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The Arrival Time Brachistochrones in General Relativity

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THE ARRIVAL TIME BRACHISTOCHRONES IN GENERAL RELATIVITY

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ABSTRACT. We consider a general relativistic version of the classical brachistochrone problem, whose solutions are causal curves, parameterized by a constant multiple of their proper time and with 4-acceleration perpendicular to a given observer field, that extremize the *arrival time* measured by an observer at the final endpoint. This kind of brachistochrones presents characteristics different from the *travel time brachistochrones*, that were studied in [8, 9, 10]. In this paper we formulate the variational problem in a general context; moreover, in the case of a stationary metric, we prove two variational principles and we determine the second order differential equation satisfied by the arrival time brachistochrone. Using these variational principles and techniques from Critical Point Theory we establish some results concerning the existence and the multiplicity of travel time brachistochrones with a given energy between an event and an observer.

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1. Introduction and Statement of the Main Results

The classical brachistochrone[1] problem dates back to the end of the seventeenth century, when Johann Bernoulli challenged his contemporaries to solve the following problem.

> If in a vertical plane two points A and B are given, then it is required to specify the orbit AMB of the movable point M, along which it, starting from A, and under the influence of its own weight, arrives at B in the shortest possible time. *Acta Eruditorum, June 1696*

This problem attracted the attention of many important mathematicians of the time, including Newton, Leibniz, L'Hôpital, and Johann's brother, Jackob Bernoulli. The papers written on the subject may be considered the fundaments of a new field in mathematics, the *Calculus of Variations*. A beautiful historical exposition of the brachistochrone problem may be found in Reference [25], where the authors' thesis is that the brachistochrone problem also *marks the birth of Optimal Control.*

Still now the classical brachistochrone problem is very popular, and its importance is witnessed by the fact that there is hardly any book on Calculus of Variations that does not use this problem as a takeoff point. The well known solution to the brachistochrone problem is a cycloid, which is the curve described by a point P on a circle that rolls without slipping.

The cycloid curve was introduced by Galileo, who was actually the first scientist to formulate the brachistochrone problem several decades before Bernoulli, in his *Discorsi e dimostrazioni matematiche intorno a due nuove scienze*, of 1638. Curiously enough, Galileo did not find the correct answer to the problem; apparently, he simply noticed that an arc of a circle joining A and B would give a faster travel time than the straight segment.

Huygens had discovered another remarkable property of the cycloid: it is the only curve such that a body, falling under its own weight, is guided by this curve so as to oscillate with a period that is independent of the initial point where the body is released. For this reason, Huygens called this curve the *tautochrone*.[2]

The classical brachistochrone problem has several generalizations, e.g., the homogeneous gravitational field could be replaced with an arbitrary Newtonian potential, and instead of releasing the particle from rest one could prescribe an arbitrary value for the initial speed, leaving the initial direction of the velocity undetermined.

In modern terminology, the Newtonian brachistochrone problem can be stated as follows. Given a manifold $\mathcal{M}_0$ endowed with a Riemannian metric g_0, to be interpreted as the configuration space, and a smooth function $V : \mathcal{M}_0 \longmapsto \mathbb{R}$, representing the gravitational potential, a brachistochrone of energy $E > 0$ between

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[1] from the greek: $\beta\rho\alpha\chi\iota\sigma\tau o\varsigma$=shortest, $\chi\rho o\nu o\varsigma$=time.

[2] from the greek, $\tau\alpha\upsilon\tau o\varsigma$=equal or same, and $\chi\rho o\nu o\varsigma$=time.

two points x_0 and x_1 of $\mathcal{M}_0$ is a curve $x : [0, T_x] \longmapsto \mathcal{M}$ joining x_0 and x_1 that extremizes the travel time T_x in the space of all unit speed curves y joining x_0 and x_1 and satisfying the conservation of energy law:

$$\frac{1}{2} g(\dot{x}, \dot{x}) + V(x) \equiv E. \tag{1.1}$$

(throughout this paper we will consider the motion of particles with unit mass) A well known variational principle states that a curve x joining x_0 and x_1 is a brachistochrone of fixed energy E if and only if x is a geodesic with respect to the conformal Riemannian metric $\phi_E \cdot g_0$, with conformal factor $\phi_E = (E - V)^{-1}$.

The first relativistic versions of the brachistochrone problem appear in [11] and [13]. V. Perlick (see [20]) has determined the brachistochrone equation in a *regular* stationary Lorentzian manifold, i.e., in a time-independent split gravitational field according to general relativity, and Giannoni, Piccione and Verderesi in [10] have generalized Perlick's result to the case of a possibly non regular stationary Lorentzian manifold by reformulating the brachistochrone problem in the context of sub-Riemannian geometry. The variational principle proven in [10] was then used in [8] to prove some results concerning the existence and the multiplicity of relativistic brachistochrones with respect to the travel time, having fixed energy, between a fixed event and a fixed observer of a stationary spacetime.

The general relativistic brachistochrone problems can be formulated on Lorentzian manifolds in the following way.

Let $(\mathcal{M}, g)$ be a 4-dimensional Lorentzian manifold, i.e., an arbitrary spacetime in the sense of general relativity and fix a timelike smooth vector field Y on $\mathcal{M}$. The integral curves of Y can be interpreted as the worldlines of *observers*. Please note that we do not require Y to be normalized, i.e., in general the worldlines of our observers are not parameterized by proper time. The reason is that in the stationary case, i.e., if $(\mathcal{M}, g)$ admits a timelike Killing vector field, it is convenient to choose this Killing vector field for Y and not a renormalized version of it, that may fail to be Killing.

To formulate the brachistochrone problem with respect to our arbitrarily chosen observer field Y, we fix a point p in $\mathcal{M}$, a (maximal) integral curve $\gamma : \mathbb{R} \longmapsto \mathcal{M}$ of Y and a real number $k > 0$. The *trial paths* for our variational problem are all timelike smooth curves $\sigma : [0, 1] \longmapsto \mathcal{M}$ which are nowhere tangent to Y and satisfy the following conditions.

(1.2) $\sigma(0) = p;$

(1.3) $\sigma(1) \in \gamma(\mathbb{R});$

(1.4) $g(\dot\sigma(0), Y(\sigma(0))) = -k\,\big(-g(\dot\sigma(0),\dot\sigma(0))\big)^{1/2};$

(1.5) $g(\nabla_{\dot\sigma}\dot\sigma, \dot\sigma) = 0;$

(1.6) $g(\nabla_{\dot\sigma}\dot\sigma, Y) = 0.$

Here ∇ denotes the Levi-Civita connection of the Lorentzian metric g.

If we interpret each integral curve of Y as a "point in space", (1.2) and (1.3) mean that all trial paths connect the same two given points in space, where the starting time is fixed whereas the arrival time is not. Condition (1.4) says that all trial paths start with the same speed with respect to the observer field Y. By condition (1.5), the quantity $\mathcal{T}_\sigma$ defined by $-\mathcal{T}_\sigma{}^2 = g(\dot\sigma,\dot\sigma)$ is a constant for each trial path σ (but takes different values for different trial paths). This implies that the curve parameter s along σ is related to proper time τ by an affine transformation, $\tau = \mathcal{T}_\sigma s + const.$ As a consequence, the 4-velocity along each trial path is given by $\mathcal{T}_\sigma{}^{-1}\dot\sigma$, whereas the 4-acceleration is given by $\mathcal{T}_\sigma{}^{-2}\nabla_{\dot\sigma}\dot\sigma$. Hence, conditions (1.5) and (1.6) require the 4-acceleration to be perpendicular to the plane spanned by $\dot\sigma$ and Y. In other words, with respect to the observer field Y there are only forces perpendicular to the direction of motion. Such forces can be interpreted as constraint forces supplied by a frictionless slide which is at rest with respect to the observer field Y. The quantity $\mathcal{T}_\sigma$ will be called the *travel time* of the curve σ; the set of trial paths for our variational problem will be denoted with the symbol $\mathcal{B}^+_{p,\gamma}(k)$.

The two brachistochrone problems can now be formulated in the following way.

The *travel time brachistochrones* of energy k between p and γ are those curves in $\mathcal{B}^+_{p,\gamma}(k)$ for which the travel time is stationary.

The *arrival time brachistochrones* of energy k between p and γ are defined to be the stationary points in $\mathcal{B}^+_{p,\gamma}(k)$ for the *arrival time* functional, given by

$$\tau(\sigma) = \gamma^{-1}(\sigma(1)).$$

In other words, $\tau(\sigma)$ is the value of the time of the receiver at the arrival event; this is the proper time if and only if Y is normalized along γ. In order for τ to be well defined, we need to assume that γ does not have self-intersections, i.e., that γ is injective. Observe that if one reparameterizes smoothly the curve γ, then clearly the values of the arrival time functional τ are affected by this change; nevertheless, the stationary points of τ do *not* depend on the parameterization of γ.

The *arrival time functional* was introduced by Kovner in [14], and its properties were further investigated for the study of causal geodesics in Lorentzian manifold by other authors (see e.g. [5, 19]).

In physical terms, the two brachistochrone problems differ by the way of measuring time: in the first case the time is measured by a watch traveling along the trajectory of the mass, in the second case the time is measured by the observer that receives the mass at the end of its trajectory.

For a physical interpretation of our brachistochrone problem, the timelike vector field Y should be related to some observable quantities, i.e., Y should be co-moving with some bodies. For instance, if we are in the solar system and Y is comoving with the planets, the solutions to our brachistochrone problem will give worldlines of particles that minimize the arrival time time among all curves that have a fixed specific energy in the rest system of the planets. If Y is at rest with respect to the sun and to the distant stars, then the brachistochrones will be the worldlines of massive objects that minimize the arrival time among all curves that have fixed energy in a reference system oriented at distant stars.

It is also possible to return to the original interpretation of of the brachistochrone problem and think of the body guided by a frictionless slide, in which case Y is determined by being the rest system of the slide.

If $(\mathcal{M}, g)$ is a stationary spacetime and Y is a *Killing* vector field, i.e., the flow of Y preserves the metric g, then the condition (1.6) means that the product $g(\dot\sigma, Y)$ is constant along σ. The value of this constant can be easily computed using condition (1.4), that gives $g(\dot\sigma, Y) \equiv -k\mathcal{T}_\sigma$. Hence, in the stationary case, the conditions (1.4) and (1.6) can be resumed in the condition:

$$g(\dot\sigma, Y) = -k\mathcal{T}_\sigma. \tag{1.7}$$

The condition (1.7) is the relativistic counterpart of the energy conservation law in the Newtonian case. Although physically meaningful, the mathematical approach to the general relativistic brachistochrone problem in the non stationary case presents difficulties of higher order than in the stationary case. For instance, it is not even clear whether the non stationary brachistochrones are solutions to a second order differential equation; in Reference [21], the authors used a Lagrange multiplier technique to derive a system of differential equations for the travel time brachistochrones and for the Lagrangian multipliers. Unfortunately, it does not seem to be possible to eliminate the Lagrangian multipliers from the system without introducing integrals, unless in the stationary case. Thus, it looks as if the brachistochrones in the non-stationary case are not determined by a second-order differential equation, but rather by an integro-differential equation.

The travel time brachistochrones in stationary manifolds have already been studied from a variational point of view, and the main results may be found in References [6, 8, 9, 10]. This kind of brachistochrones are characterized by the second order differential equation:

$$\nabla_{\dot\sigma}\dot\sigma+\frac{2k\,T_\sigma}{g(Y,Y)}\,\nabla_{\dot\sigma}Y+2k\,\frac{g(\nabla_{\dot\sigma}Y,Y)}{g(Y,Y)\,(k^2+g(Y,Y))}\,(k\dot\sigma-T_\sigma Y)=0. \tag{1.8}$$

In this paper we want to develop a similar theory for the arrival time brachistochrones, in particular we determine the differential equation that characterizes the arrival time brachistochrones, and we give conditions that guarantee the existence and the multiplicity of brachistochrones of given energy between an event and an observer of a stationary Lorentzian manifold.

