LIMIT CYCLES FOR TWO CLASSES OF CONTROL PIECEWISE LINEAR DIFFERENTIAL SYSTEMS

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ABSTRACT. We study the bifurcation of limit cycles from the periodic orbits of $2n$–dimensional linear centers $\dot{x} = A_0 x$ when they are perturbed inside classes of continuous and discontinuous piecewise linear differential systems of control theory of the form $\dot{x} = A_0 x + \varepsilon(Ax + \phi(x_1)b)$, where ϕ is a continuous or discontinuous piecewise linear function, A_0 is a $2n \times 2n$ matrix with only purely imaginary eigenvalues, ε is a small parameter, A is an arbitrary $2n \times 2n$ matrix, and b is an arbitrary vector of $\mathbb{R}^n$.

1. Introduction and statement of the main results

In control theory it is relevan to study the *continuous piecewise linear differential systems* of the form

$$\dot{x} = Ax + \varphi(x_1)b, \tag{1}$$

with A a $m \times m$ matrix, $x, b \in \mathbb{R}^m$, $\varphi : \mathbb{R} \to \mathbb{R}$ is the continuous piecewise linear function

$$\varphi(x_1) = \begin{cases} -1 & \text{if } x_1 \in (-\infty, -1), \\ x_1 & \text{if } x_1 \in [-1, 1], \\ 1 & \text{if } x_1 \in (1, \infty), \end{cases} \tag{2}$$

where $x = (x_1, \ldots, x_m)^T$, and the dot denotes the derivative with respect to the independent variable t, the time.

The *discontinuous piecewise linear differential systems* of the form (1) where instead of the function φ we have the discontinuous piecewise linear function

$$\psi(x_1) = \begin{cases} -1 & \text{if } x_1 \in (-\infty, 0), \\ 1 & \text{if } x_1 \in (0, \infty). \end{cases} \tag{3}$$

are also importante in control theory. For more details on these continuous and discontinuous piecewise linear differential systems see for instance [1, 3, 4, 12, 13, 18, 20].

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The analysis of discontinuous piecewise linear differential systems goes back mainly to Andronov and coworkers [2] and nowadays still continues to receive attention by many researchers. In particular, discontinuous piecewise linear differential systems appear in a natural way in control theory and in the study of mechanical systems, electrical circuits, ... see for instance the book [5] and the references quoted there. These systems can present complicated dynamical phenomena such as those exhibited by general nonlinear differential systems.

One of the main ingredients in the qualitative description of the dynamical behavior of a differential system is the number and the distribution of its limit cycles. The goal of this paper is to study analytically the existence of limit cycles for a class of continuous and a class of discontinuous piecewise linear differential of the form (1).

More precisely, first we consider the class of continuous piecewise linear differential systems

$$\dot{x} = A_0 x + \varepsilon\big(Ax + \varphi(x_1)b\big), \tag{4}$$

where $|\varepsilon| \neq 0$ is a sufficiently small real parameter, A_0 is the $2n \times 2n$ matrix having on its principal diagonal the following 2×2 matrices

$$\begin{pmatrix} 0 & -(2k-1) \\ 2k-1 & 0 \end{pmatrix} \qquad \text{for } k = 1, \ldots, n,$$

and zeros in the complement, A is an arbitrary $2n \times 2n$ matrix and $b \in \mathbb{R}^{2n} \setminus \{0\}$. Note that for $\varepsilon = 0$ system (4) becomes

$$\dot{x}_1 = -x_2, \quad \dot{x}_2 = x_1, \quad \ldots \quad , \dot{x}_{2n-1} = -(2n-1)x_{2n}, \quad \dot{x}_{2n} = (2n-1)x_{2n-1}. \tag{5}$$

Moreover, the origin of (5) is a *global isochronous center* in $\mathbb{R}^{2n}$, i.e. all its orbits different from the origin are periodic with period 2π.

In a similar way we consider the discontinuous piecewise linear differential systems

$$\dot{x} = A_0 x + \varepsilon\big(Ax + \psi(x_1)b\big). \tag{6}$$

Our main results on the limit cycles of the continuous and discontinuous piecewise linear differential systems (4) are the following ones.

Theorem 1. *For $|\varepsilon| > 0$ sufficiently small the averaging theory of first order provides the existence of at least one limit cycle γ_ε for the continuous piecewise linear differential system* (4) *that bifurcates from the periodic orbits of system* (5)*, i.e. γ_ε tends to a periodic solution of system* (5) *when $\varepsilon \to 0$. Moreover there are systems* (4) *with $|\varepsilon| > 0$ sufficiently small having such a limit cycle.*

Theorem 1 is proved in section 3.

Theorem 2. *For $|\varepsilon| > 0$ sufficiently small the averaging theory of first order provides the existence of at least one limit cycle γ_ε for the discontinuous piecewise linear differential system (6) that bifurcates from the periodic orbits of system (5). Moreover there are systems (6) with $|\varepsilon| > 0$ sufficiently small having such a limit cycle.*

Theorem 2 is proved in section 4.

Since the differential systems of Theorems 1 and 2 are non-smooth the stability or instability of the periodic solutions provided in those theorems is not easy to study, but for the continuous piecewise differential systems can be studied using results of the reference [8].

According to the results of the averaging theory used it follows that for the control differential systems here studied, the limit cycles that we obtain bifurcate from some periodic orbit of the $2n$-dimensional linear differential center (5). This technique of finding limit cycles bifurcating from centers has been intensively studied in dimension 2, see for instance the book of Christopher and Li [11] and the hundreds of references quoted therein.

Other results different to the ones presented here, but which also study the limit cycles of control systems of the form (1) using averaging theory, can be found in [7, 9, 10, 14, 15].

