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Polar Multiplicities and Euler Obstruction of the Stable Types in Weighted Homogeneus Map Germs from $\mathbb{C}^n$ to $\mathbb{C}^3$, $n \geq 3$

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Polar Multiplicities and Euler obstruction of the stable types in weighted homogeneus map germs from $\mathbb{C}^n$ to $\mathbb{C}^3$, $n \geq 3$

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Resumo

Neste artigo são mostradas formulas para calcular as multiplicidades polares dos tipos estáveis $f(\Delta(f)$ and $f(\Sigma^{n-2,1}(f)$, de germes semi quase homogêneos finitamente determinados $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^3, 0)$, com $n \geq 4$, em termos dos pesos e graus de quse homogeneidade de f. Como consequência é mostrado com obter a obstrução de local de Euler destes tipos estáveis, em termos dos pesos e graus.

Abstract

In this article we show that for corank 1, quasi-homogeneous and finitely determined map germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^3, 0)$, $n \geq 4$ one can obtain formulae for the polar multiplicities defined on the following stable types of f, $f(\Delta(f)$ and $f(\Sigma^{n-2,1}(f)$, in terms of the weights and degrees of f. As a consequence we show how to compute the Euler obstruction of such stable types, also in terms of the weights and degrees of f.

1 Introduction

Teissier introduced in some key papers in singularity theory ([22], [23]) the notions of polar varieties and polar multiplicities. His work was taken up by several authors which show that the polar varieties and its multiplicities are powerful tools for solving some problems in singularity theory. Gaffney in [4] states that, for an important class of finitely determined map germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, if all the polar multiplicities of the strata in the source and target of a well chosen stratification are constant along a family of such germs, then this family is Whitney equisingular. With the aid of Gonzales-Sprinberg's purely algebraic interpretation of the local Euler obstruction for singular varieties, Lê and Teissier proved a formula showing that the local Euler obstruction is an alternate sum of the polar multiplicities of the local polar varieties.

However these invariants have not been used extensively. This could be due to the fact that they are difficult to compute in practise.

In this paper we show that for, quasi-homogeneous and finitely determined map germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^3, 0)$ with $n \geq 4$, of corank 1, one can obtain formulae for the polar multiplicities defined on the stable types of a generic unfolding of f. In addition, we apply the results shown in [12] to show how to compute the Local Euler obstruction of such stable types in terms of the weights and degrees.

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2 Stable types and polar multiplicities

We denote by $\mathcal{O}(n,p)$ the set of origin preserving germs of holomorphic mappings from $\mathbb{C}^n$ to $\mathbb{C}^p$, $\mathcal{O}_e(n,p)$ denotes the set of germs at the origin but not necessarily origin preserving.

For a germ $f \in \mathcal{O}_e(n,p)$, $J(f)$ denotes the ideal generated by the set of $p \times p$ minors of the derivative of f. The critical set $\Sigma(f)$ of f is the set of points $x \in \mathbb{C}^n$ such that $J(f)(x) = 0$. The discriminant $\Delta(f)$ of f is the image of $\Sigma(f)$ by f. The determinant of the derivative of $f \in \mathcal{O}_e(n,n)$ is denoted by $J[f]$.

2.1 The Stable types

A map-germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is **stable** in a finite set S if, by composition with families of holomorphic diffeomorphisms in source and target, every deformation is $\mathcal{A}$-trivial, where $\mathcal{A}$ denotes the usual Mather group of germs of holomorphic difeomorphisms in the source and in the target.

We call **stable type**, an equivalence class of stable map germs.

If $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, y)$ is a stable multi germ, we say $y \in \mathbb{C}^p$ is of stable type $\mathcal{Q}$ if f is a representative of this stable type, we denote the set of stable points in $\mathbb{C}^p$ of type $\mathcal{Q}$ by $\mathcal{Q}(f)$. The set $f^{-1}(\mathcal{Q}(f)) \cap \Sigma(f)$ is denoted by $\mathcal{Q}_\Sigma(f)$ and $\mathcal{Q}_S(f)$ denotes the set $f^{-1}(\mathcal{Q}(f)) - \mathcal{Q}_\Sigma(f)$, where $\Sigma(f)$ is the critical set of f.

Our interest in this article is to compute the polar multiplicities which appear in a versal unfolding of a $\mathcal{A}$-finitely determined weighted homogeneous map-germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^3, 0)$, with $n \geq 3$.

A germ f in $\mathcal{O}(n,p)$ is k-$\mathcal{A}$-determined if any $g \in \mathcal{O}(n,p)$ with the same k-jet as f, i.e. $j^k g = j^k f$, is $\mathcal{A}$-equivalent to f. The germ f is said to be finitely $\mathcal{A}$-determined if it is k-$\mathcal{A}$-determined for some k.

Mather and Gaffney showed the characterization of finitely determined map germs in terms of stable germs.

Proposition 2.1. *Suppose $f \in \mathcal{O}(n,p)$. Then f is finitely determined if, and only if, for each representative $\tilde{f}$ of f, there exist neighborhoods of the origin $U \subset \mathbb{C}^n$, $V \subset \mathbb{C}^p$ such that $\tilde{f}^{-1}(0) \cap U \cap \Sigma(\tilde{f}) = 0$ and for each $y \in V$, $y \neq 0$, the germ $\tilde{f}_y : (\mathbb{C}^n, S_y) \to (\mathbb{C}^p, y)$ is stable, where $S_y = \tilde{f}^{-1}(y) \cap U \cap \Sigma(\tilde{f})$ and $\Sigma(\tilde{f})$ denotes the critical set of $\tilde{f}$.*

See [5] for a proof.

A finitely determined germ f has **discrete stable type** if there exist a versal unfolding of f in which only a finite number of stable types occur. If the numbers (n,p) are in Mather's "nice dimensions" (which is our focus here) or on the boundary thereof, then every finitely determined germ $f \in \mathcal{O}(n,p)$ has discrete stable type.

For finitely determined map germs, we need to consider the stable types which appear in a versal unfolding of such germs.

We denote by $G : (\mathbb{C}^s \times \mathbb{C}^n, 0) \to (\mathbb{C}^s \times \mathbb{C}^p, 0)$ a versal unfolding of such an f.

Definition 2.2. *A stable type $\mathcal{Q}$ appears in G if for any representative $\overline{G} = (id(u), g_u(x))$ of G, there exists a point $(u,y) \in \mathbb{C}^s \times \mathbb{C}^p$ such that the multi germ $g_u : (\mathbb{C}^n, S_y) \to (\mathbb{C}^p, y)$ is a stable multi germ of type $\mathcal{Q}$, here $S_y \subset \mathbb{C}^n$ denotes the image of the restriction to $S := G^{-1}(u,y) \cap \Sigma(G)$ of the projection of $\mathbb{C} \times \mathbb{C}^n$ on the second factor $\mathbb{C}^n$.*

If f is finitely determined, we write $\overline{\mathcal{Q}(f)} = (\{0\} \times \mathbb{C}^p) \cap \overline{\mathcal{Q}(G)}$, $\overline{\mathcal{Q}_S(f)} = (\{0\} \times \mathbb{C}^n) \cap \overline{\mathcal{Q}_S(G)}$ and $\overline{\mathcal{Q}_\Sigma(f)} = (\{0\} \times \mathbb{C}^n) \cap \overline{\mathcal{Q}_\Sigma(G)}$, where the bar over a set means the closure of this set.

