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PATTERN'S RELIABILITY IMPORTANCE UNDER DEPENDENCE CONDITION AND DIFFERENT INFORMATION LEVELS

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ABSTRACT. Using martingale methods in reliability theory we develop the Barlow and Proschan reliability importance of a pattern to the system reliability under dependence conditions. We allows several levels of information to calculate the importance measure.

Keywords: Martingale methods in reliability theory, compensator process, reliability importance measure of patterns.

INTRODUCTION

There seems to be two main reasons for given an importance measure of system's component. Firstly it permits the analyst to determine which component merit the most additional research and development to improve overall system reliability at minimum cost or effort. Secondly, it may suggest the most efficient way to diagnose system failure by generating a repair checklist for an operator to follows. Birnbaum (1969) defined the reliability importance of a component in a system as follows. Let T and T_i denote the random lifetimes of the system and component respectively. Then the importance of T_i for T at time t is

$$I^B(i,t) = P(T > t | T_i > t) - P(T > t | T_i \leq t). \tag{0.0.1}$$

This measure depends on a given point t in time and it is not quite relevant for most design or redesign decisions. Several time independent importance measures have been suggested, and most of them are weighted integrals of $I^B(i,t)$ over t. Barlow and Proschan (1975) defined the reliability importance of component i to the system reliability as $\int_0^\infty I^B(i,t)dF_i(t)$ where $F_i(t)$ is the absolutely continuous distribution function of T_i, $1 \leq i \leq n$ which are independent. Iyer (1992) generalize this quantity considering dependent components which are jointly absolutely continuous and therefore, rules out ties among the failure times. In this paper we generalize the definition of Iyer (1992) considering pattern's failures, that is, failures of a set of components at the same time t.

Its well known that there exists a bijective relation between the class of all distribution functions and the class of the $\Im_t$-compensator processes, where $\Im_t$ is the natural σ-algebra (Norros (1986a)). To consider the dependence between the components, the structural dependence are the pattern's failure we use this relation to formulate the Barlow and Proschan measure through compensator processes.

Key words and phrases. Martingale methods in reliability theory, compensator process, reliability importance measure of patterns.

1. IMPORTANCE MEASURES OF INDEPENDENT COMPONENTS

We consider a collection of n components, $C_1, C_2, ..., C_n$. These are often assumed to form a large system ϕ. Each component C_i has a positive lifetime T_i after 0, where 0 can be thought of as the time at which Φ is installed. We let T_i, $1 \leq i \leq n$, be random variables in a complete probability space $(\Omega, \Im, P)$. If the vector of the component's lifetime is denoted by $\mathbf{T} = (T_1, ..., T_n)$, the system lifetime can be set as its series-parallel decomposition

$$T = \Phi(\mathbf{T}) = \min_{1 \leq j \leq k} \max_{i \in K_j} T_i, \tag{1.0.2}$$

where $K_j, 1 \leq j \leq k$ are minimal cut sets, that is, a minimal set of components whose joint failure causes the system's failure. Φ is the system structure function.

We also consider such decomposition on the equivalent notation

$$X(t) = \Phi(\mathbf{X}(t)) = \min_{1 \leq j \leq k} \max_{i \in K_j} X_i(t),$$

where $(X(t))_{t \geq 0}$ and $(X_i(t))_{t \geq 0}$ are right continuous stochastic processes with $X(t) = I(T > t)$, $X_i(t) = I(T_i > t)$ and $\mathbf{X}(t) = (X_1(t), X_2(t), ..., X_n(t))$.

A component i is critical to the system at time t if

$$\Phi(1_i, \mathbf{X}(t)) - \Phi(0_i, \mathbf{X}(t)) = 1,$$

where $(\cdot_i, \mathbf{X}(\mathbf{t})) = (\mathbf{X_1}(\mathbf{t}), ..., \mathbf{X_{i-1}}(\mathbf{t}), \cdot_i, \mathbf{X_{i+1}}(\mathbf{t}), ..., \mathbf{X_n}(\mathbf{t}))$.

Definition 1.0.1. (Birnbaum (1969)). The reliability importance of the component i to the system's reliability at time t is the probability that this component is critical to the system at time t, i.e.

$$\begin{aligned} I^B(i,t) &= P(\Phi(1_i, \mathbf{X}(t)) - \Phi(0_i, \mathbf{X}(t)) = 1) \\ &= P(T > t | T_i > t) - P(T > t | T_i \leq t). \end{aligned}$$

If the components are independent, the system's reliability is a multilinear function of the component's lifetimes distribution functions, $\overline{F}(t) = h(\overline{\mathbf{F}}(t))$, where $\overline{\mathbf{F}}(t) = (\overline{F}_1(t), ..., \overline{F}_n(t)), F_1, ..., F_n$ are the distribution function of $T_1, ..., T_n$ respectively and F is the system's lifetime distribution function. For a fixed t, it is easy to prove that

$$I^B(i,t) = \frac{dh(\overline{\mathbf{F}}(t))}{d\overline{F}_i(t)},$$

i.e., $I^B(i,t)$ is the rate at which the system's reliability grows relatively to the component reliability grow. The Birnbaum reliability importance of a component is a measurable function of the given point t in time. Barlow and Proschan provides a time independent importance measure:

Definition 1.0.2. (Barlow and Proschan (1975)) Let $T = \Phi(\mathbf{T})$ be the lifetime of a coherent system of n independent component lifetimes $T_1, ..., T_n$ with absolutely continuous distributions functions $F_1, ..., F_n$, respectively. The Barlow and Proschan reliability importance of component i to the system reliability is defined by

$$I^{BP}(i) = \int_0^\infty I^B(i,t) dF_i(t),$$

where $I^B(i,t)$ is the Birnbaum reliability importance.

