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**STANDBY REDUNDANCY ALLOCATION FOR A
COHERENT SYSTEM UNDER ITS SIGNATURE
POINT PROCESS REPRESENTATION**

by

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Palavras-Chaves: Martingale methods in reliability theory, signature point process, standby redundancy coherent systems.

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**Standby redundancy allocation for a coherent system
under its signature point process representation.**

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Abstract Willing to work in reliability theory under dependence condition, in a general set up, we intend to characterize standby operation through compensator transform and to use a signature point process representation of a coherent system to optimize system reliability.

Keywords: Martingale methods in reliability theory, signature point process, standby redundancy, coherent system.

1.Introduction.

There are two common types of redundancy that are used in reliability theory, namely active redundancy, which stochastically leads to consideration of maximum of random variables, and standby redundancy, which stochastically leads to consideration of the convolution of random variables.

In reliability theory, the main application of redundancy is to allocate a redundant spare in a system component position in order to optimize system lifetime. For instance, see Boland et al. (1992), Singh and Misra (1994), Meng (1996), Kuo et al. (2000),(2001), Prasad et al. (1999), Bueno (2005), Bueno et al. (2007),Belzunce et al. (2013), Zhao et al.(2012, 2013 ,2015), among others.

For a k -out-of- n :G system, Boland et al.(1992) consider likelihood ratio ordering and gives sufficient conditions to ensure that in a series system the allocation of a standby spare should go to the weakest component while in a parallel system it should go to the strongest.

Singh and Misra (1994) consider the same problem with another criterion of optimality and proves that, for a k -out-of- n :G system, it is preferable to allocate the active redundant component to the stochastically weakest component. In both above papers the components lifetimes is stochastically independent and the observations are at system level.

Few papers had attained in the case where the components are stochastically dependent. Belzunce, et al.(2013) analyses redundancies for a k -out-of- n system of dependent components. Bueno and Carmo (2007) study active redundancy allocation for a k -out-of- n system of dependent components without simultaneous failures. Bueno (2005) define a particular form of standby redundancy, which we call minimal standby redundancy and gives to the lifetime an additional lifetime as it had just before the failure. For the case of dependent components, Bueno (2005), observe the system at component level and uses the reverse rule of order 2 (RR_2) property between compensator processes to investigate the problem of where to allocate a spare, through a minimal standby redundancy, in a k -out-of- $n:F$.

In this paper, we intend to analyse standby redundancy allocation for a coherent system of dependent components without simultaneous failures, characterize standby redundancy of dependent component lifetimes at component level and, under the signature point process representation of a coherent system, proves that it is optimal to perform active redundancy on the weakest component of a coherent system.

In Section 2 we characterize standby redundancy through compensator transform for dependent components. In Section 3 we resumes mathematical details of the signature point process representation of a coherent system and in Section 4 we investigate the best standby redundancy allocation in a coherent system of dependent components in order to optimize system reliability.

2. Standby operation through compensator transform

Generally, standby redundancy gives to the lifetime an additional new unit lifetime.

Each component in standby redundancy has two phases, standby and operation under which they can fail. Depending on component failures characteristics during these phases, standby redundancy is classified into the following three types:

1) *Hot standby*: Each component has the same failure rate regardless of whether it is in standby or in operation. Since the failure rate of one component is unique and is not affected by the other components, the hot standby redundancy consists of stochastically independent components.

2) *Warm standby*: A standby component can fail, but it has smaller failure rate than the principal component. Failure characteristics of the component are affected by the other, and warm standby induces dependent component failures.

3) *Cold standby*: Components does not fail when they are in standby. The components have non-zero failure rates only when they are in operation. A failure of one principal component forces a standby component to start operation and to have a non-zero failure rate. Thus, failure characteristics of one component are affected by the others, and the cold standby redundancy results in mutually dependent component failures.

In our general set up and in order to simplify the notation, in this paper we assume that relations such as $\subset, =, \leq, <, \neq$ between random variables and measurable sets, always hold with probability one, which means that the term P -a.s., is suppressed.

We consider to observe two component lifetimes T and S , which are finite positive random variables defined in a complete probability space $(\Omega, \mathfrak{F}, P)$ through the family of sub σ -algebras $(\mathfrak{F}_t)_{t \geq 0}$ of \mathfrak{F} , where

$$\mathfrak{F}_t = \sigma\{1_{\{S > s\}}, 1_{\{T > s\}}, 0 \leq s \leq t\}$$

satisfies Dellacherie's conditions. In what follows we assume that S and T are totally inaccessible \mathfrak{F}_t -stopping time and that $P(S \neq T) = 1$, that is, the lifetimes can be dependent but simultaneous failures are ruled out.

