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Products of (weakly) discretely generated spaces ☆

Ofelia T. Alas [a], Richard G. Wilson [b,*]

[a] Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66281, 05314-970 São Paulo, Brazil

[b] Departamento de Matemáticas, Universidad Autónoma Metropolitana, Unidad Iztapalapa, Avenida San Rafael Atlixco, 186, Apartado Postal 55-532, 09340 México, DF, Mexico

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ABSTRACT

A space X is discretely generated at a point $x \in X$ if for any $A \subseteq X$ with $x \in \mathrm{cl}(A)$, there exists a discrete set $D \subseteq A$ such that $x \in \mathrm{cl}(D)$. The space X is *discretely generated* if it is discretely generated at every point $x \in X$. We say that X is *weakly discretely generated* if for any non-closed set $A \subseteq X$, there exists a discrete set $D \subseteq A$ such that $\mathrm{cl}(D) \setminus A \neq \emptyset$. We obtain new results concerning products of spaces belonging to these classes.

1. Introduction and terminology

A space X is discretely generated at a point $x \in X$ if for any $A \subseteq X$ with $x \in \mathrm{cl}(A)$, there exists a discrete set $D \subseteq A$ such that $x \in \mathrm{cl}(D)$. The space X is *discretely generated* if it is discretely generated at every point $x \in X$. We say that X is *weakly discretely generated* if for any non-closed set $A \subseteq X$, there exists a discrete set $D \subseteq A$ such that $\mathrm{cl}(D) \setminus A \neq \emptyset$. Obviously a discretely generated space is weakly discretely generated.

Discretely generated spaces were introduced in [2] where it was established, among many other results, that each compact Hausdorff space of countable tightness, each monotonically normal space and each regular space with a nested local base at every point is discretely generated and every compact Hausdorff space is weakly discretely generated. Many of these results were generalized in [1]. In [4], Ivanov and Osipov constructed, under CH, an example of a compact discretely generated space X such that $X \times X$ is not discretely generated. Also, in a recent article [6], it was shown that finite (Tychonoff) products of l-nested spaces and arbitrary box products of monotonically normal spaces are discretely generated. Although the class of weakly discretely generated spaces is, in some sense, "large" (it is not an entirely trivial exercise to find a Hausdorff space which is not weakly discretely generated), in general, it seems difficult to prove that a given family of spaces has this property (two classes of spaces which are known to be discretely generated are those of sequential and scattered Hausdorff spaces). For instance, many open problems exist regarding the preservation of (weak) discrete generability under

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* Corresponding author.
E-mail addresses: alas@ime.usp.br (O.T. Alas), rgw@xanum.uam.mx (R.G. Wilson).

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products and we address some of these problems in this paper. Specifically, we show that the countable product of spaces in some fairly large classes is discretely generated. We also show that the product of an ordinal and a weakly discretely generated space is weakly discretely generated.

If $x \in A \subseteq X$, $t(x, A)$ will denote the *tightness at x in A*, that is to say $t(x, A) = \min\{\kappa\colon$ if $B \subseteq A$ and $x \in \mathrm{cl}(B)$, then for some $C \in [B]^{\leqslant\kappa}$, $x \in \mathrm{cl}(C)\}$. Similarly, the *character of x in* $A \subseteq X$ (respectively, *the pseudocharacter of x in* $A \subseteq X$) will be denoted by $\chi(x, A)$, (respectively $\psi(x, A)$). A space is said to be *l*-nested if each point possesses a local base which is nested. All spaces are assumed to be Hausdorff and all undefined terms can be found in [3] or [5].

2. Preservation of (weak) discrete generability under products

Theorem 2.1. *If X is a regular space with a nested local base at a and Y is regular and discretely generated at b and* $t(b, Y) < \chi(a, X)$, *then* $X \times Y$ *is discretely generated at* (a, b).

Proof. Suppose that $A \subseteq X \times Y$ and $(a, b) \in \mathrm{cl}(A)$. Let $\{U_\alpha\colon \alpha < \kappa\}$ (where κ is a regular cardinal) be a nested local base of open sets at a with the property that if $\alpha < \beta$, then $\mathrm{cl}(U_\beta) \subseteq U_\alpha$ and without loss of generality, we assume that $U_0 = X$. If $(a, b) \in \mathrm{cl}(A \cap (\{a\} \times Y))$ (or $(a, b) \in \mathrm{cl}(A \cap (X \times \{b\}))$), then since both X and Y are discretely generated, it follows that there is a discrete subset of $A \cap (\{a\} \times Y)$ (or $A \cap (X \times \{b\})$) which contains (a, b) in its closure. Thus we may assume from now on that $(a, b) \notin \mathrm{cl}(A \cap (\{a\} \times Y)) \cup \mathrm{cl}(A \cap (X \times \{b\}))$. But then, there exists an open neighbourhood $U \times V$ of (a, b) such that $(U \times V) \cap [(A \cap (\{a\} \times Y)) \cup (A \cap (X \times \{b\}))] = \emptyset$ and so without loss of generality we assume that $U = X$ and $V = Y$.

