal number of fixed points in the homotopy class of f.

A presentation of Nielsen Fixed Point Theory for maps defined in simplicial complexes can be found in [9, 39] and [41].

It is clear that $N(f) = 0$ implies $L(f) = 0$ and there are plenty of examples where $L(f) = 0$ but $N(f) \neq 0$. Therefore Nielsen number represents, a sharper homotopy invariant with respect to fixed points associated to a homotopy class of a map. The original question, namely, under what conditions we may have a converse of the Lefschetz Fixed Point Theorem, is then replaced by deciding wheather $N(f) = 0$ suffices to guarantee that f can be deformed to a fixed point free map.

Under this perspective, many other questions arise and we mention some of them focusing in those that are connected to the contribuitions related to the Algebraic Topology group of IME-USP.

1. Is it possible to stablish settings where the Nielsen number of a map f, $N(f)$, may represent exactly the minimum number of fixed points in the homotopy class of f? In case of a positive answer for all maps $f : X \to X$, we say that the space X satisfies the Wecken property. This question includes the original one when we assume $N(f) = 0$.
2. Can we provide answers to the original question when we look at specific homotopy classes of maps? A special case of interest is looking at deformations, i.e., the homotopy class of the identity map.
3. Can we provide ways of evaluating the Nielsen number?

Similar notions and questions may be asked in the coincidence context, that is, when we take a pair of maps $f, g : X \to Y$ and look at the coincidence set $C(f, g) = \{x \in X \mid f(x) = g(x)\}$.

Nevertheless it should be pointed out that to set up the correspondent coincidence theory, the degree of complexity is much higher, due to the fact that the domain and the target do not have to be the same and may have a quite different nature. So far, besides the case where the spaces involved are orientable manifolds of the same dimension n, with n bigger than or equal to three, there is not a well defined coincidence theory even when we assume that both spaces are finite complexes, let alone for the more general situation where the spaces are ENR's.

There is a handfull of works providing, under certain conditions, positive answers to these questions and, in most of them, the spaces involved are manifolds or finite simplicial complexes. In particular, question 1 has a positive answer for maps $f : M \to M$, where M is a compact triangulable manifold of dimension different from 2, see [51].

This survey is divided into four sections besides this introduction. In Sect. 2, connected with the question of evaluating Nielsen numbers, some results on Reidemeister Theory are presented. Section 3 is devoted to some aspects of equivariant fixed point theory. We turn our attention to coincidence theory in Sect. 4 . An appendix constitutes Sect. 5 where a proof of Theorem 3.1 using obstruction methods is presented.

2 On Reideimeister numbers for fixed points

The results and the example that will be presented in this section were taken from[11, 12] and [13] and their statements were copied almost verbatim from these references.

As we mentioned before, for a compact simplicial complex X, the Nielsen number of a map $f : X \to X$ is a homotopy invariant and a lower bound to the minumum number of fixed points in the homotopy class of f, and under certain conditions, these two numbers coincide. Therefore, one important question in Nielsen theory is to evaluate Nielsen numbers. One possible approach to evaluate the Nielsen number is through the Reidemeister number of f, $\mathrm{R}(f)$, defined as the number of equivalence classes, under conjugation, of liftings of f to the universal covering space of X. Introduced in 1936 by K. Reidemeister [45], it is an upper bound for the Nielsen number, $\mathrm{N}(f) \leq \mathrm{R}(f)$. B. Jiang (in [39]) defined a subgroup, $J(X)$, of the fundamental group of X, $\pi_1(X)$, known as the Jiang subgroup and studied the spaces for which $J(X) = \pi_1(X)$. Spaces with this property are called Jiang-spaces and for them all Nielsen fixed point classes have the same index. If X is a Jiang space and $L(f) \neq 0$, we have $\mathrm{R}(f) = \mathrm{N}(f)$. Since the Reidemeister number is easier to compute than the Nielsen number, it is an useful concept in fixed point theory.

In different settings such as maps of pairs or fiber maps, analogs of the Nielsen number are defined. Developed in works of Brown [9], Schirmer [47], Jiang [39] and Zhao [57], the relative Nielsen number and its related numbers, such as the Nielsen number on the complement, the Nielsen number of the closure, the Nielsen number of the triad, the surplus Nielsen number, among others, have made the study of fixed points more accurate, in the sense that those Nielsen numbers are better bounds for the respective minimum numbers of fixed points.

Initially, let us consider a pair of spaces (X, A), where X is a compact, connected polyhedron, $A \subset X$ is a finite subpolyhedron, not necessarily connected, and let $f : (X, A) \longrightarrow (X, A)$ be a map of the pair, that is $f : X \longrightarrow X$ such that $f(A) \subset A$.

By extending the concept of a universal covering space, using the conjugacy by deck transformations and the index of fixed point classes as defined by Jiang (see [39]), we classify and count the number of classes of liftings of a map (see [11]). This approach, more geometrical, suggests the definition of a Reidemeister number of the complement, $R(f;X-A)$, and a relative Reidemeister number, $R(f;X,A)$, both of which are homotopy invariants for maps of the pair and satisfy many properties that the usual Reidemeister number does. We have that $R(f;X-A)$ is an upper bound for the Nielsen number of the complement $N(f;X-A)$ defined in [57] and $R(f;X-A)=N(f;X-A)$ when (X, A) is a Jiang pair (i.e., (X, A) is a pair of compact polyhedra, where X is a connected Jiang space) and the Lefschetz number of f is not zero. Similarly, $R(f;X,A)$ is an upper bound for the relative Nielsen number $N(f;X,A)$ defined in [47] and $R(f;X,A)=N(f;X,A)$ when (X, A) is a Jiang pair and the Lefschetz number of f, $L(f)$, is not zero.

To illustrate, we consider the example showed in [11].

Example 1 Let $X=P^2$ be the real projective space, with universal covering space $\tilde{X}=S^2$ and covering projection $p: S^2 \longrightarrow P^2$ defined by $p(x,y,z)=\{(x,y,z),(-x,-y,-z)\}$; its covering transformations $\gamma_i: \tilde{X} \longrightarrow \tilde{X}$ are $\gamma_1(x,y,z)=(x,y,z)=\mathrm{id}_{\tilde{X}}$ and $\gamma_2(x,y,z)=(-x,-y,-z)$ (the antipodal map). Let $A=\{(1,0,0),(-1,0,0)\} \subset X$, a point in P^2; thus, its universal covering space is $\tilde{A}=A$, its covering projection is $p_A=\mathrm{id}_A$, and it has covering transformation $\gamma: \tilde{A} \longrightarrow \tilde{A}$ given by $\gamma_{\tilde{A}}=\mathrm{id}_{\tilde{A}}$.

Let $f: X \longrightarrow X$ be defined by

$$f(\{(x,y,z),(-x,-y,-z)\})=\{(-x,y,-z),(x,-y,z)\} \quad .$$

Then, we have two liftings $\tilde{f}_1(x,y,z)=(-x,y,-z)$ and $\tilde{f}_2(x,y,z)=(x,-y,z)$. Since $\tilde{f}_1 \neq \gamma_1 \circ \tilde{f}_2 \circ \gamma_1^{-1}$ and $\tilde{f}_1 \neq \gamma_2 \circ \tilde{f}_2 \circ \gamma_2^{-1}$, we have two classes of liftings, defined by conjugation by deck transformations; the number of such classes is the geometric definition of the Reidemeister number, therefore $R(f)=2$. Likewise, observe that

$$f(\{(1,0,0),(-1,0,0)\})=\{(-1,0,0),(1,0,0)\}=\mathrm{id}_A \quad ,$$

therefore $f_A=\mathrm{id}_A$ and $R(f_A)=1$.

Since we are dealing with lifting classes, we want somehow to count the classes of liftings of f that are related to the classes of liftings of f_A. The usual way of relating these two types of classes, is well explained in [39, Chapter 3]. In order to do that, consider the inclusion

$$\begin{gathered} i: A \hookrightarrow X \\ \{(1,0,0),(-1,0,0)\} \mapsto \{(1,0,0),(-1,0,0)\}\ , \end{gathered}$$

a lifting of i is a map $\tilde{i}: \tilde{A} \longrightarrow \tilde{X}$ satisfying the equality $p_X \circ \tilde{i} = i \circ p_A$. This gives us two possibilities:

$$\tilde{i}_1(\{(1,0,0),(-1,0,0)\})=(1,0,0) \text{ or } \tilde{i}_2(\{(1,0,0),(-1,0,0)\})=(-1,0,0)$$

We will, then, look at the liftings $\tilde{f}$, of f, that will make the following diagram

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{\tilde{f}_A} & \tilde{A} \\ \tilde{i}_j \downarrow & & \downarrow \tilde{i}_j \\ \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X} \end{array}$$

commutative, for $j = 1$ or 2, because this is a correspondence that doesn't depend on the lifting we pick, only on the map i itself. One can see, for $\tilde{i}_1$, that the lifting $\tilde{f}_1$ of f is the one such that $\tilde{f}_1 \circ \tilde{i}_1 = \tilde{i}_1 \circ \tilde{f}_A$. Also, the above equality doesn't hold for $\tilde{f}_2$ in the place of $\tilde{f}_1$. In this particular example, we are identifying the liftings and the lifting classes. So, the number of lifting classes of f that are not related to the lifting classes of f_A is 1 and this will be our Reidemeister number of the complement, for this particular map. Again, looking at the relative Nielsen number, we will say that the relative Reidemeister number is the number of lifting classes of f_A plus the number of lifting classes of f that are not related to them; in this case, we have that the relative Reidemeister number is 2.

For an alternative algebraic formulation of the Reidemeister number (see [12]), notice that there is a group automorphism, φ, of the fundamental group of a space, $\pi = \pi(X)$, that gives a one-to-one correspondence between Reidemeister classes and lifting classes. This originates a Reidemeister action of π on π, where the Reidemeister classes are the orbits of this action. The Reidemeister number of a map, $\mathrm{R}(f)$, thus is the cardinality of the set of orbits, $\#\mathcal{R}(\varphi, \pi)$. Similarly, for every f-invariant component A_k of A, we have, for each k, $\mathrm{R}(f_k) = \#\mathcal{R}(\varphi_k, \pi_k)$ and we define $\mathrm{R}(f_A) = \#\mathcal{R}(\varphi_A, \pi_A) = \sum_k \#\mathcal{R}(\varphi_k, \pi_k)$.

It is possible to amplify the class of spaces where these definitions can be useful, although many of the results are also valid for more general spaces such as compact ANRs or spaces which admit a fixed point index with the usual properties and for which universal covering spaces exist.

A space X is a Jiang-type space, as defined by P. Wong (see [55]), if the following conditions are satisfied for all selfmaps $f : X \longrightarrow X$:

(C1) $\mathrm{L}(f) = 0 \Rightarrow \mathrm{N}(f) = 0$;
(C2) $\mathrm{L}(f) \neq 0 \Rightarrow \mathrm{N}(f) = \mathrm{R}(f)$.