The main tools for proving the previous theorems are the extensions of the classical averaging theory for computing periodic solutions of $\mathcal{C}^2$ differential systems to continuous and discontinuous differential systems. In section 2 we summarize the extensions of the averaging theory that we shall use here in the proofs of our results.

2. First order averaging theory

For the classical averaging theory used to study the existence of periodic orbits in differential systems of class $\mathcal{C}^2$ see, for instance, the chapter 11 of the book of Verhulst [19].

We begin this section presenting the result on the continuous averaging theory used in Theorem 1. This theory uses the Brouwer degree for continuous function and its proof can be found in [6].

Theorem 3. *We consider the following differential system*

$$\dot{x} = \varepsilon H(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{7}$$

where $H : \mathbb{R} \times D \to \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are continuous functions, T-periodic in the first variable, and D is an open bounded subset of $\mathbb{R}^n$. We define $h : D \to \mathbb{R}^n$ as

$$h(z) = \int_0^T H(s,z)ds, \tag{8}$$

and assume that

(i) *H and R are locally Lipschitz with respect to x;*
(ii) *for $p \in D$ with $h(p) = 0$, there exists a neighborhood V of p such that $h(z) \neq 0$ for all $z \in \bar{V} \setminus \{p\}$ and the Brouwer degree $d_B(h, V, 0) \neq 0$.*

Then, for $|\varepsilon| \neq 0$ sufficiently small, there exists an isolated T- periodic solution $x(t,\varepsilon)$ of system (7) such that $x(0,\varepsilon) \to p$ as $\varepsilon \to 0$.

Remark 4. *Let $h : D \to \mathbb{R}^n$ be a C^1 function with $h(p) = 0$, where D is an open bounded subset of $\mathbb{R}^n$ and $p \in D$. If the Jacobian of h at p is not zero, then there exists a neighborhood V of p such that $h(z) \neq 0$ for all $z \in \bar{V} \setminus \{p\}$, and the Brouwer degree $d_B(h, V, p) \in \{-1, 1\}$.*

For a proof of Remark 4 see [17].

In the proof of Theorem 2 an extension of the averaging theory for discontinuous differential systems is used, see [16] for details. Before state this result we present some definitions about discontinuous differential systems.

Considering $D \subset \mathbb{R}^n$ an open subset and $g : \mathbb{R} \times D \to \mathbb{R}$ a C^1-function having the origin as regular value. Let $F^1, F^2 : \mathbb{R} \times D \to \mathbb{R}^n$ be continuous functions and $\Sigma = g^{-1}(0)$. We define the discontinuous differential system

$$\dot{x}(t) = F(t, x) = \begin{cases} F^1(t, x) & \text{if } (t, x) \in \Sigma^+, \\ F^2(t, x) & \text{if } (t, x) \in \Sigma^-, \end{cases}$$

where $\Sigma^+ = \{(t, x) \in \mathbb{R} \times D \, : \, g(t, x) > 0\}$ and $\Sigma^- = \{(t, x) \in \mathbb{R} \times D \, : \, g(t, x) < 0\}$.

The manifold Σ can be split as the closure of two disjoint regions, that is, $\Sigma = \overline{\Sigma^c} \cup \overline{\Sigma^s}$ where Σ^c is the *Crossing region* and Σ^s is called *Sliding region*, respectively defined as

$$\Sigma^c = \{p \in \Sigma \, : \, \langle \nabla g(p), (1, F^1(p)) \rangle \cdot \langle \nabla g(p), (1, F^2(p)) \rangle > 0\},$$
$$\Sigma^s = \{p \in \Sigma \, : \, \langle \nabla g(p), (1, F^1(p)) \rangle \cdot \langle \nabla g(p), (1, F^2(p)) \rangle < 0\}.$$

Theorem 5. *We consider the following discontinuous differential system*

$$x'(t) = \varepsilon H(t, x) + \varepsilon^2 R(t, x, \varepsilon), \tag{9}$$

with

$$H(t, x) = H_1(t, x) + \text{sign}(g(t, x)) H_2(t, x),$$
$$R(t, x, \varepsilon) = R_1(t, x, \varepsilon) + \text{sign}(g(t, x)) R_2(t, x, \varepsilon),$$

where $H_1, H_2 : \mathbb{R} \times D \to \mathbb{R}^n$, $R_1, R_2 : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ and $g : \mathbb{R} \times D \to \mathbb{R}$ are continuous functions, T–periodic in the variable t and D is an open subset of $\mathbb{R}^n$. We also suppose that g is a $\mathcal{C}^1$ function having 0 as a regular value.

Define the average function $h : D \to \mathbb{R}^n$ as

$$h(x) = \int_0^T H(t, x) dt. \tag{10}$$

We assume the following conditions.

(*i*) *H_1, H_2, R_1, R_2 are locally Lipschitz with respect to x;*
(*ii*) *there exists an open bounded subset $C \subset D$ such that, for $|\varepsilon| > 0$ sufficiently small, every orbit starting in $\overline{C}$ reaches the set of discontinuity only at its crossing regions.*
(*iii*) *for $a \in C$ with $h(a) = 0$, there exists a neighbourhood $U \subset C$ of a such that $h(z) \neq 0$ for all $z \in \overline{U} \setminus \{a\}$ and $d_B(h, U, 0) \neq 0$.*

Then, for $|\varepsilon| > 0$ sufficiently small, there exists a T–periodic solution $x(t, \varepsilon)$ of system (9) such that $x(0, \varepsilon) \to a$ as $\varepsilon \to 0$.

3. Proof of Theorem 1

The main tool for proving Theorem 1 is the averaging theory of first order for continuous differential systems as stated in Theorem 3. In order to use this result we need to apply some changes of variables in the differential system (4) to re-write it in the normal form (7).