We note that, if the components are independent we have

$$\frac{dF(t)}{dt} = -\frac{dh(\overline{\mathbf{F}}(t))}{dt} = -\sum_{i=1}^{n}\frac{dh(\overline{\mathbf{F}}(t))}{d\overline{F}_i(t)}\frac{d\overline{F}_i(t)}{dt} = \sum_{i=1}^{n} I^B(i,t)\frac{dF_i(t)}{dt}$$

and therefore

$$E[I(T \leq t)] = P(T \leq t) = \sum_{i=1}^{n}\int_0^t I^B(i,s)dF_i(s),$$

i.e. we can improve system reliability through $I^{BP}(i), 1 \leq i \leq n$.

Iyer(1992) consider the case in which $T_1,...,T_n$ are dependent lifetime with jointly absolutely continuous distribution, where $P(T_i = T_j) = 0$ for every $i \neq j$.

Definition 1.0.3. (Iyer (1992)) Let $T = \Phi(T_1,...,T_n)$ be the lifetime of the system, where $T_i, i = 1,...n$ are jointly absolutely continuous. The reliability importance of component i to the system reliability is

$$I(i) = \int_0^{+\infty} P[\Phi(1_i, \mathbf{X}(t)) - \Phi(0_i, \mathbf{X}(t)) = 1 | T_i = t]dF_i(t).$$

This measure correspond to the Barlow and Proschan importance measure when the components are independents and absolutely continuous.

2. IMPORTANCE MEASURES OF PATTERNS WITH DEPENDENT COMPONENTS.

2.1. Mathematical Formulation: The system is monitored at component's level, that is, at each instant t the observer knows if the events $\{T_i \leq t\}, 1 \leq i \leq n$, either occurred or not and if its does, he knows exactly the value of T_i. The mathematical formulation is given through a family of sub σ-algebras of $\Im$, denoted $(\Im_t)_{t\geq 0}$, where

$$\Im_t = \sigma(I(T_i > s), 1 \leq i \leq n,\ s \leq t),$$

satisfies the Dellacherie's condition of right continuity and completeness. We next describe the failures of $C_1,...,C_n$ as they appear in advancing time, as a stochastic process. This is conveniently done in terms of a multivariate (or marked) point process. The failure of a system consisting of $C_1,...,C_n$ can be thought of as a simple point process derived from the multivariate process.

For any outcome $T_1(w),...,T_n(w)$ of the lifetimes of $C_1,...,C_n$ let $q(w)$ be the number of distinct values in the set $\{T_i(w); 1 \leq i \leq n\}$. We denote the strictly increasing order statistics of this set by $T_{(k)}$, having then

$$T_{(1)} < T_{(2)} < ... < T_{(q(w))}.$$

Also let

$$J_{(k)}(w) = \{i : T_i(w) = T_{(k)}(w), 1 \leq i \leq n,\}$$

be the index set of the components failing at the kth smallest failure time $T_{(k)}$. If there are no multiples failures, the value of $J_{(k)}$ is one of the singletons $\{i\}, 1 \leq i \leq n$. In general, however, $J_{(k)}$ is a Λ-valued random variable, where Λ is the power set of $\{1, 2, ..., n\}$. We call $T_{(k)}$ the kth failure time and $J_{(k)}$ the kth failure pattern.

The random sequence $(T_{(k)}, J_{(k)})_{1\leq k\leq q}$ (of random length q)describes completely how the components $C_1,...,C_n$ fail. We let

$$T_{(q+1)} = T_{(q+2)} = ... = \infty$$

and

$$J_{(q+1)} = J_{(q+2)} = \dots = \emptyset$$

and call the multivariate point process $(T_{(k)}, J_{(k)})_{k\geq 1}$ the failure process of $C_1, \dots, C_n$.

Another equivalent way to describe the failures is by a multivariate counting process: For each fixed $J \in \Lambda$, let T_J and $N_J(w;t)$ be defined by

$$T_J = \inf\{T_{(k)} : J_{(k)} = J\},$$

where $\inf \emptyset = \infty$ and

$$N_J(w,t) = I(T_J \leq t) = \begin{cases} 0 & \text{if } t < T_J(w) \\ 1 & \text{if } t \geq T_J(w). \end{cases}$$

Each $(T_i(w))_{1\leq i\leq n}$ determines a sample path of the process $(N_J(t); J \in \Lambda)_{t\geq 0}$ and conversely. Therefore

$$\Im_t = \sigma(N_J(s); J \in \Lambda, s \leq t)$$

is equivalent to

$$\Im_t = \sigma(I(T_i > s); 1 \leq i \leq n,\ s \leq t).$$

The stopping times T_i or T_J are rarely of direct concern in reliability theory. One is more interested in system failures times, which depend on the cumulative pattern of failed components. In more detail, let Φ a monotone (or coherent) system with lifetime T. We let

$$D(t) = \begin{cases} J_{(1)} \cup \dots \cup J_{(k)}, & \text{if } T_{(k)} \leq t < T_{(k+1)} \\ 0 & \text{if } t < T_{(1)} \end{cases}$$

be the cumulative pattern of failed components up to time t. The sample paths $t \to D(w,t)$ are then right continuous and increasing in the natural partial order of Λ. We let $D(t-) = \lim_{s\uparrow t} D(s)$. If

$$\Lambda_\Phi = \{K_1, \dots, K_{k_0}\}, k_0 \geq k,$$

is the collection of all the cut sets of Φ, we clearly have

$$T = \inf\{t \geq 0 : D(t) \in \Lambda_\Phi\} = \min\{T_{(k)} : J_{(1)} \cup \dots \cup J_{(k)} \in \Lambda_\Phi\}.$$

We can therefore think that the point process with its only point at T, or equivalently the counting process

$$N_\Phi(t) = I(T \leq t), t \geq 0,$$

has been derived from the multivariate point process $(T_{(k)}, J_{(k)})_{k\geq 1}$.