As examples of Jiang-type spaces we have the classical Jiang spaces, nilmanifolds, and certain classes of solvmanifolds and homogeneous spaces.

In [12], a Jiang-type result was proven

Theorem 2.1 [12, *Theorem* 3.2] *Suppose that* (X, A) *is a pair of Jiang-type spaces, such that* $\mathrm{L}(f) \cdot (\prod_k \mathrm{L}(f_k)) \neq 0$, *then* $\mathrm{N}(f; X, A) = \mathrm{R}(f; X, A)$.

Turning our attention to fibrations, let $p : E \to B$ be a Hurewicz fibration where each fiber $F = p^{-1}(b)$ over $b \in B$, E and B are 0-connected compact ANRs. A selfmap $f : E \to E$ is fiber-preserving if f induces a map $\bar{f} : B \to B$ such that $\bar{f} \circ p = p \circ f$.

The Nielsen-type number for fiber-preserving maps, denoted by $N_{\mathcal{F}}(f,p)$, can be realized as a sharp lower bound for the number of fixed points in the fiberwise homotopy class of f (see [36]). Under Jiang-type conditions it is possible to calculate it as the relative Reidemeister number $\mathrm{R}(f;E,F_{\xi})$, as established in [12] as follows

Theorem 2.2 [12, *Theorem* 5.1] *Let $p : E \longrightarrow B$ be a Hurewicz fibration with typical fiber $F = p^{-1}(b)$, $b \in B$, E and B 0−connected compact ANRs. Suppose that E and F are of Jiang-type. For any set ξ of essential representatives of fixed points of $\bar{f}$, if*

$$L(f) \cdot \prod_{b_i \in \xi} L(f_{b_i}) \neq 0$$

then

$$N_{\mathcal{F}}(f,p) = \mathrm{R}(f;E,F_{\xi}) \,.$$

Here $\bar{f}$ denotes the induced map in the base and f_b the restriction of f on the fiber over $b \in \mathrm{Fix}(\bar{f})$.

We further explore algebraic conditions under which the computation of the relative Reidemeister number may be simplified, leading to the following theorem (see [12])

Theorem 2.3 [12, *Theorem* 5.2] *Let $p : E \longrightarrow B$ be a Hurewicz fibration with typical fiber $F = p^{-1}(b)$, $b \in B$, E and B 0−connected compact ANRs. Suppose that $\pi_2(B)$ is trivial. Let $f : E \longrightarrow E$ be a fiber-preserving map with induced map $\bar{f} : B \longrightarrow B$ and ξ be a set of essential representatives of fixed points of $\bar{f}$. For $b \in \xi$, let $\bar{f}_{\#,b} : \pi_1(B,b) \longrightarrow \pi_1(B,b)$ be the induced homomorphism. If for any $b \in \xi$, $\mathrm{Fix}(\bar{f}_{\#,b}) = 1$, then $\mathrm{R}(f;E,F_{\xi}) = \mathrm{R}(f)$. If, in addition, F and E are Jiang-type spaces and $L(f) \cdot \prod_{b_i \in \xi} L(f_{b_i}) \neq 0$, then*

$$N(f;E,F_{\xi}) = N_{\mathcal{F}}(f,p) = \mathrm{R}(f;E,F_{\xi}) = \mathrm{R}(f) \,.$$

As a remark (we reproduce [12, Remark 6]) observe that the condition $\mathrm{Fix}(\bar{f}_{\#,b}) = 1$ for every $b \in \xi$ is the same as the “essentially fix trivial” condition as in [36]. Essentially fix trivial spaces include the class of solvmanifolds and therefore the class of nilmanifolds.

Let $p : E \longrightarrow B$ be a Hurewicz fibration and $f : E \longrightarrow E$ a fiber preserving map. R. Brown [8] initiated the study of the Nielsen fixed point theory for fiber-preserving maps and gave conditions for which $\mathrm{N}(f) = \mathrm{N}(f_b)\mathrm{N}(\bar{f})$, where $\bar{f}$ denotes the induced map in the base and f_b the restriction of f on the fiber over $b \in \mathrm{Fix}(\bar{f})$. Such a product formula was further studied by Fadell in [19] and necessary and sufficient conditions for its validity were given by You in [56].

One of the standing assumptions in these works is that the fibration p be orientable. By relaxing this assumption, an addition formula, rather than a product formula has been obtained by Heath, Keppelmann and Wong in [36]. The main objective of these formulas is to compute the Nielsen number of f in terms of possibly simpler Nielsen type invariants of $\overline{f}$ and of f_b.

In his thesis [48], A. Schusteff established the product formula for the relative Nielsen number of a fiber preserving map of pairs. More precisely, given a commutative diagram

$$\begin{array}{ccc} (E,E_0) & \xrightarrow{f,f_0} & (E,E_0) \\ p\downarrow p_0 & & p\downarrow p_0 \\ (B,B_0) & \xrightarrow{\overline{f},\overline{f_0}} & (B,B_0) \end{array}$$

conditions were given to ensure that the product formula holds, i.e.,

$$\mathrm{N}(f;E,E_0)=\mathrm{N}(f_b;F_b,F_{0b})\,\mathrm{N}(\overline{f};B,B_0) \quad ,$$

where $E_0\xrightarrow{p_0}B_0$ is a sub-fibration of a Hurewicz fibration $E\xrightarrow{p}B$ and F_b, F_{0b} are the fibers of p and p_0, respectively, over $b\in \mathrm{Fix}(\overline{f})|_{B_0}$. Furthermore, a relative Reidemeister number $\mathrm{R}(f;E,E_0)$ was introduced in [48] to give computational results when the spaces are Jiang spaces.

The main objective of the work by F. Cardona and P. Wong, see [13], is to compute the relative Reidemeister number $\mathrm{R}(f;E,E_0)$ and the relative Reidemeister number on the complement $\mathrm{R}(f;E-E_0)$ of fiber-preserving maps of pairs. In order to give an algebraic formulation of the relative Reidemeister numbers for fiber preserving maps, we adapted what was done in [34] for the Reidemeister number for coincidences via an algebraic approach, for our fixed-point settings.

Let π_X denote the group of deck transformations of the universal cover of X; thus π_X is also identified with $\pi_1(X)$ with one appropriate basepoint.

Let $(E,\ p,\ B)$ be a Hurewicz fibration with the typical fiber $F=p^{-1}(b)$ for $b\in B$, with all spaces being 0-connected. Let $f:E\longrightarrow E$ be a fiber preserving map. Suppose $K=\ker i$, where $i_\#:\pi_F\longrightarrow\pi_E$ is induced by $i:F\hookrightarrow E$. We will denote by $i_{\#K}$ the induced map on the quotient, $i_{\#K}:\pi_F/K\longrightarrow\pi_E$. Also, the set of orbits of the Reidemeister action of φ' in π_F/K will be denoted by $\mathcal{R}_K(\varphi',\pi_F)$. Moreover, we can suppose, without loss of generality, that $i_{\#K}$ is the inclusion map (notice that $\pi_F/K\cong \mathrm{Im}\, i_{\#K}<\pi_E$). In what follows we will indicate the conjugation map by $\tau_\alpha(\beta)=\alpha\beta\alpha^{-1}$, without being explicit where it is defined, since the context will make it clear. The following theorem (see [13]) gives a general formula for a fiber-preserving map.

Theorem 2.4 [13, *Theorem* 2.1] *Let* (E,p,B) *be a fibration as described above*; *let* f *be a fiber-preserving map. Then*, *there is a one-to-one correspondence between the sets*

$$\mathcal{R}(\varphi,\pi_E)\leftrightarrow\coprod_{[\overline{\alpha}]\in\mathcal{R}(\overline{\varphi},\pi_B)}\widehat{i_{\alpha K}}\mathcal{R}_K(\tau_\alpha\,\varphi',\pi_F),$$

where $\widehat{i_{\alpha K}}$ *is induced by* $i_{\#K}$ *defined above, for any* $[\alpha] \in \widehat{p}^{-1}([\overline{\alpha}])$. *If the cardinalities of the sets involved are finite, we have*

$$\mathrm{R}(f) = \sum_{[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \pi_B)} \sum_{[\beta]} \frac{1}{[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_{\#}(\mathrm{Fix}(\tau_{\beta\alpha}\varphi))]}$$

where $[\beta] \in \mathcal{R}_K(\tau_\alpha \varphi', \pi_F)$ *and* $[\alpha] \in \widehat{p}^{-1}([\overline{\alpha}])$.

Let $\alpha \in \mathcal{R}(\varphi, \pi_E)$. The *index of* α is simply index$(f, \eta_E \operatorname{Fix}(\alpha\widetilde{f}))$, the usual fixed point index, where $\eta_E : \widehat{E} \to E$ denotes the universal covering. So, if index of α is nonzero then α is said to be *essential*. Denote by $\mathcal{N}(\varphi, \pi_E)$ the set of essential $\alpha \in \mathcal{R}(\varphi, \pi_E)$. Therefore, $\mathrm{N}(f) = \#\mathcal{N}(\varphi, \pi_E)$. Similarly, we denote by $\mathcal{N}_K(\varphi', \pi_F)$ the set of essential $\alpha' \in \mathcal{R}_K(\varphi', \pi_F)$, and $\mathrm{N}_K(f') = \#\mathcal{N}_K(\varphi', \pi_F)$.

Definition 2.1 Let f be a fiber-preserving map, let φ and $\overline{\varphi}$ be the homomorphisms induced by f and $\overline{f}$. We say that f is *locally* (resp. *essentially locally*) *Fix group uniform* if

$$[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_{\#}(\mathrm{Fix}(\tau_\alpha \varphi))]$$

does not depend on $[\alpha] \in \widehat{p}^{\,-1}([\overline{\alpha}])$ (resp. $[\alpha] \in \widehat{p}^{\,-1}([\overline{\alpha}]) \cap \mathcal{N}(\varphi, \pi_E)$). Similarly, we say that f is *globally* (resp. *essentially globally*) *Fix group uniform* if

$$[\mathrm{Fix}(\tau_{\overline{\alpha}}\overline{\varphi}) : p_{\#}(\mathrm{Fix}(\tau_\alpha \varphi))]$$

does not depend on $[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \pi_B)$ (resp. $\mathcal{N}(\overline{\varphi}, \pi_B)$).

A *fiber map of the pair* is a pair of fiber preserving maps $(f,f_0) : (E, E_0) \to (E, E_0)$ with $f_0 = f|_{E_0}$ where (E_0, p_0, B_0) is a Hurewicz sub-fibration of a Hurewicz fibration (E, p, B). Also, we assume that E, E_0, B, B_0 and the typical fibers are all 0−connected spaces.