Let's assume from now on that (M,g) is a stationary Lorentzian manifold, and that Y is a given timelike Killing vector field on $\mathcal{M}$. For all $q\in\mathcal{M}$, we will denote by $\langle\cdot,\cdot\rangle$ the Lorentzian inner product in the tangent spaces $T_q\mathcal{M}$ induced by the metric g. As it is natural to expect, in order to obtain existence results for brachistochrones, one needs to assume the *completeness* of $\mathcal{M}$ with respect to some Riemannian structure which is related to our variational setup. To this purpose, we introduce an auxiliary Riemannian structure on $\mathcal{M}$, denoted by g_{R}, defined by means of the timelike field Y as follows:

$$g_{\mathrm{R}}(v,v)=\langle v,v\rangle_{(\mathrm{R})}=\langle v,v\rangle-2\frac{\langle v,Y\rangle^2}{\langle Y,Y\rangle}, \tag{1.9}$$

for all tangent vector $v\in T\mathcal{M}$. The positive definiteness of g_{R} is proven easily using the *wrong-way Schwartz's inequality* satisfied by the Lorentzian inner products. It is easy to see that Y is Killing also in the metric g_{R}; moreover, the restriction of g and g_{R} on the orthocomplement of Y coincide.

For all $k\in\mathbb{R}^+$, we consider the open subset $U_k\subseteq\mathcal{M}$ defined by:

$$U_k=\Big\{q\in\mathcal{M}:\langle Y(q),Y(q)\rangle+k^2>0\Big\}. \tag{1.10}$$

Since Y is Killing, then the quantity $\langle Y,Y\rangle$ is constant along each flow line of Y; it follows that U_k is Y-invariant, i.e., U_k is invariant by the flow of Y.

We have the following existence result for the arrival time brachistochrones:

Theorem 1.1. *Let $(\mathcal{M},g)$ be a stationary Lorentzian manifold, Y be a timelike Killing vector field on $\mathcal{M}$ and $k\in\mathbb{R}^+$ be a fixed positive constant. Let p be a fixed point in U_k and $\gamma:\mathbb{R}\mapsto U_k$ be a maximal integral line of Y, which is assumed to be injective.*

Suppose that the following hypotheses are satisfied:

(1) *$-k^2$ is a regular value for the function $\langle Y,Y\rangle$ on $\mathcal{M}$;*

(2) *Y is bounded away from 0, i.e., there exists a positive constant $\nu>0$ such that:*

$$0<\nu\le-\langle Y(q),Y(q)\rangle,\quad\forall q\in U_k; \tag{1.11}$$

(3) *the closure* $\overline{U}_k = U_k \bigcup \partial U_k$ *is complete with respect to the Riemannian metric* g_{R}*;*
(4) $p \notin \gamma(\mathbb{R})$.

Then, there exists at least one arrival time brachistochrone of energy k *between* p *and* γ.

If the topology of U_k is non trivial, then we can prove the existence of arrival time brachistochrones of arbitrary large arrival time:

Theorem 1.2. *Under the hypotheses of Theorem 1.1, if* $\overline{U}_k$ *is not contractible, then there exists a sequence of arrival time brachistochrones* $\{\sigma_n\}_{n\in\mathbb{N}}$ *in* $B^+_{p,\gamma}(k)$ *such that:*

$$\lim_{n\to\infty} \tau(\sigma_n) = +\infty. \tag{1.12}$$

Theorem 1.2 is the analogue of Serre's Theorem (Ref. [23]) concerning the multiplicity of geodesics joining two fixed points in a complete, non contractible Riemannian manifold.

We outline briefly the structure of the paper.

In Section 2 we give the basic definition and properties of our functional setup, and we prove the existence of an infinite dimensional differentiable structure for the brachistochrone variational problem.

In Section 3 we prove the regularity of the solutions of our variational problem, and we give a characterization of the arrival time brachistochrone in terms of a second order nonlinear differential equation.

In Section 4, we introduce a sort *deformation map* which plays the role of a spatial projection (observe that we do not assume any topological space-time splitting on the Lorentzian manifold $(\mathcal{M}, g)$), which is used to prove a first variational principle for brachistochrones. Such principle relates the arrival time brachistochrones of a given energy k to the local minimizers of a Lipschitz functional, denoted by τ_k, defined on the set of curves that are horizontal with respect to the orthogonal distribution of Y. The functional τ_k lacks smoothness. Moreover, it is invariant by reparameterization, which means that the set of its *critical points*, i.e., minimizers for a set of curves joining events and integral curves of Y sufficiently close, is acted upon by the infinite dimensional group of diffeomorphism of the interval $[0, 1]$. Then it is quite hard to study with global variational techniques.

Under this point of view, it appears an evident analogy with the problem of lightlike geodesics in stationary manifolds. In reference [4], under the assumption of space-time splitting for the stationary metric g, the authors prove a Fermat principle for lightlike geodesics, which reduces the null geodesic problem to the study of critical points of a smooth functional defined on the set of spatial curves. Such functional, which is not invariant by reparameterization, can be obtained from the arrival time functional by *interchanging* the position of a *square root* and an *integral sign* of a suitable Riemannian metric (see formulas (4.5) and (5.2)).

In order to overcome the problems of the lack of regularity for τ_k and its parameterization invariance, using the same artifice employed in reference [4], in Section 5 we prove a second variational problem for arrival time brachistochrone of a given energy k with the introduction of a smooth functional, denoted by G_k, defined on the set of horizontal curves, and which is not invariant by reparameterization. We prove the existence of a bijection between the set of critical points of G_k and the set of critical points of τ_k that are parameterized in such a way that a suitable conservation law is satisfied. Thanks to this principle, we reduce the proof of Theorems 1.1 and 1.2 the the proof of analogous results of existence and multiplicity of critical points of the smooth functional G_k. The situation here is very different from the travel time brachistochrones, where the problem was reduced to the search of geodesics in a convex subset of a suitable Riemannian metric (see [8, 10]).

To prove the existence of critical points for G_k, we use well known techniques from Critical Point Theory. Under the assumptions of Theorem 1.1, the functional G_k does *not* satisfy the Palais–Smale compactness condition, because of the presence of the boundary ∂U_k. To deal with this problem, we use a *penalization* technique which was introduced to study unidimensional variational problems in manifolds with convex boundary. In Section 6 we present this technique with the introduction of a family $G_{k,\varepsilon}$ of smooth functional, parameterized by a positive constant ε, which *approximate* the functional G_k as $\varepsilon \to 0$, that are bounded from below and that satisfy the Palais–Smale condition. In Section 7 we prove some *a priori estimates* on the critical points of the penalized functionals $G_{k,\varepsilon}$, and we prove Theorem 1.1. Finally, in Section 8, we obtain a Ljusternik–Schnirelman theory for the critical points of G_k by a limit process that involves the estimates proved in Section 7, and that will yield the proof of Theorem 1.2.

In Appendix A we discuss the abstract theory in the particular case of the Schwarzschild metric. Here, it appears a remarkable difference between the travel time and the arrival time brachistochrones. Namely, thanks to a suitable convexity property, the hypothesis of Theorem 1.1 are satisfied in the Schwarzschild spacetime, so that every event p and every observer γ can be joined by a travel time brachistochrone of energy k, for any positive value of k. On the contrary, due the presence of the *events horizon*, there are pairs (p, γ) that *cannot* be joined by any travel time brachistochrone (see [20]).

For the basic geometric notions used in this paper we refer to standard textbooks of semi-Riemannian geometry, like for instance [1, 17]; the classical books [1, 12, 17, 22] provide excellent references for the background physical knowledge assumed in this paper.

2. The Functional Framework

Throughout this paper we will denote by $(\mathcal{M}, g)$ a stationary Lorentzian manifold, with g a Lorentzian metric tensor on $\mathcal{M}$, and Y is a smooth timelike Killing vector field on $\mathcal{M}$.

Moreover, *we will assume throughout the paper that the hypotheses 1, 2, 3 and 4 of Theorem 1.1 are satisfied by* (M, g), Y, k, p *and* γ. The role of each single hypothesis in the proof of our results will be pointed out at every occurrence.

The symbol $\langle \cdot, \cdot \rangle$ will denote the bilinear form induced by g on the tangent spaces of $\mathcal{M}$; the usual nabla symbol ∇ will denote the covariant derivative relative to the Levi–Civita connection of g. Given a smooth function ϕ on $\mathcal{M}$, for $q \in \mathcal{M}$ we denote by $\nabla\phi(q)$ the gradient of ϕ at q with respect to g, which is the vector in $T_q\mathcal{M}$ defined by $\langle \nabla\phi(q), \cdot \rangle = \mathrm{d}\phi(q)[\,\cdot\,]$; the Hessian $H^\phi(q)$ of ϕ at q is the symmetric bilinear form on $T_q\mathcal{M}$ given by $H^\phi(q)[v_1, v_2] = \langle \nabla_{v_1} \nabla\phi, v_2 \rangle$, for $v_1, v_2 \in T_q\mathcal{M}$.

The Killing property of Y, which is crucial in most of the results presented in this paper, will be used systematically in our computations through the following two facts:

(1) the quantity $\langle Y, Y \rangle$ is constant along each flow line of Y,
(2) $\langle \nabla_v Y, w \rangle = -\langle \nabla_w Y, v \rangle$ for all pair of vectors v and w; in particular, for all $v \in T\mathcal{M}$, it is $\langle \nabla_v Y, v \rangle = 0$.

Observe the second condition above is in fact *equivalent* to the Killing property of Y (see [17, Proposition 9.25]). Moreover, since the open set U_k is invariant by the flow of Y and $\overline{U}_k$ is g_{R}-complete, it follows that Y is complete in U_k, i.e., the flow lines of Y in U_k are defined over the whole real line.

We set:

$$m = \dim(\mathcal{M});$$

the physical interesting case is $m = 4$.

Given any two smooth manifolds M_1 and M_2, possibly with boundary, and an integer $n \in \mathbb{N}$, we denote by $C^n(M_1, M_2)$ the set of all maps of class C^n between M_1 and M_2. As customary, for $1 \le q \le +\infty$, $L^q([0,1], \mathbb{R})$ will denote the space of Lebesgue q-integrable real functions; for $n \in \mathbb{N}$, $H^n([0,1], \mathbb{R})$ will denote the Sobolev space of functions of class C^{n-1} and having weak n-th derivative in $L^2([0,1], \mathbb{R})$.

For all $q \ge 1$, we define the spaces $L^q([0,1], T\mathcal{M})$ of q-integrable $T\mathcal{M}$-valued functions:

(2.1)

$$L^q([0,1], T\mathcal{M}) = \left\{ \zeta : [0,1] \longmapsto T\mathcal{M} \text{ measurable} : \int_0^1 \langle \zeta(t), \zeta(t) \rangle_{(\mathrm{R})}^{\frac{q}{2}} \, \mathrm{d}t < +\infty \right\}.$$

The set $L^\infty([0,1],T\mathcal{M})$ is defined similarly:

$$L^\infty([0,1],T\mathcal{M}) = \Big\{\zeta : [0,1] \longmapsto T\mathcal{M} \text{ measurable} : \langle\zeta,\zeta\rangle_{(\mathrm{R})} \in L^\infty([0,1),\mathbb{R})\Big\}.$$

Let $\pi : T\mathcal{M} \longmapsto \mathcal{M}$ be the canonical projection. Given any curve $\sigma : I \subseteq \mathbb{R} \longmapsto A$, a *vector field along* σ is a map $\zeta : I \longmapsto T\mathcal{M}$ such that $\pi \circ \zeta = \sigma$. Let A be any open set of $\mathcal{M}$; for all $j \in \mathbb{N}$ and all $1 \le q \le +\infty$ we define the Sobolev space $W^{j,q}([0,1],A)$ as:

(2.2)

$$W^{j,q}([0,1],A) = \Big\{\sigma \in C^{j-1}([0,1],A) : \nabla_{\dot\sigma}^{j-1}\sigma \text{ is absolutely continuous and } \nabla_{\dot\sigma}^{j}\sigma \in L^q([0,1],A)\Big\};$$

we set:

$$H^1([0,1],A) = W^{1,2}([0,1],A). \tag{2.3}$$

It is not too difficult to prove that the definition of the spaces $W^{j,q}([0,1],A)$ does *not* indeed depend on the choice of the Riemannian metric g_{R}, that appears in formula (2.1). As a matter of fact, $W^{j,q}([0,1],A)$ can be defined intrinsically for any differentiable manifold A using local charts (see [18]) or, equivalently, using auxiliary structures on A, like for instance a Riemannian metric. The definition of the spaces $W^{j,q}([0,1],T\mathcal{M})$ is given similarly.