Changing the point of view slightly, we can fix a time t and then look what immediate failure patterns in dt would result in a system failure. We call the set of such patterns,

$$\Gamma_\Phi(t) = \{J \in \Lambda : D(t^-) \notin \Lambda_\Phi, D(t^-) \cap J = \emptyset, D(t^-) \cup J \in \Lambda_\Phi\}$$

the set of critical failure patterns at time t. We see that $t \to \Gamma_\Phi(t)$ is increasing in the natural partial order of Λ for $t \leq T$ and left continuous. Furthermore,

$$\{T \in dt\} = \bigcup_{J \in \Gamma_\Phi(t)} \{T_J \in dt\}.$$

Will be convenient to define the critical level of the failure pattern J, the time from which onwards a failure of pattern J leads to system failure, that is, the $\Im_t$-stopping time $Y_\Phi(J)$ by

$$\begin{aligned} Y_\Phi(J) &= \inf\{t \geq 0 : J \in \Gamma_\Phi(t)\} \\ &= \inf\{t \geq 0 : D(t^-) \cup J \in \Lambda_\Phi\}. \end{aligned}$$

We see clearly that

$$\{J \in \Gamma_\Phi(t)\} = \{Y_\Phi(J) < t \leq T\}.$$

Having treated the component and system failures as point processes, a reader familiar with martingale methods in point process theory already expects that the notion of hazard will be in terms of the stochastic intensity of such processes, or, in an integral form, the compensator of the counting process $(N_J(t); J \in \Lambda)_{t\geq 0}$ (or of $(N_\Phi(t))_{t\geq 0}$). This notion of hazard is developed in the following. We remark that little of what follows depends on the particular structure we have postulated for the mark space Λ, the family of all subsets of $\{1, 2, ..., n\}$.

We start by considering a fixed pattern $J \in \Lambda$ and introduce the corresponding pattern-specific hazard process. The family of such processes, indexed by $J \in \Lambda$, is called the multivariate hazard process. We then go on by studying an arbitrary system lifetime and derive the connection between its hazard process and the multivariate process.

The $\Im_t$- compensator $(A_J(t))_{t\geq 0}$ of the univariate counting process $(N_J(t))_{t\geq 0}$ is the a.s. unique right continuous increasing predictable process such that, for each $k \geq 1$, the difference process stopped at $T_{(k)}$,

$$N_J(t \wedge T_{(k)}) - A_J(t \wedge T_{(k)})),$$

is an $(\Im_t)$-martingale. In view of the fact that $T_{(n+1)} = \infty$ then we have that $N_J(t) - A_J(t)$, is an $(\Im_t)$-martingale.

The compensator, when understood as a measure on the real line, is well known to have the interpretation

$$A_J(dt) = P(T_J \in dt | \Im_{t-}).$$

Intuitively, this corresponds to predicting if T_J is going to occur "now", based on all observations available up to the present, but not including it. Motivated by this we call $(A_J(t))_{t\geq 0}$ the hazard process of failure pattern J and $(A_J(t); J \in \Lambda)_{t\geq 0}$ the multivariate hazard process.

We now go on by studying the $\Im_t$-compensator of the counting process $(N_\Phi(t))_{t\geq 0}$ of system failure, denoting it by $(A_\Phi(t))_{t\geq 0}$. For obvious reasons we call this compensator the system hazard process.

It is natural to ask what is the contribution of the failure's component propensity for predicting the system 's failure propensity. To answer this question, in Theorem 2.1.1 below, Arjas (1981) characterize the relationship between the component's $\Im_t$- compensator and the system's $\Im_t$-compensator processes.

Theorem 2.1.1. *Under the above notation, the $\Im_t$-compensator of $M(t) = I(T \leq t)$ is*

$$A(t) = \sum_{J \in \Gamma\Phi(t)} [A_J(t \wedge T) - A_J(Y_\Phi(J))]^+) a.s.$$

where $[a]^+ = \max\{a, 0\}$.

2.2. **Importance of Patterns:** In what follows, we generalize the notation of Section 1. For $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $J = \{j_1, ..., j_r\} \subset \{1, ..., n\}$ the vector $(1_J, x)$ and $(0_J, x)$ denote this n-dimensional state vectors in which the components $x_{j_1}, ..., x_{j_r}$ of x are replaced by 1's and 0's respectively.

A pattern J is critical to the system at time t if

$$\Phi(\mathbf{1_J}, \mathbf{X(t)}) - \Phi(\mathbf{0_J}, \mathbf{X(t)}) = 1$$

Clearly we have the equivalence

$$\{Y_\Phi(J) < t \leq T\} = \{\Phi(\mathbf{1_J}, \mathbf{X(t)}) - \Phi(\mathbf{0_J}, \mathbf{X(t)}) = 1\}. \tag{2.2.1}$$

On the set $\{T_J = t\}$ we can define

Definition 2.2.1. The reliability importance of the pattern J (on the set $\{T_J = t\}$), to the system's reliability at time t is the probability that J is critical to the system at time t, that is

$$I_C^B(J, t) = P(Y_\Phi(J) < t \leq T | T_J = t).$$

Remark 2.2.2. In the case where simultaneous failure are ruled out, the values of $J_{(k)}$ is one of the singletons $\{i\}, 1 \leq i \leq n$, and

$$I_C^B(\{i\}, t) = P(\Phi(\mathbf{1_i}, \mathbf{X(t)}) - \Phi(\mathbf{0_i}, \mathbf{X(t)}) = 1 | T_i = t),$$

which appear in Iyer(1992). If the components are independents, $I_C^B(\{i\}, t)$ is the Birnbaum reliability importance.

We apply the notion of criticality to generalize the Barlow and Proschan importance of component's reliability to the system's reliability.

Definition 2.2.3. Let T be the lifetime of a coherent system and let T_J be the pattern lifetime with $\Im_t$-compensator processes $A_J(t)$. The reliability importance of a pattern J to the system reliability is defined by

$$I^\Phi(J) = E[[A_J(t \wedge T) - A_J(Y_\Phi(J))]^+]$$

where $J \in \Gamma_\Phi(t)$ and $Y_\Phi(J)$ is the critical level of pattern J to the system.