Since $(a, b) \in \mathrm{cl}(A \cap (U_0 \times Y))$, it follows that $b \in \mathrm{cl}(\pi_Y[A \cap (U_0 \times Y)])$ and since Y is discretely generated and $t(Y) < \kappa$, there is some discrete subset $E_0 \subseteq \pi_Y[A \cap (U_0 \times Y)]$ such that $b \in \mathrm{cl}(E_0)$ and $|E_0| < \kappa$. For each $d \in E_0$, we may choose $x_d \in U_0$ so that $(x_d, d) \in (U_0 \times Y) \cap A$. Let $D_0 = \{(x_d, d)\colon d \in E_0\}$; clearly D_0 is a discrete subset of A of cardinality less than κ. We are assuming that $A \cap (\{a\} \times Y) = \emptyset$ and since there is a nested local base at a of size κ, it follows that $(a, b) \notin \mathrm{cl}(D_0)$.

Suppose that for some $\alpha < \kappa$ we have chosen basic open neighbourhoods U_{γ_β} of a and subsets $E_\beta \subseteq Y$ and $D_\beta \subseteq A$ for each $\beta < \alpha$, with the following properties:

(1) $\gamma_0 = 0$ and $\gamma_\beta < \gamma_\delta$ if $\beta < \delta < \alpha$,
(2) Both D_β and $E_\beta = \pi_Y[D_\beta]$ are discrete, $|D_\beta| = |E_\beta| < \kappa$ and $b \in \mathrm{cl}(E_\beta)$ for each $\beta < \alpha$,
(3) $D_\beta \subseteq (U_{\gamma_\beta} \times Y) \cap A$ for each $\beta < \alpha$,
(4) If $\mu < \beta < \alpha$, then $\mathrm{cl}(U_{\gamma_\beta} \times Y) \cap \mathrm{cl}(\bigcup\{D_\xi\colon \xi \leqslant \mu\}) = \emptyset$, and
(5) $\bigcup\{D_\mu\colon \mu \leqslant \beta\}$ is discrete for each $\beta < \alpha$.

Since $|\bigcup\{D_\beta\colon \beta < \alpha\}| < \kappa$, it follows as before that $(a, b) \notin \mathrm{cl}(\bigcup\{D_\beta\colon \beta < \alpha\})$; then since $|\bigcup\{D_\beta\colon \beta < \alpha\}| < \kappa$ and κ is regular, we may find a neighbourhood U_{γ_α} of a, where $\gamma_\alpha > \gamma_\beta$ for each $\beta < \alpha$ so that $\mathrm{cl}(\bigcup\{D_\beta\colon \beta < \alpha\}) \cap \mathrm{cl}(U_{\gamma_\alpha} \times Y) = \emptyset$. Since $(a, b) \in \mathrm{cl}((U_{\gamma_\alpha} \times Y) \cap A)$, it follows that $b \in \mathrm{cl}(\pi_Y[(U_{\gamma_\alpha} \times Y) \cap A])$ and since Y is discretely generated, there is a discrete set $E_\alpha \subseteq \pi_Y[(U_{\gamma_\alpha} \times Y) \cap A]$ of cardinality less than κ such that $b \in \mathrm{cl}(E_\alpha) \setminus E_\alpha$. For each $d \in E_\alpha$, we may choose $x_d \in U_{\gamma_\alpha}$ so that $(x_d, d) \in (U_{\gamma_\alpha} \times Y) \cap A$. Let $D_\alpha = \{(x_d, d)\colon d \in E_\alpha\}$; clearly D_α is a discrete subset of A of cardinality less than κ. Properties (1) to (4) are clearly satisfied for $\beta < \alpha + 1$ and furthermore, if $\bigcup\{D_\beta\colon \beta < \alpha\}$ is discrete, then so is $\bigcup\{D_\beta\colon \beta \leqslant \alpha\}$.

Hence to show that $D = \bigcup\{D_\alpha\colon \alpha < \kappa\}$ is a discrete subset of A, we need to show that $\bigcup\{D_\beta\colon \beta < \alpha\}$ is discrete for each $\alpha \leqslant \kappa$. To this end, we suppose that $(x, y) \in \bigcup\{D_\beta\colon \beta < \alpha\}$; then there is some $\xi < \alpha$ so that $(x, y) \in D_\xi \subseteq (U_{\gamma_\xi} \times Y) \cap A$. Since D_ξ is discrete, we need only show that $(x, y) \notin \mathrm{cl}(\bigcup\{D_\mu\colon \mu < \xi\}) \cup \mathrm{cl}(\bigcup\{D_\mu\colon \mu > \xi\})$.

