STATISTICS DEPARTAMENT

Technical Report

RT-MAE-2016-02

Preservation of a decreasing failure rate distributions under a coherent system signature point process representation

by

Vanderlei da Costa Bueno

Institute of Mathematics and Statistics

University of São Paulo
São Paulo - Brazil

Preservation of a decreasing failure rate distributions under a coherent system signature point process representation

Vanderlei da Costa Bueno

Abstract We analyse preservation properties of multivariate conditioned decreasing non-parametric distribution classes under formation of coherent system through a point process signature representation.

Keywords: Decreasing failure rate distribution relative to $(\Im_t)_{t\geq 0}$; New worse than used distribution relative to $(\Im_t)_{t\geq 0}$; Signature point processes.

AMS Classification: 60G55, 60G44.

1 Introduction

Classes of non-parametric distributions, such as decreasing failure rate (DFR) distributions, new worse than used (NWU) distributions and others, have been extensively investigated in Reliability Theory.

Several extensions of these concepts appeared in the literature, e.g. Harris (1970), Barlow and Proschan (1981), Marshall (1975) and others. However, they all have in common that they don't order the lifetime vectors in the sense of stochastic order as the univariate concept does. Arjas (1981), considered to observe the components, continuously in time, based on a family of sub σ-algebras $(\Im_t)_{t\geq 0}$ to introduce the notion of decreasing failure rate distribution and new worse than used distribution relative to σ-algebras $(\Im_t)_{t\geq 0}$, denoted by DFR$|\Im_t$ and NWU$|\Im_t$, respectively, generalizing the conventional definition of DFR and NWU and extending these classes into a multivariate form, denoted by MDFR$|\Im_t$ and MNWU$|\Im_t$. Bueno (2008) analyse theses distribution classes under a stopping time shift. Using these results, we intend to analyse the preservation properties of these non-parametric distribution classes under the formation of coherent system through a point process signature representation.

As in Barlow and Proschan (1981) a complex engineering system is completely characterized by its structure function ϕ which relates its lifetime T and its components lifetimes T_i, $1 \leq i \leq n$, defined in a complete probability space $(\Omega, \Im, P)$, by example, the series parallel decomposition:

$$T = \phi(\mathbf{T}) = \min_{1\leq j\leq k} \max_{i\in K_j} T_i,$$

where $K_j, 1 \leq j \leq k$ are minimal cut sets, that is, a minimal set of components whose failures causes the system fail. The performance of a coherent system can be measured from this structural relationship and the joint distribution function of its component lifetimes. The structures functions offer a way of indexing the class of coherent system but such representations make the distribution function of the system lifetime analytically very complicated. An alternative representation for the coherent system distribution function is through the system signature, as in Samaniego (1985), that, while narrower in scope than the structure function, is substantially more useful:

Definition 1.1 Let T be the lifetime of a coherent with component lifetimes $T_1, ..., T_n$ which are independent and identically distributed (i.i.d.) random variables with continuous distribution F. Then the signature vector is defined as $\mathbf{s} = (s_1, ..., s_n)$ where $s_i = P(T = T_{(i)})$ and $\{T_{(i)}, 1 \leq i \leq n\}$ are the order statistics of $\{T_i, 1 \leq i \leq n\}$.

Under the definition assumptions the events $\{T = T_{(i)}\}$ $1 \leq i \leq n$ form a (P-a.s.) partition of the probability space and

$$P(T \leq t) = \sum_{i=1}^{n} P(T = T_{(i)}) P(T_{(i)} \leq t | T = T_{(i)}) = \sum_{i=1}^{n} P(T = T_{(i)}) P(T_{(i)} \leq t).$$

A detailed treatment of the theory and applications of system signature may be found in Samaniego (2007). This reference gives detailed justification for the i.i.d. assumption used in the system signature definition. An important feature of system signature is the fact that, in the context of i.i.d. continuous component lifetimes, they are distribution free measures of system quality, depending solely on the design characteristics of the system and independent of the system components behavior.

Bueno (2013) define the signature point process to introduce the above structure under a more general condition. Under a complete information level $(\Im_t)_{t \geq 0}$, where $P(T_i = T_j) = 0, \forall i \neq j$, we have

$$P(T > t | \Im_t) = \sum_{i=1}^{n} 1_{\{T = T_{(i)})\}} 1_{\{T_{(i)} > t\}}.$$

This representation is suitable to analyse coherent systems of dependent components. In this context, to generalize classical results using the properties of non-parametric classes of distributions, we need to verify if these properties are preserved under such representation.

