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GENERALIZED GEOMETRY AND TRANSITION PROBABILITIES

JURIAANS, S. O.[1,†] & QUEIROZ, P.C.[2,‡]

IME-USP[1], UFMA, Brazil[2]

ostanley@usp.br[†] pqueiroz@ime.usp.br[‡]

Abstract

Theories of Generalized Functions have been used to obtain solutions of differential equations containing singularities and nonlinearities and also to explain phenomena in physical reality such as in General Relativity and Fluid Dynamics. However the milieus in which this has always been done are the classical environments although results and developments in the generalized environments were used. Here we change the underlying structure $\mathbb{R}$ to $\widetilde{\mathbb{R}}$ and, relying on a Generalized Differential Calculus already developed, propose a Generalized Differential Geometry together with several other tools useful to fall back to classical milieus or classical solutions. In particular, given a classical oriented Riemannian manifold M, we embed M discretely into a generalized manifold M^* in such a way that M and its differential structure are the shadow of the differential structure of M^*. Among the tools we propose are a Fixed Point Theorem, based on the notion of hypersequences, the notion of support and ways to calculate generalized probabilities and transition probabilities. We extend an existing embedding theorem by proving that there exists an algebra embedding $\kappa : \hat{\mathcal{G}}(M) \longrightarrow \mathcal{C}^\infty(M^*, \widetilde{\mathbb{R}}_f)$, thus relating the generalized construction on classical manifolds to the classical construction on generalized manifolds.

1 Introduction

The theory of generalized functions goes back to Schwartz. More recently, J.F. Colombeau and E. Rosinger has undertaken the challenge of developing a nonlinear theory of generalized functions, thus extending Schwartz premier work. Colombeau's proposal has been extensively used. Several mayor contributions were given by prominent researchers in the field. In spite of these important contributions and development, somehow the underlying algebraic structure remained $\mathbb{R}$ or $\mathbb{C}$. It might be that one of the reasons to sticks to the classical underlying structure is the concern that introducing another underlying structure might lead to controversies either about the existence and rigor or about how much of nonstandard tools one needs to know to understand these structures. However, this should not really be concern since, for example, in [1], the Fermat reals ${}^\bullet\mathbb{R}$ were used as the algebraic underlying structure of a generalized differential calculus. The totally ordered topological ring ${}^\bullet\mathbb{R}$ is basically the union of halows of real numbers, each halow consisting of unique real number ${}^\circ x$ and elements $y = {}^\circ x + dt_a$, with $dt_a, a \in [1, \infty[$ being nilpotent elements. In particular, the group of invertible elements $Inv({}^\bullet\mathbb{R})$ is open but not dense and zero and nonzero infinitesimals are precisely the noninvertible elements.

It happens though that in applications and certain areas one must deal with infinitesimals and infinities which, in certain situations, are cancelled out by each other and thus suggesting that they are invertible elements. Can an environment be constructed in which infinitesimals and infinities coexists and some of which are invertible elements or at least invertible in some sense? There are several of such milieus and most of which are non-Archimedean rings. Recall that such non-Archimedean rings somehow originate with J. Tate. Here we focus on $\widetilde{\mathbb{R}}$ which was constructed in Colombeau's approach to generalized functions. Originally, it was just a ring where generalized functions took values, but, over time, it turned out to have a very rich topological and algebraic structure making it suitable to be the underlying algebraic milieu of a new Differential Calculus, a Generalized Differential Calculus. Let's sum up some of its features. Infinitesimals and infinities live like ebony and ivory in $\widetilde{\mathbb{R}}$ and when rendezvous occurs an interleaving of real numbers may be the result. Moreover, $Inv(\widetilde{\mathbb{R}})$ is open and dense, $\mathcal{B}(\widetilde{\mathbb{R}})$, its Boolean algebra of idempotent elements, consists of the characteristic functions of subsets of the real interval $I =]0,1]$ and if $x \notin Inv(\widetilde{\mathbb{R}})$ then there exist $e, f \in \mathcal{B}(\widetilde{\mathbb{R}})$ such that $e \cdot x = 0$ and $f \cdot x \in Inv(f \cdot \widetilde{\mathbb{R}})$.