If $\mu < \xi < \alpha$ then since $D_\xi \subseteq U_{\gamma_\xi} \times Y$, and $\mathrm{cl}(U_{\gamma_\xi} \times Y) \cap D_\mu = \emptyset$, it follows that $(x, y) \notin \mathrm{cl}(\bigcup\{D_\mu\colon \mu < \xi\})$. On the other hand, for each $\xi < \mu < \alpha$ we have $D_\mu \subseteq U_{\gamma_{\xi+1}} \times Y$ and since $\mathrm{cl}(U_{\gamma_{\xi+1}} \times Y) \cap D_\xi = \emptyset$ it follows that $(x, y) \notin \mathrm{cl}(\bigcup\{D_\mu\colon \mu > \xi\})$.

Thus we have proved that $D = \bigcup\{D_\alpha\colon \alpha < \kappa\}$ is a discrete subset of A and we claim finally that $(a, b) \in \mathrm{cl}(D)$. To see this, suppose that $U \times W$ is a neighbourhood of (a, b); then there is some $\gamma_\alpha < \kappa$ such that $U \supseteq U_{\gamma_\alpha}$. For this $\alpha \in \kappa$, $W \cap E_\alpha \neq \emptyset$, since $b \in \mathrm{cl}(E_\alpha) \setminus E_\alpha$ and so there is some $d \in W \cap E_\alpha$. But then, $(x_d, d) \in U_{\gamma_\alpha} \times W \subseteq U \times W$, showing that $D \cap (U \times W) \neq \emptyset$. □

Corollary 2.2. *If Y is a discretely generated regular space and* $t(Y)$ *is less than the character of each non-isolated point of a regular l-nested space X, then* $X \times Y$ *is discretely generated.*

In case X is first countable, the condition on tightness of Y can be omitted. The proof is quite similar to that of Theorem 2.1.

Theorem 2.3. *The product of a regular discretely generated space with a first countable regular space is discretely generated.*

Proof. Suppose that X is a first countable regular space and let $Z = X \times Y$, where Y is a regular discretely generated space. Suppose that $A \subseteq Z$ and $(a, b) \in \mathrm{cl}(A)$; as in the previous theorem, without loss of generality, we assume that

$A \cap (\{a\} \times Y) = A \cap (X \times \{b\}) = \emptyset$. Let $\{U_n\colon n \in \omega\}$ be a local base at a with the property that for each $n \in \omega$, $\mathrm{cl}(U_{n+1}) \subseteq U_n$. Since $U_0 \times Y$ is an open neighbourhood of (a, b), it follows that $(a, b) \in \mathrm{cl}(A \cap (U_0 \times Y))$ and so $b \in \mathrm{cl}(\pi_Y[A \cap (U_0 \times Y)])$. Then we may find a discrete subset $E_0 \subseteq \pi_Y[A \cap (U_0 \times Y)]$ such that $b \in \mathrm{cl}(E_0)$ and for each $d \in E_0$, pick $x_d \in X$ such that $(x_d, d) \in A \cap (U_0 \times Y)$; let $D_0 = \{(x_d, d)\colon d \in E_0\}$. Clearly D_0 is a discrete subset of Z and if $(a, b) \in \mathrm{cl}(D_0)$ then we are done.

Suppose now that for some $m \in \omega$ and each non-negative integer $k < m$, we have chosen a basic open neighbourhood U_{n_k} of a, an open neighbourhood V_k of b and subsets E_k of Y and D_k of Z with the following properties:

(1) $n_0 = 0$, $n_{k-1} < n_k$ for each $1 \leqslant k < m$,
(2) E_k is discrete, $b \in \mathrm{cl}(E_k)$ and $D_k \subseteq \pi_Y^{-1}[E_k]$, for each $k < m$,
(3) $\mathrm{cl}(V_k) \subseteq V_{k-1}$ for each $1 \leqslant k < m$,
(4) $D_k \subseteq (U_{n_k} \times V_k)$ for each $k < m$,
(5) $\mathrm{cl}(D_{k-1}) \cap \mathrm{cl}(U_{n_k} \times V_k) = \emptyset$ for each $1 \leqslant k < m$, and
(6) $\bigcup\{D_k\colon 0 \leqslant k < m\}$ is discrete.

If $(a, b) \in \mathrm{cl}(\bigcup\{D_k\colon 0 \leqslant k < m\})$ then we are done. If not, there is some open neighbourhood V_m of b such that $\mathrm{cl}(V_m) \subseteq V_{m-1}$ and $n_m \in \omega$ such that $n_m > n_{m-1}$ and $\mathrm{cl}(\bigcup\{D_k\colon 0 \leqslant k < m\}) \cap \mathrm{cl}(U_{n_m} \times V_m) = \emptyset$.