Suppose $\mathcal{Q}(G) = \{\mathcal{Q}_1, \ldots, \mathcal{Q}_r\}(G)$ is the set of points of a 0-dimensional stable singularity type of a versal unfolding G of f.

Definition 2.3. *The* **0-stable invariant** *of type* $\mathcal{Q}$ *of* f, *denoted by* $m(f, \mathcal{Q})$ *is the multiplicity of the ideal* $m_s\mathcal{O}_{\overline{\mathcal{Q}(G)},(0,0)}$ *in* $\mathcal{O}_{\overline{\mathcal{Q}(G)},(0,0)}$

This multiplicity is just the degree of the map obtained by projecting $\overline{\mathcal{Q}(G)}$ onto $\mathbb{C}^s$ using the projection of $\mathbb{C}^s \times \mathbb{C}^p$ on the first factor $\mathbb{C}^s$. This projection gives a finite map because the fiber over zero is just $\overline{\mathcal{Q}(f)}$, which is either the origin or empty, since f is finitely determined and $\mathcal{Q}$ is a zero dimensional singularity type. See [20] p. 121, for details on the relationship between degree and multiplicity.

Since $\mathcal{Q}(G)$ lies in $G(\Sigma(G))$, if $\mathcal{Q}(G)$ involves r algebras, $G|\overline{\mathcal{Q}_\Sigma(G)}$ dominates $\overline{\mathcal{Q}(G)}$ and is generically r to 1. Them, $r \cdot m(f, \mathcal{Q})$ is the multiplicity of $m_s\mathcal{O}_{\overline{\mathcal{Q}_\Sigma(G)}}$. When $\mathcal{O}_{\mathcal{Q}_\Sigma(G)}$ is Cohen-Macaulay for a stable map G, $r \cdot m(f, \mathcal{Q})$ is the length of $\mathcal{O}_{\overline{\mathcal{Q}_\Sigma(G)}}/m_s\mathcal{O}_{\overline{\mathcal{Q}_\Sigma(G)}}$.

We remark that $m(f, \mathcal{Q})$ is independent of the versal unfolding of f, and is invariant under coordinate changes on f, see Proposition 3.4 of [4].

We remark that we can deal with the entire class of cases $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^3, 0)$, for all $n \geq 3$, in consequence of the fact that for all $n \geq 3$, the stable singularities are precisely the same type, namely double points curve, cuspidal edge, triple point, swallowtail, cuspidal edge crossing transversally with a plane and the regular part.

We can also use the Thom-Boardman stratification in the source to describe the stratification by **stable types** which appear in the discriminant $\Delta(f)$ of f. For any Boardman symbol $i = (i_1, \ldots, i_r)$, we denote by $\Sigma^i(f)$ the set of points in $\Sigma(f)$ of type i.

Below we show the r-dimensional stables types, for $r = 1, 2$, which appear in any finitely determined map germ germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^3, 0)$. These stable types are in in $\Delta(f)$ and are formed by the smooth parts of the following sets.

1. The discriminant $\Delta(f) = f(\Sigma(f))$, which is 2-dimensional;
2. The 1-dimensional set $f(\Sigma^{(n-2,1)}(f))$;
3. The image of the double points of f, which is 1-dimensional and denoted by $f(D_1^2(f|\Sigma(f)))$.

The first purppose of this article is to describe formulae to compute the polar multiplicities, which we describe next, of the stable types $f(\Delta(f))$ and $f(\Sigma^{n-2,1}(f))$, in terms of the weights and degrees of weighted homogeneous map germ f of corank 1.

2.2 Polar multiplicities

The polar multiplicities are the multiplicities of the polar varieties, notion developed as a means of studying the singularities of any pure d-dimensional analytic germ $(X, x) \subset (\mathbb{C}^n, x)$.

The k^{th}, with $0 \leq k \leq d-1$, polar varieties of X were defined by Teissier in [23], using the Nash blowup and by Henry and Merle in [9] and [10], using the conormal modification.

These k^{th} polar varieties of X are obtained by taking the closure of the critical set of the restriction of a generic projection $p : \mathbb{C}^N \to C^{d-k+1}$ to the regular part of X. Roughly speaking, "generic projection" means that the projections which define the polar varieties are the least singular of projections.

We denote the k^{th} polar variety of X by $P_k(X,x,p)$. The key invariant of $P_k(X,x,p)$, for $k=0,\dots,d-1$ is its multiplicity at 0, called **k-polar multiplicity**, which we denote by $m_k(X,p)$. Since it is constant for an open set of projections, we denote it by $m_0(P_k(X))$ or $m_k(X)$.

Gaffney in [4] page 195, also defines a d-polar multiplicity associated to a stable set $\mathcal{Q}(f)$ of a finitely determined map germ f, which should be given by the "polar variety of greatest codimension". But as this variety is zero-dimensional, it is not even well defined, Gaffney defines the d-polar multiplicity (of codimension d) in each stratum as following:

Take a versal unfolding $G:(\mathbb{C}^s\times\mathbb{C}^n,(0,0))\to(\mathbb{C}^s\times\mathbb{C}^p,(0,0))$ of f. Specify a stratum $\mathcal{Q}=\mathcal{Q}(G)$ in target (or $\mathcal{Q}=\mathcal{Q}_\Sigma(G)$ or $\mathcal{Q}_S(G)$ in source), such that $\dim\overline{\mathcal{Q}(f)}\geq 1$. Select D a linear subspace of $(\mathbb{C}^p,0)$ (or of $(\mathbb{C}^n,0)$) of dimension 1 and form the relative polar variety on $\mathcal{Q}$, denoted $P_d(\overline{\mathcal{Q}},\pi_s)$, where π_s is the projection to $\mathbb{C}^s$ and $d=\dim(\overline{\mathcal{Q}})-s$.

Definition 2.4. *The d^{th}-stable multiplicity of f of type $\mathcal{Q}(f)$, denoted $m_d(f,\mathcal{Q})$, is the multiplicity of $m_s\mathcal{O}_{P_d(\overline{\mathcal{Q}}(G),\pi_s),(0,0)}$ in $\mathcal{O}_{P_d(\overline{\mathcal{Q}}(G),\pi_s),(0,0)}$.*

In [12] it is shown the relation between the polar multiplicities of $\Delta(f)$ in terms of the Milnor number of the singular set.