We recall that an \mathfrak{F}_t -stopping time Y is called predictable if an increasing sequence $(Y_n)_{n \geq 0}$, of \mathfrak{F}_t -stopping time, $Y_n < Y$, exists such that, $Y_n \rightarrow Y$ as $n \rightarrow \infty$ and an \mathfrak{F}_t -stopping time Y is totally inaccessible if $P(Y = Z < \infty) = 0$ for all predictable \mathfrak{F}_t -stopping time Z . For a mathematical basis of stochastic processes applied to reliability theory see the book of Aven and Jensen (1999), Bremaud (1981).

In our context the standby operation of S by T is defined by the improvement of S by $(T - S)^+$ and denoted by S^{HS}

$$S^{HS} = S + (T - S)^+,$$

where $(T - S)^+ = T - S$ in $\{T > S\}$, and is equal to 0 in $\{T \leq S\}$.

In relation to $(\mathfrak{F}_t)_{t \geq 0}$ we consider the predictable compensator processes $(A_t)_{t \geq 0}$ and $(B_t)_{t \geq 0}$ such that $1_{\{T \leq t\}} - A_t$ and $1_{\{S \leq t\}} - B_t$ are 0 mean \mathfrak{F}_t -martingales. From the totally inaccessibility of S and T , A_t and B_t are continuous.

The compensator process is expressed in terms of the conditional probability, given the available information. Its generalize the classical notion of hazard. Intuitively, this corresponds to product whether the failure is going to occur now, on the basis of all observations available up to, but not including, the present.

Follows, by the well known equivalence between the distribution functions and the compensator processes, see Arjas and Yashin (1988), that $P(T > t | \mathfrak{F}_t) = e^{-A_t}$ and $P(S > t | \mathfrak{F}_t) = e^{-B_t}$.

In the case of independent lifetimes, the survival function of the improved lifetime S^{HS} is

$$P(S + (T - S)^+ > t | \mathfrak{F}_t) = P(S > t) + \int_0^t P((T - s)^+ > t - s | \mathfrak{F}_t) e^{-B_s} dB_s = e^{-(A_t + B_t)} \{e^{A_t} + e^{B_t} - 1\}.$$

Therefore the \mathfrak{F}_t -compensator of $1_{\{S^{HS} \leq t\}}$ is

$$B_t^{HS} = A_t + B_t - \ln[e^{A_t} + e^{B_t} - 1] = A_t + B_t - \int_0^t \frac{e^{A_s} dA_s + e^{B_s} dB_s}{e^{A_s} + e^{B_s} - 1} = \int_0^t \frac{e^{B_s} - 1}{e^{A_s} + e^{B_s} - 1} dA_s + \int_0^t \frac{e^{A_s} - 1}{e^{A_s} + e^{B_s} - 1} dB_s.$$

In this fashion, and preserving this interpretation, we define, for dependent lifetimes, the \mathfrak{F}_t -compensator of S^{HS} as the sum of the compensator transformations of A_t , A_t^* , and of B_t , B_t^* , that is

$$A_t^* = \int_0^t \alpha_s dA_s, \quad \alpha_s = \frac{e^{B_s} - 1}{e^{A_s} + e^{B_s} - 1},$$

and

$$B_t^* = \int_0^t \beta_s dB_s, \quad \beta_s = \frac{e^{A_s} - 1}{e^{A_s} + e^{B_s} - 1}.$$

We observe that $0 < \alpha_s < 1$ and $0 < \beta_s < 1$ implying $A_t^* \leq A_t$ and $B_t^* \leq B_t$ getting an improvement of the lifetimes.

Following this thinking, as a predictable compensator is unique we are going to find a probability measure, under which, A_t^* is the \mathfrak{F}_t -compensator of $1_{\{T \leq t\}}$ and B_t^* is the \mathfrak{F}_t -compensator of $1_{\{S \leq t\}}$.