In this paper, in Section 2 we discuss the point process approach and the Arjas (1981) concepts of conditioned distribution classes. In Section 3 we analyse preservation results of the classes MDFR$|\Im_t$ and MNWU$|\Im_t$ under the formation of coherent system through a point process signature representation. We actualize the reader that the author analyse the increasing failure rate distribution relative to $(\Im_t)_{t \geq 0}$ and new better than used distribution relative to $(\Im_t)_{t \geq 0}$ through signature point processes in Proceedings of the 60th ISI World Statistics Comgress, Bueno (2015).

2 Preliminaries

2.1 Signature point processes.

In our general setup, we consider the vector $(T_1, ..., T_n)$ of n component lifetimes which are finite and positive random variables defined in a complete probability space $(\Omega, \Im, P)$, with $P(T_i \neq T_j) = 1$, for all $i \neq j, i, j$ in $C = \{1, ..., n\}$, the index set of components. The lifetimes can be dependent but simultaneous failures are ruled out.

The evolution of components in time define a marked point process given through the failure times and the corresponding marks.

We denote by $T_{(1)} < T_{(2)} < ... < T_{(n)}$ the ordered lifetimes $T_1, T_2, ..., T_n$, as they appear in time and by $X_i = \{j : T_{(i)} = T_j\}$ the corresponding marks. As a convention we set $T_{(n+1)} = T_{(n+2)} = ... = \infty$ and $X_{n+1} = X_{n+2} = ... = e$ where e is a fictitious mark not in C, the index set of the components. Therefore the sequence $(T_n, X_n)_{n\geq 1}$ defines a marked point process.

The mathematical description of our observations, the complete information level, is given by a family of sub σ-algebras of $\Im$, denoted by $(\Im_t)_{t\geq 0}$, where

$$\Im_t = \sigma\{1_{\{T_{(i)}>s\}}, X_i = j, 1 \leq i \leq n, j \in C, 0 < s \leq t\},$$

satisfies the Dellacherie conditions of right continuity and completeness .

Intuitively, at each time t the observer knows if the event $\{T_{(i)} \leq t, X_i = j\}$ have either occurred or not and if it had, he knows exactly the value $T_{(i)}$ and the mark X_i. Follows that the component and the system lifetimes T are $\Im_t$ stopping times.

Remark 2.1.1 In what follows we assume that relations between random variables and measurable sets, respectively, always hold with probability one, which means that the term P-a.s., is suppressed. An extended and positive random variable τ is an $\Im_t$-stopping time if, and only if, $\{\tau \leq t\} \in \Im_t$, for all $t \geq 0$; an $\Im_t$-stopping time τ is called predictable if an increasing sequence $(\tau_n)_{n\geq 0}$ of $\Im_t$-stopping time, $\tau_n < \tau$, exists such that $\lim_{n\to\infty} \tau_n = \tau$; an $\Im_t$-stopping time τ is totally inaccessible if $P(\tau = \sigma < \infty) = 0$ for all predictable $\Im_t$-stopping time σ. For a mathematical basis of stochastic processes applied to reliability theory see the book of Aven and Jensen(2009), Bremaud (1981).

We consider the lifetimes $T_{(i),j}$ defined by the failure event $\{T_{(i)}, X_i = j\}$ with their sub-distribution function $F_{(i),j}(t) = P(T_{(i),j} \leq t) = P(T_{(i)} \leq t, X_i = j)$ suitable standardized.

Under the complete information level the behavior of the process $P(T \leq t|\Im_t)$, as the information flows continuously in time is given by the following Theorem

Theorem 2.1.2 Let $T_1, T_2, ..., T_n$ be the component lifetimes of a coherent system with lifetime T. Then,

$$P(T \leq t|\Im_t) = \sum_{k,j=1}^{n} 1_{\{T=T_{(k),j}\}} 1_{\{T_{(k),j}\leq t\}}.$$

Proof From the total probability rule we have $P(T \leq t|\Im_t) =$

$$\sum_{k,j=1}^{n} P(\{T \leq t\} \cap \{T = T_{(k),j}|\Im_t) = \sum_{k,j=1}^{n} E[1_{\{T=T_{(k),j}\}} 1_{\{T_{(k),j} \leq t\}}|\Im_t].$$

As T and $T_{(k),j}$ are $\Im_t$-stopping time and it is well known that the event $\{T = T_{(k),j}\} \in \Im_{T_{(k),j}}$ where

$$\Im_{T_{(k),j}} = \{A \in \Im_\infty : A \cap \{T_{(k),j} \leq t\} \in \Im_t, \forall t \geq 0\},$$

we conclude that $\{T = T_{(k),j}\} \cap \{T_{(k),j} \leq t\}$ is $\Im_t$-measurable.