Theorem 2.5. ([12], p. 05) *Let $f\in\mathcal{O}(n,3)$, $n\geq 3$ be a finitely determined map germ. Then:*

$$m_2(\Delta(f))-m_1(\Delta(f))+m_0(\Delta(f)=\mu(\Sigma(f))+1 \qquad (I)$$

The relation between the polar multiplicities of $f(\Sigma^{n-2,1}(f))$ is given in terms of the Milnor number of the set $\Sigma^{n-2,1}(f)$ and also of the number of singularities of type A_3, denoted $\sharp A_3$ which appears in a generic unfolding of a finitely determined corank one map germ in $\mathcal{O}(n,3)$ with $n\geq 4$.

Theorem 2.6. ([12], p. 09) *Let $f\in\mathcal{O}(n,3), n>3$ be a finitely determined corank one map germ. Then:*

$$m_0(f(\Sigma^{n-2,1}(f)))-m_1(f(\Sigma^{n-2,1}(f)))=\sharp A_3-\mu(\Sigma^{n-2,1}(f))+1 \qquad (II)$$

We remark that it is possible to compute the number $\sharp A_3$ in consequence of the results [4.6-item(6)], 4.3 and 2.5 (in this order) of [3]. Then we have

$$\sharp A_3=\dim_{\mathbb{C}}\frac{\mathcal{O}_n}{\langle g_{x_3},g_{x_4},\dots,g_{x_n},M,M_{x_3},M_{x_4},\dots,M_{x_n}\rangle},$$

where M is the determinant $\begin{vmatrix} g_{x_3^2} & \cdots & g_{x_3x_n}\\ g_{x_4x_3} & \cdots & g_{x_4x_n}\\ \vdots & \ddots & \vdots\\ g_{x_nx_3} & \cdots & g_{x_n^2}\end{vmatrix}$ and M_{x_i} denotes the partial derivative of M in the variable x_i.

As consequence by above results, in [12] it is shown a Corollary which involves the polar multiplicity $m_1(\Delta(f))$ with the polar multiplicity $m_0(f(\Sigma^{n-2,1}(f)))$.

Corollary 2.7. *[12]*

$$m_1(\Delta(f))=m_0(f(\Sigma^{n-2,1}(f))). \qquad (III)$$

3 Weighted homogeneous map germs

In order to compute the polar multiplicities of the stable types we need to know the Milnor numbers $\mu(\Sigma(f))$, and $\mu(\Sigma^{n-2,1})(f)$, which are ICIS. We describe how to compute these numbers.

An analytic map-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $f = (f_1, \ldots, f_p)$ is said to be *quasi-homogeneous*, or *weighted homogeneous*, of the type $(w_1, \ldots, w_n; d_1, \ldots, d_p)$ if there are positive integers $w_1, \ldots, w_n$, called weights, and positive integers $d_1, \ldots, d_p$, called degrees, such that $f_i(\lambda^{w_1} x_1, \ldots, \lambda^{w_n} x_n) = \lambda^{d_i} f_i$ for all $i = 1, \ldots, p$, $x \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$.

A map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $f = (f_1, \ldots, f_p)$, is said to be *semi quasi homogeneous* of the type $(w_1, \ldots, w_n; d_1, \ldots, d_p)$ if f can be expressed as a sum $f = f^0 + f'$, where $f^0 = (f_1^0, \ldots, f_p^0)$ is a finitely determined weighted homogeneous map germ of weights $w_1, \ldots, w_n$ and degrees $d_1, \ldots, d_p$; $f' = (f_1', \ldots, f_p')$ and each f_i' is a germ of function such that $w_1 k_1 + \ldots w_n k_n > d_i$ for all $k = (k_1, \ldots, k_n)$ of any monomial x^k in the Taylor series of f'. The germ f^0 is called *the initial part* of f.

First we see that we only need to compute the Milnor number of the initial parts.

Theorem 3.1. ([8], p. 86) *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a semi quasi-homogeneous germ that defines a complete intersection isolated singularity. Then the space $f^{-1}(0)$ is a complete intersection isolated singularity of dimension $(n-p)$ and $\mu(f) = \mu(f^0)$.*

We shall need then use the following result of Greuel and Hamm.

Theorem 3.2. ([8], p. 77) *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $n \geq p$, $f = (f_1, \ldots, f_p)$ be a quasi-homogeneous germ that defines a complete intersection with isolated singularity of weights $w_1, \ldots, w_n$ and degrees $d_1, \ldots, d_p$. Then*

(a) If $d_1 = \ldots = d_p = d$:

$$\mu(f) = (-1)^{n-p+1} + (-1)^{n-p} \frac{d^n}{w_1 \cdot \ldots \cdot w_n} \sum_{\substack{0 \leq l \leq n-p \\ 1 \leq \nu_1 < \ldots < \nu_l \leq n}} (-1)^l \prod_{\lambda=1}^{l} \left(1 - \frac{w_{\nu_\lambda}}{d}\right)$$

(b) If $d_i \neq d_j$, when $i \neq j$:

$$\mu(f) = \sum_{\rho=1}^{p} \prod_{\nu=1}^{n} \left(\frac{d_\rho}{w_\nu} - 1\right) \prod_{\substack{\kappa=1 \\ \kappa \neq \rho}}^{p} \left(\frac{1}{\frac{d_\rho}{d_\kappa} - 1}\right)$$

We shall also need to apply the formula below to compute the polar multiplicities.

Theorem 3.3. ([1]) *Let $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, $h = (h_1, \ldots, h_n)$ be a semi quasi-homogeneous map germ of weights $w_1, \ldots, w_n$ and degrees $d_1, \ldots, d_n$. Suppose that $h_1, \ldots, h_n$ is a system of generators of an ideal I of finite codimension, then*

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I} = \frac{d_1 \ldots d_n}{w_1 \ldots w_n}.$$

4 Polar Multiplicities in $\Delta(f)$ in terms of weights and degrees

In this section we compute the polar multiplicities of the strata in the target associated to a finitely determined, quasi-homogeneous, corank one germ $f \in O(n,3)$, $n \geq 3$. We start dealing the set $\Delta(f)$.