If A is a smooth submanifold of $\mathcal{M}$, in particular if A is an open subset, then $H^1([0,1],A)$ has the structure of an infinite dimensional Hilbertian manifold, modeled on the Sobolev space $H^1([0,1],\mathbb{R}^m)$; for $\sigma \in H^1([0,1],A)$, the tangent space $T_\sigma H^1([0,1],A)$ can be identified with the Hilbert space:

$$T_\sigma H^1([0,1],A) = \Big\{\zeta \in H^1([0,1],T\mathcal{M}) : \zeta \text{ vector field along } \sigma\Big\}. \tag{2.4}$$

The inner product in $T_\sigma H^1([0,1],A)$ is given by:

$$\langle\zeta,\zeta\rangle_* = \int_0^1 \Big(\langle\zeta,\zeta\rangle_{(\mathrm{R})} + \langle\nabla^{(\mathrm{R})}_{\dot\sigma}\zeta,\nabla^{(\mathrm{R})}_{\dot\sigma}\zeta\rangle_{(\mathrm{R})}\Big)\,\mathrm{d}t. \tag{2.5}$$

Note that, if we require $\zeta(0) = 0$ in $T_\sigma H^1([0,1],A)$, then the inner product (2.5) is equivalent to:

$$\langle\zeta,\zeta\rangle_0 = \int_0^1 \langle\nabla^{(\mathrm{R})}_{\dot\sigma}\zeta,\nabla^{(\mathrm{R})}_{\dot\sigma}\zeta\rangle_{(\mathrm{R})}\,\mathrm{d}t. \tag{2.6}$$

Recalling the definition of the open set U_k given in (1.10), where k is a fixed positive constant, we now choose an event $p \in U_k$ and $\gamma : \mathbb{R} \longmapsto U_k$ a maximal integral line of Y whose image does not contain p. We introduce the space:

$$\Omega^{(1)}_{p,\gamma} = \Big\{w \in H^1([0,1],U_k) : w(0) = p,\ w(1) \in \gamma(\mathbb{R})\Big\}; \tag{2.7}$$

it is well known (see [18]) that $\Omega^{(1)}_{p,\gamma}$ is a smooth submanifold of $H^1([0,1],U_k)$.

For $w \in \Omega^{(1)}_{p,\gamma}$, the tangent space $T_w\Omega^{(1)}_{p,\gamma}$ is identified with the Hilbert subspace of $T_wH^1([0,1],U_k)$ given by:

$$T_w\Omega^{(1)}_{p,\gamma} = \Big\{\zeta \in T_wH^1([0,1],U_k) : \zeta(0) = 0,\ \zeta(1) \in \mathbb{R}\cdot Y(w(1))\Big\}. \tag{2.8}$$

Given any absolutely continuous curve $w : [a,b] \longmapsto \mathcal{M}$ with $\dot w \in L^1([0,1],T\mathcal{M})$, and any continuous vector field V along w, the *covariant integral* of V along w is an absolutely continuous vector field $\tilde V$ along w, denoted by:

$$\tilde V(t) = {}^{w}\!\!\int_a^t V \tag{2.9}$$

which is (uniquely) determined by the conditions:

$$\tilde V(a) = 0 \quad \text{and} \quad \nabla_{\dot w}\tilde V \equiv V \text{ a.e. on } [a,b]. \tag{2.10}$$

In local coordinates, the covariant integral of V along w is obtained as the solution of a first order liner differential equation that involves the Christoffel symbols of the metric g, which are smooth functions, evaluated at the points of w. It is easy to see that, if w is a curve of class H^1 and V is continuous, then $\tilde V$ is a vector field of class H^1.

We consider the *arrival time* functional τ on $\Omega^{(1)}_{p,\gamma}$, given by:

$$\tau(w) = \gamma^{-1}(w(1)). \tag{2.11}$$

Observe that τ is well defined because γ is assumed to be injective, by the causality of $\mathcal{M}$. The value of the functional τ at a given curve may be interpreted physically as the time measured by an observer at the final endpoint of the trajectory of w.

We have the following easy regularity result for τ:

Lemma 2.1. *The functional τ is smooth on $\Omega^{(1)}_{p,\gamma}$. For $w \in \Omega^{(1)}_{p,\gamma}$ and $\zeta \in T_w\Omega^{(1)}_{p,\gamma}$, the Gateaux derivative* $\mathrm{d}\tau(w)[\zeta]$ *is given by:*

$$\mathrm{d}\tau(w)[\zeta] = \frac{\langle \zeta(1), Y(w(1))\rangle}{\langle Y(w(1)), Y(w(1))\rangle}. \tag{2.12}$$

Proof. Let $w \in \Omega^{(1)}_{p,\gamma}$ and $\zeta \in T_w\Omega^{(1)}_{p,\gamma}$ be fixed. Let $s \longmapsto w_s$ be a smooth variation of w with variational vector field ζ, i.e., $s \longmapsto w_s$ is a smooth map from $]-\varepsilon,\varepsilon[$ to $\Omega^{(1)}_{p,\gamma}$, with $\varepsilon > 0$, with $w_0 = w$ and $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} w_s = \zeta$. It is:

$$\gamma(\tau(w_s)) = w_s(1), \quad \forall\, s \in]-\varepsilon,\varepsilon[. \tag{2.13}$$

Differentiating (2.13) with respect to s and evaluating at $s = 0$, since $\dot\gamma(\tau(w)) = Y(w(1))$, we get:

$$\mathrm{d}\tau(w)[\zeta]\cdot Y(w(1)) = \zeta(1). \tag{2.14}$$

Formula (2.12) follows easily from (2.14), keeping in mind that $\langle Y, Y\rangle \neq 0$. Observe that (2.12) defines a smooth function in ζ, because the evaluation at the point 1 is smooth. It follows that τ is smooth and we are done. □

We now introduce formally the space of candidates for our variational problem, which is defined by:

(2.15)
$$\mathcal{B}^{(1)}_{p,\gamma}(k) = \Big\{\sigma \in \Omega^{(1)}_{p,\gamma} : \exists\, \mathcal{T}_\sigma > 0 \text{ such that } \langle \dot\sigma, Y\rangle \equiv -k\mathcal{T}_\sigma \text{ and } \langle \dot\sigma, \dot\sigma\rangle \equiv -\mathcal{T}_\sigma{}^2\Big\}.$$

Observe that $\dot\sigma$ is only defined as an L^2-function; therefore, $\langle \dot\sigma, Y\rangle \equiv -k\mathcal{T}_\sigma$ and $\langle \dot\sigma, \dot\sigma\rangle \equiv -\mathcal{T}_\sigma{}^2$ are to be interpreted as almost everywhere identities. In order to avoid bothering the reader, in the rest of the paper we will omit to emphasize such remarks and we will tacitly mean almost everywhere equalities whenever necessary.

Due to the presence of the double constraint, it is not clear whether $\mathcal{B}^{(1)}_{p,\gamma}(k)$ is a smooth submanifold of $\Omega^{(1)}_{p,\gamma}$. Nevertheless, we can prove that $\mathcal{B}^{(1)}_{p,\gamma}(k)$ admits a dense open subset, denoted by $\mathcal{A}^{(1)}_{p,\gamma}(k)$, which has the structure of a smooth submanifold of $\Omega^{(1)}_{p,\gamma}$.

Proposition 2.2. *There exists an open dense subset $\mathcal{A}^{(1)}_{p,\gamma}(k)$ of $\mathcal{B}^{(1)}_{p,\gamma}(k)$ that has the structure of a C^1 submanifold of $\Omega^{(1)}_{p,\gamma}$. The set $\mathcal{A}^{(1)}_{p,\gamma}(k)$ contains all the curves $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ that are of class C^1; moreover, if $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ is a curve of class C^1, then the tangent space $T_\sigma\mathcal{A}^{(1)}_{p,\gamma}(k)$ is given by:*

(2.16)
$$T_\sigma\mathcal{A}^{(1)}_{p,\gamma}(k) = \Big\{\zeta \in T_\sigma\Omega^{(1)}_{p,\gamma} : \textit{there exists } C_\zeta \in I\!R \textit{ such that } \langle \nabla_{\dot\sigma}\zeta, \dot\sigma\rangle \equiv \frac{\mathcal{T}_\sigma C_\zeta}{k}, \textit{ and } \langle \nabla_{\dot\sigma}\zeta, Y\rangle - \langle \zeta, \nabla_{\dot\sigma} Y\rangle \equiv C_\zeta\Big\}.$$

Proof. We will show that, given a curve $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ of class C^1, then there is an open neighborhood of σ in $\mathcal{B}^{(1)}_{p,\gamma}(k)$ which has the desired structure.

Let $k \in I\!R^+$ be a fixed constant; we consider the following map:

$$\mathcal{F} : \Omega^{(1)}_{p,\gamma} \longmapsto L^2([0,1], I\!R) \times L^2([0,1], I\!R) \tag{2.17}$$

given by:

$$\mathcal{F}(\sigma) = \left(\langle \dot\sigma, Y\rangle, \sqrt{k^2\langle \dot\sigma, \dot\sigma\rangle_{(R)} + 1} - \sqrt{1 - \left(1 + \frac{2k^2}{\langle Y, Y\rangle}\right)\langle \dot\sigma, Y\rangle^2}\right). \tag{2.18}$$

Observe that $1 + \frac{2k^2}{\langle Y,Y\rangle} < 0$ in $\overline{U}_k$.

Let $\mathcal{C}$ denote the subspace of $L^2([0,1],\mathbb{R})$ given by all the functions which are constant almost everywhere, and let $\mathcal{C}^-$ denote the open submanifold of $\mathcal{C}$ consisting of negative functions. It is easy to see that $\mathcal{B}^{(1)}_{p,\gamma}(k) = \mathcal{F}^{-1}(\mathcal{C}^- \times \{0\})$.

It is not difficult to prove that $\mathcal{F}$ is a map of class C^1 and that, for $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ and $V \in T_\sigma\Omega^{(1)}_{p,\gamma}$, the Gateaux derivative $\mathrm{d}\mathcal{F}(\sigma)[V]$ is given by:

$$\mathrm{d}\mathcal{F}(\sigma)[V] =$$

$$\tag{2.19} \Big(\langle\nabla_{\dot\sigma}V,Y\rangle - \langle V,\nabla_{\dot\sigma}Y\rangle, \quad \big[\langle\dot\sigma,Y\rangle(\langle\nabla_{\dot\sigma}V,Y\rangle-\langle V,\nabla_{\dot\sigma}Y\rangle) + k^2\langle\nabla_{\dot\sigma}V,\dot\sigma\rangle\big]\left(k^2\langle\dot\sigma,\dot\sigma\rangle_{(\mathbb{R})}+1\right)^{-\frac{1}{2}}\Big).$$

Here we have used the fact that Y is Killing, thus $\langle\dot\sigma,\nabla_V Y\rangle = -\langle V,\nabla_{\dot\sigma}Y\rangle$.

Let $\tilde{L}^2([0,1],\mathbb{R})$ denote the quotient space $L^2([0,1],\mathbb{R})/\mathcal{C}$, which is naturally identified with the set of functions with null average in $[0,1]$:

$$\tag{2.20} \tilde{L}^2([0,1],\mathbb{R}) = L^2([0,1],\mathbb{R})/\mathcal{C} \simeq \Big\{f \in L^2([0,1],\mathbb{R}) : \int_0^1 f = 0\Big\}.$$

Let $\Pi : L^2([0,1],\mathbb{R}) \times L^2([0,1],\mathbb{R}) \longmapsto \tilde{L}^2([0,1],\mathbb{R}) \times L^2([0,1],\mathbb{R})$ be given by the quotient map on the first factor and the identity on the second factor.