Therefore $P(T \leq t|\Im_t) =$

$$\sum_{k,j=1}^{n} E[1_{\{T=T_{(k),j}\}} 1_{\{T_{(k),j} \leq t\}}|\Im_t] = \sum_{k,j=1}^{n} 1_{\{T=T_{(k),j}\}} 1_{\{T_{(k),j} \leq t\}}.$$

The above decomposition allows us to define the signature process at component level.

Definition 2.1.3 The vector $(1_{\{T=T_{(k),j}\}}, 1 \leq k, j \leq n)$ is defined as the marked point signature process of the system ϕ.

Remark 2.1.4 As $P(T_i = T_j) = 0$ for all i, j, the collection $\{\{T = T_{(i),j}\}, 1 \leq i \leq n,\ \ 1 \leq j \leq n\}$ form a partition of Ω and $\sum_{k,j=1}^{n} \Sigma_{j=1}^{n} 1_{\{T=T_{(k),j}\}} = 1$. Therefore

$$P(T > t|\Im_t) = 1 - P(T \leq t|\Im_t) = \sum_{k,j=1}^{n} 1_{\{T=T_{(k),j}\}} - \sum_{k,j=1}^{n} 1_{\{T=T_{(k),j}\}} 1_{\{T_{(k),j} \leq t\}} =$$

$$\sum_{k,j=1}^{n} 1_{\{T=T_{(k)j}\}}[1 - 1_{\{T_{(k)j} \leq t\}}] = \sum_{k,j=1}^{n} 1_{\{T=T_{(k),j}\}} 1_{\{T_{(k),j} > t\}}.$$

Remark 2.1.5 Using Remark 2.1.4 we can calculate the system reliability as

$$P(T > t) = E[P(T > t|\Im_t)] =$$

$$E[\sum_{k,j=1}^{n} 1_{\{T=T_{(k),j}\}} 1_{\{T_{(k),j} > t\}}] = \sum_{k,j=1}^{n} P(\{T = T_{(k),j}\} \cap \{T_{(k),j} > t\}).$$

If the component lifetimes are totally inaccessible, independent and identically distributed we have,

$$P(T > t) = \sum_{k,j=1}^{n} P(T = T_{(k),j}) P(T_{(k),j} > t)$$

recovering the classical result as in Samaniego (1985).

Remark 2.1.6. Inspecting the system at a fixed time t.

As in Samaniego et al.(2009), in time dynamics we can observe the event $\{T > t\} \cap \{T_{(i)} \leq t \leq T_{(i+1)}\}$. However it is well know that

$$\Im_t \cap \{T_{(i)} \leq t \leq T_{(i+1)}\} = \Im_{T_{(i)}} \cap \{T_{(i)} \leq t \leq T_{(i+1)}\},$$

that is, the information up to t is the same information up to $T_{(i)}$. It means that, after the i-th failure we continue to observe $(\Im_{T_{(i)}+t})_{t\geq 0}$, where

$$\Im_{T_{(i)}+t} = \{A \in \Im_\infty : A \cap \{T_{(i)} \leq s - t\} \in \Im_s, \forall s > 0\}.$$

Using conveniently the Random Sample Theorem, after the i-th failure, we can consider the signature process

$$((1_{\{T=T_{(i+1)j_{i+1}}\}}|\Im_{T_{(i)}}), (1_{\{(T=T_{(i+2)j_{i+2}}\}}|\Im_{T_{(i)}}), ..., 1_{\{(T=T_{(n)j_n}\}}|\Im_{T_{(i)}})$$

and the representation

$$P(T - T_{(i)} > t|\Im_{T_{(i)}+t}) = E[\sum_{k,j_k=i+1}^{n} 1_{\{(T=T_{(k)j_k}\}} 1_{\{T_{(k)j_k}-T_{(i)}>t\}}|\Im_{T_{(i)}}].$$

2.2 Decreasing failure rate conditional classe of distribution

We let θ_t be a shift in time defined by

$$\theta_t T_i = (T_i - t)^+ = \max\{T_i - t, 0\}, \quad 1 \leq i \leq n.$$

We may think of $\theta_t T_i$ as the residual lifetime of T_i at time t. Let $\theta_t \mathbf{T} = (\theta_t T_1, .., \theta_t T_n)$.