Since $U_{n_m} \times V_m$ is an open neighbourhood of (a, b), it follows that $(a, b) \in \mathrm{cl}(A \cap (U_{n_m} \times V_m))$ and so $b \in \mathrm{cl}(\pi_Y[A \cap (U_{n_m} \times V_m)])$. Then we may find a discrete subset $E_m \subseteq \pi_Y[A \cap (U_{n_m} \times V_m)]$ such that $b \in \mathrm{cl}(E_m)$ and for each $d \in E_m$, pick $x_d \in A$ such that $(x_d, d) \in A \cap (U_{n_m} \times V_m)$; let $D_m = \{(x_d, d)\colon d \in E_m\}$. Clearly D_m is a discrete subset of Z and the conditions (1) to (6) are satisfied for all $k \leqslant m$ by construction.

If $(a, b) \in \mathrm{cl}(\bigcup\{D_k\colon 1 \leqslant k \leqslant l\})$ for some $l \in \omega$, then the recursive construction terminates and we are done. If not, then we obtain discrete sets $D_k \subseteq A$ for each $k \in \omega$ and from the construction, it is easy to see that $D = \bigcup\{D_k\colon k \in \omega\}$ is a discrete subset of A; we claim that $(a, b) \in \mathrm{cl}(D)$. To see this, suppose that $U \times V$ is a neighbourhood of (a, b); then there is some $m \in \omega$ such that $U \supseteq U_{n_m}$. For this $m \in \omega$, $V \cap E_m \neq \emptyset$, since $b \in \mathrm{cl}(E_m) \setminus E_m$ and so there is some $d \in V \cap E_m$. But then, $(x_d, d) \in U_{n_m} \times V \subseteq U \times V$, showing that $D \cap (U \times V) \neq \emptyset$. □

Theorem 2.4. *If X is a regular space with a nested local base at a and Y is regular and discretely generated at b and $\chi(a, X) = \chi(b, Y)$, then $X \times Y$ is discretely generated at (a, b).*

Proof. Suppose that $\chi(a, X) = \chi(b, Y) = \kappa$. If $\kappa = \omega$, then both spaces are first countable at a and b and the result follows immediately. Thus we assume that κ is uncountable and must be regular since there is a nested local base $\{U_\alpha\colon \alpha < \kappa\}$ of open sets at a. We also assume without loss of generality, that whenever $\alpha < \beta$, $\mathrm{cl}(U_\beta) \subseteq U_\alpha$. Let $\{W_\alpha\colon \alpha < \kappa\}$ be a local base of open sets at b. Then $\{U_\alpha \times W_\alpha\colon \alpha < \kappa\}$ is a local base at (a, b), since if $S \times T$ is an open neighbourhood of (a, b), there is $\alpha < \kappa$ such that $U_\alpha \subseteq S$ and, since b is not isolated, there is $\beta > \alpha$ such that $W_\beta \subseteq T$; then $U_\beta \times W_\beta \subseteq S \times T$.

Suppose now that $(a, b) \in \mathrm{cl}(A) \setminus A$; since both X and Y are discretely generated, we may assume also that $(a, b) \notin \mathrm{cl}(A \cap (\{a\} \times Y)) \cup \mathrm{cl}(A \cap (X \times \{b\}))$. We proceed as in the proof of Theorem 2.1.

Pick $(x_0, y_0) \in (U_0 \times W_0) \cap A$; suppose that for some $0 < \beta < \kappa$, points (x_α, y_α) for $0 \leqslant \alpha < \beta$ and ordinals γ_α for $0 < \alpha < \beta$ have been chosen so that

(1) For each $\alpha < \beta$, $(x_\alpha, y_\alpha) \in (U_\alpha \times W_\alpha) \cap A$, and
(2) If $0 < \lambda < \mu < \beta$, then $\lambda < \gamma_\lambda < \gamma_\mu$, and
(3) $\mathrm{cl}(\{(x_\alpha, y_\alpha)\colon 0 \leqslant \alpha < \mu\}) \cap \mathrm{cl}(U_{\gamma_\mu} \times Y) = \emptyset$, whenever $1 \leqslant \mu < \beta$.