Theorem 4.1. *Let $f(x_1, x_2, \ldots, x_n) = (x_1, x_2, g(x_1, x_2, \ldots, x_n))$ be a finitely determined, quasi-homogeneous, corank one map germ with weights $w_1, \ldots, w_n$, where $w_1 \leq w_2$ and $w_3 \leq \ldots \leq w_n$ and let d be the degree of g, $d \geq w_1$ and $d \geq w_2$. Then,*

$$\mathbf{m_0(\Delta(f))} = \prod_{j=3}^{n}\left(\frac{D_j}{w_j}\right)$$

$$\mathbf{m_1(\Delta(f))} = \sum_{\rho=1}^{n}\prod_{\nu=1}^{n}\left(\frac{D_\rho}{w_\nu} - 1\right)\prod_{\substack{\kappa=1\\ \kappa\neq\rho}}^{n}\left(\frac{1}{\frac{D_\rho}{D_\kappa} - 1}\right) + 1$$

$$\mathbf{m_2(\Delta(f))} = \sum_{\rho=3}^{n}\prod_{\nu=1}^{n}\left(\frac{D_\rho}{w_\nu} - 1\right)\prod_{\substack{\kappa=3\\ \kappa\neq\rho}}^{n}\left(\frac{1}{\frac{D_\rho}{D_\kappa} - 1}\right) - \prod_{j=3}^{n}\left(\frac{D_j}{w_j}\right) + \sum_{\rho=1}^{n}\prod_{\nu=1}^{n}\left(\frac{D_\rho}{w_\nu} - 1\right)\prod_{\substack{\kappa=1\\ \kappa\neq\rho}}^{n}\left(\frac{1}{\frac{D_\rho}{D_\kappa} - 1}\right) + 2.$$

with $D_1 = w_1, D_2 = (n-2)d - 2\sum_{i=3}^{n} w_i, D_l = d - w_l, 3 \leq l \leq n$.

We remark that when $n = 3$ this result was done by Jorge-Perez in the following.

Theorem 4.2. *[11] Let $f = (x, y, g(x, y, z)) \in \mathcal{O}(3,3)$ be a finitely determined, quasi-homogeneous, corank 1 map germ with weights w_1, w_2, w_3 and d denotes the degree of g. Then,*

$$m_0(\Delta(f)) = \frac{(d-w_3)}{w_3},$$

$$m_1(\Delta(f)) = \frac{(d-w_3)(d-2w_3)}{w_1.w_3} \wedge \frac{(d-w_3)(d-2w_3)}{w_2.w_3},$$

$$m_2(\Delta(f)) = \frac{(d-w_3)(d-2w_3)}{w_1.w_3} \wedge \frac{(d-w_3)(d-2w_3)}{w_2.w_3} + \frac{(d-w_3)(d-2w_3)(d-w_1-w_2-w_3)}{w_1.w_2.w_3}.$$

where the symbol $a \wedge b$ represents the minimum integer number of a and b (where at least one of these is an integer), that is,

$$a \wedge b = \begin{cases} min(a,b) & if \;\; a, b \in \mathbb{Z} \\ a & if \;\; b \notin \mathbb{Z} \\ b & if \;\; a \notin \mathbb{Z} \end{cases}$$

For the proof of this result, Jorge Pérez used the formula $m_0(\Delta(f)) = \delta(f) - 1$, where $\delta(f)$ is the degree of the map germ f. He obtained this from the following result due to Looijenga [[17], p.78]:" Let $f : \mathbb{C}^k, 0 \to \mathbb{C}^k, 0$ be a germ such that $X = f^{-1}(0)$ is an ICIS. Then, $\mu(X, 0) = \delta(f) - 1$". However we cannot apply these results here, since here we also are considering $n > 3$.

For the particular case $n = 4$ and $p = 3$ we derive

Corollary 4.3. *Let $f(x_1, x_2, x_3, x_4) = (x_1, x_2, g(x_1, x_2, x_3, x_4))$ be a finitely determined, quasi-homogeneous, corank one map germ with weights w_1, w_2, w_3, w_n, where $w_1 \leq w_2$ and $w_3 \leq w_4$ and let d be the degree of g, $d \geq w_1$ and $d \geq w_2$. Then,*

$$\mathbf{m_0(\Delta(f))} = \frac{(d-w_3)(d-w_4)}{w_3w_4},$$

$$\mathbf{m_1(\Delta(f))} = \frac{(2d-w_2-2w_3-2w_4)(2d-3w_3-2w_4)(2d-2w_3-3w_4)(d-w_3)(d-w_4)}{w_2w_3w_4(d-2w_3-w_4)(d-w_3-2w_4)}+$$

$$\frac{(d-w_2-w_3)(2d-2w_3-2w_4)(d-w_3-w_4)(d-2w_3)(d-w_4)}{w_2w_3w_4(-d+w_3+2w_4)(w_4-w_3)}+$$

$$\frac{(d-w_2-w_4)(2d-2w_3-2w_4)(d-w_3-w_4)(d-w_3)(d-2w_4)}{w_2w_3w_4(-d+2w_3+w_4)(w_4-w_3)}+1$$

$$\mathbf{m_2(\Delta(f))} = \frac{(d-w_1-w_3)(d-w_2-w_3)(d-2w_3)(d-w_3-w_4)(d-w_4)}{w_1w_2w_3w_4(w_4-w_3)}-$$

$$\frac{(d-w_1-w_4)(d-w_2-w_4)(d-w_3-w_4)(d-2w_4)}{w_1w_2w_3w_4(w_4-w_3)}-\frac{(d-w_3)(d-w_4)}{w_3w_4}+$$

$$\frac{(2d-w_2-2w_3-2w_4)(2d-3w_3-2w_4)(2d-2w_3-3w_4)(d-w_3)(d-w_4)}{w_2w_3w_4(d-2w_3-w_4)(d-w_3-2w_4)}+$$

$$\frac{(d-w_2-w_3)(2d-2w_3-2w_4)(d-w_3-w_4)(d-2w_3)(d-w_4)}{w_2w_3w_4(-d+w_3+2w_4)(w_4-w_3)}+$$

$$\frac{(d-w_2-w_4)(2d-2w_3-2w_4)(d-w_3-w_4)(d-w_3)(d-2w_4)}{w_2w_3w_4(-d+2w_3+w_4)(w_4-w_3)}+2$$

Proof of Theorem 4.1 The first polar multiplicity that we consider is $m_0(\Delta(f))$. Using the relation $m_0(\Delta(f)) = deg(p_1 \circ f, J[f])$ given in 2.5, where p_1 is a linear generic projection $p_1 : \mathbb{C}^3 \to \mathbb{C}^2$ and without loss of generality we may assume that this projection p_1 is given by $p_1(x,y,z) = (a_1x+a_2y+a_3z, b_2y+b_3z))$ and that $J[f] = \langle g_{x_3}, \dots, g_{x_n}\rangle$.

Then,

$$m_0(\Delta(f)) = deg(a_1x_1+a_2x_2+a_3g(x_1,\dots,x_n), b_2x_2+b_3g(x_1,\dots,x_n), g_{x_3}, g_{x_4}\cdots, g_{x_n}) =$$

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle a_1x_1+a_2x_2+a_3g(x_1,\dots,x_n), b_2x_2+b_3g(x_1,\dots,x_n), g_{x_3}, g_{x_4}\cdots, g_{x_n}\rangle} \qquad (*)$$

Note that the n-generators of the ideal $I = \langle a_1x_1+a_2x_2+a_3g(x_1,\dots,x_n), b_2x_2+b_3g(x_1,\dots,x_n), g_{x_3}, g_{x_4}\cdots, g_{x_n}\rangle$ form a semi quasi homogeneous map germ whose degrees are $D_1 = w_1, D_2 = w_2, D_3 = d-w_3, D_4 = d-w_4, \dots, D_n = d-w_n$, respectively. Therefore, from the Theorem 3.3 it follows that the dimension (*) is given by $\frac{D_1D_2\dots D_n}{w_1w_2\dots w_n}$. But $D_1 = w_1$ and $D_2 = w_2$, so this dimension is equal to $\frac{D_3D_4\dots D_n}{w_3w_4\dots w_n}$.