Using a stochastic process approach to multivariate reliability systems, Arjas (1981) introduced classes of non-parametric distributions concepts based on conditional stochastic order.

In the decreasing failure rate case, it attempting to the fact that, for any w, the residual lifetimes $\theta_t T_i(w) = (T_i(w)-t)^+$ are decreasing in t. Since that we are looking for a definition where each $\theta_t T_i$ is stochastically increasing in t (given $\Im_t$),we must hole out the possibility that T_i, and hence $\theta_t T_i$, are actually fixed by $\Im_t$ and w.

Definition 2.2.1 We say that the lifetime T is decreasing failure rate relative to $(\Im_t)_{t\geq 0}$, denoted by DFR$|\Im_t$, if for all $0 \leq t \leq t^*$ and all real number $u \in \Re$

$$P\{\theta_t T > u|\Im_t\} \leq P\{\theta_{t^*} T > u|\Im_{t^*}\}, \quad on \quad \{T > t\}$$

Follows that a lifetime T is DFR if and only if T is DFR$|\sigma\{1_{\{T>s\}} 0 \leq s \leq t\}$. That means, we consider to observe the system lifetime, only. This fact follows easily noticing that the function

$$f(u,s;w = \frac{P(T>t+u)}{P(T>t)}, if P(T>t)>0; w \in \{T>t\}; 0, otherwise,$$

for $t > 0$ is a version of $P\{\theta_t T > u | \Im_t\}$.

Therefore the DFR$|\Im_t$ definition extend the classical definition of DFR. We now come to the definition of multivariate increasing failure rate distribution relative to $\Im_t$.

To define the multivariate version we consider $\Im_t$-stopping times $T_i, \quad 1 \leq i \leq n$ and the observable sets in $t, \quad J_t$ the index set of components which have not failed up to time t, i.e. $J_t = \{i : 1 \leq i \leq n, T_i > t\}$, in the way that J_t is $\Im_t$- measurable and we can think of events $\{J_t = I\}$ as observables at time t.

In the multivariate version $MDFR|\Im_t$ we simply restrict the monotonicity to components which have not failed during the time observed:

Definition 2.2.2 We say that $\mathbf{T}$ is multivariate decreasing failure rate relative to $(\Im_t)_{t\geq 0}$, denoted by MDFR$|\Im_t$, if for all $0 \leq t \leq t^*$

$$P((\theta_t T_i)_{i\in I_0} \in U|\Im_t) \leq P((\theta_t^* T_i)_{i\in I_0} \in U|\Im_{t^*}),$$

holds on $\{J_t = I_0\}$ for all subsets $I_0 \subset \{1, 2, ..., n\}$ and d_0-dimensional upper set $U \subset \Re^n$, $d_0 = card(I_0)$. U is an upper set if, $\mathbf{x} \in U$ and $\mathbf{y} \geq \mathbf{x}$ implies $\mathbf{y} \in U$.

Remark 2.2.3 The MDFR$|\Im_t$ distribution class have most of what could be called "desirable properties" of any extension of the conventional DFR class. In particular the marginalization property, that is, if $\mathbf{T} = (T_1, ..., T_n)$ is MDFR$|\Im_t$ and $I_0 \subset C$ is a subset index of the components, then $\mathbf{T}_0 = (T_i)_{i\in I_0}$ is MDFR$|\Im_t$.

Concerning the preservation of MDFR$|\Im_t$ distribution class under formation of coherent systems, the system should not have some of their components fail between t and t^*, $t \leq t^*$, while the other components are getting stochastically longer residual lives. If one, however, restricts the attention to the trace σ-fields $\Im_t \cap \{T_i > t, 1 \leq i \leq n\}$ (so that $J_t = \{1, 2, ..., n\}$) it can be proved that the DFR$|\Im_t$, class is preserved.