As in Theorem 2.1, since there is a nested local base at a and we are assuming that $(a, b) \notin \mathrm{cl}(A \cap (\{a\} \times Y))$, it follows that $(a, b) \notin \mathrm{cl}(\{(x_\alpha, y_\alpha)\colon \alpha < \beta\})$. Then since κ is regular, we may find $\gamma_\beta > \max\{\sup\{\gamma_\alpha\colon \alpha < \beta\}, \beta\}$ such that $\mathrm{cl}((\{(x_\alpha, y_\alpha)\colon \alpha < \beta\}) \cap \mathrm{cl}(U_{\gamma_\beta} \times Y) = \emptyset$ and then pick $(x_\beta, y_\beta) \in [U_{\gamma_\beta} \times (W_{\gamma_\beta} \cap W_\beta)] \cap A \subseteq (U_\beta \times W_\beta) \cap A$. It is clear that if $\{(x_\alpha, y_\alpha)\colon \alpha < \beta\}$ is discrete then so is $\{(x_\alpha, y_\alpha)\colon \alpha \leqslant \beta\}$ and satisfies condition (1) above.

Let $D = \{(x_\alpha, y_\alpha)\colon \alpha < \kappa\}$; clearly, $(a, b) \in \mathrm{cl}(D)$, since D meets every basic open neighbourhood $U_\alpha \times W_\alpha$ of (a, b) and it remains only to show that D is discrete. To this end, suppose that $\mu < \kappa$; we must prove that $(x_\mu, y_\mu) \notin \mathrm{cl}(\{(x_\alpha, y_\alpha)\colon \alpha < \mu\}) \cup \mathrm{cl}(\{(x_\alpha, y_\alpha)\colon \kappa > \alpha > \mu\})$. However, if $\mu < \alpha < \kappa$, then $(x_\alpha, y_\alpha) \in \mathrm{cl}(U_{\gamma_\alpha} \times Y) \subseteq \mathrm{cl}(U_{\gamma_{\mu+1}} \times Y)$ and since $(x_\mu, y_\mu) \notin \mathrm{cl}(U_{\gamma_{\mu+1}} \times Y)$ it follows that $(x_\mu, y_\mu) \notin \mathrm{cl}(\{(x_\alpha, y_\alpha)\colon \kappa > \alpha > \mu\})$. If $\alpha < \mu$, then since $(x_\mu, y_\mu) \in U_{\gamma_\mu} \times Y$ and $\mathrm{cl}(\{(x_\alpha, y_\alpha)\colon \alpha < \mu\}) \cap \mathrm{cl}(U_{\gamma_\mu} \times Y) = \emptyset$, the result is proved. □

We now consider infinite products of discretely generated spaces. We restrict attention to countable products, since an uncountable product of spaces each with more than one point contains a copy of 2^{ω_1} and hence cannot be discretely generated (Theorem 2.1 of [6]).

Theorem 2.5. *Let $\{X_n\colon n \in \omega\}$ be a countable family of regular discretely generated spaces with the property that for each $m \in \omega$, the finite product $\Pi\{X_k\colon 0 \leqslant k \leqslant m\}$ is discretely generated; then $\Pi\{X_k\colon k \in \omega\}$ is discretely generated.*

Proof. Suppose that $B \subseteq \Pi\{X_k\colon k \in \omega\}$ is not closed and $b = \langle b_n \rangle \in \mathrm{cl}(B)\backslash B$. For each $m \in \omega$, let Π_m denote the projection map from $\Pi\{X_k\colon k \in \omega\}$ to the finite subproduct $\Pi\{X_k\colon 0 \leqslant k \leqslant m\}$; since this map is continuous, it follows that $(b_0, \dots, b_m) \in \mathrm{cl}(\Pi_m[B])$ for each $m \in \omega$.

Step 0: Π_0 is continuous and so $b_0 \in \mathrm{cl}(\Pi_0[B])$; thus there is a discrete subset $D_0 \subseteq \Pi_0[B] \subseteq X_0$ such that $b_0 \in \mathrm{cl}(D_0)$. For each $d \in D_0$, pick $e_d \in B$ such that $\Pi_0(e_d) = d$. Clearly the set $M_0 = \{e_d\colon d \in D_0\}$ is discrete in $\Pi\{X_k\colon k \in \omega\}$.

If $b \in \mathrm{cl}(M_0)$ then we are done. If not, then there is a basic open set U_0 containing b whose closure misses $\mathrm{cl}(M_0)$, say $U_0 = \Pi\{V_n^0\colon n \in \omega\}$ where for each $n \in \omega$, $b_n \in V_n^0$ and $\mathrm{cl}(U_0) \cap \mathrm{cl}(M_0) = \emptyset$. Furthermore, $b \in \mathrm{cl}(U_0 \cap B)$ and there is some $m_1 \in \omega$ such that for all $n > m_1$, $V_n^0 = X_n$.