Lemma 6. *Doing the change of variables $(x_1, x_2, \dots, x_{2n}) \mapsto (\theta, r, \theta_1, r_1, \dots, \theta_{n-1}, r_{n-1})$ defined by*

$$\begin{aligned} x_1 &= r\cos\theta, \\ x_2 &= r\sin\theta, \\ x_{2j-1} &= r_{j-1}\cos((2j-1)\theta + \theta_{j-1}), \\ x_{2j} &= r_{j-1}\sin((2j-1)\theta + \theta_{j-1}), \end{aligned}$$

for $j = 2, \dots, n$ system (4) is transformed into the system

$$\begin{aligned} \frac{dr}{d\theta} &= \varepsilon H_1(\theta, r, \theta_1, r_1, \dots, \theta_{n-1}, r_{n-1}) + \mathcal{O}(\varepsilon^2), \\ \frac{dr_{j-1}}{d\theta} &= \varepsilon H_{2(j-1)}(\theta, r, \theta_1, r_1, \dots, \theta_{n-1}, r_{n-1}) + \mathcal{O}(\varepsilon^2), \\ \frac{d\theta_{j-1}}{d\theta} &= \varepsilon H_{2j-1}(\theta, r, \theta_1, r_1, \dots, \theta_{n-1}, r_{n-1}) + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{11}$$

where

$$H_1 = \sum_{l=1}^{n} r_{l-1}\Big(F_{1,l}\cos\theta + F_{2,l}\sin\theta\Big) + \varphi(r\cos\theta)(b_1\cos\theta + b_2\sin\theta),$$

and for $j = 2, 3, \dots, n$ we have

$$\begin{aligned} H_{2(j-1)} = &\sum_{l=1}^{n} r_{l-1}\Big(F_{2j-1,l}\cos((2j-1)\theta + \theta_{j-1}) + F_{2j,l}\sin((2j-1)\theta + \theta_{j-1})\Big) \\ &+\varphi(r\cos\theta)\big[b_{2j-1}\cos((2j-1)\theta + \theta_{j-1}) + b_{2j}\sin((2j-1)\theta + \theta_{j-1})\big], \end{aligned}$$

$$\begin{aligned} H_{2j-1} = &\sum_{l=1}^{n} \frac{r_{l-1}}{r_{j-1}}\Big(F_{2j,l}\cos((2j-1)\theta + \theta_{j-1}) - F_{2j-1,l}\sin((2j-1)\theta + \theta_{j-1})\Big) \\ &+(2j-1)\sum_{l=1}^{n} \frac{r_{l-1}}{r}\Big(F_{1,l}\sin\theta - F_{2,l}\cos\theta\Big) \\ &+\varphi(r\cos\theta)\Big(\frac{b_{2j}}{r_{j-1}}\cos((2j-1)\theta + \theta_{j-1}) - \frac{b_{2j-1}}{r_{j-1}}\sin((2j-1)\theta + \theta_{j-1})\Big) \\ &-(2j-1)\varphi(r\cos\theta)\Big(\frac{b_2}{r}\cos\theta - \frac{b_1}{r}\sin\theta\Big), \end{aligned}$$

with

$$F_{i,l} = F_{i,l}(r,\theta,\theta_{l-1}) = a_{i(2l-1)}\cos((2l-1)\theta+\theta_{l-1}) + a_{i(2l)}\sin((2l-1)\theta+\theta_{l-1}).$$

We take ε_0 sufficiently small, m arbitrarily large and

$$D_m = \left\{(r,\theta_1,r_1,\ldots,\theta_{n-1},r_{n-1}) \in \left(\frac{1}{m},m\right) \times \left[\mathbb{S}^1 \times \left(\frac{1}{m},m\right)\right]^{n-1}\right\}.$$

Then the vector field associated with the differential system (11) *is well defined and continuous on $\mathbb{S}^1 \times D_m \times (-\varepsilon_0,\varepsilon_0)$. Moreover the system is 2π-periodic with respect to variable θ and locally Lipschitz with respect to variables $(r,\theta_1,r_1,\ldots,\theta_{n-1},r_{n-1})$.*

Proof. In the variables $(\theta,r,\theta_1,r_1,\ldots,\theta_{n-1},r_{n-1})$ the differential system (4) becomes

$$\begin{aligned}
\dot\theta =&\ 1+\frac{\varepsilon}{r}\left[\sum_{l=1}^{n} r_{l-1}\left(F_{2,l}\cos\theta - F_{1,l}\sin\theta\right) + \varphi(r\cos\theta)(b_2\cos\theta - b_1\sin\theta)\right],\\
\dot r =&\ \varepsilon H_1(\theta,r,\theta_1,r_1,\ldots,\theta_{n-1},r_{n-1}),\\
\dot r_{j-1} =&\ \varepsilon H_{2(j-1)}(\theta,r,\theta_1,r_1,\ldots,\theta_{n-1},r_{n-1}),\\
\dot\theta_{j-1} =&\ \varepsilon H_{2j-1}(\theta,r,\theta_1,r_1,\ldots,\theta_{n-1},r_{n-1}),
\end{aligned}$$

for $j = 2,3,\ldots,n$. Note that for $\varepsilon = 0$, $\dot\theta(t) > 0$ and hence for $|\varepsilon| \neq 0$ sufficiently small this property remains valid for each t when $(\theta,r,\theta_1,r_1,\ldots,\theta_{n-1},r_{n-1}) \in \mathbb{S}^1 \times D_m$. Now we take θ as the new independent variable. The right-hand side of the new system is well defined and continuous in $\mathbb{S}^1 \times D_m \times (-\varepsilon_0,\varepsilon_0)$ and it is 2π-periodic with respect to the new variable θ and locally Lipschitz with respect to $(r,\theta_1,r_1,\ldots,\theta_{n-1},r_{n-1})$. Now system (6) can be obtained doing a Taylor series expansion in the parameter ε around $\varepsilon = 0$. □