Let now σ be a C^1-curve in $\mathcal{B}^{(1)}_{p,\gamma}(k)$; then, the maps $\langle\dot\sigma,Y\rangle$, $\langle\dot\sigma,Y\rangle^2$ and $\langle\dot\sigma,\dot\sigma\rangle$ are in $C^0([0,1],\mathbb{R})$. To prove the Proposition we use the *Inverse Mapping Theorem* (see [15]). According to this Theorem, there exists an open neighborhood of σ in $\mathcal{B}^{(1)}_{p,\gamma}(k)$ which is a smooth submanifold of $\Omega^{(1)}_{p,\gamma}$ provided that the map $\mathcal{F}$ be *transversal* over $\mathcal{C}^- \times \{0\}$ at σ, i.e., if the composite map:

$$\tag{2.21} \Pi \circ \mathrm{d}\mathcal{F}(\sigma) : T_\sigma\Omega^{(1)}_{p,\gamma} \longmapsto \tilde{L}^2([0,1],\mathbb{R}) \times L^2([0,1],\mathbb{R})$$

is surjective. This amounts to saying that, for all $h_1, h_2 \in L^2([0,1],\mathbb{R})$ there exists a constant $c \in \mathbb{R}$ such that the system of differential equations:

$$\tag{2.22} \langle\nabla_{\dot\sigma}V,Y\rangle - \langle V,\nabla_{\dot\sigma}Y\rangle = h_1 + c$$

$$\tag{2.23} \langle\dot\sigma,Y\rangle(\langle\nabla_{\dot\sigma}V,Y\rangle - \langle V,\nabla_{\dot\sigma}Y\rangle) + k^2\langle\nabla_{\dot\sigma}V,\dot\sigma\rangle = h_2$$

has at least one solution $V \in T_\sigma\Omega^{(1)}_{p,\gamma}$. Using the fact that $\langle\dot\sigma,Y\rangle$ is constant, we can rewrite (2.23) as:

$$\tag{2.24} \langle\nabla_{\dot\sigma}V,\dot\sigma\rangle = h_3,$$

where

$$h_3 = \frac{h_2 + 2k\,\mathcal{T}_\sigma(h_1 + c)}{2k^2}$$

is in $L^2([0,1],\mathbb{R})$.

Let $Z \in C^1([0,1], T\mathcal{M})$ be a vector field along σ satisfying

$$\langle Y, Z \rangle \equiv 0, \quad \text{and} \quad \langle Z, \dot\sigma \rangle \neq 0. \tag{2.25}$$

To prove the existence of such a vector field Z, consider first the vector field along σ given by $\dot\sigma^{\perp}$, which is the orthogonal projection of $\dot\sigma$ onto the distribution $\Delta = Y^{\perp}$ orthogonal to Y. Formally, we have:

$$\dot\sigma^{\perp} = \dot\sigma - \frac{\langle \dot\sigma, Y \rangle}{\langle Y, Y \rangle} Y = \dot\sigma + \frac{k\,\mathcal{T}_\sigma}{\langle Y, Y \rangle} Y. \tag{2.26}$$

Obviously, we have:

$$\langle \dot\sigma^{\perp}, \dot\sigma \rangle = -\mathcal{T}_\sigma \frac{k^2 + \langle Y, Y \rangle}{\langle Y, Y \rangle} \neq 0. \tag{2.27}$$

Observe that $\dot\sigma^{\perp} \in C^0$, and it does not have the required C^1-regularity. Now, let Z be any section of class C^1 of Δ which is *uniformly* close to $\dot\sigma^{\perp}$, in such a way that $\langle Z, \dot\sigma \rangle \neq 0$ as well. For the approximation theorem, we can use a C^1 parallel referential of Δ along σ, so that sections of Δ along σ will be identified with curves in the Euclidean space, and standard approximation results apply.

Observe in particular that, since $\langle Z, \dot\sigma \rangle$ is continuous, then $\langle Z, \dot\sigma \rangle^{-1}$ is a function in $L^\infty([0,1], \mathbb{R})$.

In order to solve equations (2.22) and (2.24), we set

$$V = \varphi_1 Y + \varphi_2 Z,$$

where $\varphi_1, \varphi_2 \in H^1([0,1], \mathbb{R})$ are to be determined. Observe that such a V belongs to $T_\sigma \Omega^{(1)}_{p,\gamma}$ provided that φ_1 and φ_2 satisfy the boundary conditions:

$$\varphi_1(0) = \varphi_2(0) = 0, \quad \text{and} \quad \varphi_2(1) = 0. \tag{2.28}$$

Since $\langle Z, Y \rangle = 0$, equations (2.22) and (2.24) are translated into:

$$\varphi_1{}' \langle Y, Y \rangle + 2\varphi_2 \langle \nabla_{\dot\sigma} Z, Y \rangle = h_1 + c \tag{2.29}$$

$$-kT_\sigma \varphi_1{}' + \varphi_2{}' \langle Z, \dot\sigma \rangle + \varphi_2 \langle \nabla_{\dot\sigma} Z, \dot\sigma \rangle = h_3. \tag{2.30}$$

We solve for $\varphi_1{}'$ equation (2.29) obtaining:

$$\varphi_1{}' = \langle Y, Y \rangle^{-1} \left[h_1 + c - 2\varphi_2 \langle \nabla_{\dot\sigma} Z, Y \rangle \right]; \tag{2.31}$$

substituting (2.31) in (2.30) gives:

$$\varphi_2{}' + \alpha \varphi_2 = \beta + c\theta, \tag{2.32}$$

where

$$\alpha = \frac{\langle Y, Y \rangle \langle \nabla_{\dot\sigma} Z, \dot\sigma \rangle + 2kT_\sigma \langle \nabla_{\dot\sigma} Z, Y \rangle}{\langle Z, \dot\sigma \rangle \langle Y, Y \rangle},$$

and

$$\beta = \frac{kT_\sigma h_1 + h_3\langle Y,Y\rangle}{\langle Z,\dot\sigma\rangle\langle Y,Y\rangle}, \quad \theta = \frac{kT_\sigma}{\langle Z,\dot\sigma\rangle\langle Y,Y\rangle}.$$

Observe that α and θ are in $L^2([0,1],\mathbb{R})$, while $\beta \in L^2([0,1],\mathbb{R})$. Thus, the unique solution φ_2 of (2.32) satisfying $\varphi_2(0) = 0$, given by:

$$\varphi_2(t) = e^{-\int_0^t \alpha}\left[\int_0^t \beta e^\alpha + c\int_0^t \theta e^\alpha\right], \tag{2.33}$$

is in $H^1([0,1],\mathbb{R})$. Observe that $\theta \neq 0$ in $[0,1]$, and so $\int_0^1 \theta e^\alpha \neq 0$. In particular, there exists $c \in \mathbb{R}$ such that $\varphi_2(1) = 0$.

Finally, φ_1 can be chosen as the unique solution of (2.31) satisfying $\varphi_1(0) = 0$. Observe that the right hand side of (2.31) is in $L^2([0,1],\mathbb{R})$, so $\varphi_1 \in H^1([0,1],\mathbb{R})$ and $\mathcal{F}$ is transversal over $\mathcal{C}^-$ at σ. Hence, there exists an open neighborhood of σ in $\mathcal{B}^{(1)}_{p,\gamma}(k)$ which is a smooth submanifold of $\Omega^{(1)}_{p,\gamma}$.

By the Inverse Mapping Theorem, for $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ of class C^1, the tangent space $T_\sigma\mathcal{B}^{(1)}_{p,\gamma}(k)$ is identified with the kernel of the map $\Pi \circ \mathrm{d}\mathcal{F}(\sigma)$, which consists of the vector fields $\zeta \in T_\sigma\Omega^{(1)}_{p,\gamma}$ such that $\mathrm{d}\mathcal{F}(\sigma)[\zeta] \in \mathcal{C} \times \{0\}$.

Recalling (2.22) and (2.23), we have that $\zeta \in T_\sigma\Omega^{(1)}_{p,\gamma}$ belongs to $T_\sigma\mathcal{B}^{(1)}_{p,\gamma}(k)$ if and only if there exists $C_\zeta \in \mathbb{R}$ such that ζ satisfies the equations:

$$\langle\nabla_{\dot\sigma}\zeta, Y\rangle - \langle\zeta, \nabla_{\dot\sigma}Y\rangle = C_\zeta, \tag{2.34}$$

$$-2kT_\sigma C_\zeta + 2k^2\langle\nabla_{\dot\sigma}\zeta, \dot\sigma\rangle = 0. \tag{2.35}$$

From (2.34) and (2.35) we easily obtain (2.16) and we are done. □

In $\mathcal{B}^{(1)}_{p,\gamma}(k)$, we can define the *travel time* functional T, given by:

$$T(\sigma) = T_\sigma. \tag{2.36}$$

We now proceed to the formal definition of arrival time brachistochrone. Observe that, since $\mathcal{B}^{(1)}_{p,\gamma}(k)$ is not a manifold, then we cannot define as brachistochrones the critical points of τ in $\mathcal{B}^{(1)}_{p,\gamma}(k)$. However, we can define minima for the arrival time functional in $\mathcal{B}^{(1)}_{p,\gamma}(k)$, without the need of a differential structure.

If q is any point in U_k, we denote by γ_q the maximal integral line of Y through q. Moreover, if $I = [a,b] \subseteq [0,1]$ is any interval, and if q_1, q_2 are any two points in U_k, we define $\mathcal{B}^{(1)}_{q_1,\gamma_{q_2}}(k,I)$ as the space of curves $\ell \in H^1(I,U_k)$ such that $\ell(a) = q_1$, $\ell(b) \in \gamma_{q_2}(\mathbb{R})$, and satisfying $\langle\dot\ell, Y\rangle \equiv -kT_\ell$, $\langle\dot\ell,\dot\ell\rangle \equiv -T_\ell^2$ for some $T_\ell \in \mathbb{R}^+$.

Remark 2.3. Observe that if $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$, then, for every $I = [a,b] \subseteq [0,1]$, the restriction of σ to I is a curve in $\mathcal{B}^{(1)}_{\sigma(a),\gamma_{\sigma(b)}}(k,I)$. Due to the double constraint in $\mathcal{B}^{(1)}_{p,\gamma}(k)$, the converse of this statement does *not* hold in general, i.e., not every curve in $\mathcal{B}^{(1)}_{\sigma(a),\gamma_{\sigma(b)}}(k,I)$ is the restriction to I of some curve in $\mathcal{B}^{(1)}_{p,\gamma}(k)$.

To make this point clear, we consider the following simple but instructive example. Let $\mathcal{M} = \mathbb{R}^2$ be endowed with the Minkowski metric given in the coordinates (x, y) by $g = \mathrm{d}x^2 - \mathrm{d}y^2$; let $Y = \frac{\partial}{\partial y}$ be the chosen timelike Killing vector field, let γ be the y-axis, $k > 1$, $p = (-\sqrt{k^2-1}, 0)$, and

$$\sigma(t) = (x(t), y(t)) = ((t-1)\sqrt{k^2-1}, kt),$$

$t \in [0,1]$. Then, $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ with $\mathcal{T}_\sigma = 1$; consider the interval $[a,b] = [\frac{1}{2}, 1]$, so that $\gamma_{\sigma(b)} = \gamma$ and $q = \sigma(a) = (-\frac{1}{2}\sqrt{k^2-1}, \frac{k}{2})$.

An obvious calculation shows that every curve $\tilde{\sigma} \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ passing through q at $t = \frac{1}{2}$ is such that $\mathcal{T}_{\tilde{\sigma}} = 1$. Namely, if $\tilde{\sigma} = (x, y)$ is a curve in $\mathcal{B}^{(1)}_{p,\gamma}(k)$, then $y(t) = k\,\mathcal{T}_{\tilde{\sigma}}\,t$, and the passage through q at $t = \frac{1}{2}$ implies $\mathcal{T}_{\tilde{\sigma}} = 1$. On the other hand, one can easily construct curves in $\mathcal{B}^{(1)}_{q,\gamma}(k, [\frac{1}{2}, 1])$ with arbitrary large travel time, which proves the claim.