To proceed, we consider the compensator transform

$$A_t^* = \int_0^t \frac{e^{B_s} - 1}{e^{A_s} + e^{B_s} - 1} dA_s = \int_0^t \left\{ 1 - \left[\frac{e^{A_s}}{e^{A_s} + e^{B_s} - 1} \right] \right\} dA_s =$$

$$A_t - \int_0^t \frac{e^{A_s}}{e^{A_s} + e^{B_s} - 1} dA_s.$$

To prove the main Theorem of this note we are going to use the following Lemma:

Lemma 2.1 The following process

$$L_{A_t} = \left(\frac{e^{B_T} - 1}{e^{A_T} + e^{B_T} - 1} \right) 1_{\{T \leq t\}} e^{\int_0^t \frac{e^{A_s}}{e^{A_s} + e^{B_s} - 1} dA_s}$$

is a nonnegative local \mathfrak{F}_t -martingale and $E[L_{A_t}] = 1$.

Proof We consider the \mathfrak{F}_t -stopping time defined by

$$V_n = \inf\{t \geq 0 : A_t \geq n \text{ or } B_t \geq n\}.$$

It is sufficient to prove that the process

$$L_{A_t}^n = \left(\frac{e^{B_T} - 1}{e^{A_T} + e^{B_T} - 1} \right) 1_{\{T \leq t \wedge V_n\}} e^{\int_0^{t \wedge V_n} \frac{e^{A_s}}{e^{A_s} + e^{B_s} - 1} dA_s}$$

is a bounded \mathfrak{F}_t -martingale.

For any \mathfrak{F}_t -stopping time $V \leq V_n$ we can write

$$L_{A_V}^n = 1 - \int_0^V e^{\int_0^s \frac{e^{A_u}}{e^{A_u} + e^{B_u} - 1} dA_u} \frac{e^{A_s}}{e^{A_s} + e^{B_s} - 1} d(N_s - A_s)$$

where $N_t = 1_{\{T \leq t\}}$. The procedure is easy:

On the set $\{V < T\}$ we have

$$1 + \int_0^V e^{\int_0^s \frac{e^{A_u}}{e^{A_u} + e^{B_u} - 1} dA_u} \frac{e^{A_s}}{e^{A_s} + e^{B_s} - 1} dA_s =$$

$$e^{\int_0^V \frac{e^{A_u}}{e^{A_u} + e^{B_u} - 1} dA_u} = L_{A_V^n}.$$

Otherwise, on the set $\{V \geq T\}$

$$1 - \int_0^V e^{\int_0^s \frac{e^{A_u}}{e^{A_u} + e^{B_u} - 1} dA_u} \frac{e^{A_s}}{e^{A_s} + e^{B_s} - 1} d(N_s - A_s) =$$

$$e^{\int_0^T \frac{e^{A_u}}{e^{A_u} + e^{B_u} - 1} dA_u} \left[1 - \frac{e^{A_T}}{e^{A_T} + e^{B_T} - 1} \right] = L_{A_T^n}.$$

As the integrand $e^{\int_0^s \frac{e^{A_u}}{e^{A_u} + e^{B_u} - 1} dA_u} \frac{e^{A_s}}{e^{A_s} + e^{B_s} - 1}$ is an \mathfrak{F}_t -predictable process and $N_s - A_s$ is an \mathfrak{F}_t -martingale, $L_{A_T^n}$ is an \mathfrak{F}_t -martingale with $E[L_{A_T^n}] = 1$.

Secondly, we consider the compensator transform

$$B_t^* = \int_0^t \frac{e^{A_s} - 1}{e^{A_s} + e^{B_s} - 1} dB_s = B_t - \int_0^t \frac{e^{B_s}}{e^{A_s} + e^{B_s} - 1} dB_s$$

and with the same argument to prove Lemma 2.1 we can prove Lemma 2.2:

Lemma 2.2 The following process

$$L_{B_t} = \left(\frac{e^{A_S} - 1}{e^{A_S} + e^{B_S} - 1} \right)^{1_{\{S \leq t\}}} e^{\int_0^t \frac{e^{B_s}}{e^{A_s} + e^{B_s} - 1} dB_s}$$

is a nonnegative local \mathfrak{F}_t -martingale with $E[L_{B_t}] = 1$.

Now, we can write the main Theorem:

Theorem 2.3 The following process

$$L_t = L_{A_t} L_{B_t} = (\alpha_T)^{1_{\{T \leq t\}}} (\beta_S)^{1_{\{S \leq t\}}} [e^{A_t} + e^{B_t} - 1]$$

is a nonnegative local \mathfrak{F}_t -martingale and $E[L_t] = 1$.