The next step is to find the corresponding average function (8) of system (6) that we denoted by $h = (h_1,h_2,\ldots,h_{2(n-1)},h_{2n-1}) : D_m \to \mathbb{R}^{n-1}$ and it is defined by

$$\begin{aligned}
h_1 =&\ h_1(r,\theta_1,r_1,\ldots,\theta_{n-1},r_{n-1}) = \int_0^{2\pi} H_1(r,\theta_1,r_1,\ldots,\theta_{n-1},r_{n-1})d\theta,\\
h_{2(j-1)} =&\ h_{2(j-1)}(r,\theta_1,r_1,\ldots,\theta_{n-1},r_{n-1}) = \int_0^{2\pi} H_{2(j-1)}(r,\theta_1,r_1,\ldots,\theta_{n-1},r_{n-1})d\theta,\\
h_{2j-1} =&\ h_{2j-1}(r,\theta_1,r_1,\ldots,\theta_{n-1},r_{n-1}) = \int_0^{2\pi} H_{2j-1}(r,\theta_1,r_1,\ldots,\theta_{n-1},r_{n-1})d\theta,
\end{aligned}$$

for $j = 1,2,\ldots,n$. To calculate these integrals we will use the following equalities

$$\int_0^{2\pi} \cos((2j-1)\theta+\theta_{j-1})\sin((2l-1)\theta+\theta_{l-1})d\theta = 0 \qquad \text{for all integers } l,j>1,$$

$$\int_0^{2\pi} \cos((2j-1)\theta+\theta_{j-1})\cos((2l-1)\theta+\theta_{l-1})d\theta = \begin{cases}\pi & \text{if } l=j,\\ 0 & \text{if } l\neq j,\end{cases}$$

$$\int_0^{2\pi} \sin((2j-1)\theta+\theta_{j-1})\sin((2l-1)\theta+\theta_{l-1})d\theta = \begin{cases} \pi & \text{if } l=j, \\ 0 & \text{if } l\neq j, \end{cases}$$

and the next lemma.

For $r>0$ and $j=1,2,\dots,n$ we denote

$$\begin{aligned} I_j(r) = & \int_0^{2\pi} \varphi(r\cos\theta)\cos((2j-1)\theta)d\theta, \\ J_j(r) = & \int_0^{2\pi} \varphi(r\cos\theta)\sin((2j-1)\theta)d\theta, \end{aligned}$$

where φ is the piecewise linear function (2).

Lemma 7. *The integrals I_j and $J_j(r)$ satisfy*

$$I_j(r) = \begin{cases} \pi r & \text{if } j=1 \text{ and } 0<r\le 1, \\ 0 & \text{if } j>1 \text{ and } 0<r\le 1, \\ K(r) & \text{if } j=1 \text{ and } r>1, \\ L_j(r) & \text{if } j>1 \text{ and } r>1; \end{cases}$$

$$J_j(r) = \quad 0 \qquad \text{for all } j=1,2,\dots,n \text{ and } r>0.$$

where

$$\begin{aligned} L_j(r) = & \frac{2}{j(2j-1)^2}\Big((2j-1)\sqrt{-1+r^2}\cos((2j-1)\arctan\sqrt{-1+r^2}) \\ & -\sin((2j-1)\arctan\sqrt{-1+r^2})\Big), \\ K(r) = & \pi r + \frac{2}{r}\sqrt{r^2-1} - 2r\arctan(\sqrt{r^2-1}). \end{aligned}$$

Proof. We consider two cases: $0<r\le 1$ and $r>1$.

Case 1: $0<r\le 1$ In this case $|r\cos\theta|\le 1$ and hence $\varphi(r\cos\theta)=r\cos\theta$ for all $\theta\in[0,2\pi]$. Then if $j=1$

$$\int_0^{2\pi} \varphi(r\cos\theta)\cos\theta d\theta = r\int_0^{2\pi}\cos^2\theta d\theta = \pi r,$$

and

$$\int_0^{2\pi} \varphi(r\cos\theta)\sin\theta d\theta = r\int_0^{2\pi}\cos\theta\sin\theta d\theta = 0.$$

And if $j>1$ then

$$\int_0^{2\pi} \varphi(r\cos\theta)\cos((2j-1)\theta)d\theta = r\int_0^{2\pi}\cos\theta\cos((2j-1)\theta)d\theta = 0,$$

$$\int_0^{2\pi} \varphi(r\cos\theta)\sin((2j-1)\theta)d\theta = r\int_0^{2\pi}\cos\theta\sin((2j-1)\theta)d\theta = 0.$$

Case 2: $r > 1$ In this case choose $\theta_c \in (0, \pi/2)$ such that $\cos\theta_c = 1/r$. If $j = 1$ we have

$$\begin{aligned} I_1(r) = & \int_0^{\theta_c} \cos\theta d\theta + r\int_{\theta_c}^{\pi-\theta_c} \cos^2\theta d\theta - \int_{\pi-\theta_c}^{\pi+\theta_c} \cos\theta d\theta \\ & + r\int_{\pi+\theta_c}^{2\pi-\theta_c} \cos^2\theta d\theta + \int_{2\pi-\theta_c}^{2\pi} \cos\theta d\theta \\ = & \pi r + \frac{2}{r}\sqrt{r^2-1} - 2r\arctan(\sqrt{r^2-1}). \end{aligned}$$

The same reasoning can be applied to see that $J_1(r) = 0$. If $j > 1$ then

$$\begin{aligned} I_j(r) = & \int_0^{\theta_c} \cos((2j-1)\theta)d\theta + r\int_{\theta_c}^{\pi-\theta_c} \cos\theta\cos((2j-1)\theta)d\theta - \int_{\pi-\theta_c}^{\pi+\theta_c} \cos((2j-1)\theta)d\theta \\ & + r\int_{\pi+\theta_c}^{2\pi-\theta_c} \cos\theta\cos((2j-1)\theta)d\theta + \int_{2\pi-\theta_c}^{2\pi} \cos((2j-1)\theta)d\theta \\ = & \frac{2}{j(2j-1)^2}\Big((2j-1)\sqrt{-1+r^2}\cos((2j-1)\arctan\sqrt{-1+r^2}) \\ & - \sin((2j-1)\arctan\sqrt{-1+r^2})\Big), \end{aligned}$$

and $J_j(r) = 0$. □

With the results presented previously we are able to prove Theorem 1. Since we can choose m sufficiently large to find the zeroes of the average function h in D_m it is sufficient to look for them in $(0, \infty) \times [\mathbb{S}^1 \times (0, \infty)]^{n-1}$. To calculate the expression of the average function we consider again two cases.