A similar case is treated in Lemma 4.5 ahead.

We can now define the localized minimizers for the arrival time functional:

Definition 2.4. A curve $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ is said to be a *localized minimizer for the arrival time* if, for all $0 \le a < b \le 1$ such that $b - a$ is sufficiently small, the restriction of σ to the interval $I = [a,b]$ is a minimum point for the arrival time functional τ in the space $\mathcal{B}^{(1)}_{\sigma(a),\gamma_{\sigma(b)}}(k, I)$.

A curve σ is said to be an *arrival time brachistochrone of energy k between p and* γ if it is a critical point of class C^2 for the restriction of τ to $\mathcal{A}^{(1)}_{p,\gamma}(k)$.

We will show in Section 5 that the concepts of localized minimizer and critical point for the arrival time functional coincide. To prove this, we will use *horizontal curves* with respect to the orthogonal distribution of Y, which will allow to reduce the brachistochrone problem to the search of critical points for a functional subject to only one constraint. The main motivation for this approach is the lack of regularity for the critical points of τ on $\mathcal{B}^{(1)}_{p,\gamma}(k)$; to realize this we discuss a simple but instructive example.

Example 2.5. Let $(\mathcal{M}, g)$ be the four-dimensional Minkowski spacetime; i.e., $\mathcal{M} = \mathbb{R}^3 \times \mathbb{R}$ and $g = g_0 - \mathrm{d}x_4^2$, where $g_0 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2$ is the Euclidean metric in $\mathbb{R}^3$. Let Y be the Killing vector field $\frac{\partial}{\partial x_4}$, $p = (0,0,0,0)$, $\gamma(s) = (1,0,0,s)$, $s \in \mathbb{R}$, and let $k > 1$ be fixed.

In this example, the set $\mathcal{B}^{(1)}_{p,\gamma}(k)$ can be described explicitly as:

$$\mathcal{B}^{(1)}_{p,\gamma}(k) = \Big\{\sigma = (x_1, x_2, x_3, x_4) \in H^1([0,1], \mathbb{R}^4) : \exists\, \mathcal{T}_\sigma > 0 \text{ such that } \dot{x}_4 = k\mathcal{T}_\sigma \text{ and } \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 = (k^2-1)\,\mathcal{T}_\sigma^2,\ x_4(0) = 0,\ (x_1,x_2,x_3)(0) = (0,0,0),\ (x_1,x_2,x_3)(1) = (1,0,0)\Big\}.$$

We consider the curve $\sigma_0 \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ given by:

$$\sigma_0(t) = \begin{cases} (3t,0,0), & \text{if } t \in [0,\frac{1}{3}]; \\ (2-3t,0,0) & \text{if } t \in]\frac{1}{3},\frac{2}{3}]; \\ (3t-2,0,0) & \text{if } t \in]\frac{2}{3},1]. \end{cases}$$

Moreover, we set $x_4(t) = k\,\mathcal{T}_\sigma\, t$, where $\mathcal{T}_\sigma = 3(k^2-1)^{-\frac{1}{2}}$, so that $\sigma(t) = (\sigma_0(t), x_4(t))$ belongs to $\mathcal{B}^{(1)}_{p,\gamma}(k)$.

We claim that σ has an open neighborhood in $\mathcal{B}^{(1)}_{p,\gamma}(k)$ which has the structure of a manifold of class C^1. To prove this, arguing as in Proposition 2.2, it suffices to show that, for every $h \in L^2([0,1],\mathbb{R})$, there exists $c \in \mathbb{R}$ and $\xi = (\xi_1,\xi_2,\xi_3) \in H^1([0,1],\mathbb{R}^3)$ such that:

$$g_0(\dot\sigma_0,\dot\xi) = h + c, \quad \xi(0) = \xi(1) = 0.$$

A direct computation gives the following solution for the problem above:

$$\xi(t) = (\lambda(t),0,0)$$

with

$$\lambda(t) = \begin{cases} \frac{1}{3}\int_0^t (h(r)+c)\,\mathrm{d}r, & \text{if } t \in [0,\frac{1}{3}]; \\ \frac{1}{3}\int_0^{1/3}(h(r)+c)\,\mathrm{d}r - \frac{1}{3}\int_{1/3}^t (h(r)+c)\,\mathrm{d}r, & \text{if } t \in]\frac{1}{3},\frac{2}{3}]; \\ \frac{1}{3}\int_0^{1/3} h(r)\,\mathrm{d}r - \frac{1}{3}\int_{1/3}^{2/3} h(r)\,\mathrm{d}r + \frac{1}{3}\int_{2/3}^t (h(r)+c)\,\mathrm{d}r, & \text{if } t \in]\frac{2}{3},1], \end{cases}$$

and the constant c is given by:

$$c = -9\left(\int_0^{\frac{1}{3}} h(r)\,\mathrm{d}r - \int_{\frac{1}{3}}^{\frac{2}{3}} h(r)\,\mathrm{d}r + \int_{\frac{2}{3}}^1 h(r)\,\mathrm{d}r\right).$$

Now, the arrival time τ of σ is easily computed as:

$$\tau(\sigma) = \int_0^1 \dot x_4(t)\,\mathrm{d}t = k\,\mathcal{T}_\sigma = k\int_0^1 \sqrt{\dot x_1(t)^2 + \dot x_2(t)^2 + \dot x_3(t)^2}\,\mathrm{d}t.$$

Then, σ is a critical point for τ in $\mathcal{B}^{(1)}_{p,\gamma}(k)$ if and only if the following condition is satisfied:

(2.37)

$$\int_0^1 g_0(\dot\sigma_0,\dot\xi)\,\mathrm{d}t = 0, \quad \forall \xi \text{ of class } C^1 \text{ with } \xi(0)=\xi(1)=0 \text{ and } g_0(\dot\sigma_0,\dot\xi) \text{ constant};$$

in other words, σ is a critical point for τ in $\mathcal{B}^{(1)}_{p,\gamma}(k)$ if and only if given any ξ of class C^1 with $\xi(0) = \xi(1) = 0$ and such that $g_0(\dot\sigma_0,\dot\xi)$ is constant, then the value of such constant is zero.

Now, if $\xi = (\xi_1,\xi_2,\xi_3)$ is any map of class C^1, the condition $g_0(\dot\sigma_0,\dot\xi)$ constant is satisfied if and only if $\xi_0 \equiv 0$. Therefore, $g_0(\dot\sigma_0,\dot\xi) \equiv 0$, so that (2.37) is satisfied and σ is a non smooth critical point of τ in $\mathcal{B}^{(1)}_{p,\gamma}(k)$.

This example shows that we cannot expect to have regularity results for the critical points of τ, if we work directly in $B^{(1)}_{p,\gamma}(k)$.

To introduce the set of horizontal curves, we denote by Δ the smooth distribution on $\mathcal{M}$ given by the orthocomplement of the vector field Y. Observe that, since Y is timelike, the wrong way Schwartz's inequality implies that Δ is *spacelike*, i.e., the restriction of the Lorentzian metric g on Δ is positive definite.

Let $\psi : A \longmapsto \mathcal{M}$ be the flow of Y defined on the open subset A of $\mathcal{M} \times I\!R$, i.e., for $q \in \mathcal{M}$ and $t \in I\!R$ such that $(q,t) \in A$, $\psi(q,t)$ is the value $\gamma_q(t)$, where γ_q is the maximal integral line of Y satisfying $\gamma_q(0) = q$. As we have observed, Y is complete in U_k, which implies that the open set A contains the product $U_k \times I\!R$. Since Y is Killing, then $\psi(\cdot,t)$ is a local isometry for all $t \in I\!R$; moreover, it is easy to see that the distribution Δ is ψ-invariant, which means that $\psi_x(q,t_0)(\Delta_q) = \Delta_{\psi(q,t_0)}$, where $\psi_x(q,t_0)$ denotes the differential of the map $\psi(\cdot,t_0)$ at the point q. A function $\phi : \mathcal{M} \longmapsto I\!R$ is said to be Y-invariant if it is constant along the flow lines of Y; if ψ is C^1, this amounts to saying that $\langle Y, \nabla\phi\rangle \equiv 0$.

We define $\Omega^{(1)}_{p,\gamma}(\Delta)$ to be the subset of $\Omega^{(1)}_{p,\gamma}$ consisting of curves with tangent vector at each point lies in Δ:

$$\Omega^{(1)}_{p,\gamma}(\Delta) = \left\{ w \in \Omega^{(1)}_{p,\gamma} : \dot w(t) \in \Delta_{\dot w(t)},\ \forall t \in [0,1] \right\}. \tag{2.38}$$

Using the language of sub-Riemannian geometry, we will call *horizontal* the curves in $\Omega^{(1)}_{p,\gamma}$. By the same arguments of Proposition 2.2, one checks immediately that, since $\langle Y, Y\rangle$ is never vanishing, $\Omega^{(1)}_{p,\gamma}(\Delta)$ is a smooth submanifold of $\Omega^{(1)}_{p,\gamma}$, and that, for $w \in \Omega^{(1)}_{p,\gamma}(\Delta)$, the tangent space $T_w\Omega^{(1)}_{p,\gamma}(\Delta)$ is given by:

$$T_w\Omega^{(1)}_{p,\gamma}(\Delta) = \left\{ V \in T_w\Omega^{(1)}_{p,\gamma} : \langle \nabla_{\dot w} V, Y\rangle - \langle V, \nabla_{\dot w} Y\rangle = 0 \right\}. \tag{2.39}$$

The set $\Omega^{(1)}_{p,\gamma}(\Delta)$ is *closed* in $\Omega^{(1)}_{p,\gamma}$ with respect to the metric (2.6); namely, if $\{w_n\}_n$ is a sequence in $\Omega^{(1)}_{p,\gamma}(\Delta)$ that converges to a curve w in $\Omega^{(1)}_{p,\gamma}$, then, since $\dot w(t)$ is pointwise limit almost everywhere of $\dot w_n(t)$, it is $\langle \dot w(t), Y(w(t))\rangle = 0$ almost everywhere, and $w \in \Omega^{(1)}_{p,\gamma}(\Delta)$.

Observe that, due to the presence of the boundary ∂U_k, the manifolds $\Omega^{(1)}_{p,\gamma}$ and $\Omega^{(1)}_{p,\gamma}(\Delta)$ are not complete. Namely, if z_n is a Cauchy sequence in either of these spaces, then z_n converges to a curve z whose image may contain points of ∂U_k.

3. The Arrival Time Brachistochrone Differential Equation

The aim of this section is to characterize the arrival time brachistochrones in terms of a differential equation and suitable initial conditions.

We start with the following easy observation, that follows immediately from Lemma 2.1 and the definition of arrival time brachistochrone:

Lemma 3.1. *A curve $\sigma \in \mathcal{A}^{(1)}_{p,\gamma}(k)$ of class C^2 is an arrival time brachistochrone of energy k between p and γ if and only if, for all $\zeta \in T_\sigma \mathcal{A}^{(1)}_{p,\gamma}(k)$, it is $\zeta(1) = 0$.* □

Based on this fact, we can now prove the following:

Proposition 3.2. *Let $k > 0$ be fixed. A curve $\sigma : [0,1] \longmapsto U_k$ of class C^2 joining p and γ is an arrival time brachistochrone of energy k between p and γ if and only if σ is a curve that satisfies the second order differential equation:*

$$\nabla_{\dot\sigma}\dot\sigma - \frac{2\mathcal{T}_\sigma}{k}\nabla_{\dot\sigma}Y - \frac{2}{k}\frac{\langle \nabla_{\dot\sigma}Y, Y\rangle}{k^2 + \langle Y, Y\rangle}\,(k\dot\sigma - \mathcal{T}_\sigma Y) = 0, \tag{3.1}$$

and whose initial tangent vector $\dot\sigma(0)$ is timelike, future pointing and it satisfies the condition:

$$\langle \dot\sigma(0), Y(\sigma(0))\rangle^2 = -k^2\langle \dot\sigma(0), \dot\sigma(0)\rangle. \tag{3.2}$$

Proof. We start proving that a curve σ that satisfies the equation (3.1) and the initial condition (3.2) belongs to $\mathcal{B}^{(1)}_{p,\gamma}(k)$. Given such a σ, denote by $\mathcal{T}_\sigma$ the quantity $-k^{-1}\langle \dot\sigma(0), Y(\sigma(0))\rangle$, which is positive by definition; observe that $\langle \dot\sigma(0), \dot\sigma(0)\rangle = -\mathcal{T}_\sigma{}^2$.