If $\mathbf{T}$ is MDFR$|\Im_t$, using the Random Sample Theorem we can say that for $0 \leq t \leq t^*$ and any open upper set $U \in \Re^n$, we have

$$P(\theta_t(\mathbf{T} - T_{(i)})^+ \in U|\Im_{T_{(i)}+t}) \leq P(\theta_{t^*}(\mathbf{T} - T_{(i)})^+ \in U|\Im_{T_{(i)}+t^*}),$$

where

$$\Im_{T_{(i)}+t^*} = \{A \in \Im_\infty : A \cap \{T_{(i)} + t \leq s\} \in \Im_s, \quad \forall s \geq 0\},$$

preserving the MIFR$|\Im_t$ classes of distributions. Bueno (2008) analyse this case in detail. In our context, there is no failure in the interval $[T_{(i-1)}, T_{(i)})$ and in this interval we can apply the definition o MDFR$|\Im_t$, without the restriction of the observable sets. As $\Re^+ = \cup_{i=1}^n + 1[T_{9i-1)}, T_{(i)})$, $T_{(0)} = 0 \quad T(n+1) = \infty$, we apply this approach in the next section.

This argument appears in a different fashion, in a classical way, where Samaniego, et al.(2009) define uniformly new better than used distributions.

3 Preservation results

In the next theorem we discuss the preservation of the component lifetimes MDFR$|\Im_t$ property under the point process signature representation. In fact, the MDFR$|\Im_t$ is carried over the system lifetime.

Theorem 3.1 Let $\mathbf{T} = (T_1, ..., T_n)$ be $\Im_t$-stopping times, representing the component lifetimes of a coherent system with lifetime T. If $\mathbf{T}$ is MDFR$|\Im_t$, then T is DFR$|\Im_t$.

Proof As $T_{(k)j}, 1 \leq k \leq n, 1 \leq j \leq n$ are increasing functions of $\mathbf{T}$, it can be proved that $T_{(k)j}$ is DFR$|\Im_{T_{k-1}+t}$. Therefore, if $T_{(k-1)} \leq t < t^* \leq T_{(k)}$, on $\{T_{(k)j} > u\}$, we have

$$P(\theta_{T_{(k-1)}+t} T_{(k)j} > u|\Im_{T_{(k-1)}+t}) \leq P(\theta_{T_{k-1)}+t^*} T_{(k)j} > u|\Im_{T_{(k-1)}+t^*}),$$

which means that

$$P(T_{(k)j} > u + T_{(k-1)} + t, \Delta) = \int_\Delta P(\theta_{T_{(k-1)}+t} T_{(k)j} > u|\Im_{T_{(k-1)}+t})dP \leq$$

$$\int_\Delta P(\theta_{T_{(k-1)}+t^*} T_{(k)j} > u|\Im_{T_{(k-1)}+t^*})dP = P(T_{(k)j} > u + T_{(k-1)} + t^*, \Delta)$$

for all $\Delta \in \Im_{T_{(k-1)}+t}$.

As

$$\{T = T_{(k)j}\} \in \Im_{T^-_{(k)j}} = \sigma\{A \cap \{T_{(k)j} > t\}, A \in \Im_t, , t \geq 0\}$$

we conclude that, for $u \in [T_{(k-1)}, T_{(k)})$, $\{T = T_{(k)j}\} \cap \{T_{(k)j} > u\} \in \Im_u$ and $\Im_u = \Im_{T_{k-1}}$. Follows that

$$\{T = T_{(k)j}\} \cap \{T_{(k)j} > t + u\} \in \Im_{T_{(k-1)}+t}$$

Replacing $\Delta = \{T = T_{(k)j}\} \cap \{T_{(k)j} > t + u\} \in \Im_{T_{(k-1)}+t}$ we have

$$P(T_{(k)j} > T_{k-1} + t + u, T_{(k)j} = T) = P(T_{(k)j} > T_{k-1} + t + u, T_{(k)j} = T, T_{(k)j} > t + u) \leq$$

$$P(T_{(k)j} > T_{k-1} + t^* + u, T_{(k)j} = T, T_{(k)j} > t^* + u) = P(T_{(k)j} > T_{k-1} + t^* + u, T_{(k)j} = T).$$