Proof. Using Lemma 2.1, Lemma 2.2 and the Stieltjes differentiation rule we have

$$L_{A_t} L_{B_t} - 1 = \int_0^t L_{A_{s-}} dL_{B_s} + \int_0^t L_{B_{s-}} dL_{A_s} + \sum_{s \leq t} \Delta L_{A_s} \Delta L_{B_s}.$$

As by assumption, A_t and B_t are continuous and $P(S = T) = 0$ we have $\sum_{s \leq t} \Delta L_{A_s} \Delta L_{B_s} = 0$ and therefore $L_{A_t} L_{B_t}$ in an local \mathfrak{F}_t -martingale with $E[L_{A_t} L_{B_t}] = 1$ and the theorem is proved.

We are looking for a probability measure Q , such that, under Q , $C_t^* = A_t^* + B_t^*$ becomes the \mathfrak{F}_t -compensator of $1_{\{S \wedge T \leq t\}}$ with respect to this modified probability measure.

Under certain conditions, it is possible to find Q . Indeed, assume that the process L_t is uniformly integrable. Then it follows from well known results on point process martingales (Bremaud (1981)) that the desired measure Q is given by the Radon Nikodyn derivative $\frac{dQ}{dP} = L_\infty$.

The random variable $L(\infty)$ is given by

$$L_\infty = \left(\frac{e^{B_T} - 1}{e^{A_T} + e^{B_T} - 1} \right) \left(\frac{e^{A_S} - 1}{e^{A_S} + e^{B_S} - 1} \right) [e^{A_T} + e^{B_S} - 1] = \frac{(e^{B(S \wedge T)} - 1)(e^{A(S \wedge T)} - 1)}{e^{A(S \wedge T)} + e^{B(S \wedge T)} - 1}$$

where $S \wedge T = \min\{S, T\}$.

Remark 2.4.

In reference to the first paragraph of this section, in the above setting we can identify the measure L_∞ with warm standby in which case the component in standby can fail before the component in operation.

In the case of cold standby redundancy, T does not fail before S , we can consider $S < T$ almost surely and we have

$$L_\infty = L_\infty = \frac{(e^{B_S} - 1)(e^{A_S} - 1)}{e^{A_S} + e^{B_S} - 1}$$

In the case where T and S are identically distributed, we have $A_t = B_t$ and the compensator transform is given by

$$A_t^* = 2 \int_0^t \frac{e^{A_s} - 1}{2e^{A_s} - 1} dA_s = \int_0^t \frac{2 - 2e^{-A_s}}{2 - e^{-A_s}} dA_s.$$

which can be used to define a hot standby redundancy, through compensator transform when the component and the standby component and the component in operation are stochastically dependent but identically distributed.

3. Results in signature point process

In our general setup, we consider the vector (T_1, \dots, T_n) of n component lifetimes which are finite and positive random variables defined in a complete probability space $(\Omega, \mathfrak{F}, P)$, with $P(T_i \neq T_j) = 1$, for all $i \neq j, i, j$ in $C = \{1, \dots, n\}$, the index set of components. Therefore, the lifetimes can be dependent but simultaneous failures are ruled out.

The evolution of components in time define a marked point process given through the failure times and the corresponding marks.

We denote by $T_{(1)} < T_{(2)} < \dots < T_{(n)}$ the ordered lifetimes T_1, T_2, \dots, T_n , as they appear in time and by $X_i = \{j : T_{(i)} = T_j\}$ the corresponding marks. As a convention we set $T_{(n+1)} = T_{(n+2)} = \dots = \infty$ and $X_{n+1} = X_{n+2} = \dots = e$ where e is a fictitious mark not in C , the index set of the components. Therefore the sequence $(T_n, X_n)_{n \geq 1}$ defines a marked point process.

The mathematical description of our observations, the complete information level, is given by a family of sub σ -algebras of \mathfrak{F} , denoted by $(\mathfrak{F}_t)_{t \geq 0}$, where

$$\mathfrak{F}_t = \sigma\{1_{\{T_{(i)} > s\}}, X_i = j, 1 \leq i \leq n, j \in C, 0 < s \leq t\},$$

satisfies the Dellacherie conditions of right continuity and completeness.

Intuitively, at each time t the observer knows if the event $\{T_{(i)} \leq t, X_i = j\}$ have either occurred or not and if it had, he knows exactly the value $T_{(i)}$ and the mark X_i . Follows that the component and the system lifetimes T are \mathfrak{F}_t stopping times.