Step 1: The map Π_{m_1} is continuous and so

$$(b_0, b_1, \dots, b_{m_1}) \in \mathrm{cl}\big(\Pi_{m_1}\big[\Pi\big\{V_n^0\colon n \in \omega\big\} \cap B\big]\big).$$

Thus there is a discrete subset $D_1 \subseteq \Pi_{m_1}[\Pi\{V_n^0\colon n \in \omega\} \cap B]$ such that $(b_0, b_1, \dots, b_{m_1}) \in \mathrm{cl}(D_1)$. We proceed as in the previous step: For each $d \in D_1$, pick $e_d \in \Pi\{V_n^0\colon n \in \omega\} \cap B$ such that $\Pi_{m_1}(e_d) = d$. Clearly the set $M_1 = \{e_d\colon d \in D_1\}$ is discrete in $\Pi\{X_k\colon k \in \omega\}$ and furthermore, $M_0 \cup M_1$ is also discrete in $\Pi\{X_k\colon k \in \omega\}$.

As before, if $b \in \mathrm{cl}(M_0 \cup M_1)$ then we are done. If not, then we may find a basic open neighbourhood $U_1 = \Pi\{V_n^1\colon n \in \omega\}$ of b, where $b_n \in V_n^1$ for each $n \in \omega$, such that $\mathrm{cl}(U_1) \subseteq U_0$, $\mathrm{cl}(U_1) \cap \mathrm{cl}(M_0 \cup M_1) = \emptyset$ and for some $m_2 > m_1$, $V_n^1 = X_n$ for all $n > m_2$.

The recursive construction is now clear and either the process stops at finite step j when $b \in \mathrm{cl}(M_0 \cup M_1 \cup \cdots \cup M_j)$ or we determine:

(i) A strictly increasing sequence of integers $\langle m_k \rangle_{k\in\omega}$, and
(ii) For each $k \in \omega$, sets $M_k \subseteq \Pi\{V_n^k\colon n \in \omega\} \cap B$, and
(iii) Basic open sets $U_k = \Pi\{V_n^k\colon n \in \omega\}$ such that
  (a) $\mathrm{cl}(\Pi\{V_n^k\colon n \in \omega\}) \cap \mathrm{cl}(\bigcup\{M_j\colon 0 \leqslant j \leqslant k-1\}) = \emptyset$,
  (b) $b \in \Pi\{V_n^{k+1}\colon n \in \omega\} \subseteq \mathrm{cl}(\Pi\{V_n^{k+1}\colon n \in \omega\}) \subseteq \Pi\{V_n^k\colon n \in \omega\}$, and
  (c) $V_n^k = X_n$ for all $n > m_k$, and
  (d) $\bigcup\{M_k\colon 0 \leqslant k \leqslant n\}$ is discrete for each $n \in \omega$.

It is clear from the construction that $\bigcup\{M_k\colon k \in \omega\}$ is discrete and so it only remains to show that $b \in \mathrm{cl}(\bigcup\{M_k\colon k \in \omega\})$. To this end, suppose that $\Pi\{W_n\colon n \in \omega\}$ is a basic open neighbourhood of b. There is some $t \in \omega$ such that $W_n = X_n$ for all $n \geqslant t$ and so if we choose s large enough so that $m_s > t$, it follows that $(b_0, b_1, \dots, b_{m_s}) \in W_0 \times \cdots \times W_{m_s}$ and $W_0 \times \cdots \times W_{m_s} \cap D_s \neq \emptyset$ since $(b_0, b_1, \dots, b_{m_s}) \in \mathrm{cl}(D_s)$, showing that $\Pi\{W_n\colon n \in \omega\} \cap M_s \neq \emptyset$. □

Note that any subspace of a discretely generated space is discretely generated. Furthermore, any finite product of monotonically normal spaces, any finite product of l-nested spaces and any finite product of generalized ordered spaces is discretely generated (see Theorem 2.6, Corollaries 2.8 and 2.17 respectively of [6]). The following corollaries now follow immediately from the above theorem, the first three give answers to Questions 3.6, 3.7 and 3.8 of [6].

Corollary 2.6. *Any subspace of a countable product of monotonically normal spaces is discretely generated.*

Corollary 2.7. *Any subspace of a countable product of generalized ordered (GO) spaces is discretely generated.*

Corollary 2.8. *Any subspace of a countable product of regular l-nested spaces is discretely generated.*

Since any finite product of scattered spaces is scattered, we also have:

Corollary 2.9. *Any subspace of a countable product of regular scattered spaces is discretely generated.*

Question 2.10. Is Theorem 2.5 valid in the class of Hausdorff spaces?

Finally we consider finite products of weakly discretely generated spaces. In spite of the fact that this class is much larger than that of the discretely generated spaces, it seems much more difficult to prove results concerning products and there are only two closely related main results.

Theorem 2.11. *If Y is weakly discretely generated and κ is an ordinal, then $\kappa \times Y$ is weakly discretely generated.*

Proof. Clearly the result is true if $\kappa = 1$. Suppose then that $\lambda \times Y$ is weakly discretely generated whenever $\lambda < \kappa$. We will show that the result is true at κ.