We introduce the two functions ρ_1 and ρ_2 in $C^1([0,1], \mathbb{R})$ given by:

$$\rho_1(t) = \langle \dot\sigma(t), Y(\sigma(t))\rangle, \quad \text{and} \quad \rho_2(t) = \frac{1}{2}\langle \dot\sigma(t), \dot\sigma(t)\rangle. \tag{3.3}$$

By construction, we have:

$$\rho_1(0) = -k\mathcal{T}_\sigma, \quad \rho_2(0) = -\frac{1}{2}\mathcal{T}_\sigma{}^2; \tag{3.4}$$

the curve σ belongs to $\mathcal{B}^{(1)}_{p,\gamma}(k)$ if and only if ρ_1 and ρ_2 are constant on $[0,1]$. To prove this, we multiply the differential equation (3.1) by Y and by $\dot\sigma$, and we obtain the system of differential equation:

$$\begin{cases} \dot\rho_1 + \Phi\,\rho_1 + k\mathcal{T}_\sigma\,\Phi = 0, \\ \dot\rho_2 + 2\Phi\,\rho_2 - \frac{\mathcal{T}_\sigma}{k}\Phi\,\rho_1 = 0, \end{cases} \tag{3.5}$$

where Φ is the function:

$$\Phi = -2\frac{\langle \nabla_{\dot\sigma}Y, Y\rangle}{k^2 + \langle Y, Y\rangle}. \tag{3.6}$$

Then, an immediate calculation shows that the constant functions $\rho_1 \equiv -k\mathcal{T}_\sigma$ and $\rho_2 \equiv -\frac{1}{2}\mathcal{T}_\sigma{}^2$ are the unique solutions of the system (3.5) with initial conditions (3.4), which proves that $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$.

We now prove that a curve $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ of class C^2 is a critical point for the arrival time functional if and only if σ satisfies (3.1). Observe that (3.2) is satisfied by every curve in $\mathcal{B}^{(1)}_{p,\gamma}(k)$.

To this aim, given any $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ of class C^2 and any vector field V along σ, with $V \in H^1([0,1], T\mathcal{M})$ and $V(0) = V(1) = 0$, we define a vector field $\zeta \in T_\sigma \mathcal{B}^{(1)}_{p,\gamma}(k)$ by setting:

$$\zeta(t) = V(t) + \lambda(t) \cdot Y(\sigma(t)) + \mu(t) \cdot \dot\sigma(t), \tag{3.7}$$

where λ and μ are functions satisfying the boundary conditions:

$$\lambda(0) = \mu(0) = \mu(1) = 0, \tag{3.8}$$

and the differential equations:

$$\lambda' = -\frac{\langle \nabla_{\dot\sigma} V, T_\sigma Y - k\dot\sigma \rangle - \langle V, T_\sigma \nabla_{\dot\sigma} Y \rangle}{T_\sigma (k^2 + \langle Y, Y \rangle)}, \tag{3.9}$$

$$\mu' = -\frac{C_\zeta}{kT_\sigma} + \frac{\langle Y, Y \rangle \langle \nabla_{\dot\sigma} V, \dot\sigma \rangle + kT_\sigma C_V}{{T_\sigma}^2 (k^2 + \langle Y, Y \rangle)}, \tag{3.10}$$

where the function C_V is:

$$C_V = \langle \nabla_{\dot\sigma} V, Y \rangle - \langle V, \nabla_{\dot\sigma} Y \rangle,$$

and the constant C_ζ is defined as:

$$C_\zeta = \frac{k}{T_\sigma} \int_0^1 \frac{\langle Y, Y \rangle \langle \nabla_{\dot\sigma} V, \dot\sigma \rangle + kT_\sigma C_V}{k^2 + \langle Y, Y \rangle} \mathrm{dt}.$$

We observe that, using Lemma 3.1, a C^2-curve $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ is a critical point for τ if and only if, for every $\zeta = W + \lambda \cdot Y$ vector field in $T_\sigma \mathcal{A}^{(1)}_{p,\gamma}(k)$, with W vector field along σ of class H^1 such that $W(0) = W(1) = 0$, and λ any function of class H^1 on $[0,1]$, it is $\lambda(1) = 0$. Hence, the curve σ is an arrival time brachistochrone if and only if $\lambda(1) = 0$, which, using (3.9), is the same as:

$$\int_0^1 \frac{\langle \nabla_{\dot\sigma} V, -k\dot\sigma + T_\sigma Y \rangle - \langle V, T_\sigma \nabla_{\dot\sigma} Y \rangle}{k^2 + \langle Y, Y \rangle} \, dt = 0, \tag{3.11}$$

for all vector field V of class H^1 along σ such that $V(0) = V(1) = 0$.

Integrating by parts the first term of (3.11) gives:

$$\int_0^1 \langle V, \nabla_{\dot\sigma} \left(\frac{k\dot\sigma - T_\sigma Y}{k^2 + \langle Y, Y \rangle} \right) - \frac{T_\sigma \nabla_{\dot\sigma} Y}{k^2 + \langle Y, Y \rangle} \rangle \, \mathrm{dt} = 0, \tag{3.12}$$

for all V. Hence, the Fundamental Lemma of Calculus of Variations implies that:

$$\nabla_{\dot\sigma} \left(\frac{k\dot\sigma - T_\sigma Y}{k^2 + \langle Y, Y \rangle} \right) - \frac{T_\sigma \nabla_{\dot\sigma} Y}{k^2 + \langle Y, Y \rangle} = 0, \tag{3.13}$$

which is equivalent to (3.1), and we are done. □

If $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$, we can reparameterize σ using its proper time by setting:

$$z : [0, \mathcal{T}_\sigma] \longmapsto U_k, \quad z(t) = \sigma(\frac{t}{\mathcal{T}_\sigma}).$$

Then, a curve z parameterized by proper time is an arrival time brachistochrone if and only if it satisfies the differential equation:

$$\nabla_{\dot z}\dot z - \frac{2}{k}\nabla_{\dot z}Y - 2\frac{\langle \nabla_{\dot z}Y, Y\rangle}{k^2 + \langle Y, Y\rangle}\dot z + \frac{2}{k}\frac{\langle \nabla_{\dot z}Y, Y\rangle}{k^2 + \langle Y, Y\rangle}Y = 0, \tag{3.14}$$

and with initial tangent vector $\dot z(0)$ satisfying:

$$\langle \dot z(0), Y(z(0))\rangle = -k.$$

4. Localized Minimizers of the Arrival Time

Let $k > 0$ be fixed and let U_k be the open subset of $\mathcal{M}$ defined in (1.10). We introduce the following smooth functions $\Psi_k : \mathcal{M} \longmapsto I\!R$ and $\phi_k : U_k \longmapsto I\!R$:

$$\Psi_k(q) = \langle Y(q), Y(q)\rangle + k^2, \quad \text{and} \quad \phi_k(q) = -\frac{\langle Y(q), Y(q)\rangle}{\langle Y(q), Y(q)\rangle + k^2}. \tag{4.1}$$

Observe that ϕ_k and Ψ_k are positive in U_k; moreover, it is:

$$\partial U_k = \Psi_k^{-1}(0).$$

The assumption that k^2 be a regular value for the function $-\langle Y, Y\rangle$ implies that the derivative of Ψ_k is non vanishing on the boundary of U_k:

$$\mathrm{d}\Psi_k \neq 0 \text{ on } \partial U_k.$$

Hence, ∂U_k is a smooth submanifold of $\mathcal{M}$.

In order to state properly our variational principle, we introduce an operator $\mathcal{D}$ that *deforms* curves in $\Omega^{(1)}_{p,\gamma}$ into horizontal curves using the flow of Y.

Let $\mathcal{D}$ be the map:

$$\mathcal{D} : \Omega^{(1)}_{p,\gamma} \longmapsto \Omega^{(1)}_{p,\gamma}(\Delta)$$

defined by $\mathcal{D}(\sigma) = w$, where

$$w(t) = \psi(\sigma(t), \mathfrak{r}_\sigma(t)), \tag{4.2}$$

and $\mathfrak{r}_\sigma$ is the unique solution on $[0, 1]$ of the Cauchy problem:

$$\dot{\mathfrak{r}}_\sigma = -\frac{\langle \dot\sigma, Y\rangle}{\langle Y, Y\rangle} = \frac{k\,\mathcal{T}_\sigma}{\langle Y, Y\rangle}, \qquad \mathfrak{r}_\sigma(0) = 0, \tag{4.3}$$

namely,

$$r_\sigma(t) = \int_0^t \frac{k\,\mathcal{T}_\sigma}{\langle Y, Y\rangle}.$$

Using the Killing property of Y it is easily checked that $\mathcal{D}$ is well defined, i.e., the maximal solution of (4.3) is defined on the entire interval $[0,1]$ and the corresponding curve w given by (4.2) is horizontal.

The differentiability of $\mathcal{D}$ and a formula for the differential $\mathrm{d}\mathcal{D}$ is established in the next:

Proposition 4.1. *The map $\mathcal{D}$ is a smooth deformation retract between the manifolds $\Omega^{(1)}_{p,\gamma}$ and $\Omega^{(1)}_{p,\gamma}(\Delta)$.*

Proof. The smooth dependence on σ of the solution r_σ of (4.3) proves that $\mathcal{D}$ is a smooth map.

To prove that $\mathcal{D}$ is a deformation retract, we consider the map

$$H : \Omega^{(1)}_{p,\gamma} \times [0, +\infty] \longmapsto \Omega^{(1)}_{p,\gamma}$$

given by:

$$H(\sigma, r)(t) = \begin{cases} \psi(\sigma(t), r), & \text{if } r \le r_\sigma(t); \\ \psi(\sigma(t), r_\sigma(t)), & \text{if } r > r_\sigma(t). \end{cases} \tag{4.4}$$

Such a map H is clearly continuous, and it is a homotopy between the maps $H(\cdot, 0)$, which is the identity on $\Omega^{(1)}_{p,\gamma}$, and $H(\cdot, \infty) = \mathcal{D}$. □

Remark 4.2. Observe that, by the last statement of Proposition 4.1, the spaces $\Omega^{(1)}_{p,\gamma}$ and $\Omega^{(1)}_{p,\gamma}(\Delta)$ have the same homotopy type.

We define the following functional on $\Omega^{(1)}_{p,\gamma}(\Delta)$:

$$\tau_k(w) = \tau(w) - k\int_0^1 \frac{\sqrt{\phi_k(w)\cdot\langle \dot w, \dot w\rangle}}{\langle Y(w), Y(w)\rangle}\,\mathrm{d}t. \tag{4.5}$$

Observe that every curve $w \in \Omega^{(1)}_{p,\gamma}(\Delta)$ is *spacelike*, i.e., $\langle \dot w, \dot w\rangle \ge 0$ almost everywhere, and τ_k is well defined in $\Omega^{(1)}_{p,\gamma}(\Delta)$.

It is not difficult to prove that τ_k is Lipschitz continuous in $\Omega^{(1)}_{p,\gamma}(\Delta)$ and that it is differentiable at those points w for which the following condition is satisfied:

$$\exists\, \nu_w > 0 \text{ such that } \phi_k(w)\langle \dot w, \dot w\rangle \ge \nu_w \text{ a.e. on } [0,1]. \tag{4.6}$$

By *critical point of* τ_k we will mean a curve w that satisfies (4.6) and $\mathrm{d}\tau_k(w) = 0$. Observe also that τ_k is *invariant by reparameterizations*, and so is the space $\Omega^{(1)}_{p,\gamma}(\Delta)$. By that, we mean that if $w \in \Omega^{(1)}_{p,\gamma}(\Delta)$ is given and w_0 is any reparameterization of w of class H^1 on the interval $[0,1]$, then $w_0 \in \Omega^{(1)}_{p,\gamma}(\Delta)$ and $\tau_k(w_0) = \tau_k(w)$.