Consequently, $m_0(\Delta(f)) = \dfrac{D_3D_4\dots D_n}{w_3w_4\dots w_n}$.

Now, to exhibit $m_1(\Delta(f))$ in terms of weights and degrees of f we use part of the demonstration of the Theorem 2.5 to get the following equality

$$m_1(\Delta(f)) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle p_2\circ f, J[f], J[p_1\circ f, J[f]]\rangle} = \mu(p_2\circ f, J[f], J[p_1\circ f, J[f]])+1$$

where p_2 is a linear generic projection from $\mathbb{C}^3$ to $\mathbb{C}$. We can suppose, without loss of generality, that p_2 is given by $p_2(x, y, z) = c_1x + c_2y + c_3z$.

The next step is to show that the ideal $J = \langle p_2 \circ f, J[f], J[p_1 \circ f, J[f]]\rangle$ is semi quasi homogeneous and apply the Theorem 3.1. To get this it is necessary to study each generator of this ideal. First we note that

$$(p_2 \circ f)(x_1, x_2, \ldots, x_n) = p_2(x_1, x_2, g(x_1, x_2, \ldots, x_n)) = c_1x_1 + c_2x_2 + c_3g(x_1, x_2, \ldots, x_n)$$

and $J(f)$ is generated by the system $\{g_{x_3}, \ldots, g_{x_n}\}$.

For the ideal $J[p_1 \circ f, J(f)]$, we remark that that

$$(p_1 \circ f)(x_1, \ldots, x_n) = (a_1x_1 + a_2x_2 + a_3g(x_1, \ldots, x_n), b_2x_2 + b_3g(x_1, \ldots, x_n))$$

Then,

$$J[p_1 \circ f, J[f]] = J[a_1x_1 + a_2x_2 + a_3g(x_1, \ldots, x_n), b_2x_2 + b_3g(x_1, \ldots, x_n), g_{x_3}, \ldots, g_{x_n}].$$

This last ideal is generated by determinant of the following matrix of the order n:

$$\begin{bmatrix} a_1 + a_3g_{x_1} & a_2 + a_3g_{x_2} & a_3g_{x_3} & a_3g_{x_4} & \cdots & a_3g_{x_n} \\ 0 & b_2 & b_3g_{x_3} & b_3g_{x_4} & \cdots & b_3g_{x_n} \\ g_{x_3x_1} & g_{x_3x_2} & g_{x_3^2} & g_{x_3x_4} & \cdots & g_{x_3x_n} \\ g_{x_4x_1} & g_{x_4x_2} & g_{x_4x_3} & g_{x_4^2} & \cdots & g_{x_4x_n} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ g_{x_nx_1} & g_{x_nx_2} & g_{x_nx_3} & g_{x_4x_n} & \cdots & g_{x_n^2} \end{bmatrix}.$$

Consequently, $J[p_1 \circ f, J[f]] = \langle a_1b_2g_{x_3^2}g_{x_4^2}\cdots g_{x_n^2} +$ terms with upper degree$\rangle$ and $J = \langle c_1x_1 + c_2x_2 + c_3g(x_1, x_2, \ldots, x_n), g_{x_3}, \ldots, g_{x_n}, a_1b_2g_{x_3^2}g_{x_4^2}\cdots g_{x_n^2} +$ terms with upper degree$\rangle$.

Now we consider the ideal $J_0 = \langle x_1, a_1b_2g_{x_3^2}g_{x_4^2}\cdots g_{x_n^2}, g_{x_3}, \ldots, g_{x_n}\rangle$, whose degree of the generators are $D_1 = w_1$, $D_2 = (n-2)d - 2\sum_{i=3}^{n} w_i$, $D_l = d - w_l, 3 \leq l \leq n$, respectively.

By hypothesis, we suppose $w_1 \leq, w_2$ and $w_3 \leq \ldots \leq w_n$, so that J is semi quasi homogeneous. Moreover as p_1 is generic $a_1b_2 \neq 0$, hence the initial part is equal to J_0. From the Theorem 3.1, it follows that $\mu(J) = \mu(J_0)$ and we obtain $m_1(\Delta(f)) = \mu(J) + 1 = \mu(J_0) + 1$.

We use the Theorem 3.2-(b) to calculate $\mu(J_0)$ in terms of the weights and of the degrees of the map germ f, namely:

$$\mu(J_0) = \sum_{\rho=1}^{n}\prod_{\nu=1}^{n}\left(\frac{D_\rho}{w_\nu} - 1\right)\prod_{\substack{\kappa=1\\ \kappa\neq\rho}}^{n}\left(\frac{1}{\frac{D_\rho}{D_\kappa} - 1}\right)$$

Hence,

$$m_1(\Delta(f)) = \sum_{\rho=1}^{n}\prod_{\nu=1}^{n}\left(\frac{D_\rho}{w_\nu} - 1\right)\prod_{\substack{\kappa=1\\ \kappa\neq\rho}}^{n}\left(\frac{1}{\frac{D_\rho}{D_\kappa} - 1}\right) + 1,$$

with $D_1 = w_1$, $D_2 = (n-2)d - 2\sum_{i=3}^{n} w_i$, $D_l = d - w_l, 3 \leq l \leq n$.

Now, it remains to express $m_2(\Delta(f))$ in terms of the weights and of the degrees of the map germ f. For this note that from the Theorem 2.5 we obtain the following equality,

$$m_2(\Delta(f)) - m_1(\Delta(f)) + m_0(\Delta(f) = \mu(\Sigma(f)) + 1$$

Therefore

$$m_2(\Delta(f)) = \mu(\Sigma(f)) - m_0(\Delta(f) + m_1(\Delta(f)) + 1 \qquad (**)$$

We observe that $\Sigma(f) = V(g_{x_3}, \ldots, g_{x_n})$ and each generator g_{x_l} has degree $D_l = d - w_l, 3 \leq l \leq n$, then if we apply again the theorem 3.1, we deduce

$$\mu(\Sigma(f)) = \sum_{\rho=3}^{n} \prod_{\nu=1}^{n} \left(\frac{D_\rho}{w_\nu} - 1\right) \prod_{\substack{\kappa=3\\ \kappa\neq\rho}}^{n} \left(\frac{1}{\frac{D_\rho}{D_\kappa} - 1}\right)$$

Finally, it follows by replacing in the relation above $\mu(\Sigma(f))$, $m_0(\Delta(f))$, $m_1(\Delta(f))$ by their values, the last desired equality

$$m_2(\Delta(f)) = \sum_{\rho=3}^{n} \prod_{\nu=1}^{n} \left(\frac{D_\rho}{w_\nu} - 1\right) \prod_{\substack{\kappa=3\\ \kappa\neq\rho}}^{n} \left(\frac{1}{\frac{D_\rho}{D_\kappa} - 1}\right) - \prod_{j=3}^{n} \left(\frac{D_j}{w_j}\right) +$$

$$\sum_{\rho=1}^{n} \prod_{\nu=1}^{n} \left(\frac{D_\rho}{w_\nu} - 1\right) \prod_{\substack{\kappa=1\\ \kappa\neq\rho}}^{n} \left(\frac{1}{\frac{D_\rho}{D_\kappa} - 1}\right) + 2$$

with $D_1 = w_1$, $D_2 = (n-2)d - 2\sum_{i=3}^{n} w_i$, $D_l = d - w_l, 3 \leq l \leq n$.