In the case where $T_J, J \in \Gamma_\Phi(t)$ are unpredictable $\Im_t$-stopping time, the $\Im_t$-compensator are continuous and

$$\begin{aligned} E[A_J(t \wedge T) - A_J(Y_\Phi(J))]^+ &= E[\int_{Y_\Phi(J)}^{T \wedge t} dA_J(s)] \\ &= E[\int_0^t I(Y_\Phi(J) < s \leq T) dA_J(s)]. \end{aligned}$$

If the components are independent, there are no multiples failures and the value of $J_{(k)}$ is one of the singletons $\{i\}, 1 \leq i \leq n$. $A_i(t) = -\log \overline{F}_i(t \wedge T_i)$, where $\overline{F}_i(t) = 1 - F_i(t)$, the survival function of component i, is the reliability of component i for a fixed mission time t. We have the following proposition:

Proposition 2.2.4. *If the components are independent with absolutely continuous distribution functions, the reliability importance of component i to the system's reliability is*

$$I^\Phi(\{i\}) = I^{BP}(i) = \int_0^\infty I^B(i, t) dF_i(t).$$

Proof. Under the proposition assumption, $I_C^B(\{i\},t) = I^B(i,t)$ and $Y_\Phi(\{i\}) = Y_\Phi(i)$

$$
\begin{aligned}
I^\Phi(\{i\}) &= E[\int_0^\infty I(Y_\Phi(\{i\}) < t \leq T) d(-\log \overline{F}_i(t \wedge T_i)] \\
&= E[\int_0^{T_i} I(Y_\Phi(\{i\}) < t \leq T) d(-\log(\overline{F}_i(t))] \\
&= \int_0^\infty [\int_0^s I(Y_\Phi(\{i\}) < t \leq T) \frac{dF_i(t)}{\overline{F}_i(t)}] dF_i(s) \\
&= \int_0^\infty [\int_t^\infty dF_i(s)] I(Y_\Phi(\{i\}) < t \leq T) \frac{dF_i(t)}{\overline{F}_i(t)} \\
&= \int_0^\infty I(Y_i < t \leq T) dF_i(t) \\
&= \int_0^\infty P(\Phi(1_i, \mathbf{X(s)}) - \Phi(0_i, \mathbf{X(s)}) = 1 | T_i = s) dF_i(s) \\
&= \int_0^\infty I^B(i,s) dF_i(s).
\end{aligned}
$$

□

However the independence is not a necessary condition for the both measure's equality, as we can see in Proposition 2.2.5 below.

Proposition 2.2.5. *Under the above notation* $I^\Phi(J) = \int_0^\infty I_C^B(J,s) dF_J(s)$.

Proof. Firstly we note that

$$\int_0^t I(Y_\Phi(J) < s \leq T) d[N_J(s) - A_J(s)]$$

is an $\Im_t$-martingale because $I(Y_\Phi(J) < s \leq T)$ is left continuous and $\Im_t$-predictable. Therefore

$$
\begin{aligned}
I^\Phi(J) &= E[\int_0^\infty I(Y_\Phi(J) < s \leq T) dA_J(s)] \\
&= E[\int_0^\infty I(Y_\Phi(J) < s \leq T) dN_J(s)] = P(Y_\Phi(J) < T_J \leq T) \\
&= \int_0^\infty P(\Phi(1_J, \mathbf{X(s)}) - \Phi(0_J, \mathbf{X(s)}) = 1 | T_J = s) dF_J(s) \\
&= \int_0^\infty I_C^B(J,s) dF_J(s).
\end{aligned}
$$

□

Definition 2.2.6. Let $T = \Phi(T_1, ..., T_n)$ the lifetime of a coherent system. If T_J is an unpredictable $\Im_t$-stopping time, the reliability importance of the pattern J to the system is $I_C^{BP}(J) = \int_0^\infty I_C^B(J,s) dF_J(s)$.

Remark 2.2.7. In the case where simultaneous failure are ruled out, the value of $J_{(k)}$ is one of singleton $\{i\}, 1 \leq i \leq n$. If the distribution of $T = (T_1, ..., T_n)$ are jointly absolutely continuous we have :

$$I^\Phi(\{i\}) = \int_0^\infty P(\Phi(1_i, \mathbf{X(s)}) - \Phi(0_i, \mathbf{X(s)}) = 1 | T_i = s) dF_i(s),$$

and we conclude that $I^{\Phi}(J)$ generalize the importance measure of Iyer (1992).

Example 1: We analyze a system of two dependent components C_1, C_2. To this purpose the two-dimensional exponential distribution of Marshall and Olkin with parameters given by $\beta_1, \beta_2 > 0$ and $\beta_{12} \geq 0$ is used, with bivariate distribution function given by

$$\begin{aligned} P(T_1 \leq s, T_2 \leq t) &= 1 - e^{-(\beta_1+\beta_{12})s} - e^{-(\beta_2+\beta_{12})t} \\ &+ e^{-(\beta_1 s+\beta_2 t+\beta_{12}(s \vee t))}, \end{aligned}$$

where T_1 and T_2 are the component's lifetimes. An interpretation of this distribution is as follows. Three independent exponential random variables Z_1, Z_2, Z_{12} with corresponding parameters β_1, β_2 and β_{12} describe the time points when a shock causes failure of component 1 or 2 or all intact components at the same time respectively. Then the components lifetimes are given by $T_1 = Z_1 \wedge Z_{12}$ and $T_2 = Z_2 \wedge Z_{12}$. The three different patterns to distinguish are $\{1\}$, $\{2\}$, $\{12\}$. Note that $T_{\{1\}} \neq T_1$ as we have for example $T_{\{1\}} = \infty$ on $\{T_1 = T_2\}$, i.e., on $\{Z_{12} < Z_1 \wedge Z_2\}$. Calculations then yield

$$A_{\{1\}}(t) = \begin{cases} \beta_1 t & \text{on } \{T_1 > t, T_2 > t\} \\ (\beta_1 + \beta_{12})t & \text{on } \{T_1 > t, T_2 \leq t\} \\ 0 & \text{elsewhere.} \end{cases}$$

$A_{\{2\}}(t)$ is given by obvious index interchanges, and

$$A_{\{12\}}(t) = \begin{cases} \beta_{12} t & \text{on } \{T_1 > t, T_2 > t\} \\ 0 & \text{elsewhere.} \end{cases}$$

The marginal distributions functions are exponential random variables , $P(T_i \leq t) = 1 - e^{-(\beta_i+\beta_{12})s}, i = 1, 2$. If the system is series, its lifetime is $T = T_1 \wedge T_2$ with distribution function $P(T \leq t) = 1 - e^{-(\beta_1+\beta_2+\beta_{12})t}$.