Just as before, consider $K_0 = \ker i_{0\#}$, where $i_{0\#} : \pi_{F_0} \longrightarrow \pi_{E_0}$ is induced by $i_0 : F_0 \hookrightarrow E_0$. Denote by $i_{0\#_{K_0}}$ the induced map on the quotient, the set of orbits of the respective Reidemeister action by $\mathcal{R}_{K_0}(\varphi_0', \pi_{F_0})$, and the respective cardinality, $\#\mathcal{R}_{K_0}(\varphi_0', \pi_{F_0})$, by $\mathrm{R}_{K_0}(f_0')$. We will denote the set of orbits of the Reidemeister action of φ' which are in the image of the orbits of the Reidemeister action of φ_0' under $\pi_{F_0}/K_0 \longrightarrow \pi_F/K$ by $\mathcal{R}_{K,K_0}(\varphi', \varphi_0')$, and the respective cardinality, $\#\mathcal{R}_{K,K_0}(\varphi', \varphi_0')$, by $\mathrm{R}_{K,K_0}(f',f_0')$. Moreover, we can suppose, without loss of generality, that $i_{\#K}$ is the inclusion map (notice that $\pi_F/K \cong \operatorname{Im} i_{\#K} < \pi_E$).

Similar to $\mathcal{N}_K$ and N_K, we can define $\mathcal{N}_{K,K_0}(\varphi', \varphi_0')$ as the set of essential $\alpha' \in \mathcal{R}_{K,K_0}(\varphi', \varphi_0')$, and $\mathrm{N}_{K,K_0}(f',f_0') = \#\mathcal{N}_{K,K_0}(\varphi', \varphi_0')$.

The following theorem (see [13]) in which $\mathrm{R}(f;E, E_0)$ and $\mathrm{R}(f;E - E_0)$ are computed or estimated in terms of the relative Reidemeister numbers of f and of f_b generalizes some of the results of [48].

Theorem 2.5 [13, *Theorem* 3.4] *Let* (f, f_0) *be a fiber map of the pair. Let* $K = \ker i_\#$ *and* $K_0 = \ker i_{0\#}$. *Suppose* f *is globally Fix group uniform and let* $s = [\mathrm{Fix}(\overline{\varphi}) : p_\#(\mathrm{Fix}(\varphi))]$, *then*

$$\mathrm{R}(f; E - E_0) = \frac{1}{s}\left\{ \sum_{[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \pi_B)} \#\mathcal{R}_K(\tau_\alpha \varphi', \pi_F) - \sum_{[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \overline{\varphi}_0)} \#\mathcal{R}_{K,K_0}(\tau_\alpha \varphi', \tau_{\alpha_0} \varphi_0') \right\}$$

and,

$$\mathrm{R}(f; E, E_0) = \sum_{[\overline{\alpha_0}] \in \mathcal{R}(\overline{\varphi_0}, \pi_{B_0})} \frac{\#\mathcal{R}_{K_0}(\tau_{\alpha_0} \varphi_0', \pi_{F_0})}{[\mathrm{Fix}(\tau_{\overline{\alpha_0}} \overline{\varphi_0}) : p_{0\#}(\mathrm{Fix}(\tau_{\alpha_0} \varphi_0))]} + \frac{1}{s}\left\{ \sum_{[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \pi_B)} \#\mathcal{R}_K(\tau_\alpha \varphi', \pi_F) - \sum_{[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi}, \overline{\varphi_0})} \#\mathcal{R}_{K,K_0}(\tau_\alpha \varphi', \tau_{\alpha_0} \varphi_0') \right\}.$$

Also, if (E, E_0) *is a Jiang-type pair with nonzero Lefschetz numbers*, $\mathrm{L}(f) \cdot \mathrm{L}(f_0) \neq 0$, *then we have the respective Nielsen numbers*

$$\mathrm{N}(f; E - E_0) = \frac{1}{s}\left\{ \sum_{[\overline{\alpha}] \in \mathcal{N}(\overline{\varphi}, \pi_B)} \#\mathcal{N}_K(\tau_\alpha \varphi', \pi_F) - \sum_{[\overline{\alpha}] \in \mathcal{N}(\overline{\varphi}, \overline{\varphi}_0)} \#\mathcal{N}_{K,K_0}(\tau_\alpha \varphi', \tau_{\alpha_0} \varphi_0') \right\}$$

and,

$$\mathrm{N}(f; E, E_0) = \sum_{[\overline{\alpha_0}] \in \mathcal{N}(\overline{\varphi_0}, \pi_{B_0})} \frac{\#\mathcal{N}_{K_0}(\tau_{\alpha_0} \varphi_0', \pi_{F_0})}{[\mathrm{Fix}(\tau_{\overline{\alpha_0}} \overline{\varphi_0}) : p_{0\#}(\mathrm{Fix}(\tau_{\alpha_0} \varphi_0))]} + \frac{1}{s}\left\{ \sum_{[\overline{\alpha}] \in \mathcal{N}(\overline{\varphi}, \pi_B)} \#\mathcal{N}_K(\tau_\alpha \varphi', \pi_F) - \sum_{[\overline{\alpha}] \in \mathcal{N}(\overline{\varphi}, \overline{\varphi_0})} \#\mathcal{N}_{K,K_0}(\tau_\alpha \varphi', \tau_{\alpha_0} \varphi_0') \right\}.$$

As an application, using the relative Reidemeister number on the complement and equivariant fixed point theory, in the last section of [4] we estimated the asymptotic Nielsen type number, denoted by $\mathrm{NI}^\infty(f)$ (for more information on the later, see [40]), when f is a fiber-preserving map on a compact polyhedron. This is Theorem 4.1. For the sake of simplicity let us state only a consequence of this theorem, which is proved there:

Corollary 2.1 [13, *Corollary* 4.2] *If* X *is a solvmanifold and* $\mathrm{R}(\overline{f^n}) < \infty$ *for all* n, *then for any prime* p,

$$\mathrm{NI}^\infty(f) \geq \mathrm{Growth}_{r \to \infty} \sum_{[\overline{\alpha}] \in \mathcal{R}(\overline{\varphi_r}, \pi_{B_r}) - \mathcal{R}(\overline{\varphi_r}, \overline{\varphi_{0r}})} \#\mathcal{R}(\tau_\alpha \varphi_r', \pi_{F_r})$$

where f *is a fiber-preserving map of a Mostow fibration of* X.

3 Some aspects of equivariant fixed point theory

In this section we intend to describe some results related to the questions mentioned at the introduction when considering the equivariant setting.

Some of the results presented in this section were taken from [1] and from [23] and their statements were copied verbatim from these references.

We will start by setting some notation. Consider a finite group G and an n-dimensional smooth compact G-manifold. For a subgroup H of G, we define $M^H = \{x \in M \mid hx = x, \forall h \in H\}$ and for $f : M \to M$ a G-map we denote by $f^H : M^H \to M^H$ the restriction of f to M^H.

We say that f is G-deformable to a fixed point free map if there exists a map f_1, G-homotopic to f, with no fixed points. Observe that if this is the case, then every $f^H : M^H \to M^H$ is deformed to a fixed point free map, for every subgroup H of G.

In the work G-Deformation to Fixed Point Free Maps via Obstruction Theory, by L. D. Borsari e D. L. Gonçalves, see [2], the converse of the above statement is obtained under certain conditions, namely:

Theorem 3.1 *Let M be a compact differentiable manifold and assume the action of G on M satisfies one of the following conditions*:

(a) *Given any two isotropy groups H and K, with $H \leq K$, then the codimension of any connected component of M^K in M^H is different from one. Furthermore, the dimension of each component of M^H is different from 2, for every $H \leq G$.*
(b) *Each component of M^H is simply connected, for all $H \leq G$.*

Then if $f : M \to M$ is a G–map such that, for every $H \leq G$, $f^H : M^H \to M^H$ is deformable to a fixed point free map it follows that f is equivariantly deformable to a fixed point free map.

This result is proved in a pre-print which has not been submitted to publication because by the time it was being written down, back in 1987, a work by E. Fadell and P. Wong, see [22], was published with the same main results as ours, although proved with different techniques. The proof we gave was based on obstruction theory methods, where an appropriate local system of coefficients for cohomology is set up. We will take this opportunity to present this work in the appendix section, since it seems to us that the technique is interesting in its own and may be useful to treat other cases.

Observe that when a space is simply connected, there will be only one Nielsen class and therefore the Nielsen number is either zero or one. If the Lefschetz number is zero, then the Nielsen number will also be zero.

Moreover, under the conditions stated in the theorem above and assuming also that each component of M^H has dimension bigger than or equal to three, for all $H \leq G$, it is true that $N(f^H) = 0$ implies that f^H is deformable to fixed point free map and therefore f is equivariantly deformed to a fixed point free map.

Turning to the question involving deformations, i.e., maps homotopic to the identity, we begin by quoting a classical theorem of H. Hopf (see [37]) which states that a closed connected orientable smooth manifold M admits a non-singular vector field if and only if the Euler characteristic of M, $\chi(M)$, vanishes. R. Brown, in [7], extended this result to topological manifolds, by replacing vector fields with path fields, a concept first introduced by J. Nash in [43]. R. Brown showed that a compact topological manifold admits a non-singular path field if and only if $\chi(M) = 0$.

The non-singular path field problem is equivalent to the fixed point free deformation problem, that is, M admits a non-singular path field if and only if the identity map on M is homotopic to a fixed point free map. Since the Euler characteristic of a manifold M coincides with the Lefschetz number of a map homotopic to the identity on M, the converse of the Lefschetz Fixed Point Theorem holds true for deformations on topological closed, orientable manifolds.

Moreover, the existence of a path field allows one to show the Complete Invariance Property (CIP). A topological space M is said to have CIP if for any non-empty closed subset A of M, there exists a map $f : M \to M$ having A as its fixed point set. Similarly, M has CIP with respect to deformation (denoted by CIPD) if $f : M \to M$ is homotopic to the identity on M.

L. D. Borsari, F. Cardona and P. Wong, in the work Equivariant Path Fields on Topological Manifolds, see [1], an equivariant analog of Brown's results in [7] are given for locally smooth G-manifolds, for G a finite group. More specifically, the following theorems hold true:

Theorem 3.2 [1, *Theorem* 3.7] *Let G be a finite group and M a compact locally smooth G-manifold. Then there exists a G-path field on M having at most one singular orbit in the closure of each component of M_H. Moreover, M admits a non singular G-path field if and only if $|\chi|(M_H) = 0$, for all $H \leq G$.*

Theorem 3.3 [1, *Theorem* 4.1]) *Let G be a finite group and M a compact locally smooth G-manifold. Suppose for each isotropy type (H), M^H has dimension at least 2. Let $A \subset M$ be a non-empty closed invariant subset. Then the following are equivalent*:

(a) *There exists a G-deformation $\phi : M \to M$ such that $A = Fix(\phi)$.*
(b) *$A \cap \overline{C} \neq \emptyset$ whenever $\chi(C) \neq \emptyset$ for any connected component C of M_H and $\overline{C}$ denotes the closure of C n M^H.*

Finally, related to the question of computing Nielsen numbers, as we mentioned in the previous section, in the non-equivariant case, for a compact, connected manifold M, B. Jiang (in [39]) defined a subgroup, $J(M)$, of the fundamental group of M, $\pi_1(M)$, and studied the spaces for which $J(M) = \pi_1(M)$, the Jiang spaces. For these, all Nielsen fixed point classes have the same index and if $L(f) = 0$ then $N(f) = 0$. In case $L(f)$ is not zero then the Nielsen number coincides with the Reidemeister number of f.