We consider the lifetimes $T_{(i),j}$ defined by the failure event $\{T_{(i)}, X_i = j\}$ with their sub-distribution function, suitable standardized

$$F_{(i),j}(t) = P(T_{(i),j} \leq t) = P(T_{(i)} \leq t, X_i = j).$$

At the 26-th European Conference on Operational Research, Rome, Italy, 2013, Bueno presented results in marked point signature processes and proved, under a complete information level the behavior of the point process $P(T \leq t | \mathfrak{F}_t)$, as the information flows continuously in time is given by the following Theorem

Theorem 3.1 Let T_1, T_2, \dots, T_n be the component lifetimes of a coherent system with lifetime T . Then,

$$P(T \leq t | \mathfrak{S}_t) = \sum_{k,j=1}^n 1_{\{T=T_{(k),j}\}} 1_{\{T_{(k),j} \leq t\}}.$$

The above decomposition allows us to define the signature process at component level.

Definition 3.2 The vector $(1_{\{T=T_{(k),j}\}}, 1 \leq k, j \leq n)$ is defined as the marked point signature process of the system ϕ .

Remark 3.3 As $P(T_i = T_j) = 0$ for all i, j , the collection $\{\{T = T_{(i),j}\}, 1 \leq i \leq n, 1 \leq j \leq n\}$ form a partition of Ω and $\sum_{k,j=1}^n \sum_{j=1}^n 1_{\{T=T_{(k),j}\}} = 1$. Therefore

$$\begin{aligned} P(T > t | \mathfrak{S}_t) &= \sum_{k,j=1}^n 1_{\{T=T_{(k),j}\}} - \sum_{k,j=1}^n 1_{\{T=T_{(k),j}\}} 1_{\{T_{(k),j} \leq t\}} = \\ &= \sum_{k,j=1}^n 1_{\{T=T_{(k),j}\}} [1 - 1_{\{T_{(k),j} \leq t\}}] = \sum_{k,j=1}^n 1_{\{T=T_{(k),j}\}} 1_{\{T_{(k),j} > t\}}. \end{aligned}$$

Remark 3.4 Using Remark 3.3 we can calculate the system reliability as

$$\begin{aligned} P(T > t) &= E[P(T > t | \mathfrak{S}_t)] = \\ &= E\left[\sum_{k,j=1}^n 1_{\{T=T_{(k),j}\}} 1_{\{T_{(k),j} > t\}}\right] = \sum_{k,j=1}^n P(\{T = T_{(k),j}\} \cap \{T_{(k),j} > t\}). \end{aligned}$$

If the component lifetimes are totally inaccessible, independent and identically distributed we have,

$$P(T > t) = \sum_{k=1}^n P(T = T_{(k)}) P(T_{(k)} > t)$$

recovering the classical result as in Samaniego (1985).

Remark 3.5 The marked point $N_t((i), j) = 1_{\{T_{(i)} \leq t, X_t = j\}}$ is an \mathfrak{F}_t -submartingale, that is, $T_{(i), j}$ is \mathfrak{F}_t -measurable and $E[N_t((i), j) | \mathfrak{F}_s] \geq N_s((i), j)$ for all $0 \leq s \leq t$.

From Doob-Meyer decomposition, there exists a unique \mathfrak{F}_t -predictable process, $(A_t((i), j))_{t \geq 0}$, $A_0((i), j) = 0$, called the \mathfrak{F}_t -compensator of $N_t((i), j)$, such that $M_t((i), j) = N_t((i), j) - A_t((i), j)$ is a zero mean uniformly integrable \mathfrak{F}_t -martingale. We assume that $T_i, 1 \leq i \leq n$ are totally inaccessible \mathfrak{F}_t -stopping time and, under this assumption, $A_t((i), j)$ is continuous. In certain sense, an absolutely continuous lifetime is totally inaccessible.

The compensator process $(A_t((i), j))_{t \geq 0}$ generalizes the classical hazard function notion, on the basis of all observations available up to, but not including, the present.

As $N_t((i), j)$ can only count on the time interval $(T_{(i-1)}, T_{(i)})$, the corresponding compensator differential $dA_t((i), j)$ must vanish outside this interval.

Note that, to count the i -th failure we let $N_t((i)) = \sum_{j \geq 1} N_t((i), j)$ with \mathfrak{F}_t -compensator process $A_t((i)) = \sum_{j \geq 1} A_t((i), j)$. $N_t(j) = \sum_{i \geq 1} N_t((i), j)$, counts the component failure and it has \mathfrak{F}_t -compensator process $A_t(j) = \sum_{i \geq 1} A_t((i), j)$.