In particular, given any $w \in \Omega^{(1)}_{p,\gamma}(\Delta)$ satisfying (4.6), there exists a unique reparameterization $w_0 \in \Omega^{(1)}_{p,\gamma}(\Delta)$ of w for which the quantity $\phi_k(w_0(t))\langle \dot{w}_0(t), \dot{w}_0(t)\rangle$ is constant (positive) on $[0,1]$. In the proof of the following Lemma we will see that, if $w \in \mathcal{D}(\mathcal{B}^{(1)}_{p,\gamma}(k))$, then w is parameterized in such a way that $\phi_k(w)\langle \dot{w}, \dot{w}\rangle$ is constant (and positive).

The maps τ, τ_k and $\mathcal{D}$ are related by the following:

Lemma 4.3. *For all $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$, we have:*

$$\tau_k\left(\mathcal{D}(\sigma)\right) = \tau(\sigma). \tag{4.7}$$

Proof. Let $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ be fixed and let w_σ denote the curve $\mathcal{D}(\sigma)$. We start with the following calculation, that relates the Riemannian length of $\dot{w}_\sigma$ with the travel time $\mathcal{T}_\sigma$:

$$\begin{aligned}
&\phi_k(w_\sigma)\langle \dot{w}_\sigma, \dot{w}_\sigma\rangle = \\
&\quad = -\frac{\langle Y,Y\rangle}{k^2+\langle Y,Y\rangle}\langle \mathrm{d}_x\psi[\dot{\sigma}] + \dot{\mathrm{r}}_\sigma Y, \mathrm{d}_x\psi[\dot{\sigma}] + \dot{\mathrm{r}}_\sigma Y\rangle = \\
&\quad = -\frac{\langle Y,Y\rangle}{k^2+\langle Y,Y\rangle}\left(\langle\dot{\sigma},\dot{\sigma}\rangle + 2\,\dot{\mathrm{r}}_\sigma\langle Y,\dot{\sigma}\rangle + \dot{\mathrm{r}}_\sigma^2\langle Y,Y\rangle\right) = \\
&\quad = -\frac{\langle Y,Y\rangle}{k^2+\langle Y,Y\rangle}\left(\langle\dot{\sigma},\dot{\sigma}\rangle - \frac{\langle\dot{\sigma},Y\rangle^2}{\langle Y,Y\rangle}\right) = \\
&\quad = -\frac{\langle Y,Y\rangle}{k^2+\langle Y,Y\rangle}\left(-\mathcal{T}_\sigma{}^2 - \mathcal{T}_\sigma{}^2\frac{k^2}{\langle Y,Y\rangle}\right) = \mathcal{T}_\sigma{}^2 \equiv -\langle\dot{\sigma},\dot{\sigma}\rangle
\end{aligned} \tag{4.8}$$

Then, using (4.2), (4.3), (4.5) and (4.8), we compute easily:

$$\begin{aligned}
\tau_k(w_\sigma) &= \tau(w_\sigma) - k\int_0^1 \frac{\sqrt{\phi_k(w_\sigma)\langle \dot{w}_\sigma, \dot{w}_\sigma\rangle}}{\langle Y,Y\rangle}\,\mathrm{d}t = \\
&= \tau(\sigma) + \mathrm{r}_\sigma(1) - k\int_0^1 \frac{\mathcal{T}_\sigma}{\langle Y,Y\rangle}\,\mathrm{d}t = \\
&= \tau(\sigma) + k\int_0^1 \frac{\mathcal{T}_\sigma}{\langle Y,Y\rangle}\,\mathrm{d}t - k\int_0^1 \frac{\mathcal{T}_\sigma}{\langle Y,Y\rangle}\,\mathrm{d}t = \tau(\sigma),
\end{aligned} \tag{4.9}$$

which concludes the proof. □

The above Lemma explains the introduction of the functional τ_k. However, in order to prove the results of existence and multiplicity, it turns out to be more convenient to use another functional denoted by G_k, constructed starting from τ_k, that will be introduced in Section 5.

Remark 4.4. Observe that, from (4.8) it follows that, if $w_\sigma = \mathcal{D}(\sigma)$ for some $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$, then $\phi_k(w_\sigma)\langle \dot w_\sigma, \dot w_\sigma\rangle$ is constant along w_σ. Moreover, if $w \in \Omega^{(1)}_{p,\gamma}(\Delta)$ is such that $\phi_k(w)\langle \dot w, \dot w\rangle$ is constant, then there exists $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ such that $w = \mathcal{D}(\sigma)$. To see this, one can introduce the following map:

$$\mathcal{G} : \Omega^{(1)}_{p,\gamma} \longmapsto \Omega^{(1)}_{p,\gamma}, \tag{4.10}$$

given by:

$$\mathcal{G}(w)(t) = \psi(w(t), h_w(t)), \tag{4.11}$$

where

$$h_w(t) = -k\int_0^t \frac{\sqrt{\phi_k(w(0))\langle \dot w(0), \dot w(0)\rangle}}{\langle Y, Y\rangle}\, dr.$$

It is easy to check that if $w \in \Omega^{(1)}_{p,\gamma}(\Delta)$ is such that $\phi_k(w)\langle \dot w, \dot w\rangle$ is constant, then $\mathcal{G}(w) \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ and $\mathcal{D}(\mathcal{G}(w)) = w$. In particular, $\mathcal{D}$ gives a bijection between $\mathcal{B}^{(1)}_{p,\gamma}(k)$ and the set:

$$\left\{ w \in \Omega^{(1)}_{p,\gamma} : \phi_k(w)\langle \dot w, \dot w\rangle \text{ is constant a.e. on } [0,1] \right\}.$$

In order to define the concept of *localized minimizer* for the functional τ_k we need to give a *localized* version of the space $\Omega^{(1)}_{p,\gamma}(\Delta)$ and of the functional τ_k. This is done as follows. For $z_1, z_2 \in U_k$ and $I = [a,b] \subseteq [0,1]$, we define the space $\Omega^{(1)}_{z_1,\gamma_{z_2}}(\Delta, I)$ as

$$\Omega^{(1)}_{z_1,\gamma_{z_2}}(\Delta, I) = \left\{ w \in H^1(I, U_k) : w(a) = z_1,\ w(b) \in \gamma_{z_2}(I\!R),\ w \text{ horizontal} \right\}.$$

Recall that γ_z denotes the maximal integral line of Y such that $\gamma_z(0) = z$.

If $w \in \Omega^{(1)}_{p,\gamma}(\Delta)$ and $I = [a,b] \subseteq [0,1]$, then the restriction of w to I is an element of $\Omega^{(1)}_{w(a),\gamma_{w(b)}}(\Delta, I)$. Conversely, we have the following simple Lemma:

Lemma 4.5. *Let $q_1, q_2 \in U_k$ be such that there exists $w \in \Omega^{(1)}_{p,\gamma}(\Delta)$ with $w(a) = q_1$ and $w(b) = q_2$ for some $0 \le a \le b \le 1$. Then setting $I = [a,b]$, every curve $u \in \Omega^{(1)}_{q_1,\gamma_{q_2}}(\Delta, I)$ is the restriction to I of some curve $w_1 \in \Omega^{(1)}_{p,\gamma}(\Delta)$.*

Proof. Given any $u \in \Omega^{(1)}{}_{q_1,\gamma_{q_2}}(k, I)$, let $r_0 = r_0(u) \in I\!R$ be defined by the relation:

$$\psi(w(b), r_0) = u(b).$$

Define w_1 as follows:

$$w_1(t) = \begin{cases} w(t), & \text{if } t \in [0,a]; \\ u(t), & \text{if } t \in]a,b]; \\ \psi(w(t), r_0), & \text{if } t \in]b,1]. \end{cases}$$

Obviously, $u = w_1\big|_I$. To see that $w_1 \in \Omega^{(1)}_{p,\gamma}(\Delta)$, observe that w_1 is continuous, and since both w and u are of class H^1, then also w_1 is of class H^1. Clearly, $w_1(0) = p$ and $w_1(1) \in \gamma(\mathbb{R})$. Finally, to check that $\langle \dot w_1, Y\rangle = 0$ observe that $\langle \dot w, Y\rangle = \langle \dot u, Y\rangle \equiv 0$, and that $\mathrm{d}_x\psi$ is an isometry. □

The localized functional $\tau_k(I)$ on $\Omega^{(1)}_{z_1,\gamma_{z_2}}(\Delta, I)$ is defined by:

$$\tau_k(I)(w) = \gamma_{z_2}^{-1}(w(b)) - k\int_a^b \frac{\sqrt{\phi_k(w)\langle \dot w, \dot w\rangle}}{\langle Y, Y\rangle}\,\mathrm{d}t.$$

We can now define the local minimizers of τ_k as follows.

Definition 4.6. *A curve $w \in \Omega^{(1)}_{p,\gamma}(\Delta)$ is a* localized minimizer *for the functional τ_k if for every interval $I = [a, b] \subseteq [0, 1]$ with a and b sufficiently close, the restriction $w|_I$ is a minimum for the functional $\tau_k(I)$ in $\Omega^{(1)}_{w(a),\gamma_{w(b)}}(\Delta, I)$.*

By Lemma 4.3, we see immediately that the following result holds:

Proposition 4.7. *If σ is a localized minimizer for τ in $B^{(1)}_{p,\gamma}(k)$, then $w = \mathcal{D}(\sigma)$ is a localized minimizer for τ_k in $\Omega^{(1)}_{p,\gamma}(\Delta)$.*

Conversely (see Remark 4.4), if w is a localized minimizer for τ_k in $\Omega^{(1)}_{p,\gamma}(\Delta)$ parameterized by $\phi_k(w)\langle \dot w, \dot w\rangle = const.$, then there exists a unique $\sigma \in B^{(1)}_{p,\gamma}(k)$ which is a localized minimizer for τ in $B^{(1)}_{p,\gamma}(k)$ and such that $w = \mathcal{D}(\sigma)$. □

Remark 4.8. Using Proposition 4.7, we could try to prove the existence results for the arrival time brachistochrones by proving the existence of smooth localized minimizers for the functional τ_k in $\Omega^{(1)}_{p,\gamma}(\Delta)$. Towards this goal, two main difficulties arise. In first place, the square root under the integral sign defining τ_k (formula (4.5)) involves many analytical difficulties. This problem will be solved by the introduction of a smooth functional G_k in $\Omega^{(1)}_{p,\gamma}(\Delta)$, whose critical points are precisely the critical points of τ_k with a suitable parameterization (see Section 5). The second problem is the presence of the constraint given by the distribution Δ. In [8] the authors managed to *remove* the constraint using the Killing property of Y and the fact that, in the case of the travel time brachistochrones, the functional to extremize in $\Omega^{(1)}_{p,\gamma}(\Delta)$ was given by an integral of a Y-invariant function (cf. [8, Proposition 2.11]). In the case of the arrival time brachistochrones, the reduction to a variational problem in $\Omega^{(1)}_{p,\gamma}$ is *not* possible. Indeed, the critical points of τ in $\Omega^{(1)}_{p,\gamma}$ do not in general belong to $\Omega^{(1)}_{p,\gamma}(\Delta)$, and the critical points of τ_k in $\Omega^{(1)}_{p,\gamma}(\Delta)$ are not critical points of τ in $\Omega^{(1)}_{p,\gamma}$.

We conclude this section by showing the following crucial property of the functional τ_k.