Case 1: $0 < r \leq 1$. In this case the system whose zeros can provide limit cycles of system (4) is

$$\begin{aligned} h_1 = & (a_{11} + a_{22} + b_1)\pi r, \\ h_2 = & (a_{33} + a_{44})\pi r_1, \\ h_3 = & (a_{43} - a_{34} + 3(a_{12} - a_{21} - b_2))\pi, \\ & \vdots \\ h_{2(n-1)} = & (a_{(2n-1)(2n-1)} + a_{(2n)(2n)}))\pi r_{n-1}, \\ h_{2n-1} = & (a_{(2n)(2n-1)} - a_{(2n-1)(2n)} + (2n-1)(a_{12} - a_{21} - b_2)\pi. \end{aligned} \tag{12}$$

Note that the variables $\theta_1, \theta_2, \ldots, \theta_{n-1}$ does not appear explicitly into system (12). Hence, if this system has zeros, it has a continuum of zeros. Therefore the assumption (ii) of the averaging theory, presented in Theorem 3, is not satisfied and this theorem does not provide any information about the limit cycles of system (11).

Case 2: $r > 1$. Now the system whose zeros can provide limit cycles of system (11) is

$$\begin{aligned}
h_1 = &\quad (a_{11}+a_{22})\pi r + b_1 K(r),\\
h_2 = &\quad (a_{33}+a_{44})\pi r_1 + (b_3\cos\theta_1 + b_4\sin\theta_1)L_2(r),\\
h_3 = &\quad (a_{43}-a_{34}+3(a_{12}-a_{21}))\pi - \\
&\quad \frac{3b_2 r_1 K(r) - r(b_4\cos\theta_1 - b_3\sin\theta_1)L_2(r)}{r r_1},\\
&\quad \vdots \\
h_{2(n-1)} = &\quad (a_{(2n-1)(2n-1)} + a_{(2n)(2n)})\pi r_{n-1} + \\
&\quad (b_{2n-1}\cos\theta_{n-1} + b_{2n}\sin\theta_{n-1})L_n(r),\\
h_{2n-1} = &\quad (a_{(2n)(2n-1)} - a_{(2n-1)(2n)} + (2n-1)(a_{12}-a_{21}))\pi - \\
&\quad \frac{(2n-1)b_2 r_{n-1}K(r) - r(b_{2n}\cos\theta_{n-1} - b_{2n-1}\sin\theta_{n-1})L_n(r)}{r r_{n-1}},
\end{aligned} \tag{13}$$

For each $j \in \{2, 3, \dots, n\}$ we will study the zeros of the system

$$\begin{aligned}
h_1 = &\quad (a_{11}+a_{22})\pi r + b_1 K(r),\\
h_{2(j-1)} = &\quad (a_{(2j-1)(2j-1)} + a_{(2j)(2j)})\pi r_{j-1} + \\
&\quad (b_{2j-1}\cos\theta_{j-1} + b_{2j}\sin\theta_{j-1})L_j(r),\\
h_{2j-1} = &\quad (a_{(2j)(2j-1)} - a_{(2j-1)(2j)} + (2j-1)(a_{12}-a_{21}))\pi - \\
&\quad \frac{(2j-1)b_2 r_{j-1}K(r) - r(b_{2j}\cos\theta_{j-1} - b_{2j-1}\sin\theta_{j-1})L_j(r)}{r r_{j-1}},
\end{aligned}$$

Claim: *The function $K : (1,\infty) \to (\pi, 4)$ is a diffeomorphism.* Indeed note that K is twice differentiable with

$$K'(r) = \pi - 2\frac{\sqrt{r^2-1}}{r^2} - 2\arctan\sqrt{r^2-1},$$

and

$$K''(r) = -\frac{4}{r^3\sqrt{r^2-1}} < 0$$

which implies that K' is a strictly decreasing function. Moreover $\lim_{r\to\infty} K'(r) = 0$ what means that $K'(r)$ has a horizontal asymptote given by the axis r and then $K'(r) \geq 0$. Suppose that there exists an $r_0 \in (1,\infty)$ such that $K'(r_0) = 0$. Then for all $r > r_0$ we have $K'(r) < K'(r_0) = 0$, contradiction. Therefore it follows that $K'(r) > 0$ for all $r \in (1,\infty)$ and the Inverse Function Theorem guarantees that K is a local diffeomorphism and since that K is an injective function we obtain the global diffeomorphism, ending the proof of this claim.

First we note that in order to $h_1 = 0$ has solutions for $r > 1$ it is necessary that $b_1(a_{11} + a_{22}) < 0$. Moreover since $K'(r) > 0$ and $K''(r) < 0$ the graphic of $K(r)$ in $(1, \infty)$ is increasing and convex, consequently the graphic of $K(r)$ and the straight line $(a_{11} + a_{22})\pi r$ can intersect the curve $K(r)$ at most in two points.