The critical level of component 1 is $Y_{\Phi}(1) = 0$ and $A_1(t) = (\beta_1 + \beta_{12})t$ on $\{T_1 > t\}$. Therefore

$$\begin{aligned} I^{\Phi}(1) &= E[\int_0^{\infty} I(0 < s \leq T_1 \wedge T_2) dA_1(s)] \\ &= E[\int_0^{T_1} I(0 < s \leq T_1 \wedge T_2)(\beta_1 + \beta_{12}) ds] \\ &= \frac{(\beta_1 + \beta_{12})}{(\beta_1 + \beta_2 + \beta_{12})}. \end{aligned}$$

The critical level of pattern $\{1\}$ is also 0, however

$$A_{\{1\}}(t) = \beta_1 t I(T_1 > t, T_2 > t) + (\beta_1 + \beta_{12}) t I(T_1 > t, T_2 \leq t),$$

and therefore

$$\begin{aligned} I^{\Phi}(\{1\}) &= E[\int_0^{\infty} I(0 < s \leq T_1 \wedge T_2) dA_{\{1\}}(s)] \\ &= E[\int_0^{\infty} I(0 < s \leq T_1 \wedge T_2)[\beta_1 I(T_1 > s, T_2 > s) \\ &+ (\beta_1 + \beta_{12}) I(T_1 > s, T_2 \leq s)] ds] \\ &= \beta_1 E[T_1 \wedge T_2] \\ &= \frac{(\beta_1)}{(\beta_1 + \beta_2 + \beta_{12})}. \end{aligned}$$

The critical level of pattern $\{12\}$ is also 0, and $A_{\{12\}}(t) = \beta_{12} t I(T_1 > t, T_2 > t)$.

$$\begin{aligned} I^{\Phi}(\{12\}) &= E[\int_0^{\infty} I(0 < s \leq T_1 \wedge T_2) dA_{\{12\}}(s)] \\ &= E[\int_0^{\infty} I(0 < s \leq T_1 \wedge T_2) \beta_{12} t I(T_1 > s, T_2 > s) ds \\ &= \beta_{12} E[T_1 \wedge T_2] \\ &= \frac{(\beta_{12})}{(\beta_1 + \beta_2 + \beta_{12})} \end{aligned}$$

If we consider a parallel system with lifetime $T = \Phi(\mathrm{T}) = T_1 \vee T_2$, the critical level for component 2 is $Y_{\Phi}(2) = Z_1$ and the compensator process is $A_2(t) = (\beta_2 + \beta_{12}) t I(T_2 > s)$.The reliability importance of component 2 is

$$\begin{aligned} I^{\Phi}(2) &= E[\int_0^{\infty} I(0 < s \leq T_1 \vee T_2) dA_2(s)] \\ &= E[\int_0^{T_2} I(Z_1 < s \leq T_1 \vee T_2)(\beta_1 + \beta_{12}) ds] \\ &= (\beta_1 + \beta_{12}) E[T_2 - Z_1] \\ &= (\beta_1 + \beta_{12})[\frac{1}{(\beta_1 + \beta_{12})} - \frac{1}{\beta_1}]. \end{aligned}$$

The critical level for the pattern $\{2\}$ is also Z_1 and

$$A_{\{2\}}(t) = \beta_2 t I(T_1 > t, T_2 > t) + (\beta_2 + \beta_{12}) t I(T_1 \leq t, T_2 > t).$$

Therefore

$$\begin{aligned} I^{\Phi}(\{2\}) &= E[\int_0^{\infty} I(Z_1 < s \leq T_1 \vee T_2) dA_{\{2\}}(s)] \\ &= E[\int_0^{\infty} I(0 < s \leq T_1 \vee T_2)[\beta_2 I(T_1 > s, T_2 > s) \\ &+ (\beta_2 + \beta_{12}) I(T_1 \leq s, T_2 > s)] ds] \\ &= \beta_1[\frac{1}{\beta_1 + \beta_{12}} + \frac{1}{\beta_2 + \beta_{12}} - \frac{1}{\beta_1 + \beta_2 + \beta_{12}} - \frac{1}{\beta_1}] \\ &+ (\beta_2 + \beta_{12})[\frac{1}{\beta_2 + \beta_{12}} - \frac{1}{\beta_1}]. \end{aligned}$$

3. IMPORTANCE OF PATTERNS WITH CHANGE OF INFORMATION LEVEL .

In the basic probability space $(\Omega, \Im, P)$, let $(\Im_t)_{t\geq 0}$, be the pre-t-history which contains all events of $\Im$ that can be distinguished up to and including time t. In our context $(\Im_t)_{t\geq 0}$ is the natural σ-algebra generated by the components lifetime

$$\Im_t = \sigma(I(T_i > s), 1 \leq i \leq n,\ s \leq t).$$

Let $(\mathcal{A}_t)_{t\geq 0}$ be another filtration an $(\Omega, \Im, P)$ which is a subfiltration of $(\Im_t)_{t\geq 0}$, that is $\mathcal{A}_t \subset \Im_t$ for all $t \geq 0$. We can view $(\Im_t)_{t\geq 0}$ as the complete information filtration and $(\mathcal{A}_t)_{t\geq 0}$ as the actual observation an a lower level. We assume that $(\Im_t)_{t\geq 0}$, and $(\mathcal{A}_t)_{t\geq 0}$ satisfies the Dellacherie's condition of right continuity and completeness.