□

5 Polar Multiplicities in $f(\Sigma^{(n-2,1)}(f))$

We now deal with the polar multiplicities of the set $f(\Sigma^{(n-2,1)}(f))$.

Theorem 5.1. *Let $f(x_1, x_2, \ldots, x_n) = (x_1, x_2, g(x_1, x_2, \ldots, x_n))$ be a finitely determined, quasi homogeneous, corank one map germ with weights $w_1, \ldots, w_n$, where $w_1 \leq w_2$ and $w_3 \leq \ldots \leq w_n$ and let d be the degree of g, with $d \geq w_1$ and $d \geq w_2$.*

Then,

$$\mathbf{m_0(f(\Sigma^{(n-2,1)}(f)))} = \sum_{\rho=1}^{n} \prod_{\nu=1}^{n} (\frac{D_\rho}{w_\nu} - 1) \prod_{\substack{\kappa=1\\ \kappa\neq\rho}}^{n} (\frac{1}{\frac{D_\rho}{D_\kappa} - 1}) + 1$$

$$\mathbf{m_1(f(\Sigma^{(n-2,1)}(f)))} = \sum_{\rho=2}^{n} \prod_{\nu=1}^{n} (\frac{D_\rho}{w_\nu} - 1) \prod_{\substack{\kappa=2\\ \kappa\neq\rho}}^{n} (\frac{1}{\frac{D_\rho}{D_\kappa} - 1})[\prod_{\mu=2}^{n} (\frac{1}{\frac{D_\mu}{D_1} - 1}) + 1] - \sharp A_3$$

with $D_1 = w_1, D_2 = (n-2)d - 2\sum_{i=3}^{n} w_i, D_l = d - w_l, 3 \leq l \leq n$.

For a quasi-homogeneous map germ in $\mathcal{O}(3,3)$, Jorge Pérez in [11] showed how to compute the polar multiplicities in terms of the weights and degrees.

Theorem 5.2. *[11] Let $f=(x,y,g(x,y,z))\in\mathcal{O}(3,3)$ be a finitely determined, quasi-homogeneous, corank 1 germ with weights w_1, w_2, w_3 and d the degree of g. Then,*

$$m_0(f(\Sigma^{1,1}(f)))=\frac{(d-w_3)(d-2w_3)}{w_1.w_3}\wedge\frac{(d-w_3)(d-2w_3)}{w_2.w_3},$$

$$m_1(f(\Sigma^{1,1}(f)))=\sum_{j=1}^{2}\prod_{i=1}^{3}(\tfrac{d_j}{w_i}-1)\prod_{k=1,k\neq j}^{2}(\tfrac{d_k}{d_j-d_k})+\frac{(d-w_3)(d-2w_3)}{w_1.w_3}\wedge\frac{(d-w_3)(d-2w_3)}{w_2.w_3}-(\frac{w_1.w_2.w_3+\prod_{j=1}^{3}(d-jw_3)}{w_1.w_2.w_3})$$

For the demonstration of this theorem the author used strongly the fact that $n=p=3$. To quasi-homogeneous germs in $\mathcal{O}(n,n)$ there is a important result, due to Marar, Montaldi and Ruas, which shows the number of singularities of type A_k in terms of the weight and degree of f, namely:

Theorem 5.3. *[19] Let $f:(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ be a corank 1 weighted- homogeneous $\mathcal{A}$–finite map-germ with weights and degrees as above. For any stabilization of f, and any partition $\mathcal{P}$ of n,*

$$\sharp A_{\mathcal{P}}=\frac{w_n^{n-1}}{N(\mathcal{P})w}\prod_{j=1}^{n+l-1}(\frac{d}{w_n}-j)$$

where l is the length of $\mathcal{P}$, $w_n=wt(f_n)$, $d=degree(f_n)$, $w=\prod_{i=1}^{n-1}$ and $N(\mathcal{P})$ define the order of the sub group of S_l which fixes $\mathcal{P}$. Here S_l acts on $\mathbb{R}^l$ by permuting the coordinates.

Again, for the particular case $n=4$ and $p=3$ we get

Corollary 5.4. *Let $f(x_1,x_2,x_3,x_4)=(x_1,x_2,g(x_1,x_2,x_3,x_4))$ be a finitely determined, quasi-homogeneous, corank one map germ with weights w_1,w_2,w_3,w_n, where $w_1\leq w_2$ and $w_3\leq w_4$ and let d be the degree of g, $d\geq w_1$ and $d\geq w_2$.*

Then,

$$\mathbf{m_0(f(\Sigma^{(2,1)}(f)))}=\sum_{\rho=1}^{4}\prod_{\nu=1}^{4}(\frac{D_\rho}{w_\nu}-1)\prod_{\substack{\kappa=1\\ \kappa\neq\rho}}^{4}(\frac{1}{\frac{D_\rho}{D_\kappa}-1})+1$$

$$\mathbf{m_1(f(\Sigma^{(2,1)}(f)))}=\sum_{\rho=2}^{4}\prod_{\nu=1}^{4}(\frac{D_\rho}{w_\nu}-1)\prod_{\substack{\kappa=2\\ \kappa\neq\rho}}^{4}(\frac{1}{\frac{D_\rho}{D_\kappa}-1})[\prod_{\mu=2}^{4}(\frac{1}{\frac{D_\mu}{D_1}-1})+1]-\sharp A_3$$

with $D_1=w_1$, $D_2=2(d-w_3-w_4)$, $D_3=d-w_3$ and $D_4=d-w_4$.

Proof of Theorem 5.1 First, since f has corank one, it follows by Corolary 2.7 that

$$m_1(\Delta(f))=m_0(f(\Sigma^{(n-2,1)}(f)))$$

Thus, by this equality and by item (i) from the Theorem 4.1, we obtain

$$m_0(f(\Sigma^{(n-2,1)}(f)))=\sum_{\rho=1}^{n}\prod_{\nu=1}^{n}\left(\frac{D_\rho}{w_\nu}-1\right)\prod_{\substack{\kappa=1\\ \kappa\neq\rho}}^{n}\left(\frac{1}{\frac{D_\rho}{D_\kappa}-1}\right)+1 \qquad (1)$$

with $D_1=w_1, D_2=(n-2)d-2\sum_{i=3}^{n}w_i, D_l=d-w_l, 3\leq l\leq n$.