The notions of G- equivariant Nielsen classes and of G-Jiang spaces were defined by P. Wong in [53] and [54] and it is also true that for a G-Jiang space, all equivariant Nielsen classes have the same index.

Fagundes and Gonçalves in a work named Fixed Point Indices of Equivariant Maps of Certain G-spaces, see [23], consider the family of spaces X for which all maps $f : X \to X$ have the property (called J-property) that all Reidemeister classes have the same index. In many cases, spaces with this property are not G-Jiang spaces. For spaces having the J-property, they obtained the following result:

Theorem 3.4 [23, *Theorem* 3.4] *Let X satisfy the J-property and G be a finite group which acts freely on X. If the G-spaces X have the property that the fundamental group of the orbit space is torsion free, then all equivariant Nielsen classes of a given equivariant map $f : X \to X$ have the same index. Furthermore the index of each such class is $|G|$ times the index of one of the Nielsen classes of f.*

4 Coincidence theory via classical obstruction theory

Let $f, g : X \to Y$ be a pair of maps between two topological spaces. Denote by $C(f, g) = \{x \in X \mid f(x) = g(x)\}$ and let $\mu(f, g)$ be the minimal number among the cardinalities of $C(f', g')$, as f', g' varies in the homotopy classes of f, g, respectively.

When X, Y are closed orientable manifolds of the same dimension, the fact that $C(f, g) \neq \emptyset$ is guaranteed by the classical Lefschetz coincidence theorem provided the Lefschetz coincidence number $L(f, g)$ is nonzero.

Schirmer in [46] developed the coincidence Nielsen Theory in this context. Two coincidence points x_0 and x_1 are in the same coincidence Nielsen class of the pair (f, g) if there exists a path λ, from x_0 to x_1, such that $f(\lambda)$ is homotopic, as paths, to $g(\lambda)$. An index of a Nielsen coincidence class is also defined and the Nielsen coincidence number, $N(f, g)$, is the number of classes with non zero index, the essencial ones. The Lefschetz coincidence number then represents the global index of the coincidence set and $N(f, g)$ is a homotopy invariant and therefore a lower bound to the minimal number of fixed points in the homotopy class of the pair (f, g). Schirmer also proves that $N(f, g) = \mu(f, g)$, when the dimension of the manifolds are bigger than or equal to three.

Dobrenko and Jezierski [15] succeeded developing a type of Coincidence Nielsen theory for maps between manifolds of the same dimension without the hypothesis of orientability. This extension came together with a new feature when comparing with the classical case, namely, the local index, called semi-index, is defined for the coincidence Nielsen classes, and it is no longer necessarily an integer. It is, in fact, an element of either $\mathbb{Z}$ or $\mathbb{Z}_2$, the cyclic group of order 2. The index is an element of

$\mathbb{Z}_2$ for Nielsen classes called defective, and an integer for the others. It turns out that this Nielsen coincidence theory is very suitable to estimate $\mu(f, g)$. They succeed showing that a pair of maps can be deformed to coincidence free if and only if the Nielsen number is zero, in case the dimension of the manifold is at least three.

The validity of the converse of the Lefschetz Coincidence Theorem, which does not hold in general, amounts to the ability of deforming f and g to f' and g', respectively, such that the intersection between the diagonal ΔN of N and the graph of $f' \times g'$ is empty. Equivalently, it is a question of deforming $f \times g$ into the subspace $N \times N - \Delta N$. This approach to the converse of the Lefschetz theorem was first studied by Fuller in [25]. Subsequently, Fadell in [18], Faddell and Husseine in [20, 21], Dobrenko and Jezierski in [15], Gonçalves in [28], Borsari and Gonçalves in [3, 4], Gonçalves, Jiezierski and Wong in [30] further explored the connection between coincidence theory and classical obstruction to deformation, among other contributions.

In the fixed point case, for non-simply connected manifolds of dimension at least three, Fadell and Husseini [20] computed the (only) primary obstruction to deforming a selfmap to be fixed point free (see also [14]). This approach gave an obstruction-theoretic proof of a classical result of Wecken stating that if the manifold is of dimension at least three then the Nielsen number $N(f)$ is zero iff f is deformable to be fixed point free.

The purpose of this section is to describe the development of the coincidence theory that was obtained with the participation of the Algebraic Topology group of IME-USP.

We present the contributions that were made when looking at the primary obstruction for a pair of maps $f, g : K \to N$ from a finite complex K of dimension m into a manifold N of dimension n, in the cases $m = n$ and $m > n$, since it is well known that for $m < n$, all pair of maps can be deformed to a coincidence free pair of maps. We observe that the vanishing of the primary obstruction, in general, is not sufficient to guarantee that the pair can be deformed to a coincidence free pair. Accidentally this may happen and examples are given in [30, Section 5], where a pair of maps from a torus into a nilmanifold can be deformed to coincidence free if and only if the primary obstruction vanishes.

For a more systematic study of coincidence theory in positive codimension, we would like to mention works by Hatcher and Quinn [35], Dimovski and Geoghegan [17], Jezierski [38], Koschorke [42], and Dold and Gonçalves [16]. We should observe that one motivation to the study of coincidence theory between manifolds of different dimensions comes from the fact that in [33] some questions were posed where a coincidence problem of two maps between closed manifolds of the same dimension was studied via another coincidence problem of maps between manifolds of different dimensions.

The approach developed by Fadell and Husseini in [20], via obstruction theory, indicates which kind of algebraic object the index of a Nielsen coincidence class should be in more general settings. It turns out that for each Nielsen class, the index is going to be an element of an abelian group which depends on the Nielsen class. The approach via obstruction is quite suitable, in the sense that it is clear that if the index of all Nielsen classes are zero, the maps can be deformed, up to the n_0-skeleton of K to be coincidence free, where $n \leq n_0 \leq m$ and n_0 is the first integer for which the cohomology group $H^{n_0}(K)$, with a certain system of coefficients, is non trivial.

When extending the theory for maps from a simplicial complex K into a manifold N one can not expect to deform the pair of maps so that the number of coincidences is finite. Also one can no longer expect to deform the maps such that each essencial coincidence Nielsen class has only one point, even when K has nice properties, such as being n-dimensionally-connected, for $n = dimN$ greater than 2. So the problem of minimizing, in the homotopy class of the pair (f, g), the number of coincidences, in this more general setting, even in the case when K and N have the same dimension, requires a new approach in order to stablish a type of Nielsen coincidence theory. This will be explored in what follows.

4.1 General geometric and algebraic properties

The results described in this section were taken from [28]. In order to keep this paper self-contained, some of them are copied verbatim from [28].

Let $f, g : K \to N$ be a pair of maps, where K is a finite complex and N is a manifold, both of dimension n.

From classical obstruction theory, see [52], and following [20], we have a cohomology class $O^n(f, g) \in H^n(K, \mathbb{Z}[\pi])$ which represents the primary obstruction to deform (f, g) to a pair of coincidence free maps. We recall that $H^n(K, \mathbb{Z}[\pi])$ is the nth cohomology group of K with local coefficients $\mathbb{Z}[\pi]$, where $\pi = \pi_1(N)$. The action $w : \pi_1(K) \to Aut(\mathbb{Z}[\pi])$ is given by $w(\theta).\alpha = sign(f_\#(\theta))g_\#(\theta)\alpha f_\#(\theta)^{-1}$. This gives the abelian group $\mathbb{Z}[\pi]$ a structure of a $\mathbb{Z}[\pi_1(K)]$-module or, in short, a $\pi_1(K)$-module.

Let $\mathcal{R}(f, g)$ be the set of Reidemeister classes and let

$$A_{[\alpha]} \approx \oplus_{\alpha\in[\alpha]}\mathbb{Z}_\alpha,$$

where $[\alpha] \in \mathcal{R}(f, g)$. By definition of the action one can see that the subgroups $A_{[\alpha]}$ are invariants under this action. Let us denote by $\omega_{[\alpha]} : \pi_1(K) \to Aut(A_{[\alpha]})$ the action of $\pi_1(K)$ on $A_{[\alpha]}$, provided by the action ω. With respect to the above actions we have:

Proposition 4.1 [28, *Proposition* 2.1] *The* $\mathbb{Z}[\pi_1(K)]$*-module* $\mathbb{Z}[\pi]$ *is isomorphic to the direct sum of the* $\mathbb{Z}[\pi_1(K)]$*–modules* $A_{[\alpha]}$*, where the action is given in a natural way by the direct sum of the actions* $\omega_{[\alpha]} : \pi_1(K) \to Aut(A_{[\alpha]})$ *and*

$$H^n(K, \mathbb{Z}[\pi]) \approx \oplus_{[\alpha]\in\mathcal{R}(f,g)}H^n(K, A_{[\alpha]}).$$

Let $F \subset Coin(f, g)$ be a Nielsen class. This class F corresponds to a Reidemeister class which we denote by $[\alpha]$.

The above result motivates the following definition.

Definition 4.1 The index of F, denoted by $i(F)$ is $p_{[\alpha]}(O^n(f,g))$, where

$$p_{[\alpha]} : H^n(K, \mathbb{Z}[\pi]) \to H^n(K, A_{[\alpha]})$$

is the natural projection.

Definition 4.2 *F* is called essential if $i(F) \neq 0$.

Let us consider the cocycle $c^n(f,g)$ as defined in [20], using classical obstruction theory.

Proposition 4.2 [28, *Proposition* 2.2] *The cocycle $c^n(f,g)$ is the sum of cocycles $c^n_{[\alpha]}$, where the summand $c^n_{[\alpha]}$ is a cocycle of $H^n(K, A_{[\alpha]})$, for $[\alpha] \in \mathcal{R}(f,g)$.*

Consider the case where K is a n-manifold. We observe that if K has non-empty boundary, then it has the homotopy type of a $(n-1)$-complex. Therefore $H^n(K, \mathbb{Z}[\pi]) = 0$ and every pair (f, g) can be made coincidence free. So let us assume that K is a manifold without boundary.

Proposition 4.3 [28, *Proposition* 2.5] *$i(F)$ is either an element of $\mathbb{Z}$ or $\mathbb{Z}_2$.*

Theorem 4.1 [28, *Theorem* 2.6] *For $f, g : M \to N$ where M and N are manifolds of the same dimension, we have $N(f,g) = \mu(f,g)$.*

The above result, that has been proved by Schirmer [46] in the orientable case and by Dobrenko and Jezierski [15] in the non-orientable case, not only solves both cases at once, but also gives some insight on how to deal with complexes more general than a manifold.

The examples that will be presented in what follows were taken from [28, section 4] and will show that the classical Nielsen coincidence number is too weak to estimate $\mu(f,g)$. So, in order to extend the theory for maps from a more general complex into a manifold, we constructed in [4] an algorithm, using all possible cocycles representing the primary obstruction class, to find the number $\mu(f,g)$, which will be described in Sect. 4.3.