We consider that the lifetime T of the system is totally inaccessible $\Im_t$-stopping time and the process $N_t = I(T \leq t)$ has an smooth semimartingale representation (SSM),ie.

$$N_t = I(T \leq t) = \int_0^t I(T > s)\lambda_s ds + M_t, t \geq 0\ , \tag{3.0.2}$$

where $M = (M_t)_{t\geq 0}$ is a $\Im$-martingales with paths right-continuous, left-hand limits and $M_0 = 0$. One of the advantages of the semimartingale technique is the possibility of studying the random evolution of stochastic process on different information levels. This was described in general by the projection theorem (Kallianpur, G (1980)), which say in which way an SSM representation changes when changing the filtration from $\Im_t$ to a sub-filtration $\mathcal{A}_t$. This projection theorem can be applied to the lifetime indicator process $(N_t)_{t\geq 0}$. If the lifetime can be observed in $\mathcal{A}_s$, ie. $\{T \leq s\} \in \mathcal{A}_s$ for all $0 \leq s \leq t$ them the change of the information level from $\Im_t$ to $\mathcal{A}_t$ leads from (3.0.2) to the representation:

$$N_t = E[I(T \leq t)|\mathcal{A}_t] = \int_0^t I\{T > s\}\hat{\lambda}_s ds + \bar{M}_t \tag{3.0.3}$$

where $\hat{\lambda}_s ds = E[\lambda_s|\mathcal{A}_s]ds = d\hat{A}_s$ and $\bar{M}_s = E[M_s|\mathcal{A}_s]$. Note that $A(t) = \int_0^t I(T > s)\lambda_s ds$ in (3.0.2), is the system's $\Im_t$-compensator process and therefore, from Theorem 2.1.1, I can be represented by

$$N_t = I(T \leq t) = \sum_{J\in\Gamma_\Phi(t)} \int_0^t I(Y_\Phi(J) < s \leq T)dA_J(s) + M_t. \tag{3.0.4}$$

If for all $J \in \Gamma_\Phi(t), Y_\Phi(J)$ and T are $\mathcal{A}_t$-stopping time, $t \geq 0$, we can use by projection theorem to write

$$N_t = I(T \leq t) = \sum_{J\in\Gamma_\Phi(t)} \int_0^t I(Y_\Phi(J) < s \leq T)d\hat{A}_J(s) + \bar{M}_t, \tag{3.0.5}$$

where $d\hat{A}_J(s) = E[dA_J(s)|\mathcal{A}_s]$ and $\bar{M}_s = E[M_s|\mathcal{A}_s]$. Under these assumptions we can define:

Definition 3.0.8. Let T be the lifetime of a coherent system and let T_J be the lifetime of pattern J with continuous $\Im_t$-compensator process $A_J(t)$, where

$(\Im_t)_{t\geq 0}$ is the natural history. Let $(\mathcal{A}_t)_{t\geq 0}$ a sub-filtration of $(\Im_t)_{t\geq 0}$ If the critical level $Y_\Phi(J)$ and T are $\mathcal{A}_t$-stopping time, then we define the reliability importance of pattern J given $\mathcal{A}_t$ by:

(3.0.6) $$\hat{I}^\Phi(J) = E[\int_0^\infty I(Y_\Phi(J) < s \leq T) d\hat{A}_J(s).$$

Example 2: (In continuation of Example 1)
Information about T_1, Z_2 and T ie.

$$\mathcal{A}_t = \sigma\{T_1 > s, Z_2 > s, T > s, 0 \leq s \leq t\}$$

i): Parallel system ($T = \Phi(\mathrm{T}) = T_1 \vee T_2$ where $T_1 = Z_1 \wedge Z_{12}$ and $T_2 = Z_2 \wedge Z_{12}$).
The critical level for pattern $\{1\}$ is $Y_\Phi(\{1\}) = Z_2$ and the failure rate process $d\hat{A}_{\{1\}}(t) = \hat{\lambda}_{\{1\}}(t)dt$ is $E[dA_{\{1\}}(t)|\mathcal{A}_t] = E[\beta_1 I(T_1 > t) + \beta_{12} I(T_1 > t) I(T_2 \leq t)|\mathcal{A}_t] = I(T_1 > t)(\beta_1 + \beta_{12} - \beta_{12}\exp^{-\beta_2 t})$.The reliability importance of pattern $\{1\}$ given $\mathcal{A}_t$ is:

$$\begin{aligned}\hat{I}^\Phi(\{1\}) &= E[\int_0^\infty I(Z_2 < s \leq T_1 \vee T_2) d\hat{A}_{\{1\}}(s)] \\ &= (\beta_1 + \beta_{12})(\frac{1}{\beta_1 + \beta_{12}} - \frac{1}{\beta_2}) + \frac{\beta_{12}}{\beta_2}(\frac{\beta_1 + \beta_{12}}{\beta_1 + \beta_2 + \beta_{12}} - \frac{1}{\beta_2^2}).\end{aligned}$$