But if some straight line intercept the graph of $K(r)$ in two points then it cannot pass through the origin. Then the equation $h_1 = 0$ has at most one solution if $r > 1$ and we can choose the coefficients a_{11}, a_{22} and b_1 taking r_0 in the domain of K such that the equation $\dfrac{K(r_0)}{r_0} = -\dfrac{(a_{11} + a_{22})}{b_1}\pi$ holds. We denote this solution by r_0 and substitute it into the equations $h_{2(j-1)} = 0$ and $h_{2j-1} = 0$. Defining

$$
\begin{aligned}
A_j = &\quad (a_{(2j-1)(2j-1)} + a_{(2j)(2j)})\pi, \qquad B_j = b_{2j-1}L_j(r_0), \qquad C_j = b_{2j}L_j(r_0),\\
D_j = &\quad (a_{(2j)(2j-1)} - a_{(2j-1)(2j)} + (2j-1)(a_{12} - a_{21}))\pi - \frac{1}{r_0}(2j-1)b_2K(r_0),\\
u_j = &\quad \cos\theta_{j-1}, \qquad v_j = \sin\theta_{j-1}.
\end{aligned}
$$

But $h_{2(j-1)} = h_{2j-1} = 0$ is equivalent to

$$
\begin{aligned}
A_j r_{j-1} + B_j u_j + C_j v_j = &\quad 0,\\
D_j r_{j-1} + C_j u_j - B_j v_j = &\quad 0,\\
u_j^2 + v_j^2 - 1 = &\quad 0.
\end{aligned}
$$

From the first two equations we obtain

$$
u_j = -\frac{(A_jB_j + C_jD_j)r_{j-1}}{B_j^2 + C_j^2}, \qquad v_j = \frac{(B_jD_j - A_jC_j)r_{j-1}}{B_j^2 + C_j^2}.
$$

Substituting these two expressions in the third one we get

$$
(A_j^2 + D_j^2)r_{j-1}^2 - B_j^2 - C_j^2 = 0.
$$

Therefore there is at most one solution $r_{j-1} > 0$, which provide a unique u_j and v_j. Since we fixed an arbitrarily j to solve this system, the same reasoning can be applied to each pair of equations $h_{2(j-1)} = 0$ and $h_{2j-1} = 0$. So we conclude that system (13) has at most one solution. Moreover it is possible to choose conveniently the parameters of the initial system (4) such that this solution exists and if its Jacobian is not zero, so the averaging theory provides that at most one limit cycle bifurcating from the periodic orbits of the center of system (5) when we perturb it as in system (4). The following example provides a system in the assumptions of the theorem such that the corresponding solution of system (4) has nonzero Jacobian, so for that system the periodic solution provides a limit cycle and completes the proof of Theorem 1.

Now we present an explicit example of a continuous piecewise linear differential system as (4) in $\mathbb{R}^4$ and, repeating the process described in the proof of Theorem 1 we guarantee that such system has one limit cycle. Consider the following differential system

$$\dot{x} = A_0 x + \varepsilon(Ax + \varphi(x_1)b), \tag{14}$$

where

$$A_0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \end{pmatrix}, A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & \dfrac{18\pi - \sqrt{3}}{9\pi} \end{pmatrix}, b = \begin{pmatrix} -\dfrac{24\pi}{3\sqrt{3}+2\pi} \\ 1 \\ \dfrac{9(3-2\sqrt{3}\pi)}{2} \\ -1 \end{pmatrix}.$$

Doing the change or variables $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $x_3 = r_1\cos(3\theta+\theta_1)$, $x_4 = r_1\sin(3\theta+\theta_1)$ and taking θ as the new independent variable we obtain the system

$$\begin{aligned} r'(\theta) = \ & \frac{dr}{d\theta} = \varepsilon H_1(\theta, r, \theta_1, r_1) + \mathcal{O}(\varepsilon^2), \\ r_1'(\theta) = \ & \frac{dr_1}{d\theta} = \varepsilon H_2(\theta, r, \theta_1, r_1) + \mathcal{O}(\varepsilon^2), \\ \theta_1'(\theta) = \ & \frac{d\theta_1}{d\theta} = \varepsilon H_3(\theta, r, \theta_1, r_1) + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{15}$$

where

$$\begin{aligned} H_1(\theta, r, \theta_1, r_1) = \ & 2r + \varphi(r\cos\theta)\sin\theta + \cos\theta\left(r\sin\theta - \frac{24\pi\varphi(r\cos\theta)}{3\sqrt{3}+2\pi}\right), \\ H_2(\theta, r, \theta_1, r_1) = \ & -\frac{1}{18\pi}\Big(18\pi\varphi(r\cos\theta)\sin(3\theta+\theta_1) + 9\pi r_1\sin(2(3\theta+\theta_1)) \\ & +\sqrt{3}r_1 + 81\pi(2\sqrt{3}\pi - 3)\varphi(r\cos\theta)\cos(3\theta+\theta_1) - \\ & (\sqrt{3} - 36\pi)r_1\cos(2(3\theta+\theta_1))\Big), \end{aligned} \tag{16}$$

(17)

$$\begin{aligned} H_3(\theta, r, \theta_1, r_1) = \ & \sin^2(3\theta+\theta_1) + 2\sin(2(3\theta+\theta_1)) - \frac{\sin(2(3\theta+\theta_1))}{6\sqrt{3}\pi} + 3\sin^2\theta \\ & -\frac{72\pi\varphi(r\cos\theta)\sin\theta}{3\sqrt{3}r + 2\pi r} - \frac{\varphi(r\cos\theta)\cos(3\theta+\theta_1)}{r_1} - \frac{3\varphi(r\cos\theta)\cos\theta}{r} \\ & +\frac{9\pi\sqrt{3}\varphi(r\cos\theta)\sin(3\theta+\theta_1)}{r_1} - \frac{27\varphi(r\cos\theta)\sin(3\theta+\theta_1)}{2r_1}. \end{aligned}$$