Example 1 Consider n disjoint copies of the sphere S^m and connect them by strips of dimension $m-1$. Take a map f from the above complex into the m-sphere such that f restricted to any one of the spheres is homotopic to the identity. Let g be the constant map. Then certainly $\mathcal{R}(f,g)$ contains only one element and $N(f,g) = 1$. But it is quite simple to see that $\mu(f,g) = n$.

Example 2 Consider n disjoint copies of the sphere S^m and connect them, in sequence, by points and take f and g as in Example 2. Then $\mathcal{R}(f,g)$ contains only one element and $N(f,g) = 1$. But it is not hard to see that $\mu(f,g) = [(n+1)/2]$, where [ ] means the greatest integer less than or equal to the number inside of the bracket.

Example 3 Consider the bouquet of n, spheres $S^m$and take f and g as above. This space has the same homotopy type as the spaces given in Examples 2 and 3, $\mathcal{R}(f,g)$ contains only one element and $N(f,g)=1$. It is easy to see that $\mu(f,g)=1$.

Example 4 Let $K_i,\ i=1,2$, be the two 2-cell complexes obtained from S^1 by attaching a 2-cell by the maps $\varphi_i : S^1 \to S^1,\ i=1,2$, of degrees 2 and 3, respectively. (K_1 is just the two-dimensional projective space.) Take

$$\frac{K_1 \cup S^1 \times [0,1] \cup K_2}{\sim},$$

where we identify the one skeleton $S^1 \subset K_1$ with $S^1 \times 0$ and $S^1 \subset K_2$ with $S^1 \times 1$. One can show that K is simply connected and has the homology of the sphere S^2. So K has the homotopy type of the 2-sphere. If we consider $f,g : K \to S^2$ where g is the constant map and f has degree d (which we may assume greater than zero), we have:

(a) If d is relatively prime with 6, then $\mu(f,g) \geq 2$, because the maps restricted to $K_i,\ i=1,2$, must have at least one coincidence point. (We believe that $\mu(f,g)=2$.)

(b) If d is relatively prime with 2, then there exists at least one coincidence point in K_1 and of course $\mu(f,g) \geq 1$. The cases where d is relatively prime to 3 are similar.

(c) Finally, if 6 divides d, then we believe that $\mu(f,g)=1$ and the coincidence point can be located anywhere in K.

This example shows that even for a complex which has the homotopy type of a compact manifold, the situation can be quite different from the case where the domain is a compact manifold.

Comments Example 2 was known by R. Brooks, in [6], where the reader will also find some material related with this work.

The examples above show how relevant the geometry of the complex K is, in order to define a Nielsen type number to play the role of a good lower bound for $\mu(f,g)$. It also becomes clear that one should look for a Van Kampen type theorem.

4.2 Local coincidence index, the number $NO(f,g;K)$ and the minimal number of coincidences

The results in this section were taken from [4] and some of them were copied verbatim from [4].

In this section we define a homotopy invariant which coincides, under mild conditions, with the minimal number of coincidences in the homotopy class of the pair $f,g : K \to N^n$, where K is a simplicial complex of dimension n and N^n is a closed n-manifold. This invariant is constructed in terms of the primary obstruction to deform a pair of maps to coincidence free as well as in terms of the geometry of the complex K. We will start by reviewing the notion of local index as formulated by Fadell and Husseini in [21], adapted to the terminology of the coincidence context.

Let U be an open set of K and $(f, g) : U \to N^n$ be a pair of maps where the set of coincidence points are compact.

As in [21], we consider the diagonal $\triangle$ in $N^n \times N^n$ and replace the inclusion $N^n \times N^n - \triangle \hookrightarrow N^n \times N^n$ by a fiber map $p : E \to N^n \times N^n$, where $E = \{(\alpha, \beta) : \alpha(0) \neq \beta(0)\}$, and $p(\alpha, \beta) = (\alpha(1), \beta(1))$. For $b = (x, y)$ in $N^n \times N^n$ and $F_b = p^{-1}(b)$, $\pi_{m-1}(F_b)$ is a local system of coefficients on $N^n \times N^n$. There is an isomorphism of local systems on $N^n \times N^n$

$$\zeta : \pi_{m-1}(F_b, b) \to \mathcal{Z}[\pi],$$

where $\pi = \pi_1(N^n, x)$ and the action of $\pi \times \pi$ on $\mathcal{Z}[\pi]$ is given by

$$\alpha \cdot (\sigma, \tau) = sgn\sigma\sigma^{-1} \cdot \alpha \cdot \tau$$

We wil refer to this system as $\mathcal{B}$.

Let the local system on U be the one induced from $\mathcal{B}$ by $f \times g : U \to N^n \times N^n$ and denote it by $\mathcal{B}(f \times g)$. Consider the fiber space $E(f,\ g)$ obtained by pulling back $p : E \to N^n \times N^n$ over U by $f \times g$.

The obstruction to deform the pair $(f,\ g)$ to a coincidence free pair is related to the obstruction to extend sections of the fiber map $E(f, g) \to U$.

Following the steps in [21] and making the usual adaptations to the coincidence case, we end up with:

Definition 4.3 The coincidence index of $(f, g) : U \to N^n$ is the cohomology class $i(f,g)$ in $H^n_c(U;\mathcal{B}(f \times g))$ with the property that $(f,\ g)$ can be deformed by a compact homotopy to a coincidence free pair if and only if $i(f,g)$ vanishes.

Consider now F an isolated set of coincidences of $(f,\ g)$ and let V be an open set of U such that $F = V \cap coin(f, g)$. Consider the diagram

$$H^n(V, V - F;\mathcal{B}(f \times g)) \xrightarrow{j^{*-1}} H^n(U, U - F;\mathcal{B}(f \times g)) \xrightarrow{k^*} H^n(U, U - coin(f, g);\mathcal{B}(f \times g))$$

where the first arrow is the inverse of the excision isomorphism and the second is induced by the inclusion. Recall that $H^n_c(U;\mathcal{B}(f \times g))$ is the inverse limit of $H^n(U, U - C;\mathcal{B}(f \times g))$, where the limit is taken over all compact subsets C of U.

Definition 4.4 The local coincidence index of F, denoted by $i(f,g;F)$, is the element in $H^n_c(U;\mathcal{B}(f \times g))$ given by $k^*(j^*)^{-1}(\alpha)$, where α in $H^n(V, V - F;\mathcal{B}(f \times g))$ corresponds to the coincidence index of $(f, g) : V \to N^n$.

Let us consider the group $H^n(K, A)$, the n-th simplicial cohomology group of K with local coefficients, where A is a free abelian group and identified with the direct sum of $Z's$ indexed by some set J. *We call a cochain* $c_n \in C^n(K, A)$ *elementary* if c_n is nonzero in only one n-simplex, called its support, and has value in one summand Z of A indexed by $j \in J$. So we can associate to each elementary cochain a pair (Δ^n, j), where Δ^n is its support and j is the index of the summand $Z \subset A$ where the cochain assumes its value. *Two elementary cochains are disjoint* if the pairs (Δ^n, j), (Δ'^n, j') are not equal. Given an arbitrary cocycle (or cochain) $c_n \in C^n(K, A)$ we define an integer, $\ell(c_n)$, as follows:

The cocycle c_n can be uniquely written as a sum of disjoint elementary cocycles i.e. $c_n = c_{n,1} + c_{n,2} + ... + c_{n,r}$, where each $c_{n,i}$ is elementary.

Definition 4.5 A cocycle is essential if it represents a nonzero cohomology class.

Definition 4.6 A partial sum $c_{n,i_1} + ... + c_{n,i_s}$ of the decomposition of c_n is said to be combinable if the intersection of the supports of all elementary summands is nonempty and they have values in the same summand Z of A. Define $\ell(c_n)$ to be the minimal number of combinable partial summands among all decompositions of c_n.

Definition 4.7 For maps $f, g : K \to N^n$ the number $NO(f, g; K)$ is defined as the minimum of the numbers $\ell(c_n)$, where c_n runs over the set of all cocycles representing the obstruction $O^n(f, g) \in H^n(K, Z[\pi])$ to deform (f, g) to coincidence free.

Theorem 4.2 [4, *Theorem* 3.6] *$NO(f, g; K)$ is a homotopy invariant.*

In order to state the minimizing result we need to set some notation. We will define a decomposition of K in terms of a simplicial structure of K, although it can be shown that it does not dependent on the choice of the simplicial structure. For each maximal simplex Δ^n of K, let $C(\Delta^n)$ be the smallest subcomplex which contains all n-simplices Δ'^n such that there is a sequence of n-simplices starting at Δ^n and ending Δ'^n so that the intersection of two consecutive ones is a $(n-1)$-simplex which faces only these two n-simplices. This defines a covering of K by homogeneous simplicial subcomplexes which we denote by $\{K_1, ..., K_r\}$. Associated to this covering we have the subcomplex $K_0 = \bigcup_{i \neq j} K_i \cap K_j$. Observe that the points of K_0 are characterized by the property that they are not locally Euclidean in K.

Theorem 4.3 [4, *Theorem* 4.1] *Let $(f, g) : K \to N^n$ be a pair of maps, where K and N^n have dimension bigger than or equal to three. Assume every component of K_0 is of non-zero dimension. Then the minimum number of coincidences in the homotopy class of the pair (f, g) is given by $NO(f, g; K)$.*

Remark 4.1 In the case where some, if not all, components of K_0 have zero dimension, it could happen that two or more combinable partial sums have the intersection of their supports being only one point. In this case, only one set of coincidences, arising from the combinable partial sums, would be joint to this point. Therefore, we would have to add to the number $\ell(c_n)$ the number of elements of all, except the biggest, combinable partial sums for which the intersection of supports is the same single point.Then, the minimum of these numbers, as c_n runs through all possible cocycles representing the obstruction class, will give us the minimum number of coincidences in the homotopy class of the pair (f, g).

As an application of the above result, let $K' \subset K$ be any subcomplex such that the homomorphism $i^* : H^n(K, Z[\pi]) \to H^n(K', Z[\pi])$, induced by the inclusion map, is a cohomology isomorphism with local coefficients, where $\pi = \pi_1(N^n)$. Observe that if two subcomplexes have this property then their intersection does too. Hence, we may always consider the minimal one, namely, the intersection of all subcomplexes satisfying the above condition.

Theorem 4.4 [4, *Theorem* 4.3] *Given* $f, g : K \to N^n$ *then* $\mu(f, g) = \mu(f', g')$, *where* f', g' *are the restrictions of* f, g, *respectively*, *to* K'.

Remark 4.2 It is not an easy task to compute $\mu(f, g)$ since one first need to find the obstruction class and then apply the algorithm using the cocycles. Nevertheless, one can estimate some upper bound for $\mu(f, g)$. Certainly the number of n-dimensional simpleces is an upper bound. But a much better upper bound can be defined when N^n is simply connected.