Therefore

$$P(\theta_{T_{(k-1)}+t} T > u|\Im_{T_{(k-1)}+t}) = P(T > T_{(k-1)} + t + u|\Im_{T_{(k-1)}+t}) =$$

$$E[\sum_{i,j_i=k}^{n} 1_{\{(T=T_{(i)j_i}\}} 1_{\{T_{(i)j_i}-T_{(k-1)}>t+u\}}|\Im_{T_{(k-1)}}] =$$

$$\sum_{j,k=1}^{n} P(T = T_{(i)j_i}, T_{(i)j_i} > T_{(k-1)} + t + u|\Im_{T_{(k-1)}}) \leq$$

$$\sum_{j,k=1}^{n} P(T = T_{(i)j_i}, T_{(i)j_i} > T_{(k-1)} + t^* + u|\Im_{T_{(k-1)}}) =$$

$$E[\sum_{i,j_i=k}^{n} 1_{\{(T=T_{(i)j_i}\}} 1_{\{T_{(i)j_i} - T_{(k-1)} > t^* + u\}}|\Im_{T_{(k-1)}}] =$$

$$P(T > T_{(k-1)} + t^* + u|\Im_{T_{(k-1)}+t^*}) =$$

$$P(\theta_{T_{(k-1)}+t^*} T > u|\Im_{T_{(k-1)}+t^*}).$$

Remark 3.2 This result contradicts the well knows result that monotone system with independent DFR component lifetimes need not be DFR.

It would be desirable that systems inherit the DFR$|\Im_t$ property of their components relative to the σ-algebra generated by the system lifetime T, that is, $G_t = \sigma\{1_{\{T>s\}}, 0 < s \leq t\}$. However it is not always true. A special case is given by **Lemma 3.3** Let

$\mathbf{T} = (T_1, ..., T_n)$ be $\Im_t$-stopping times, representing the component lifetimes of a coherent system with lifetime T. If $\mathbf{T}$ is MDFR$|\Im_t$, $G_t \subseteq \Im_t$ and $\theta_t T$ is independent of $\Im_t$, given G_t, then T is DFR$|G_t$. **Proof** For all $T_{(k-1)} \leq t < t^* < T_{(k)}$, we have

$$P(\theta_{T_{(k-1)}+t} T > s|G_{T_{(k-1)}+t}) = P(\theta_{T_{(k-1)}+t} T > s|G_{T_{(k-1)}+t} \vee \Im_{T_{(k-1)}+t}) =$$

$$P(\theta_{T_{(k-1)}+t} T > s|\Im_{T_{(k-1)}+t}) \leq P(\theta_{T_{(k-1)}+t^*} T > s|\Im_{T_{(k-1)}+t^*}) =$$

$$P(\theta_{T_{(k-1)}+t^*} T > s|G_{T_{(k-1)}+t^*} \vee \Im_{T_{(k-1)}+t^*}) = P(\theta_{T_{(k-1)}+t^*} T > s|G_{T_{(k-1)}+t^*}).$$

However, except for the trivial choice $G_t = \Im_t$, we can think of one just concrete case where $\theta_t T$ is independent of $\Im_t$ given G_t: a series system with the complete information $\Im_t$ and system generated σ-algebra $G_t = \sigma\{1_{\{T>s\}}, 0 < s \leq t\}$, with $T = \min_{1 \leq i \leq n} T_i$.

4. New worse than used conditional distribution class.

Definition 4.1 We say that $\mathbf{T}$ is multivariate new worse than used relative to $(\Im_t)_{t \geq 0}$, denoted by MNWU$|\Im_t$, if

$$P\{(\theta_t T_i)_{i \in I_0} \in U|\Im_t\} \geq P\{(T_i)_{i \in I_0} \in U|\Im_0\},$$

holds on $\{J_t = I_0\}$ for all subsets $I_0 \subset \{1, 2, ..., n\}$ and d_0-dimensional upper set $U \subset \Re^n$ $d_0 = card(I_0)$.

Clearly, the MDFR$|\Im_t$ distributions classe is contained in the class of MNWU$|\Im_t$ distributions. As in the MDFR$|\Im_t$ case, we also consider the the stopping time shift version:

Using the Random Sample Theorem we can say that for $0 \leq t \leq t^*$ and any open upper set $U \in \Re^n$, we have

$$P(\theta_t(\mathbf{T} - T_{(i)})^+ \in U|\Im_{T_{(i)}+t}) \geq P((\mathbf{T} - T_{(i)})^+ \in U|\Im_{T_{(i)}}),$$

preserving the MNWU$|\Im_t$ classes of distributions.