The \mathfrak{F}_t -compensator of $1_{\{T \leq t\}}$, where T is the system lifetime is set in the following Theorem:

Theorem 3.6 Let T_1, T_2, \dots, T_n , be the components lifetimes of a coherent system with lifetime T . Then, the \mathfrak{F}_t -submartingale $P(T \leq t | \mathfrak{F}_t)$, has the \mathfrak{F}_t -compensator

$$\sum_{k, j=1}^n \int_0^t 1_{\{T=T_{(k), j}\}} dA_s((k), j).$$

Proof

We consider the process

$$1_{\{T=T_{(k), j}\}}(w, s) = 1_{\{s=T_{(k), j}\}}(w).$$

It is left continuous and \mathfrak{F}_t -predictable. Therefore

$$\int_0^t 1_{\{T=T_{(k), j}\}}(s) dM_s((k), j)$$

is an \mathfrak{F}_t -martingale.

As a finite sum of \mathfrak{F}_t -martingales is an \mathfrak{F}_t -martingale, we have

$$\sum_{k,j=1}^n \int_0^t 1_{\{T=T_{(k),j}\}} dM_s((k), j) =$$

$$\sum_{k,j=1}^n \int_0^t 1_{\{T=T_{(k),j}\}} d1_{\{T_{(k),j} \leq s\}} - \sum_{k,j=1}^n \int_0^t 1_{\{T=T_{(k),j}\}} dA_s((k), j).$$

is an \mathfrak{F}_t -martingale. As the compensator is unique we finish the proof.

4. Standby redundancy in a coherent system of dependent components

We are concerned with the problem of where to allocate a spare component using standby redundancy in a coherent system in order to optimize system reliability improvement. We let $T = \phi(\mathbf{T})$ be the lifetime of a coherent system with component lifetimes $\mathbf{T} = (T_1, \dots, T_n)$, $P(T_i = T_j) = 0$, for all $i \neq j$, $1 \leq i, j \leq n$ under the hypothesis and notation of Section 3. Furthermore, let $T^i = \phi(T_1, \dots, T_{i-1}, T_i + S, T_{i+1}, \dots, T_n)$ be the system lifetime resulting from a standby redundancy operation of component i through a spare with lifetime S , not identically distributed as T_i . In particular we count this system failure through $N_t^i = 1_{\{T^i \leq t\}}$, a counting process with \mathfrak{F}_t -compensator A_t^i , $1 \leq i \leq n$.

To compare the system lifetimes resulting from redundancy operations we are going to compare the components point processes compensators through cumulative hazard order as in Shaked and Shanthikumar (1993):

Definition 4.1 Consider two point processes, N_t corresponding to the component lifetime vector \mathbf{T} defined in a complete probability space $(\Omega, \mathfrak{F}, P_{\mathbf{T}})$, and N_s , corresponding to the component lifetime vector \mathbf{S} , possibly defined on a different probability spaces, with corresponding continuous compensator processes

$$A_t(n) = A_{t|t_0, t_1, \dots, t_{n-1}} \text{ on } [T_{(n-1)}, T_{(n)}];$$

$$B_t(m) = B_{t|s_0, s_1, \dots, s_{m-1}} \text{ on } [S_{(m-1)}, S_{(m)}];$$

which are, P_T almost surely, continuous in t . If for all $t \geq \max\{t_{n-1}, s_{m-1}\}$ and $n \leq m$, $A_t(n) \leq B_t(m)$, for all $0 = s_0 < s_1 < \dots < s_{n-1}$, $0 = t_0 < t_1 < \dots < t_{n-1}$ and $s_i \leq t_i$, $0 \leq i \leq n-1$, we say that S is smaller than T in cumulative hazard order, denoted as $S \leq^{ch} T$.

Also, we are going to use the following result from Kwieciński and Szekli (1991)

Theorem 4.2 Kwieciński and Szekli (1991). Consider two point processes N_t corresponding to the component lifetime vector T , defined in a complete probability space $(\Omega, \mathfrak{F}, P_T)$, and M_s , corresponding to the component lifetime vector S , possibly defined on a different probability space.

If S is smaller than T in cumulative hazard order, ($S \leq^{ch} T$), then

$$E_{P_T}[\Psi(N_t)] \leq E_{P_T}[\Psi(M_t)]$$

for all decreasing real and right continuous function with left hand limits Ψ , which is, equivalent to $N_t \leq^{st} M_t$, $t \geq 0$.

4.1. Minimal standby redundancy in a coherent system of dependent components

In this first subsection we resumes the results from Bueno (2005) intending to present a generalization of the main theorem.