Let $C = \{K_{i_1}, ..., K_{i_r}\}$ be the covering of K defined above, and assume that all components of K_0 have nonzero dimension.

Definition 4.8 A subset $\{K_{i_1}, ..., K_{i_s}\}$ of the covering $C = \{K_1, ..., K_r\}$ is called admissible if the intersection $K_{i_1} \cap ... \cap K_{i_r} \neq \emptyset$. Let $\ell(C)$ be the minimal number of admissible subsets which cover C. For the purpose of computing $\ell(C)$ we can assume, without loss of generality, that the admissible sets are maximal in the sense that for any $K_j \neq K_{i_t}$, $t = 1, ..., r$, we have $K_j \cap K_{i_1} \cap ... \cap K_{i_r} = \emptyset$.

Proposition 4.4 [4, *Theorem* 5.3] *Given* $f, g : K^n \to N^n$, *where* N^n *is simply connected*, *then* $\mu(f, g) \leq \ell(C)$.

In [4, Section 5] one can find examples which illustrate the above results.

4.3 Poincaré duality with local coefficients and the primary obstruction

The results of this section were taken from [30]. To keep this paper self-contained, some of the presented results are copied verbatim from [30].

In this section we look at coincidences of a pair of maps $f, g : K^m \to N^n$, where K and N are manifolds and $m \geq n$. We would like to provide ways of computing the primary obstruction in this more general situation. We succeed doing this when K is a closed PL-manifold and the result resembles those where the manifolds have the same dimension. For simplicity we will assume that K is an orientable PL-manifold. The results hold true without the orientability hypothesis, and they are proved using the same techniques as in the orientable case, see [29].

We will present two types of results. In the first one, we identify the primary obstruction with a certain homology class determined by *Coin*(*f*, *g*). The second one express the primary obstruction in terms of a sum of cup products of certain classes related to the Thom class, with local coefficients, of the fibration $(K \times K, K \times K \backslash \Delta) \to K$.

Let Γ be a local system on a space X. Recall from [49] that $\Delta_*(X;\Gamma)$, the chain complex with coefficients in the local system Γ, is defined as the set of finite sums $\sum g_\sigma \sigma$, where $\sigma : \Delta_q \to X$ is a singular chain and g_σ is a section of the system Γ over σ. More precisely, g_σ subordinates to each $x \in \Delta_q$ an element $g_\sigma(x) \in \Gamma_{\sigma(x)}$ so that for any path $\omega : [0,1] \to \Delta_q$ the equality $\Gamma_{\sigma\omega}(g_\sigma(\omega(0))) = g_\sigma(\omega(1))$ holds.

Then the boundary homomorphism $\partial : \Delta_q(X;\Gamma) \to \Delta_{q-1}(X;\Gamma)$ is given by

$$\partial(\sum_\sigma g_\sigma \sigma) = \sum_\sigma (\sum_{i=0}^{q} (-1)^i (g_{\sigma|\sigma^{(i)}})\sigma^{(i)})$$

where $\sigma^{(i)}$ denotes i-th face of σ and $g_{\sigma|\sigma^{(i)}}$ the restriction of g_σ to this face. This gives the homology groups with local coefficients $H_*(X;\Gamma)$.

We will often write $g_{\sigma(x)}\sigma$ instead of $g_\sigma \sigma$. Here $g_{\sigma(x)}$ denotes the value of the section g_σ at a point $x \in \Delta_q$. Since Δ_q is simply connected, the value at a point determines the section g_σ.We define cohomology with local coefficients $H^*(X;\Gamma)$ in a similar way.

Let N be a closed oriented PL-manifold which from now on will be assumed to have dimension at least three. In order to define the local system π on $N \times N$ let us recall from (1.12) Theorem in [52, p.263] that it is enough to define a group $\pi_{(x,y)}$, for a point (x, y), and the action of $\pi_1(N \times N;(x,y))$ on $\pi_{(x,y)}$. We fix a point $(x,y) \in N \times N \setminus \Delta N$ and we define $\pi_{(x,y)} = \pi_n(N \times N, N \times N \setminus \Delta N;(x,y))$. Since $\dim N \geq 3$, $\pi_1(N \times N;(x,y)) = \pi_1(N \times N \setminus \Delta N;(x,y))$ and the last group acts on $\pi_{(x,y)}$. We will describe the group $\pi_{(x,y)}$ and the above action as in ([14, 20]). The inclusion

$$i : (N, N \setminus \{x\}, y) \to (N \times N, N \times N \setminus \Delta N;(x,y))$$

given by $i(z) = (z,y)$, induces the isomorphism of homotopy groups

$$i_\# : \pi_n(N, N \setminus \{x\}, y) \to \pi_n(N \times N, N \times N \setminus \Delta N;(x,y))$$

We fix an embedding $\theta'_0 : (\Delta_n, \mathrm{bd}\Delta_n; v_0) \to (N, N \setminus \{x\}, y)$ representing the orientation of N. Let us denote $\theta'_\alpha = \alpha \circ \theta'_0$, for $\alpha \in \pi_1(N;y)$. Then

$$\pi_n(N, N \setminus \{x\};y) = \oplus_{\alpha \in \pi_1(N;y)} \mathbb{Z}\theta'_\alpha$$

and hence

$$\pi_{(x,y)} = \pi_n(N \times N \setminus \Delta N;(x,y)) = \oplus_\alpha \mathbb{Z}\theta_\alpha,$$

where $\theta_\alpha = i_\#(\theta'_\alpha)$ and α ranges over the group $\pi_1(N;y)$. Moreover the action of the group

$$\pi_1(N \times N) = \pi_1 N \times \pi_1 N$$

is then given by

$$(\sigma,\tau)\circ\theta_\alpha = \theta_{\tau\alpha\sigma^{-1}}.$$

In other words, $\pi_1 N \times \pi_1 N$ is acting on $\pi_1 N$ by $(\sigma,\tau)\circ\alpha = \tau\alpha\sigma^{-1}$.

We define the twisted Thom class τ_N as the primary obstruction to the deformation of the identity map on $N \times N$ to a map outside $\Delta N \subset N \times N$ as in [14]. We fix a triangulation of $N \times N$ such that $(n-1)$-skeleton is disjoint from the diagonal: $(N \times N)^{(n-1)} \subset N \times N \setminus \Delta N$. We write $N^\times$ to denote the pair $(N \times N, N \times N \setminus \Delta N)$. Then the obstruction belongs to $H^n(N^\times;\pi)$ where the system π was defined above (set $(X,A) = (N \times N, (N \times N)^{(n-1)})$ and $(Y,B) = (N \times N, N \times N \setminus \Delta N)$ in section 2 in [14]). This obstruction can be represented by the cocycle $c \in C^n(N \times N, N \times N \setminus \Delta N;\pi)$ defined by

$$< c, \sigma >= [\mathrm{id}(\sigma)] = [\sigma] \in \pi_n(N^\times;\sigma(v_0))$$

for each simplicial n-simplex. Since the simplicial and singular cohomology groups are isomorphic, this cocycle is defined on singular simplices and the above formula holds for any $\sigma : (\Delta_n, \partial\Delta_n) \to (N \times N, N \times N \setminus \Delta N)$.

Let $C(f,g) = \{x \in M | f(x) = g(x)\}$ denote the coincidence set and $\pi = \pi_n(N^\times)$ be the local system on $N \times N$. Denote by ${\pi_n}^*$ the local system on M induced by $(f \times g)d$. Then the primary obstruction to deforming f and g on the n-th skeleton of M off the diagonal ΔN is given by $o_n(f,g) = [j(f \times g)d]^*(\tau_N,)$ where τ_N is the twisted Thom class of N. The following generalizes a similar formula in [38] and in [35] in the case of simple coefficients.

Theorem 4.5 [30, *Theorem* 3.3] *The coincidence set* $C(f, g)$ *determines the homology element dual to the primary obstruction such that* $D_M^{-1}(o_n(f,g)) = [z^{\pi^*}_{C(f,g)}] \in H_{m-n}(M;\pi^*)$.

We now consider a pair of maps into N, where N is a compact simply-connected manifold.Then the local system

$$\pi(x,y) = \pi_n((N)^\times;(x,y)) = \pi_n(N, N \setminus \{x\};y) = \mathbb{Z}$$

is trivial, and so it is the induced system π^*.

A similar formula to the one in [18] for the Lefschetz coincidence class holds for the primary obstruction.

Theorem 4.6 [30, *Theorem* 3.4] *Let* M *be closed oriented PL-manifold*, $\dim M = m \geq n \geq 3$. *Let* $f, g : M \to N$ *be a pair of maps where N is any compact* $1-$ *connected manifold whose homology* $H_*(N)$ *is torsion free. Then*

$$o_n(f,g) = \Sigma(-1)^{|\alpha_i|} f^*(x_i) \cup g^*(y_i),$$

where y_i *is the Poincaré dual of* $x_i \in H^{|\alpha_i|}(N)$ *and* x_i *is defined by the Kronecker pairing with respect to a homogeneous basis* $\{\alpha_1, ..., \alpha_p\}$ *for* $H_*(N)$. *In particular if* $N = S^n$ *then*

$$o_n(f, g) = g^*(c_{S^n}) + (-1)^n f^*(c_{S^n}).$$

We finish this section observing that the notion of minimal number of fixed points, minimal number of roots and minimal number of coincidences, in the homotopy class of the maps involved, is a well defined concept when the spaces involved are manifolds of the same dimension or some few other situations slightly more general.

In the context of roots or coincidences of maps between two complexes where the domain has dimension strictly bigger than the dimension of the target, or fixed points for more general spaces like compact spaces, one no longer expects to deform the maps involved so that the number of either roots, coincidences or fixed points becomes finite. We present here an attempt of defining the concept of *minimal set* for each of these three cases, in a very general topological context, which we hope to be suitable mainly for the situation when the minimal number is not finite.

For topological spaces X and Y, we denote by $[X, Y]$ the set of homotopy classes of maps from X to Y.

Definition 4.9 For $\alpha, \beta \in [X, Y]$ we say that $Coin(f_0, g_0)$ is minimal in the pair of homotopy class (α, β) if $Coin(f, g)$ is not a proper subset of $Coin(f_0, g_0)$, for any $(f, g) \in (\alpha, \beta)$.

Definition 4.10 For $\alpha \in [X, X]$ we say that $Fix(f_0)$ is minimal in the homotopy class α if $Fix(f)$ is not a proper subset of $Fix(f_0)$, for any $f \in \alpha$.

Denoting by $Ro_{y_0}(f) = \{x \in X \mid f(x) = y_o\}$, we define

Definition 4.11 For $\alpha \in [X, Y]$ a homotopy class we say that $Ro_{y_0}(f_0)$ is minimal in the homotopy class α if $Ro_{y_0}(f)$ is not a proper subset of $Ro_{y_0}(f_0)$, for any $f \in \alpha$.