Note first that if κ is of the form, $\kappa = \eta + 1$, where η is a successor ordinal, then $\kappa \times Y$ is the topological union of $\eta \times Y$ and $\{\eta\} \times Y$ and this union is weakly discretely generated by the inductive hypothesis. Furthermore, if κ is a limit ordinal and $A \subseteq \kappa \times Y$ is not closed, then for some $\alpha < \kappa$, $A \cap (\alpha \times Y)$ is not closed in $\alpha \times Y$ and the result would again follow from the inductive hypothesis. Thus we may assume that κ is of the form $\lambda + 1$ where λ is a limit ordinal.

With this hypothesis, suppose that $A \subseteq \kappa \times Y$ is not closed and to simplify the notation, we set $A_\alpha = \{y\colon (\alpha, y) \in A\}$. Applying the inductive hypothesis, we may assume that $A \cap (\lambda \times Y)$ is closed in $\lambda \times Y$ and so A_α is closed for each $\alpha < \lambda$ and there is some $t \in Y$ such that $(\lambda, t) \in \mathrm{cl}(A) \backslash A$. Since Y is weakly discretely generated, we may further assume that $\{\lambda\} \times A_\lambda$ is closed in $\{\lambda\} \times Y$.

We now consider two cases:

(1) If $S = \{\alpha < \lambda\colon t \in A_\alpha\}$ is cofinal in λ, then since S is scattered, there is a dense discrete subset D of S and so $\{(\alpha, t)\colon \alpha \in D\}$ is a discrete subset of A with the property that $(\lambda, t) \in \mathrm{cl}(D)$.
(2) If the set S is not cofinal in λ, then there is some $\beta < \lambda$ such that $S \subseteq \beta + 1$. Since we have assumed that $A \cap ((\beta + 1) \times Y)$ is closed in $(\beta + 1) \times Y$, it follows that $A \cap [(\kappa \backslash (\beta + 1)) \times Y]$ is not closed in $(\kappa \backslash (\beta + 1)) \times Y$ and $t \notin A_\alpha$ for all $\alpha > \beta + 1$. In the sequel, we assume that $t \notin A_\alpha$ for all $\alpha \in \kappa$, but $\bigcup\{A_\alpha\colon \alpha \in \lambda\}$ is not closed since $t \in \mathrm{cl}(\bigcup\{A_\alpha\colon \alpha \in \kappa\}) \backslash \bigcup\{A_\alpha\colon \alpha \in \kappa\}$.

We claim now that if for some $z \in Y$, $z \in \mathrm{cl}(\bigcup\{A_\alpha\colon \alpha \in \kappa\}) \backslash \bigcup\{A_\alpha\colon \alpha \in \kappa\}$, then $(\lambda, z) \in \mathrm{cl}(A) \backslash A$. To prove the claim, suppose to the contrary that $(\xi, z) \notin \mathrm{cl}(A)$ for all $\xi \in \kappa$. Since κ is compact, there is an open neighbourhood V of z such that $(\kappa \times V) \cap A = \emptyset$ and so $V \cap A_\alpha = \emptyset$ for all $\alpha \in \kappa$, a contradiction. Thus, for such z, there is some $\xi \in \kappa$ such that $(\xi, z) \in \mathrm{cl}(A) \backslash A$ and clearly we must have $\xi = \lambda$.

Since $t \in \mathrm{cl}(\bigcup\{A_\alpha\colon \alpha \in \lambda\}) \backslash \bigcup\{A_\alpha\colon \alpha \in \lambda\}$, it follows that $\bigcup\{A_\alpha\colon \alpha \in \lambda\}$ is not closed and since Y is weakly discretely generated, there is some $y \in \mathrm{cl}(\bigcup\{A_\alpha\colon \alpha \in \kappa\}) \backslash \bigcup\{A_\alpha\colon \alpha \in \kappa\}$ and a discrete set $D \subseteq \bigcup\{A_\alpha\colon \alpha \in \kappa\}$ such that $y \in \mathrm{cl}(D)$. For each $d \in D$, the set $\{\alpha\colon d \in A_\alpha\}$, being scattered, contains a dense discrete subset E_d and we may then define

$$M = \bigl\{(\alpha, d)\colon \alpha \in E_d \text{ and } d \in D\bigr\}.$$

It is clear that M is a discrete subset of A and it remains only to show that $(\lambda, y) \in \mathrm{cl}(M)$. To this end, suppose that $U \times W$ is an open neighbourhood of (λ, y), where $U = \{\alpha \in \kappa\colon \alpha > \gamma\}$ and where, since A_λ is closed in Y, we have $W \cap A_\lambda = \emptyset$. Since $y \in \mathrm{cl}(\bigcup\{A_\alpha\colon \alpha \in \kappa\}) \backslash \bigcup\{A_\alpha\colon \alpha \in \kappa\}$, it follows that $\{\alpha\colon W \cap A_\alpha \neq \emptyset\}$ is unbounded. (If it were bounded, say by γ, then as before, $\bigcup\{A_\alpha\colon \alpha \in \gamma + 1\}$ would be closed since the projection $\pi : (\gamma + 1) \times Y \to Y$ is closed.) It now follows immediately that $\{\alpha\colon W \cap D \cap A_\alpha \neq \emptyset\}$ is cofinal in λ and so $M \cap (U \times W) \neq \emptyset$. □