A minimal standby redundancy of a component lifetime T_i is the sum $T_i + S$ where S is the standby lifetime with

$$P(S > t | T_i = s) = P(T_i > t + s | T_i > s).$$

Intuitively, a minimal standby redundancy gives to the lifetime an additional lifetime as it had just before the failure.

Let $F_i(t) = 1 - \bar{F}_i(t)$ the distribution function of the component lifetime T_i , then the resulting lifetime $T_i + S$ has a reliability function $P(T_i + S > t) =$

$$P(T_i > t) + \int_0^t P(T_i + S > t | T_i = s) dF_i(s) = \bar{F}_i(t) - \bar{F}_i(t) \ln \bar{F}_i(t).$$

In a random environment where the component i is affected by the behavior of other components, Bueno (2005) find a compensator approach of

the minimal standby redundancy considering the Girsanov's theorem argument (Bremaud (1981)) where the components compensators processes of $N_t(i), 1 \leq i \leq n$, $A_t(i), 1 \leq i \leq n$, are transformed by

$$B_j(t) = \int_0^t \alpha_s(j) dA_s(j)$$

, where $\alpha_s(i) = \frac{A_s(i)}{1+A_s(i)}$ and $\alpha_s(j) = 1$ for $j \neq i$.

The result is: under the measure Q defined by the Radon Nikodin derivative

$$\frac{dQ}{dP} = A_{T_i}(i)$$

, $B_t(i), 1 \leq i \leq n$ are the components compensators processes of $N_t(i), 1 \leq i \leq n$.

Observe that

$$B_t(i) = \int_0^t \frac{A_s(i)}{1+A_s(i)} dA_s(i) = A_t(i) - \ln(1+A_t(i)),$$

and in the absolutely continuous case, where $A_t(i) = -\ln P(T_i > t | \mathfrak{S}_t)$, (see Arjas and Yashin (19)), we can recover, for the independence case, the expression

$$P(T_i + S > t) = \bar{F}_i(t) - \bar{F}_i(t) \ln \bar{F}_i(t).$$

Recovering our setting, let $T^i = \phi(T_1, \dots, T_{i-1}, T_i + S, T_{i+1}, \dots, T_n)$ the system lifetime resulting from an minimal standby redundancy operation of the lifetime T_i , of component i . We count this system failure through $N_t^i = 1_{\{T^i \leq t\}}$, a counting process with \mathfrak{S}_t -compensator $A_t^i, 1 \leq i \leq n$.

Theorem 4.2.1 Let $T = \phi(\mathbf{T})$ be the lifetime of a coherent system with component lifetimes $\mathbf{T} = (T_1, \dots, T_n)$, $P(T_i = T_j) = 0$, for all $i \neq j, 1 \leq i, j \leq n$. Under the hypothesis and notation of Section 3, if $A_t(i) \geq A_t(j), t \geq 0, 1 \leq i < j \leq n$, then $N_t^i \leq^{st} N_t^j, t \geq 0, 1 \leq i < j \leq n$.

Proof

From Theorem 3.6 we have to compare system's compensators expectation values on the form

$$A_t^i = \sum_{k=1}^n \sum_{j=1, j \neq i}^n 1_{\{T=T_{(k),j}\}} A_t((k), j) + \sum_{k=1}^n 1_{\{T=T_{(k),i}\}} A_t((k), i) + \sum_{k=1}^n \sum_{j=i+1, j \neq i}^n 1_{\{T=T_{(k),j}\}} A_t((k), j),$$

for $1 \leq i \leq n$, where the notation $A_t((k), j)$ means the restriction of $A_t(j)$ to the interval $(T_{(k-1)}, T_{(k)})$.

Clearly, it is sufficient to prove for $i = 1$ and $j = 2$

$$\begin{aligned} A_t^1 &= \sum_{k=1}^n 1_{\{T=T_{(k),1}\}} [A_t((k), 1) - \ln(1 + A_t((k), 1))] + \\ &\sum_{k=1}^n \sum_{j=2, j \neq 1}^n 1_{\{T=T_{(k),j}\}} A_t((k), j) \leq \sum_{k=1}^n 1_{\{T=T_{(k),1}\}} A_t((k), 1) + \\ &\sum_{k=1}^n 1_{\{T=T_{(k),2}\}} [A_t((k), 2) - \ln(1 + A_t((k), 2))] + \\ &\sum_{k=1}^n \sum_{j=3}^n 1_{\{T=T_{(k),j}\}} A_t((k), j) = A_t^2 \end{aligned}$$

$$\leftrightarrow -\ln(A_t((k), 1) + 1) \leq -\ln(A_t((k), 2) + 1) \leftrightarrow A_t((k), 1) \geq A_t((k), 2).$$

The final result follows from Theorem 4.2.