Similarly, as in the MIFR$|\Im_t$ distributions class, the point process signature representation preserves the MNBU$|\Im_t$ property:

Theorem 4.2 Let $\mathbf{T} = (T_1, ..., T_n)$ be $\Im_t$-stopping times, representing the component lifetimes of a coherent system with lifetime T. If $\mathbf{T}$ is MNWU$|\Im_t$, then T is NWU$|\Im_t$. The

proof follows the same argument of the proof of Theorem 3., making $t = 0$.

Unlike the class of MDFR$|\Im_t$ distributions, the class MNWU$|\Im_t$ has a closure property which is not enjoyed by the MDFR$|\Im_t$ class: the σ-algebras can be made coarser without destroying the MNWU$|\Im_t$ property. Therefore we can prove that:

Theorem 4.3 Let $\mathbf{T} = (T_1, ..., T_n)$ be $\Im_t$-stopping times, representing the component lifetimes of a coherent system with lifetime T. If $\mathbf{T}$ is MNWU$|\Im_t$, $(G_t)_{t\geq 0}$ is a sub-σalgebra of $(\Im_t)_{t\geq 0}$, that is, $G_t \subseteq \Im_t, \quad \forall t \geq 0$ and $G_0 = \Im_0$, then $\mathbf{T}$ is MNWU$|G_t$. **Proof** If

$T_{(k-1)} \leq t < T_{(k)}$, we have

$$P(\theta_t(\mathbf{T} - T_{(k-1)})^+ \in U|\ G_{T_{(k-1)}+t}) = E[P(\theta_t(\mathbf{T} - T_{(k-1)})^+ \in U|\Im_{T_{(k-1)}+t})|G_{T_{(k-1)}+t})] \leq$$

$$E[P((\mathbf{T} - T_{(k-1)})^+ \in U|\Im_{T_{(k-1)}})|G_{T_{(k-1)}+t}] = E[P((\mathbf{T} - T_{(k-1)})^+ \in U|\Im_{T_{(k-1)}})|G_{T_{(k-1)}}] =$$

$$P((\mathbf{T} - T_{(k-1)})^+ \in U|G_{T_{(k-1)}}).$$

4 Acknowledgements

This work was partially supported by FAPESP, Proc. No.2015/02249-1.

References

[1] Aven, T. and Jensen,U. (1999). *Stochastic Models in Reliability.* Springer-verlag, N.Y.

[2] Arjas, E.(1981). A stochastic process approach to multivariate reliability system: Notions based on conditional stochastic order, Mathematics of Operations Research 6, 263 - 276.

[3] Barlow, R. and Proschan, F.(1981). *Statistical Theory of Reliability and Life Testing: Probability models.* Holt, Reinhart and Winston, Inc. Silver Spring, MD.

[4]Bremaud, P.(1981). *Point Processes and Queues: Martingale Dynamics.* Springer-Verlag, N.Y.

[5] Bueno,V.C. (2008). Reliaility properties under a stopping time shift, European Journal of Operational Research, 186, 202 - 210. Institute of Mathematics and Statistics, S.Paulo, Brasil.

[6] Bueno,V.C. (2013) Signature marked point processes. Unpublished report. Institute of Mathematics and Statistics, S.Paulo, Brasil.

[7] Bueno,V.C. (2015) Preservation of non-parametric distributions classes under a system signature point process, Proceedings of the 60th ISI World Statistics Congress, 310 -315, Rio de Janeiro, Brazil.

[8] Harris, R.(1970). A multivariate definition for increasing hazard rate distribution functions. The Annals of Mathematical Statistics 37, 713 - 717.

[9]Marshall, A. W.(1975). Multivariate distributions with monotone hazard rate. In *Reliability and Fault Tree Analysis, 259 - 284.* Edited by Barlow R.E., Fussel, J.B. and Singapurwalla, N.D., Copyright by Society for Industrial and Applied Mathematics. Philadelphia, Pa.

[10] Samaniego, F.J. (1985). On closure of the IFR class under formation of coherent systems. IEEE Trans. Reliab., 34, 69 - 72.

[11] Samaniego,F.J. (2007). *System signatures and their applications in engineering reliability.* International Series in Operation Research and Management Science, Vol 110, Springer, New York.

[12] Samaniego, F.J., et al. (2009). Dynamics signatures and their use in comparing the reliability of new and used systems. naval Research Logistic, 56, 577 - 591.