The critical level for pattern $\{12\}$ is $Y_\Phi(\{12\}) = 0$ and the failure rate process $d\hat{A}_{\{12\}}(t) = \hat{\lambda}_{\{12\}}(t)dt$ is $E[dA_{\{12\}}(t)|\mathcal{A}_t] = E[\beta_{12} I(T_1 > t) I(T_2 > t)|\mathcal{A}_t]dt = \beta_{12} I(T_1 > t)\exp^{-(\beta_2+\beta_{12})t}$.The reliability importance of pattern $\{12\}$ given $\mathcal{A}_t$ is:

$$\begin{aligned}\hat{I}^\Phi(\{12\}) &= E[\int_0^\infty I(0 < s \leq T_1 \vee T_2) d\hat{A}_{\{12\}}(s)] \\ &= \frac{\beta_{12}}{\beta_2 + \beta_{12}}(1 - \frac{(\beta_1 + \beta_{12})}{(\beta_1 + \beta_2)}).\end{aligned}$$

ii): Series system ($T = \Phi(\mathrm{T}) = T_1 \wedge T_2$ where $T_1 = Z_1 \wedge Z_{12}$ and $T_2 = Z_2 \wedge Z_{12}$).
The critical level for pattern $\{1\}$ is $Y_\Phi(\{1\}) = 0$ and the failure rate process $d\hat{A}_{\{1\}}(t) = \hat{\lambda}_{\{1\}}(t)dt$ is $E[dA_{\{1\}}(t)|\mathcal{A}_t] = E[\beta_1 I(T_1 > t) + \beta_{12} I(T_1 > t) I(T_2 \leq t)|\mathcal{A}_t]dt = \beta_1 I(T_1 > t)$.The reliability importance of pattern $\{1\}$ given $\mathcal{A}_t$ is:

$$\begin{aligned}\hat{I}^\Phi(\{1\}) &= E[\int_0^\infty I(0 < s \leq T_1 \wedge T_2) d\hat{A}_{\{1\}}(s)] \\ &= \frac{\beta_1}{\beta_1 + \beta_2 + \beta_{12}}.\end{aligned}$$

The critical level for pattern $\{2\}$ is $Y_\Phi(\{2\}) = 0$ and the failure rate process $d\hat{A}_{\{2\}}(t) = \hat{\lambda}_{\{2\}}(t)dt$ is $E[dA_{\{2\}}(t)|\mathcal{A}_t] = E[\beta_2 I(T_2 > t) + \beta_{12} I(T_2 > t) I(T_1 \leq t)|\mathcal{A}_t]dt = \beta_2 + \beta_{12} I(T_1 \leq t)$.The reliability importance of pattern $\{2\}$ given $\mathcal{A}_t$ is:

$$\begin{aligned} \hat{I}^{\Phi}(\{2\}) &= E[\int_0^{\infty} I(0 < s \le T_1 \wedge T_2) d\hat{A}_{\{2\}}(s)] \\ &= \frac{\beta_2}{\beta_1 + \beta_2 + \beta_{12}}. \end{aligned}$$

The critical level for pattern $\{12\}$ is $Y_{\Phi}(\{12\}) = 0$ and the failure rate process $d\hat{A}_{\{12\}}(t) = \hat{\lambda}_{\{12\}}(t)dt$ is $E[dA_{\{12\}}(t)|\mathcal{A}_t] = E[\beta_{12} I(T_1 > t) I(T_2 > t)|\mathcal{A}_t] = \beta_{12} I(T_1 > t)$. The reliability importance of pattern $\{12\}$ given $\mathcal{A}_t$ is:

$$\begin{aligned} \hat{I}^{\Phi}(\{12\}) &= E[\int_0^{\infty} I(0 < s \le T_1 \wedge T_2) d\hat{A}_{\{12\}}(s)] \\ &= \frac{\beta_{12}}{\beta_1 + \beta_2 + \beta_{12}}. \end{aligned}$$

4. IMPORTANCE OF A MODULE AND OF A MINIMAL CUT SET

We consider the structure function Φ represented by its series- parallel decomposition

(4.0.7) $$\Phi(\mathbf{X}(t)) = \min_{1 \le j \le k} \max_{i \in K_j} X_i(t)$$

where $K_j, 1 \le j \le k$ are minimal cut sets.

Intuitively, a module of a coherent system is a subset of components which behaves as a "supercomponent". More technically, the coherent structure Φ of n components can be represented by

(4.0.8) $$\Phi(\mathbf{X}(t)) = \Psi[\chi(\mathbf{X}_M(t)), \mathbf{X}_{M^c}(t)]$$

where M is a subset of $\{1, ..., n\}$ with complement M^c over $\{1, ..., n\}$, χ is a coherent system of the components in M, Ψ a coherent structure. Then (M, χ) is a module of Φ and Ψ is the organizing structure.

We denote by $I^{\Psi}(\chi)$ the importance of the module (M, χ) for the structure Ψ. To calculate $I^{\Psi}(\chi)$ we need the following Lemma.

Lemma 4.0.9. *Let (M, χ) be a module of the coherent system Φ with organizing structure Ψ and let $L \subset M$. Then the pattern L is critical to the system Φ at time t if, and only if, L is critical to the module (M, χ) and the module (M, χ) is critical to the organizing structure Ψ at time t.*

Proof. The pattern L is critical to the system Φ at time t if, and only if,

$$\Phi(1_L, \mathbf{X}(t)) = 1 = \Psi[\chi(1_L, \mathbf{X}_M(t)), \mathbf{X}_{M^c}(t)]$$

and

$$\Phi(0_L, \mathbf{X}(t)) = 0 = \Psi[\chi(0_L, \mathbf{X}_M(t)), \mathbf{X}_{M^c}(t)].$$

We note that if $\chi(\mathbf{X}_M(t)) = 1$, then $\Phi(\mathbf{X}_M(t), \mathbf{X}_{M^c}(t)) = \Phi(1_M, \mathbf{X}_{M^c}(t))$ and if $\chi(\mathbf{X}_M(t)) = 0$ then $\Phi(\mathbf{X}_M(t), \mathbf{X}_{M^c}(t)) = \Phi(0_M, \mathbf{X}_{M^c}(t))$.

Now, $\chi(1_L, \mathbf{X}_M(t)) = 1$, because if it isn't, $\chi(1_L, \mathbf{X}_M(t)) = 0$ and the first expression above implies that

$$1 = \Psi(0, \mathbf{X}_{M^c}(t)) = \Phi(0_M, \mathbf{X}_{M^c}(t)) \le \Phi(0_L, \mathbf{X}(t)) = 0,$$

that is a contradiction.