4.2. Standby redundancy in a coherent system of dependent components

In what follows we consider a unique spare with lifetime S , as in Section 2, with $1_{\{S \leq t\}}$ \mathfrak{S}_t -compensator B_t , to be allocated between the components, in order to optimize the system lifetime:

Theorem 4.2.1 Let $T = \phi(\mathbf{T})$ be the lifetime of a coherent system with component lifetimes $\mathbf{T} = (T_1, \dots, T_n)$, $P(T_i = T_j) = 0$, for all $i \neq j$, $1 \leq i, j \leq n$.

n . Under the hypothesis and notation of Section 3, if $A_t(i) \geq A_t(j)$, $t \geq 0$, $1 \leq i < j \leq n$, then $N_t^i \leq^{st} N_t^j$, $t \geq 0$, $1 \leq i < j \leq n$.

Proof

Follows from Theorem that the standby redundancy through compensator transform of the component i by a spare with compensator B_t is

$$A_t(i) + B_t - \int_0^t \frac{e^{A_s(i)} dA_s + e^{B_s} dB_s}{e^{A_s(i)} + e^{B_s} - 1} = A_t(i) + B_t - \ln[e^{A_t(i)} + e^{B_t} - 1].$$

Clearly, it is sufficient to prove for $i = 1$ and $j = 2$

$$\begin{aligned} A_t^1 &= \sum_{k=1}^n 1_{\{T=T_{(k),1}\}} [A_t((k), 1) + B_t - \ln[e^{A_t((k),1)} + e^{B_t} - 1]] + \\ &\quad \sum_{k=1}^n \sum_{j=2}^n 1_{\{T=T_{(k),j}\}} A_t((k), j) \leq \\ &\quad \sum_{k=1}^n 1_{\{T=T_{(k),1}\}} A_t((k), 1) + \sum_{k=1}^n 1_{\{T=T_{(k),2}\}} [A_t((k), 2) \\ &\quad + B_t - \ln[e^{A_t((k),2)} + e^{B_t} - 1]] + \sum_{k=1}^n \sum_{j=3}^n 1_{\{T=T_{(k),j}\}} A_t((k), j) = A_t^2 \\ &\quad \leftrightarrow -\ln(e^{A_t((k),1)} + e^{B_t} - 1) \leq -\ln(e^{A_t((k),2)} + e^{B_t} - 1) \\ &\quad \leftrightarrow A_t((k), 1) \geq A_t((k), 2). \end{aligned}$$

The final result follows from Theorem 4.2.

As by hypothesis, $A_t(i) \geq A_t(j)$, $1 \leq i < j \leq n$ we are considering component i weaker than component j in the sense that the hazard process for failure of component i is larger than the hazard process for failure of component j (it is, also, implies that T_i is stochastically less than T_j). Follows that, under Theorem 4.2.1, we understand that it is optimal to perform active redundancy allocation on the weakest component of a coherent system of continuous dependent components with no simultaneous failures.

We can, also consider two spares with lifetimes $S_1, S_2, 1_{\{S_1 \leq t\}}$ with \mathfrak{S}_t -compensator $B_t(1)$ and $1_{\{S_2 \leq t\}}$ with \mathfrak{S}_t -compensator $B_t(2)$, to be allocated between the components, in order to optimize the system lifetime. The following Corollary can be easily proved using the same argument of Theorem 4.2.1

Corollary 4.2.2 Let $T = \phi(\mathbf{T})$ the lifetime of a coherent system with component lifetimes $\mathbf{T} = (T_1, \dots, T_n)$, $P(T_i = T_j) = 0$, for all $i \neq j, 1 \leq i, j \leq n$ under the hypothesis and notation of Section 2. If $A_t(i) \geq A_t(j), 1 \leq i < j \leq n$ and $B_t(1) \geq B_t(2)$, then $N_t^i \leq^{st} N_t^j, 1 \leq i < j \leq n$, where $T^i = \phi(T_1, \dots, T_{i-1}, T_i \vee S_1, T_{i+1}, \dots, T_n)$ and $T^j = \phi(T_1, \dots, T_{i-1}, T_j \vee S_2, T_{i+1}, \dots, T_n)$.

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