After some computations the average function $h = (h_1, h_2, h_3)$ defined in (8) is

$$\begin{aligned} h_1(r,\theta_1,r_1) = &\quad 4\pi r - \frac{24\pi}{3\sqrt{3}+2\pi}\left(\pi r + \frac{2\sqrt{r^2-1}}{r} - 2r\arctan(\sqrt{r^2-1})\right),\\ h_2(r,\theta_1,r_1) = &\quad \frac{\sqrt{3}}{3}\sin\theta_1 + \frac{3}{2}(2\sqrt{3}\pi - 3)\sqrt{3}\cos\theta_1 - \frac{\sqrt{3}}{9}r_1,\\ h_3(r,\theta_1,r_1) = &\quad \frac{9\sqrt{3}\sin\theta_1}{2r_1} - \frac{9\pi\sin\theta_1}{r_1} + \frac{\sqrt{3}\cos\theta_1}{3r_1} - \frac{3}{2}\left(\sqrt{3} + \frac{2\pi}{3}\right) + 4\pi. \end{aligned}$$

Using the same steps described in the proof of Theorem 1 to solve system $h_1 = h_2 = h_3 = 0$ we obtaining that $(r^*, \theta_1^*, r_1^*) = (2, \pi/2, 3)$ is a zero of the average function. Moreover if $J = J(r, \theta_1, r_1)$ is the Jacobian matrix of h then $\det J(2, \pi/2, 3) \neq 0$ which implies that such zero is a simple zero. By Theorem 3 if follows that system (15) and consequently system (14) has one limit cycle for $|\varepsilon| > 0$ sufficiently small.

4. Proof of Theorem 2

This section is devoted to prove Theorem 2. Firstly we check hypothesis (ii) of Theorem 5. The function that defines the set of discontinuity of system (6) is $g(x_1, x_2, \ldots, x_{2n}) = x_1$. Applying the same change of variables made the proof of Theorem 1 we get, for $\varepsilon = 0$, $F^1(p) = F^1(r, \theta_1, r_1, \ldots, \theta_{n-1}, r_{n-1}) \equiv 0$ and $F^2(p) = F^2(r, \theta_1, r_1, \ldots, \theta_{n-1}, r_{n-1}) \equiv 0$. So

$$\begin{aligned} &\langle \nabla g(p), (1, F^1(p))\rangle \langle \nabla g(p), (1, F^2(p))\rangle = \\ &\langle (\cos\theta, 0, \ldots, 0), (1, 0, \ldots, 0)\rangle \langle (\cos\theta, 0, \ldots, 0), (1, 0, \ldots, 0)\rangle = \cos^2\theta > 0 \end{aligned}$$

For $|\varepsilon| \neq 0$ sufficiently small we are dealing with points in the crossing region and the hypothesis (ii) is satisfied. So we can apply (ii) of Theorem 5.

Here the same kind of arguments used in the proof of Theorem 1 can be applied to the discontinuous system (6) and, doing this we obtain the average function $h = (h_1, h_2, \ldots, h_{2(n-1)}, h_{2n-1}) : D_m \to \mathbb{R}^{n-1}$ as defined in (10)

$$\begin{aligned} h_1 = &\quad (a_{11} + a_{22})\pi r + b_1\widetilde{I}_1,\\ h_{2(j-1)} = &\quad (a_{(2j-1)(2j-1)} + a_{(2j)(2j)})\pi r_{j-1} + (b_{2j-1}\cos\theta_{j-1} + b_{2j}\sin\theta_{j-1})\widetilde{I}_j,\\ h_{2j-1} = &\quad (a_{(2j)(2j-1)} - a_{(2j-1)(2j)} + (2j-1)(a_{12} - a_{21}))\pi - \\ &\quad \frac{(2j-1)b_2 r_{j-1}\widetilde{I}_1 - r(b_{2j}\cos\theta_{j-1} - b_{2j-1}\sin\theta_{j-1})\widetilde{I}_j}{r r_{j-1}}, \end{aligned} \tag{18}$$

for $j = 2, 3, \ldots, n$, where

$$\widetilde{I}_j = \begin{cases} -\dfrac{4}{(2j-1)} & \text{if } j \text{ is even},\\ \dfrac{4}{(2j-1)} & \text{if } j \text{ is odd}. \end{cases}$$

In fact if we define

$$\begin{aligned}\widetilde{I}_j &= \int_0^{2\pi} \psi(r\cos\theta)\cos((2j-1)\theta)d\theta,\\ \widetilde{J}_j &= \int_0^{2\pi} \psi(r\cos\theta)\sin((2j-1)\theta)d\theta,\end{aligned}$$

where ψ is the piecewise linear function given by (3) then we get

$$\begin{aligned}\widetilde{I}_j &= \int_0^{2\pi} \psi(r\cos\theta)\cos((2j-1)\theta)d\theta\\ &= \int_0^{\pi/2} \cos((2j-1)\theta)d\theta - \int_{\pi/2}^{3\pi/2} \cos((2j-1)\theta)d\theta + \int_{3\pi/2}^{2\pi} \cos((2j-1)\theta)d\theta\\ &= -\frac{4}{(2j-1)}\cos(j\pi),\end{aligned}$$

and

$$\begin{aligned}\widetilde{J}_j &= \int_0^{2\pi} \psi(r\cos\theta)\sin((2j-1)\theta)d\theta\\ &= \int_0^{\pi/2} \sin((2j-1)\theta)d\theta - \int_{\pi/2}^{3\pi/2} \sin((2j-1)\theta)d\theta + \int_{3\pi/2}^{2\pi} \sin((2j-1)\theta)d\theta\\ &= -\frac{4}{(2j-1)}\sin(2j\pi)\cos(j\pi) = 0.\end{aligned}$$

Note that $\widetilde{I}_j$ is a constant non zero real number and hence h_1 is a straight line, consequentely system (18) has at most one positive zero. Moreover if we choose conveniently the coefficients b_1 a_{11} and a_{22} we can obtain a simple positive zero of system (18). This completes the proof of Theorem 2.