Corollary 2.12. *If Y is weakly discretely generated and X is a finite product of ordinals, then $X \times Y$ is weakly discretely generated.*

A very similar method can be used to prove the following result. We omit the details.

Theorem 2.13. *If Y is discretely generated and X is a finite product of ordinals, then $X \times Y$ is discretely generated.*

The form of the proof of Theorem 2.11 can be adapted to show that the product of a weakly discretely generated space and certain other compact spaces is weakly discretely generated. For example, if X is the one-point compactification of a discrete space, then by taking as a model the case $\kappa = \omega + 1$ in Theorem 2.11, it is easy to see that $X \times Y$ is weakly discretely generated. We omit the details.

Suppose now that $\{X_\alpha\colon \alpha \in \kappa\}$ is a family of mutually disjoint compact spaces; let Z be the disjoint topological union of the spaces X_α, that is $Z = \bigoplus\{X_\alpha\colon \alpha \in \kappa\}$ and $X = \alpha Z$, the one point compactification of Z. Using this notation, we may now state our last result:

Theorem 2.14. *If $X_\alpha \times Y$ is weakly discretely generated for each α, then $X \times Y$ is weakly discretely generated.*

Proof. Note first that each neighbourhood of ∞ must contain all but finitely many of the subspaces X_α of Z. Suppose that $A \subseteq X \times Y$ is not closed. If $A \cap (X_\alpha \times Y)$ is not closed for some $\alpha \in \kappa$, then we are done, since $X_\alpha \times Y$ is weakly discretely generated. Let $A_\alpha = \pi_Y[A \cap (X_\alpha \times Y)]$ and $A_\infty = \pi_Y[A \cap (\{\infty\} \times Y)]$; as in the proof of Theorem 2.11 we may assume that $A \cap (\{\infty\} \times Y)$ and $A \cap (X_\alpha \times Y)$ are closed for each $\alpha \in \kappa$, thus there is some $t \in Y$ such that $(\infty, t) \in \mathrm{cl}(A) \backslash A$ and $t \in \mathrm{cl}(\bigcup\{A_\alpha\colon \alpha \in \kappa\}$. Let $S = \{\alpha \in \kappa\colon t \in A_\alpha\}$. The proof now follows closely that of Theorem 2.11, but for completeness we give the details.

(1) If $S = \{\alpha < \kappa\colon t \in A_\alpha\}$ is infinite, then for each $\alpha \in S$, we may pick a single $x_\alpha \in X_\alpha$ such that $(x_\alpha, t) \in A$ and then $(\infty, t) \in \mathrm{cl}(\{(x_\alpha, t)\colon \alpha \in S\})$ and since $\{(x_\alpha, t)\colon \alpha \in S\}$ is discrete, we are done.
(2) If the set S is finite then clearly $t \in \mathrm{cl}(\{\bigcup\{A_\alpha\colon \alpha \in \kappa \backslash S\})$ and $\bigcup\{A_\alpha\colon \alpha \in S\}$ is closed in Y. Since Y is weakly discretely generated and $\{\bigcup\{A_\alpha\colon \alpha \in \kappa \backslash S\}$ is not closed, there is some $y \in \mathrm{cl}(\bigcup\{A_\alpha\colon \alpha \in \kappa \backslash S\}) \backslash \bigcup\{A_\alpha\colon \alpha \in \kappa \backslash S\}$ and a discrete set $E \subseteq \bigcup\{A_\alpha\colon \alpha \in \kappa \backslash S\}$ such that $y \in \mathrm{cl}(E)$. For each $\alpha \in \kappa \backslash S$, $E \cap A_\alpha$ is discrete and for each $d \in E \cap A_\alpha$ we may pick

$x_d \in X_\alpha$ so that $(x_d, d) \in A$. The set $D_\alpha = \{(x_d, d)\colon\ d \in A_\alpha\}$ is a discrete subset of $X_\alpha \times Y$ and so $D = \bigcup\{D_\alpha\colon\ \alpha \in \kappa \setminus S\}$ is a discrete subset of A. It is a simple matter to check that $(\infty, y) \in \mathrm{cl}(D)$. $\square$

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