There are some works where the minimal sets are considered according to these definitions as [26, Theorem 2.5], [27, Proposition 3.2], [24] and [32]. We should point out that all sort of simple questions, including very naive ones, relative to the concept defined above, have not been explored yet. Notable, it is not true in general that, in the same homotopy class of a pair of maps, two minimal sets are homeomorphic, see example 2.14 in [10]. Also, it is not true that when the minimal number exists, i.e. it is finite, the cardinality of a minimal set is the minimal number, see [31].

Appendix: G-deformation to fixed point free maps via obstruction theory

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In this section we present an alternative proof of Theorem 3.1 stated in Sect. 3. Consider G a finite group and let M be an n−dimensional compact differentiable G−manifold. Given a G−map $f : M \to M$ and a subgroup H of G, it is defined a map $f^H : M^H \to M^H$. It is easy to see that if f is equivariantly deformable to a fixed point free map then f^H is deformable to a fixed point free map, for every $H \leq G$. One would like to know if the converse of this statement holds true. A.Vidal has shown that when G acts semi-freely on a simply connected manifold M, with codim $M^G \geq 3$, then being able to deform $f : M \to M$ and $f^G : M^G \to M^G$ to fixed point free maps suffices to deform f equivariantly to a fixed point free one (see [50]). We will show that Vidal's result holds true under weaker hypothesis.

As we mentioned before, by the time this work was being written down, the authors received a copy of E. Fadell and P. Wong's paper (see [22]), with the main results basically the same as ours, although proved with different techniques. The proof that will be presented is based on obstruction theory and it seems to be interesting by its own and might be useful to treat other cases.

Equivariant cohomology with local coefficients

Let M be an n−dimensional compact, differentiable manifold with boundary, ∂M, where G acts freely. We may assume that M has a simplicial complex structure.

Definition 5.1 A G−local system over M is a local coefficient system w over M (in the classical sense) together with homomorphisms

$$g_{\#} : w(x_0) \to w(g.x_0),$$

for every $g \in G$ and $x_0 \in M$, satisfying:

a) $(g_2.g_1)_{\#} = g_{2\#} \circ g_{1\#}$,

and

b) $g_{1\#} \circ \lambda_{\#} = (g_1(\lambda))_{\#} \circ g_{1\#}$,

where $g_1, g_2 \in G$ and λ is a path in M.

It is well known that the fundamental group of the orbit space M/G is an extension of the fundamental group of M by G, i.e., there is a short exact sequence

$$0 \longrightarrow \pi_1(M) \xrightarrow{p^*} \pi_1(M/G) \longrightarrow G \longrightarrow 0,$$

where $p : M \to M/G$ is the projection.

Given a G−local system over M, w, it induces a local coefficient system over M/G as follows: for each vertex $[x] \in M/G$, choose a vertex $x_0 \in M$, with $[x] = [x_0]$, and define $\overline{w}([x]) = w(x_0)$. Given $\alpha \in \pi_1(M/G, [x])$, represented by $\lambda : I \to M/G$, consider the unique lifting $\tilde{\lambda} : I \to M$ of λ with $\tilde{\lambda}(0) = x_0$. Then $\tilde{\lambda}(1) = g.x_0$ for some $g \in G$. Let $\alpha_{\#} : \overline{w}([x]) \to \overline{w}([x])$ be the composite:

$$w(x_0)\xrightarrow{g_\#}w(g.x_0)\xrightarrow{\tilde{\lambda}^{-1}}w(x_0).$$

Since w is a local coefficient system over M, we have a well-defined map

$$\theta : \pi_1(M/G,[x]) \to \mathrm{Aut}(\overline{w}([x]))$$

given by

$$\theta(\alpha) = \alpha_\#.$$

Furthermore, θ is a group homomophism; for if α_1 and α_2 belong to $\pi_1(M/G,[x])$ and are represented by paths λ_1 and λ_2 respectively, let $\tilde{\lambda}_1$ e $\tilde{\lambda}_2$ be the unique liftings of λ_1 e λ_2 respectively with $\tilde{\lambda}_1(0) = \tilde{\lambda}_2(0) = x_0$. Consider $g_1, g_2 \in G$ such that $\tilde{\lambda}_1(1) = g_1.x_0$ and $\tilde{\lambda}_2(1) = g_2.x_0$. Then

$$\begin{aligned}(\alpha_1)_\#\circ(\alpha_2)_\# &= (\tilde{\lambda}_1^{-1}\circ g_{1\#})\circ(\tilde{\lambda}_2^{-1}\circ g_{2\#})\\ &= \tilde{\lambda}_{1\#}^{-1}\circ(g_1(\tilde{\lambda}_2^{-1}))_\#\circ g_{1\#}\circ g_{2\#}\\ &= (g_1(\tilde{\lambda}_2^{-1}) * \tilde{\lambda}_1^{-1})_\#\circ g_{1\#}\circ g_{2\#}\\ &= (\alpha_1 * \alpha_2)_\#.\end{aligned}$$

Remark 5.1 Let $p : M \to M/G$ be the projection onto the orbit space M/G. Then the composite

$$\pi_1(M,x_0)\xrightarrow{p^*}\pi_1(M/G,[x_0])\xrightarrow{\theta}\mathrm{Aut}(w(x_0))$$

defines a local coefficient system over M which agrees with the original system w.

For each k–simplex σ^k of M, write, $\sigma^K =< x_0\, x_1 \cdots x_k >$ and call x_0 the leading vertex of σ^k. Denote by $C^k(M,\partial M;w)$ the set of all functions assigning to each k–simplex σ^k of M an element of $w(x_0)$ and vanishing on every k–simplex of the boundary of M, ∂M. Since $w(x_0)$ is an abelian group, $C^k(M,\partial M;w)$ has a structure of an abelian group.

Define a G–action on $C^k(M,\partial M;w)$ by

$$(g\varphi)(\sigma^k) = (g^{-1})_\#\varphi(g\sigma^k),$$

where $g\sigma^k$ has leading vertex gx_0.

Proposition 5.1 *The subgroup of $C^k(M,\partial M;w)$ fixed by G is isomorphic to $C^k(M/G,\partial M/G;\overline{w})$.*

Proof Given $\varphi \in C^k(M,\partial M;w)^G$, define $\overline{\varphi} \in C^k(M/G,\partial M/G;\overline{w})$ by $\overline{\varphi}(\overline{\sigma}^k) = \varphi(\sigma^k)$, where σ^k is the k–simplex over $\overline{\sigma}^k$ having leading vertex chosen as in the definition of $\overline{w}$. It is easy to see that this correspondence is an isomorphism. □

We will now describe a G-local system of coefficients that will be useful for our purpose.

Consider a G–fibration (E, p, B) and a diagram of the type

$$\begin{array}{ccc} \partial M & \xrightarrow{\overline{f_1}} & E \\ \downarrow & & \downarrow p \\ M & \xrightarrow{f} & B \end{array}$$

where f is equivariant and $\overline{f_1}$ is an equivariant lifting of $f|_{\partial M}$ to E. Assume the fiber F of the fibration is simply connected, and that we have already lifted f relative to ∂M up to the $(k-1)$–skeleton of M. The obstruction to extend the lifting of f to the k–skeleton lies in the cohomology group $H^k(M, \partial M; w)$, where w denotes the local coefficient system induced by the local coefficient system $\{\pi_{k-1}(F)\}$ over B, via f.

It is straightforward to see that the obstruction to extend the lifting of f equivariantly to the k–skeleton lies in $H^k(M/G, \partial M/G; \overline{w})$, where $\overline{w}$ is the local coefficient system over M/G, induced by w, as defined above.

We now specialize to the case that applies to fixed point theory.

Let N be an n–dimensional compact differentiable G–manifold, with $n \geq 3$. Let $M \subset N$ be a connected submanifold with boundary, where G acts freely. Consider $f : M \to N$ an equivariant map and assume $f|_{\partial M}$ is fixed point free. We would like to know under which conditions f can be deformed equivariantly to a fixed point free map relative to ∂M.

Consider the diagram

$$\begin{array}{ccc} & & N \times N - \Delta \\ & & \downarrow i_1 \\ M & \xrightarrow{i \times f} & N \times N \end{array},$$

where Δ is the diagonal in $N \times N$ and the maps i and i_1 are natural inclusions.

According to [20], we have over $N \times N$ a local coefficient system with groups $\pi_n(N \times N, N \times N - \Delta) \approx \mathbb{Z}[\pi]$, where $\pi = \pi_1(N)$. The action of $\pi \times \pi$ on $\mathbb{Z}[\pi]$ is given by

$$(\sigma, \tau)\alpha = \mathrm{sgn}(\sigma).(\tau * \alpha * \sigma^{-1}),$$

for $\sigma, \tau \in \pi$ and $\alpha \in \mathbb{Z}[\pi]$.

Now, this local system induces, via $i \times f$, a local system over M, w, where the action of $\pi_1(M)$ in $\pi_n(N \times N, N \times N - \Delta) \approx \mathbb{Z}[\pi]$ is given by:

$$\begin{aligned} \sigma\alpha &= (i_*(\sigma), f_*(\sigma))\alpha = \\ &= \mathrm{sgn}(i_*(\sigma)) f_*(\sigma) \; * \; \alpha \; * \; i_*(\sigma)^{-1}, \end{aligned}$$

where $\sigma \in \pi_1(M)$, $\alpha \in \mathbb{Z}[\pi]$ and i_*, f_* are the induced maps on fundamental groups.

Proposition 5.2 *The local coefficient system*, $\overline{w}$, *over* M/G, *induced by* w *as above*, *is given by* $\overline{w}([x_0]) = \mathbb{Z}[\pi]$ *and the action of* $\pi_1(M/G)$ *on* $\mathbb{Z}[\pi]$ *is given by*:

$$\sigma\alpha = \mathrm{sgn}(\sigma) i_\#(f(\tilde{\sigma}) * \tilde{\sigma}^{-1})(\tilde{\sigma} * g\beta * \tilde{\sigma}^{-1}),$$

for $\alpha \in \pi_1(N)$, $\sigma \in \pi_1(M/G)$, $\tilde{\sigma}$ *a lifting of* σ *to* M *starting at* x_0 *and* β *representing* α *in* N.

Proof Using the general definition of $\overline{w}$ given in the beginning of the section, and following the same argument as in Theorem 3.1 of [20], the result follows. This is a left action and it coincides with the one in [20] when G is the trivial group. □

The main results

Let G be a finite group acting smoothly on a connected, differentiable n–dimensional manifold N. Let $A \subset N$ be a submanifold invariant under the G–action and consider a G–map $f : (N, A) \to (N, A)$ with $f|_A$ fixed point free.

Proposition 5.3 *Suppose G acts freely on $N - A$. If f can be deformed to a fixed point free map and one of the following conditions*

a) *codim* $(A, N) \neq 1$ *and* $\dim N \neq 2$
b) $\pi_1(N) = 0$

holds, then f can be deformed equivariantly (relative to A) to a fixed point free map.

Proof Let us assume first that condition a) holds. We may also assume that $\text{codim}(A, N) \neq 0$.