5. Additional Considerations

If instead of the matrix A_0 we consider the matrix A_1 where A_1 is the $2n\times 2n$ matrix having on its principal diagonal the following 2×2 matrices

$$\begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix} \qquad \text{for } k = 1,\dots,n,$$

and zeros in the complement, then the averaging theory of first order does not provide any information about the limit cycles of the systems. Indeed we have the following results.

Lemma 8. *The averaging theory of first order does not provide any information about the limit cycles of the continuous piecewise linear differential system*

$$\dot{x} = A_1 x + \varepsilon\big(Ax + \varphi(x_1)b\big). \tag{19}$$

Proof. Doing the change of coordinates

$$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta,$$
$$x_{2j-1} = r_{j-1}\cos(j\theta + \theta_{j-1}), \qquad x_{2j} = r_{j-1}\sin(j\theta + \theta_{j-1}) \quad j \in \{2,3,\dots,n\},$$

for $j = 2,3,\dots,n$, to the continuous piecewise linear differential system (19), and working as in the proof of Theorem 1 we obtain the averaged function $f = (f_1, f_2, \dots, f_{2n-1})$ given by

$$\begin{aligned} f_1 = &\quad (a_{11} + a_{22})\pi r + b_1 I_1(r), \\ f_{2(j-1)} = &\quad (a_{(2j-1)(2j-1)} + a_{(2j)(2j)})\pi r_{j-1} + (b_{2j-1}\cos\theta_{j-1} + b_{2j}\sin\theta_{j-1})I_j(r), \\ f_{2j-1} = &\quad (a_{(2j)(2j-1)} - a_{(2j-1)(2j)} + j(a_{12} - a_{21}))\pi - \\ &\quad \frac{jb_2 r_{j-1} I_1(r) - r(b_{2j}\cos\theta_{j-1} - b_{2j-1}\sin\theta_{j-1})I_j(r)}{r r_{j-1}}, \end{aligned} \tag{20}$$

where

$$I_j(r) = \int_0^{2\pi} \varphi(r\cos\theta)\cos(j\theta)d\theta.$$

Proceeding exactly as in the proof of Lemma 7 is possible to prove that

$$I_j(r) = \begin{cases} \pi r & \text{if } j = 1 \text{ and } 0 < r \le 1, \\ 0 & \text{if } j \text{ is even and } 0 < r \le 1, \\ L_j(r) & \text{if } j \text{ is odd and } r > 1, \end{cases}$$

where

$$L_j(r) = \frac{4}{j(j^2-1)}\Big(j\sqrt{r^2-1}\cos(j\arctan(\sqrt{r^2-1})) - \sin(j\arctan(\sqrt{r^2-1}))\Big).$$

The simple zeros of system (20) provide the existence of limit cycles for system (19) but since $I_j(r) = 0$ if j is even and $r > 1$, the variables θ_{j-1}, for $j = 2,4,6,\dots$ do not appear in the system $f_1 = f_2 = \dots = f_{2n-1} = 0$, so either this system has no zeros, or if it has zeros, then it has a continuum of zeros, and consequently the averaging theory cannot say anything about the limit cycles of system (19). The same occurs for the case $0 < r \le 1$. So we conclude that, using the averaging theory of first order, we can say nothing about the number of the limit cycles of system (19). □

Lemma 9. *The averaging theory of first order does not provide any information about the limit cycles of the discontinuous piecewise linear differential system*

$$\dot{x} = A_1 x + \varepsilon\big(Ax + \psi(x_1)b\big). \tag{21}$$

Proof. Now if we consider the discontinuous piecewise linear differential system (21), then its averaged function $f = (f_1, f_2, \dots, f_{2n-1})$ is

$$
\begin{aligned}
f_1 = &\quad (a_{11} + a_{22})\pi r + b_1 \widetilde{I}_1, \\
f_{2(j-1)} = &\quad (a_{(2j-1)(2j-1)} + a_{(2j)(2j)})\pi r_{j-1} + (b_{2j-1}\cos\theta_{j-1} + b_{2j}\sin\theta_{j-1})\widetilde{I}_j, \\
f_{2j-1} = &\quad (a_{(2j)(2j-1)} - a_{(2j-1)(2j)} + j(a_{12} - a_{21}))\pi - \\
&\quad \frac{j b_2 r_{j-1}\widetilde{I}_1 - r(b_{2j}\cos\theta_{j-1} - b_{2j-1}\sin\theta_{j-1})\widetilde{I}_j}{r r_{j-1}},
\end{aligned}
\tag{22}
$$

where

$$\widetilde{I}_j = \int_0^{2\pi} \psi(r\cos\theta)\cos(j\theta)d\theta.$$

Again we have that

$$\widetilde{I}_j = \int_0^{2\pi} \psi(r\cos\theta)\cos(j\theta)d\theta = \begin{cases} 0 & \text{if } j \text{ is even}, \\ \pm\dfrac{4}{(2j-1)} & \text{if } j \text{ is odd}, \end{cases}$$

and either there are no zeros of the function f, or a continuum of zeros, concluding that the averaging theory of first order given by Theorem 5 does not say anything about the existence of limit cycles of system (21). □

Remark 10. *Note the difference between the matrices A_0 and A_1. In the matrix A_0 the non-zero entries are only the odd numbers $1, 3, \dots, 2n-1$, while in the matrix A_1 the non-zero entries are the numbers $1, 2, \dots, n$. This difference provides that the continuous and discontinuous piecewise linear differential systems* (4) *and* (6) *can have limit cycles detected by the averaging theory, while for the continuous and discontinuous piecewise linear differential systems* (19) *and* (21) *the averaging theory cannot detect limit cycles.*

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