Our purpose is to piece the puzzle using as pieces all the important concepts resulting in an ultra-metric milieu $\widetilde{\mathbb{R}}^n$, for each $n \in \mathbb{N}$, in which $\mathbb{R}^n$ is the shadow, or support, of points of $\widetilde{\mathbb{R}}^n$. In $\widetilde{\mathbb{R}}^n$, $\mathbb{R}^n - \{\vec{0}\}$ is a grid of equidistant points sitting between infinitesimals, the elements of $B_1(\vec{0}) - \{\vec{0}\}$, and infinities and hence, algebraically, it is the result of the rendezvous of such elements which go undetected in physical reality. The notion of the support of elements can be defined in each milieu and similar discrete embedding results hold. If $\Omega \subset \mathbb{R}^n$ is open, then there exists a discrete embedding of $\mathcal{D}'(\Omega)$ into $\mathcal{C}^\infty(\widetilde{\Omega}_c, \widetilde{\mathbb{R}})$, where $\widetilde{\Omega}_c$ is a subset of $\widetilde{\mathbb{R}}^n$ consisting of those elements of $\overline{B}_1(\vec{0})$ whose support is contained in Ω and their norm is less than some real number. In particular, Dirac's infinity δ, becomes a $\mathcal{C}^\infty-$function on $\widetilde{\mathbb{R}}_c$ and $x\delta$ becomes nonzero and, when evaluated at certain infinitesimals, produces real values. Generalized Space-Time is constructed and applications to physical reality are given.

2 Main Results

Theorem 2.1 (Fixed Point Theorem). *Let $\Omega \subset \mathbb{R}^n$, $A = [(A_\varphi)_\varphi] \subset B_r(0) \cap \mathcal{G}_f(\Omega)$, $r < 2$, be an internal set, and $T : A \to A$ be a mapping with representative $(T_\varphi : A_\varphi \to A_\varphi)_{\varphi \in A_0(n)}$. If there exists $k = [(k_\varphi)_\varphi] \in \tilde{\mathbb{N}}$ such that each $T_\varphi^{k_\varphi}$ is a λ-contraction, then T^k is well-defined, continuous, and has a unique fixed point $f_0 \in A$.*

Theorem 2.2 (Down Sequencing Argument). *Let $f \in \mathcal{G}_f(\Omega)$ with $\Omega \subset \mathbb{R}^n$. If $f \in W^0_{m,r}[0]$ with $r > 0$ and $p_0 \in \mathbb{N}^n$, then $f \in W^{\|p_0\|}_{m,s}[0]$ where $s = 4^{-n\|p_0\|}r$, i.e., $W^0_{m,r}[0] \subset W^{\|p_0\|}_{m,s}[0]$.*

Theorem 2.3 (Embedding Theorem). *Let M be an $n-$dimensional orientable Riemannian manifold. There exists an $n-$dimensional $\mathcal{G}_f-$manifold M^*, in which M is discretely embedded, and an algebra monomorphism $\kappa : \hat{\mathcal{G}}(M) \longrightarrow \mathcal{C}^\infty(M^*, \widetilde{\mathbb{R}}_f)$ which commutes with derivation. Moreover, equations whose data have singularities or nonlinearities defined on M naturally extend to equations on M^* and, on M^*, these data become $\mathcal{C}^\infty-$functions.*

Theorem 2.4. *Let $\Omega \subset \mathbb{C}$ and let $f \in \mathcal{H}(\Omega)^*$ be holomorphic, $\mathcal{Z}_f = \{z \in \widetilde{\Omega}_c \ : \ f(z) = 0_{\widetilde{\mathbb{C}}}\}$ and $Z_f = \Omega \bigcap \mathcal{Z}_f$, the generalized and classical zero set of f. Then $\mathcal{Z}_f = Interl(Z_f)$. In particular, Z_f is the support of points of $\mathcal{Z}_f$. If given $f \in \mathcal{HG}(\Omega))$ a holomorphic net, $E \subset \Omega$ a set of uniqueness, such that $\widetilde{E} \subset \mathcal{Z}_f$, then $f = 0$. Consequently, $f = 0$ if and only if supp$(\mathcal{Z}_f)$ is a set of uniqueness.*

Theorem 2.5. *Let $T = [(T_\epsilon)] \in \mathcal{B}(\mathcal{G}_H)$ be a selfadjoint operator such that each T_ϵ is selfadjoint. Then supp$(\nu(T)) = \{\nu(T_0) : T_0 \in$ supp$(T)\}$. In particular, the support of the generalized transition probabilities of T equals the transition probabilities of the elements of the support of T.*

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