Let $f_1 = f|_{N-A} : N - A \to A$. It is easy to see that f and f_1 have the same Nielsen classes and therefore the Nielsen number of f, $N(f)$, coincides with the local Nielsen number of f_1 (see [21] for local Nielsen number). Since f is deformable to a fixed point free map, $N(f) = 0$ and so is $N(f_1, N - A)$. This implies, since $\dim N \geq 3$, that we can deform f_1 relative to $(N - A) - \text{int}K$, where K is some compact inside $N - A$. Hence, we can find an equivariant tubular neighborhood $V(A)$ of A, so that f can be deformed, relative to $\partial V(A)$, to a fixed point free map.

Now, the obstruction to deform f equivariantly to a fixed point free map lies on

$$H^n((N - \overset{o}{V}(A))/G, \partial V(A)/G; \overline{w}),$$

where $\overset{o}{V}(A)$ denotes the interior of $V(A)$.

We have just shown that this obstruction belongs to the kernel of

$$p^* : H^n((N - \overset{o}{V}(A))/G, \partial V(A)/G; \overline{w}) \to H^n(N - \overset{o}{V}(A), \partial V(A); w),$$

where p is the projection onto the orbit space. Therefore, is suffices to show that p^* is one to one. In order to do this, consider the transfer

$$\tau^* : H^n(N - \overset{o}{V}(A), \partial V(A); w) \to H^n((N - \overset{o}{V}(A))/G, \partial V(A)/G; \overline{w})$$

which is induced by the map τ given, at the chain level by

$$\tau(\alpha)([\sigma^n]) = \sum_{g \in G} g^{-1}\alpha(g\sigma^n).$$

From now on the proof follows Vidal's ideas (see [50]).

We certainly have $\tau^* \circ p^*$ given by multiplication by the order of G. Hence, it suffices to show that $H^n((N - \overset{o}{V}(A))/G, \partial V(A);\overline{w})$ is torsion-free. By the duality theorem given in Lemma 2.1 in [52], we have

$$H^n((N - \overset{o}{V}(A))/G, \partial V(A)/G;\overline{w}) \approx H_0((N - \overset{o}{V}(A))/G;\overline{w}^t),$$

where $\overline{w}^t$ is the $\mathbb{Z}[\pi_1((N - \overset{o}{V}(A))/G]$–module structure defined in [52]. By definition of $\overline{w}^t$, the action of $\mathbb{Z}[\pi_1((N - \overset{o}{V}(A))/G]$ on $\mathbb{Z}[\pi_1(N)]$ is the one given in Proposition 5.2, except for the term $\mathrm{sgn}(\sigma)$, i.e.,

$$\sigma\alpha = i_*(f(\tilde{\sigma}) \ * \ \tilde{\sigma}^{-1})(\tilde{\sigma} \ * \ g\beta \ * \ \tilde{\sigma}^{-1}).$$

This action defines an equivalence relation on $\pi_1(N)$, which splits $\pi_1(N)$ into disjoint equivalence classes. The quotient of the group ring $\mathbb{Z}[\pi_1(N)]$ by this action is isomorphic to a direct sum of $\mathbb{Z}$'s, indexed by the set of classes $\pi_1(N)/\sim$.

So $H^n((N - \overset{o}{V}(A))/G, \partial V(A)/G;\overline{w})$ is torsion-free and the results follows.

For the case where $\pi_1(N) = 0$, we may assume $\mathrm{codim}(A, N) = 1$, since otherwise the proof would follows the steps of the first part. From the fact that N is simply-connected, we have $w(x_0) \approx \mathbb{Z}$. By hypothesis, the Lefchetz number's $L(f)$ and $L(f|_A)$ vanish and therefore $L(f, f|_A) = 0$. This means that the obstruction to deform f equivariantly maps into zero. By the same arguments as above, we have that $H_0((N - \overset{o}{V}(A))(G;\mathbb{Z}^t) \approx \mathbb{Z}$ and hence, torsion free, and the proof is complete. □

Proposition 5.4 *Suppose G acts on a compact differentiable manifold M with only one orbit type, say G/H. Assume either $\pi_1(M^H) = 0$ or dimension of M^H is bigger than or equal to 3. Then a G–map $f : M \to M$ can be deformed equivariantly to a fixed point free map if and only if $f^H : M^H \to M^H$ can be deformed to a fixed point free map.*

Proof The assumption that the action has only one orbit type implies that $\psi : G \times_{NH} M^H \to M, \psi([g, x]) = gx$, is a G–isomorphism. Here NH denotes de normalizer of H in G (see [5]).

Since NH/H acts freely on M^H, Proposition 5.3 implies that f^H can be deformed NH/H–equivariantly to a fixed point free map. Denote by $(L_t)_{t\in I}$ a NH/H–homotopy which realizes this deformation.

Let $\varphi_t : G \times_{NH} M^H \to G \times_{NH} M^H$ be given by $\varphi_t([g, x]) = [g, L_t(x)]$ and consider the composite $\psi \circ \varphi_t \circ \psi^{-1} : M \to M$. Then it is a G–homotopy with $\psi \circ \varphi_0 \circ \psi^{-1}(x) = \psi[g, L_0(g^{-1}x)] = gf^H(g^{-1}x) = f(x)$, where $x \in M$ and $g \in G$ is such that $G_x = gHg^{-1}$. Now, if x is a fixed point of $\psi \circ \varphi_1 \circ \psi^{-1}$, then $x = \psi \circ \varphi_1([g, g^{-1}x]) = gL_1(g^{-1}x)$ which is a contradiction, since L_1 is fixed point free.

Hence, $\psi \circ \varphi_t \circ \psi^{-1}$ is a G–homotopy from f to a fixed point free map and the proof is done. □

We are now ready to prove the main result. Let us first set up some notation.

Let G act on M and consider $(H_1), \cdots, (H_r)$ the orbit types of the action ordered in a way that if $(H_i) \subset (H_j)$ then $j \leq i$. For each $i \in \{1, \cdots, r\}$, let $M_i = \{x \in M/(G_x) = (H_j), j \leq i\}$. Then $M_1 \subset M_2 \subset \cdots \subset M_r$, $M_1 = M_{(H_1)} = \{x \in M/(G_x) = (H_1)\}$, $M_r = M$ and $M_i - M_{i-1} = \{x \in M/(G_x) = (H_i)\} = M_{(H_i)}$.

Denote by $C_l(H)$ a connected component of M^H, for each subgroup H of G.

Theorem 5.1 *Let $f : M \to M$ be a G–map such that $f^H : M^H \to M^H$ can be deformed to a fixed point free map for each isotropy subgroup H of G. Assume the action satisfies one of the following conditions*:

a) $\dim C_i(H) \neq 2, \forall H \leq G$ *and whenever* $C_i(H) \subset C_j(K)$ *the codimension is different from* 1.
b) $C_i(H)$ *is simply-connected*, $\forall i$, $\forall H \leq G$.

Then f can be deformed equivariantly to a fixed point free map.

Proof We will assume condition a) holds.The proof assuming condition b) is totally analogous.

We look at $f_1 = f|_{M_1}$. Since f^{H_1} is deformable to a fixed point free map and the action on M_1 has only one orbit type, it follows from Proposition 5.4, that f_1 is equivariantly deformable to a fixed point free map. Using G–homotopy extension property, we may assume f_1 is fixed point free.

The proof proceeds by induction. Assuming $f_{i-1} = f|_{M_{i-1}}$ is fixed point free we must show that $f_i = f|_{M_i}$ can be equivariantly deformed (relative to M_{i-1}) to a fixed point free map.

Consider $\overline{f}^{H_i} = f^{H_i}|_{M^{H_i}-F} : M^{H_i} - F \to M^{H_i}$, where $F = M^{H_i} \cap M_{i-1}$, and observe that $M^{H_i} - F = \{x \in M : G_x = H_i\}$. By hypothesis, $\mathrm{codim}(F, M^{H_i}) \geq 2$ which implies that f^{H_i} and $\overline{f}^{H_i}$ have the same Nielsen classes and therefore $N(\overline{f}^{H_i}, M^{H_i} - F) = N(f^{H_i}) = 0$. So f^{H_i} can be deformed to a fixed point free map relative to $M_{i-1} \cap M^{H_i}$. Now, f^{H_i} is a NH_i/H_i–map and NH_i/H_i acts freely on $M^{H_i} - F$, hence, by Proposition 5.2.1, f^{H_i} can be deformed NH_i/H_i–equivariantly, relative to $M^{H_i} \cap M_{i-1}$, to a fixed point free map. Denote by $(H_t)_{t \in I}$, the NH_i/H_i–homotopy which deforms f^{H_i} to fixed point free map. Define

$$\overline{H}_t : G \times_{NH_i} (M^{H_i}, F) \to G \times_{NH_i} (M^{H_i}, F)$$

by

$$\overline{H}_t([g, x]) = [g, H_t(x)].$$

Then $(\overline{H}_t)_{t \in I}$ is G–homotopy with

$$\overline{H}_0([g, x]) = [g, f^{H_i}(x)]$$

and

$$\overline{H}_1([g,x]) = [g, H_1(x)].$$

Let $\psi : G \times_{NH_i} (M^{H_i}, M^{H_i} \cap M_{i-1}) \to (M^{(H_i)}, M^{(H_i)} - M_{(H_i)})$ be given by $\psi([g,x]) = gx$. Here, $M^{(H_i)} = \{x \in M : (G_x) \supset (H_i)\}$. Observe that ψ is onto and one to one when restricted to $G \times_{NH_i} (M^{H_i} - (M^{H_i} \cap M_{i-1}))$. Define, also,

$$L_t : (M^{(H_i)}, M^{(H_i)} - M_{(H_i)}) \to (M^{(H_i)}, M^{(H_i)} - M_{(H_i)})$$

by

$$L_t(x) = \psi(\overline{H}_t([g, g^{-1}x])) = gH_t(g^{-1}x),$$

where $g \in G$ is such that $g^{-1}x \in M^{H_i}$.

It is not hard to verify that L_t is well-defined and G–equivariant. Also,

$$L_0(x) = gH_0(g^{-1}x) = gf^{H_i}(g^{-1}x) = f(x) = f|_{M^{(H_i)}}(x),$$

and

$$L_1(x) = gH_1(g^{-1}x) \text{ is fixed point free.}$$

Hence, $(L_t)_{t\in I}$ is a G–homotopy (relative to $M_{i_1} \cap M^{(H_i)}$) from $f^{(H_i)}$ to a fixed point free map.

But, $M_i = M^{(H_i)} \cup M_{i-1}$ and therefore we may define $\overline{L}_t : M_i \to M_i$ by

$$\overline{L}_t(x) = \begin{cases} L_t(x), & x \in M^{(H_i)} \\ f|_{M_{i-1}(x)}, & x \in M_{i-1} \end{cases}$$

Hence, $(\overline{L}_t)_{t\in I}$ is a G–homotopy (relative to M_{i-1}) from $f_i = f|_{M_i}$ to a fixed point free map and the theorem is proved. □

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