It remains to show that $m_1(f(\Sigma^{(n-2,1)}(f)))$ also can be expressed in terms of the weights and of the degree. From the Theorem 2.6 we have the equality

$$m_0(f(\Sigma^{(n-2,1)}(f))) - m_1(f(\Sigma^{(n-2,1)}(f))) = \sharp A_3 - \mu(\Sigma^{(n-2,1)}(f)) + 1$$

Equivalently,

$$m_1(f(\Sigma^{(n-2,1)}(f))) = m_0(f(\Sigma^{(n-2,1)}(f))) + \mu(\Sigma^{(n-2,1)}(f)) - \sharp A_3 - 1 \tag{2}$$

To compute the Milnor number $\mu(\Sigma^{(n-2,1)}(f))$ we write $\Sigma^{(n-2,1)}(f) = V(J_{(n-2,1)}(f))$, where $J_{(n-2,1)}(f)$ is its associated Iterated Jacobian ideal, $J_{(n-2,1)}(f) = \langle g_{x_3}, g_{x_4}, \dots, g_{x_n}, M\rangle$, and M is the determinant $\begin{vmatrix} g_{x_3^2} & \cdots & g_{x_3x_n} \\ g_{x_4x_3} & \cdots & g_{x_4x_n} \\ \vdots & \ddots & \vdots \\ g_{x_nx_3} & \cdots & g_{x_n^2} \end{vmatrix}$, $g_{x_ix_j}$ denotes the partial derivative of g_{x_i} in the variable x_j.

Hence, $\Sigma^{(n-2,1)}(f) = V(M, g_{x_3}, g_{x_4}, \dots, g_{x_n})$, and each generator has degree $D_2 = (n-2)d - 2\sum_{i=3}^{n} w_i$, $D_l = d - w_l, 3 \le l \le n$, respectively.

Therefore, we can apply the theorem 3.2, to obtain

$$\mu(\Sigma^{(n-2,1)}(f)) = \sum_{\rho=2}^{n}\prod_{\nu=1}^{n}\left(\frac{D_\rho}{w_\nu} - 1\right)\prod_{\substack{\kappa=2\\ \kappa\neq\rho}}^{n}\left(\frac{1}{\frac{D_\rho}{D_\kappa} - 1}\right) \tag{3}$$

and $D_2 = (n-2)d - 2\sum_{i=3}^{n} w_i$, $D_l = d - w_l, 3 \le l \le n$.

Now if we replace (1) and (3) in (2) we obtain

$$m_1(f(\Sigma^{(n-2,1)}(f))) = \sum_{\rho=1}^{n}\prod_{\nu=1}^{n}\left(\frac{D_\rho}{w_\nu} - 1\right)\prod_{\substack{\kappa=1\\ \kappa\neq\rho}}^{n}\left(\frac{1}{\frac{D_\rho}{D_\kappa} - 1}\right) +$$

$$\sum_{\rho=2}^{n}\prod_{\nu=1}^{n}\left(\frac{D_\rho}{w_\nu} - 1\right)\prod_{\substack{\kappa=2\\ \kappa\neq\rho}}^{n}\left(\frac{1}{\frac{D_\rho}{D_\kappa} - 1}\right) - \sharp A_3 \tag{4}$$

and $D_1 = w_1$, $D_2 = (n-2)d - 2\sum_{i=3}^{n} w_i$, $D_l = d - w_l, 3 \le l \le n$

Since $D_1 = w_1$ we have that $\frac{D_1}{w_1} - 1 = 0$, so

$$\sum_{\rho=1}^{n}\prod_{\nu=1}^{n}\left(\frac{D_\rho}{w_\nu} - 1\right)\prod_{\substack{\kappa=1\\ \kappa\neq\rho}}^{n}\left(\frac{1}{\frac{D_\rho}{D_\kappa} - 1}\right) = \sum_{\rho=2}^{n}\prod_{\nu=1}^{n}\left(\frac{D_\rho}{w_\nu} - 1\right)\prod_{\substack{\kappa=1\\ \kappa\neq\rho}}^{n}\left(\frac{1}{\frac{D_\rho}{D_\kappa} - 1}\right)$$

Observe that for the equation (4), if we put the factor $\sum_{\rho=2}^{n}\prod_{\nu=1}^{n}\left(\frac{D_\rho}{w_\nu} - 1\right)\prod_{\substack{\kappa=2\\ \kappa\neq\rho}}^{n}\left(\frac{1}{\frac{D_\rho}{D_\kappa} - 1}\right)$ in evidence we get

$$m_1(f(\Sigma^{(n-2,1)}(f))) = \sum_{\rho=2}^{n}\prod_{\nu=1}^{n}\left(\frac{D_\rho}{w_\nu} - 1\right)\prod_{\substack{\kappa=2\\ \kappa\neq\rho}}^{n}\left(\frac{1}{\frac{D_\rho}{D_\kappa} - 1}\right)\left[\prod_{\mu=2}^{n}\left(\frac{1}{\frac{D_\mu}{D_1} - 1}\right) + 1\right] - \sharp A_3$$

with $D_1 = w_1$, $D_2 = (n-2)d - 2\sum_{j=3}^{n} w_j$, $D_l = d - w_l, 3 \le l \le n$

$\square$

6 The local Euler obstruction of the stable types

The local Euler obstruction for varieties, introduced in [18] by R. MacPherson in a purely obstructional way, is an invariant that is also associated to the polar multiplicities.

Lê and Teissier in [16], with the aid of Gonzales-Sprinberg's purely algebraic interpretation of the local Euler obstruction, showed that the local Euler obstruction is an alternate sum of the multiplicity of the local polar varieties.

The autors in [12] apply these results to obtain explicit and algebraic formulae for the Euler obstruction of the stable types which appear in mappings from $\mathbb{C}^n$ to $\mathbb{C}^3$.

In this section we apply these results to show how to compute the Local Euler obstruction of $\Delta(f)$ and $f(\Sigma^{n-2,1}(f))$, in terms of the weights and degrees, of any finitely determined quasi homogeneous map germ in $\mathcal{O}(n,3)$ with $n \geq 3$. First we recover the basic definitions and results.

Suppose that $X \subset \mathbb{C}^n$ is an analytic space of dimension d, ν the transformation of Nash of X. Let $p \in X$ and $z = (z_1, \cdots, z_n)$ be local coordinates in $\mathbb{C}^n$ such that $z_i(p) = 0$.