Also $\chi(0_L,\ \mathbf{X}_M(t)) = 0$ because if it isn't the second expression implies that

$$0 = \Psi(1, \mathbf{X}_{M^c}(t)) = \Phi(1_M, \mathbf{X}_{M^c}(t)) \geq \Phi(1_L,\ \mathbf{X}(t)) = 1,$$

and that is a contradiction too. Therefore L is critical to module $(M,\ \chi)$ at time t. As $\Psi(1, \mathbf{X}_{M^c}(t)) = 1$ and $\Psi(0, \mathbf{X}_{M^c}(t)) = 0$ the module $(M,\ \chi)$ is critical to the organizing structure Ψ at time t. Obviously, the reverse holds. □

Theorem 4.0.10. *The reliability importance of the module χ for system reliability is*

$$I^{\Phi}(\chi) = \sum_{L\in\Lambda_M} E[\int_0^{\infty} I(Y_{\Phi}(L) < s \leq T)dA_L(s)] = \sum_{L\in\Lambda_M} I^{\Phi}(L),$$

where Λ_M is the powerset of M.

Proof. We denote by T^{χ} the lifetime of χ, $A_{\chi}(t)$ its $\Im_t$-compensator process; denote by Y_{χ} the critical level of the module $(M,\ \chi)$ for the system and by $Y_{\chi}(L)$ the critical level of pattern L for χ.

By definition

$$I^{\Phi}(\chi) = I^{\Psi}(\chi) = E[\int_0^{\infty} I(Y_{\chi} < s \leq T)dA_{\chi}(s).$$

Using Theorem 2.1.1 we have

$$A_{\chi}(s) = \sum_{L\in\Lambda_M} \int_{Y_{\chi}(L)}^{s\wedge T^{\chi}} dA_L(t)$$

and

$$dA_{\chi}(s) = \sum_{L\in\Lambda_M} I(Y_{\chi}(L) < s \leq T^{\chi})dA_L(s),$$

therefore

$$I^{\Phi}(\chi) = \sum_{L\in\Lambda_M} E[\int_0^{\infty} I(Y_{\chi} < s \leq T)I(Y_{\chi}(L) < s \leq T^{\chi})dA_L(s)].$$

Applying Lemma 4.0.9 we get

$$I^{\Phi}(\chi) = \sum_{L\in\Lambda_M} E[\int_0^{\infty} I(Y_{\chi}(L) < s \leq T)dA_L(s)].$$

□

A minimal cut set is a minimal set of components whose joint failure causes the system to fail.Given a minimal cut set K_j we define the minimal cut set structure $\Phi_{K_j}(\mathbf{X}(t)) = \max_{i\in K_j} X_i(t)$ and its lifetime $S_{K_j} = \max_{i\in K_j} T_i$.

Theorem 4.0.11. *The reliability importance of the minimal cut set K_j to the system reliability is*

$$I^{\Phi}(K_j) = \sum_{L\in\Lambda_{K_j}} E[\int_0^{\infty} I(Y_{\Phi}(L) < s \leq T)dA_L(s)] = \sum_{L\in\Lambda_{K_j}} I^{\Phi}(L),$$

where Λ_{K_j} is the power set of K_j.

Proof. Firstly, we observe that the cut set structure is in series with the structure function Φ and therefore the critical level of the cut set for the system is 0.Follows that

$$\begin{aligned} I^{\Phi}(K_j) &= E[\int_0^{T\wedge S_{K_j}} dA_{K_j}(s)] \\ &= E[\int_0^{T\wedge S_{K_j}} dN_{K_j}(s)] \\ &= P(S_{K_j} \leq T). \end{aligned}$$

As $P(S_{K_j} \geq T) = 1$, in the set $\{S_{K_j} = T\}$ we have

$$dA_{K_j}(s) = \sum_{L\in\Lambda_{K_j}} I(Y_{\Phi_{K_j}}(L) < s \leq S_{K_j})dA_L(s),$$

where $Y_{\Phi_{K_j}}(L)$ is the critical level of the pattern L to the cut structure Φ_{K_j} with lifetime S_{K_j}. Therefore, using Lemma 4.0.9

$$\begin{aligned} I^{\Phi}(K_j) &= E[\sum_{L\in\Lambda_{K_j}} \int_0^{\infty} I(0 < s \leq T)I(Y_{\Phi_{K_j}}(L) < s \leq S_{K_j})dA_L(s)] \\ &= E[\sum_{L\in\Lambda_{K_j}} \int_0^{\infty} I(Y_{\Phi}(L) < s \leq T)dA_L(s)] \\ &= \sum_{L\in\Lambda_{K_j}} I^{K_j}(L). \end{aligned}$$

□

References

[1] Arjas, E., "The failure and hazard processes in multivariate reliability system" ,*Mathematics of Operations Research*, 6(1981), 551-562.

[2] Barlow and Proschan,F., "Importance of system components and fault tree events", *Stochastic Processes and their Applications*, 3(1975), 153-173.

[3] Barlow and Proschan,F., *Statistical Theory of Reliability and Life Testing: Probability models.* Hold, Rinehart and Wiston , Inc.Silver Spring,MD,1981.

[4] Birnbaum, Z. W.,"On the importance of different components in a multicomponent system",in *Multivariate Analysis II, P.R. Krishnaiah*, Academic Press, N. Y.,1969.

[5] Iyer, S., "The Barlow-Proschan importance and its generalizations with dependent components"', *Stochastic Processes and their Applications* ,42 (1992), 353-359.

[6] Jensen,U., "A general replacement model"', *Methods and Models of Operation Research* ,34 (1990), 423-439.

[7] Kallianpur,G., *Stochastic Filtering Theory.* Spriger, New York,1980.

[8] Norros, I.,"A compensator representation of multivariate lifelength distributions, with applications", *Scandinavian Journal of Statistics* ,13(1986a),99-112.

[9] Norros, I., "Notes on Natvig's measure of importance of system's components", *Journal of Applied Probability,* 23 (1986b),736-747.

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