Let $\| z \|^2 = \Sigma z_i \overline{z_i}$. Since $\| z \|^2$ is a real-valued function, $d\| z \|^2$ may be considered as a section of $(T\mathbb{C}^n)^*$ where $*$ denotes the real dual bundle retaining only its orientation from the complex structure. We can also consider $d\| z \|^2$ as a restriction to a section r of $(TX)^*$. In [2] it is showed that for small ϵ, the section r is non zero over ν^{-1} where $0 \leq \| z \| \leq \epsilon$. Therefore let $B_\epsilon = \{z / \| z \| \leq \epsilon\}$ and $S_\epsilon = \{z / \| z \| = \epsilon\}$. The obstruction to extending r as a non zero section of TX^* from $\nu^{-1}(S_\epsilon)$ to $\nu^{-1}(B_\epsilon)$, which we denote by $Eu(TX^*, r)$, lies in $H^d(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon); \mathbb{Z})$. If $O_{(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon))}$ denotes the orientation class in $H_d(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon); \mathbb{Z})$, then we define the local Euler obstruction of X at p to be $Eu(TX^*, r)$ evaluated on $O_{(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon))}$ or symbolically

$$Eu_p(X) = \left\langle Eu(TX^*, r), O_{(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon))} \right\rangle$$

to $\nu^{-1}(B_\epsilon)$ (see [18] or [7] for the definition and more details).

The following result shows how the local Euler obstruction and the polar multiplicities are related.

Theorem 6.1. (Lê Dũng Trang et Teissier, [16]) *Let X be a reduced analytic space at $0 \in \mathbb{C}^{n+1}$ of dimension d. Then*

$$Eu_0(X) = \Sigma_{i=0}^{d-1} (-1)^{d-i-1} m_i(X)$$

where $m_i(X)$ denotes the absolute polar multiplicity of the polar variety $P_i(X)$.

We see in [12] the formulae for the Euler obstructions of the stable types $\Delta(f)$ and $f(\Sigma^{n-2,1}(f))$.

Corollary 6.2. *[12] Let $f \in \mathcal{O}(n,3), n > 3$ be a finitely determined map germ. Then:*

$$Eu_0(\Delta(f)) = m_2(f(\Sigma(f))) - \mu(\Sigma(f)) - 1,$$
$$Eu_0(f(\Sigma^{n-2,1}(f))) = \sharp A_3 - \mu(\Sigma^{n-2,1}(f)) + 1 + m_1(f(\Sigma^{n-2,1}(f))),$$

Here we use these formulae to compute the Euler obstruction in terms of the weights and degrees.

Corollary 6.3. *Let $f \in \mathcal{O}(n,3), n \geq 3$ be a finitely determined map germ. Then:*

$$Eu_0(\Delta(f)) = \sum_{\rho=1}^{n} \prod_{\nu=1}^{n} \left(\frac{D_\rho}{w_\nu} - 1 \right) \prod_{\substack{\kappa=1 \\ \kappa \neq \rho}}^{n} \left(\frac{1}{\frac{D_\rho}{D_\kappa} - 1} \right) - \prod_{j=3}^{n} \left(\frac{D_j}{w_j} \right) + 1$$

$$Eu_0(f(\Sigma^{n-2,1}(f))) = \sum_{\rho=1}^{n}\prod_{\nu=1}^{n}(\frac{D_\rho}{w_\nu}-1)\prod_{\substack{\kappa=1\\ \kappa\neq\rho}}^{n}(\frac{1}{\frac{D_\rho}{D_\kappa}-1})+1$$

with $D_1 = w_1, D_2 = (n-2)d - 2\sum_{i=3}^{n} w_i, D_l = d - w_l, 3 \leq l \leq n.$

7 Euler obstruction of simple germs $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$

A classification of the $\mathcal{A}$-simple germs $(\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ is given in [14]. Also, in that paper there is shown a list of invariants associated to these germs. V. H. Jorge Pérez in [11] increases this list by computing the polar multiplicities of the discriminant and of the image of the cuspidal edge curve.

As a direct consequence of the Corollary 6.3 we obtain the following:

Corollary 7.1. *The local Euler obstruction of $\Delta(f)$, where f is one of the $\mathcal{A}$-simple germs below is as follows:*

Normal Form	*Euler Obstruction*
$(x, y, z^3 + (x^2 + y^{k+1})z), k \geq 0$	$Eu_0(\Delta(f)) = 0$
$(x, y, z^3 + (x^2y + y^{k-1})z), k \geq 4$	$Eu_0(\Delta(f)) = k - 3$
$(x, y, z^3 + (x^3 + y^4)z)$	$Eu_0(\Delta(f)) = 1$
$(x, y, z^3 + (x^3 + xy^3)z)$	$Eu_0(\Delta(f)) = 1$
$(x, y, z^3 + (x^3 + y^5)z)$	$Eu_0(\Delta(f)) = 1$
$(x, y, z^4 + xz + y^kz^2), k \geq 1$	$Eu_0(\Delta(f)) = -1$
$(x, y, z^4 + (y^2 + x^k)z + xz^2), k \geq 2$	$Eu_0(\Delta(f)) = 0$
$(x, y, z^5 + xz + yz^2)$	$Eu_0(\Delta(f)) = -1$
$(x, y, z^5 + xz + y^2z^2 + yz^3)$	$Eu_0(\Delta(f)) = -1$
$(x, y, z^5 + xz + yz^3)$	$Eu_0(\Delta(f)) = -1$

Corollary 7.2. *The local Euler obstruction of $f(\Sigma^{1,1}(f))$, where f is one of the $\mathcal{A}$-simple germs below is as follows:*

Normal Form	*Euler Obstruction*
$(x, y, z^3 + (x^2 + y^{k+1})z), k \geq 0$	$Eu_0(f(\Sigma^{1,1}(f)) = 2$
$(x, y, z^3 + (x^2y + y^{k-1})z), k \geq 4$	$Eu_0(f(\Sigma^{1,1}(f)) = k - 1$
$(x, y, z^3 + (x^3 + y^4)z)$	$Eu_0(f(\Sigma^{1,1}(f)) = 3$
$(x, y, z^3 + (x^3 + xy^3)z)$	$Eu_0(f(\Sigma^{1,1}(f)) = 3$
$(x, y, z^3 + (x^3 + y^5)z)$	$Eu_0(f(\Sigma^{1,1}(f))) = 3$
$(x, y, z^4 + xz + y^kz^2), k \geq 1$	$Eu_0 f(\Sigma^{1,1}(f)) = 2$
$(x, y, z^4 + (y^2 + x^k)z + xz^2), k \geq 2$	$Eu_0(f(\Sigma^{1,1}(f))) = 3$
$(x, y, z^5 + xz + yz^2)$	$Eu_0(f(\Sigma^{1,1}(f)) = 3$
$(x, y, z^5 + xz + y^2z^2 + yz^3)$	$Eu_0(f(\Sigma^{1,1}(f)) = 3$
$(x, y, z^5 + xz + yz^3)$	$Eu_0(f(\Sigma^{1,1}(f)) = 3$

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