Proposition 4.9. *Assume that Y is a timelike Killing vector field satisfying* (1.11). *Then, for all $c \in \mathbb{R}$ there exists two constants $D_1(c), D_2(c) > 0$ such that, for all*

$w \in \Omega^{(1)}_{p,\gamma}(\Delta)$, *the following holds:*

(4.12)

$$\tau_k(w) \le c \quad \Longrightarrow \quad \int_0^1 \sqrt{\langle \dot w, \dot w\rangle_{(R)}}\, dt \le D_1(c) \quad \textit{and} \quad |\tau_k(w)| \le D_2(c).$$

Proof. Let $w \in \Omega^{(1)}_{p,\gamma}(\Delta)$ be fixed; we use the local metric structure of a stationary Lorentzian manifold (see [7] for details), as follows. We cover the compact image of w, $w([0,1])$, with a finite number of open subsets U_i, $i = 1,\ldots,r$, which are the domains of local coordinates $(x^i_1,\ldots,x^i_{m-1},\theta^i) = (\mathrm{x}^i,\theta^i)$, in such a way that the following conditions are satisfied:

- there exists a partition $0 = a_0 < a_1 < \ldots < a_r = 1$ of $[0,1]$ such that $w([a_i,a_{i-1}]) \subset U_i$ for all $i = 1,\ldots,r$;
- each coordinate map (x^i,θ^i) gives a diffeomorphism of U_i with a product $V_i \times J_i$, where V_i is identified with a spacelike hypersurface in $\mathcal{M}$ and J_i is an open interval in $\mathbb{R}$;
- $\theta^i(w(a_{i-1})) = 0$;
- in U_i, the coordinate vector field $\frac{\partial}{\partial\theta^i}$ coincides with Y;
- in U_i the metric g is independent of the variable θ^i (because Y is Killing) and it is written in terms of the coordinates (x^i,θ^i) as:

(4.13)
$$g(\mathrm{x}^i,\theta^i)[(\Xi,\Theta),(\Xi,\Theta)] = \langle\Xi,\Xi\rangle + 2\langle\delta^i(\mathrm{x}^i),\Xi\rangle\,\Theta - \beta(\mathrm{x}^i)\,\Theta^2,$$

where $(\Xi,\Theta) \in T_{\mathrm{x}}V_i \times \mathbb{R} \simeq T_{(\mathrm{x}^i,\theta^i)}\mathcal{M}$, δ^i is a smooth vector field on V_i and $\beta = -\langle Y,Y\rangle$.

In such a coordinate system, the product $\langle Y,(\Xi,\Theta)\rangle$ is easily computed as:

$$\langle Y,(\Xi,\Theta)\rangle = \langle\delta_i(\mathrm{x}^i),\Xi\rangle - \beta(\mathrm{x}^i)\,\Theta,$$

hence, we have the following formula for the Riemannian metric g_{R}:

(4.14)

$$g_{\mathrm{R}}(\mathrm{x}^i,\theta^i)[(\Xi,\Theta),(\Xi,\Theta)] = \\ = \langle\Xi,\Xi\rangle + \frac{2}{\beta(\mathrm{x}^i)}\langle\delta_i(\mathrm{x}^i),\Xi\rangle^2 - 2\langle\delta_i(\mathrm{x}^i),\Xi\rangle\,\Theta + \beta(\mathrm{x}^i)\,\Theta^2.$$

Since γ is an integral curve of Y, by our choice of the coordinate maps θ_i, we can write (see (4.1) and (4.5)):

(4.15)

$$\tau_k(w) = \sum_{i=1}^r \int_{a_{i-1}}^{a_i} \dot\theta_i\, dt + \\ + k\int_{a_{i-1}}^{a_i} \frac{1}{\beta(\mathrm{x}^i)}\sqrt{\frac{\beta(\mathrm{x}^i)}{k^2-\beta(\mathrm{x}^i)}\Big(\langle\dot{\mathrm{x}}^i,\dot{\mathrm{x}}^i\rangle + 2\langle\delta_i(\mathrm{x}^i),\dot{\mathrm{x}}^i\rangle\,\dot\theta^i - \beta(\mathrm{x}^i)\,(\dot\theta^i)^2\Big)}\, dt.$$

Since w is horizontal, we have:

$$0 = \langle \dot{w}, Y(w) \rangle = \langle \delta_i(\mathbf{x}^i), \dot{\mathbf{x}}^i \rangle - \beta(\mathbf{x}^i)\, \dot{\theta}^i,$$

and therefore

$$\dot{\theta}_i = \frac{\langle \delta_i(\mathbf{x}^i), \dot{\mathbf{x}}^i \rangle}{\beta(\mathbf{x}^i)}, \tag{4.16}$$

observe that $k^2 - \beta(\mathbf{x}^i) = \langle Y, Y \rangle + k^2 > 0$.

Then, we compute

$$\tau_k(w) =$$

$$\sum_{i=1}^{r} \int_{a_{i-1}}^{a_i} \left[\frac{\langle \delta_i(\mathbf{x}^i), \dot{\mathbf{x}}^i \rangle}{\beta(\mathbf{x}^i)} + \frac{k}{\beta(\mathbf{x}^i)} \sqrt{\frac{\beta(\mathbf{x}^i)}{k^2 - \beta(\mathbf{x}^i)} \langle \dot{\mathbf{x}}^i, \dot{\mathbf{x}}^i \rangle + \frac{\langle \delta_i(\mathbf{x}^i), \dot{\mathbf{x}}^i \rangle^2}{k^2 - \beta(\mathbf{x}^i)}} \right] \mathrm{d}t$$

and, by (1.11), a straightforward calculation shows that the following inequalities hold:

$$\begin{aligned} \tau_k(w) &\geq \sum_{i=1}^{r} \int_{a_{i-1}}^{a_i} \left[-1 + k \sqrt{\frac{\beta(\mathbf{x}^i) + 1}{k^2 - \beta(\mathbf{x}^i)}} \right] \frac{\sqrt{\langle \dot{\mathbf{x}}^i, \dot{\mathbf{x}}^i \rangle}}{\beta(\mathbf{x}^i)}\, \mathrm{d}t \geq \\ &\geq \frac{1}{\nu} \left(-1 + k \sqrt{\frac{1+\nu}{k^2 - \nu}} \right) \sum_{i=1}^{r} \int_{a_{i-1}}^{a_i} \sqrt{\langle \dot{\mathbf{x}}^i, \dot{\mathbf{x}}^i \rangle}\, \mathrm{d}t, \end{aligned}$$

and

$$\begin{aligned} \tau_k(w) &\geq \sum_{i=1}^{r} \int_{a_{i-1}}^{a_i} \frac{1}{\beta(\mathbf{x}^i)} \left[\frac{k}{\sqrt{k^2 - \beta(\mathbf{x}^i)}} - 1 \right] \cdot |\langle \delta_i(\mathbf{x}^i), \dot{\mathbf{x}}^i \rangle|\, \mathrm{d}t \geq \\ &\geq \frac{1}{\nu} \left(\frac{k}{\sqrt{k^2 - \nu}} - 1 \right) \sum_{i=1}^{r} \int_{a_{i-1}}^{a_i} |\langle \delta_i(\mathbf{x}^i), \dot{\mathbf{x}}^i \rangle|\, \mathrm{d}t. \end{aligned}$$

Now, (4.12) is easily obtained by combining (4.14), (4.15) and (4.16). □

Remark 4.10. The same argument of the proof of Proposition 4.9 shows that τ_k is bounded from below on $\Omega^{(1)}_{p,\gamma}(\Delta)$.

5. A New Variational Principle

For all $k > 0$, we denote by Θ_k the smooth function on $\mathcal{M}$ given by:

$$\Theta_k(q) = \frac{\phi_k(q)}{\langle Y(q), Y(q) \rangle^2} = -\frac{1}{\langle Y, Y \rangle (k^2 + \langle Y, Y \rangle)}, \tag{5.1}$$

where ϕ_k was defined in (4.1). By (1.11), the functions ϕ_k and Θ_k are bounded away from zero in U_k:

$$\phi_k(q) \geq \frac{\nu}{k^2} > 0, \quad \Theta_k(q) \geq \frac{4}{k^2} > 0.$$

Moreover, we consider the following functional on $\Omega^{(1)}_{p,\gamma}(\Delta)$:

$$G_k(w) = \tau(w) + k\left(\int_0^1 \Theta_k(w)\cdot\langle\dot{w},\dot{w}\rangle\,\mathrm{d}t\right)^{\frac{1}{2}}. \tag{5.2}$$

In this case, since $p \notin \gamma(\mathbb{R})$ and $\Theta_k > 0$ in U_k, then the integral $\int_0^1 \Theta_k(w)\cdot\langle\dot{w},\dot{w}\rangle\,\mathrm{d}t$ is strictly positive for all $w \in \Omega^{(1)}_{p,\gamma}(\Delta)$, and so G is smooth. The *trick* of placing the square root outside the integral sign to obtain a smooth functional with the same critical points (cf. formulas (4.5) and (5.2)) was inspired by a result in [4], where the authors study lightlike rays between an observer and a source, which are obtained as the critical points of the arrival time in the space of lightlike curves joining an event and a timelike curve.

Remark 5.1. By Hölder's inequality, we have:

$$\tau_k(w) \leq G_k(w), \quad \forall\, w \in \Omega^{(1)}_{p,\gamma}(\Delta);$$

and since Θ_k is bounded away from zero in U_k, for all $c \in \mathbb{R}$, we obtain the existence of a positive constant $D(c)$ such that:

$$G_k(w) \leq c \quad \Longrightarrow \quad \int_0^1 \langle\dot{w},\dot{w}\rangle_{(\mathrm{R})}\,\mathrm{d}t \leq D(c); \tag{5.3}$$

since we are considering horizontal curves, in the integral of (5.3) one could consider equivalently the Lorentzian product $\langle\dot{w},\dot{w}\rangle$.

Observe that G_k is bounded from below because τ_k is bounded from below on $\Omega^{(1)}_{p,\gamma}$ (see Proposition 4.9 and Remark 4.10).

The following Lemma plays a crucial role in the proof of the relations between the arrival time brachistochrones of energy k and the critical points of G_k in $\Omega^{(1)}_{p,\gamma}(\Delta)$:

Lemma 5.2. *Let w be a critical point of G_k in $\Omega^{(1)}_{p,\gamma}(\Delta)$. Then, w is a curve of class C^2, and there exists a positive constant C_w such that:*

$$\Theta_k(w)\langle\dot{w},\dot{w}\rangle \equiv C_w^2 \quad \text{on } [0,1], \tag{5.4}$$

and the following differential equation is satisfied:

$$\begin{aligned}\frac{k}{C_w}\Big(\Theta_k(w)\nabla_{\dot{w}}\dot{w} + \langle\nabla\Theta_k(w),\dot{w}\rangle\,\dot{w} - \frac{1}{2}\langle\dot{w},\dot{w}\rangle\,\nabla\Theta_k(w)\Big) + \\ - \frac{2}{\langle Y,Y\rangle}\nabla_{\dot{w}}Y + \frac{2\,\langle Y,\nabla_{\dot{w}}Y\rangle}{\langle Y,Y\rangle^2}Y = 0.\end{aligned} \tag{5.5}$$

Conversely, if $w \in \Omega^{(1)}_{p,\gamma}(\Delta)$ is of class C^2 and it satisfies (5.4) and (5.5), then w is a critical point of G_k in $\Omega^{(1)}_{p,\gamma}(\Delta)$.

Proof. Let w be a critical point of G_k in $\Omega^{(1)}_{p,\gamma}(\Delta)$. Recalling formula (2.12), given any vector field $\zeta \in T_w\Omega^{(1)}_{p,\gamma}(\Delta)$ of class C^1, the derivative $\mathrm{d}G_k(w)[\zeta]$ is easily computed as:

(5.6)

$$\mathrm{d}G_k(w)[\zeta] = \frac{\langle Y(w(1)), \zeta(1)\rangle}{\langle Y(w(1)), Y(w(1))\rangle} + \\ + \frac{k}{\sqrt{\int_0^1 \Theta_k(w)\langle \dot w, \dot w\rangle \, \mathrm{d}t}} \int_0^1 \Big(\frac{1}{2}\langle \nabla\Theta_k(w), \zeta\rangle\langle \dot w, \dot w\rangle + \Theta_k(w)\langle \dot w, \nabla_{\dot w}\zeta\rangle\Big) \, \mathrm{d}t.$$

Let V be a vector field along w, with $V(0) = V(1) = 0$ which is the restriction to w of a C^∞ vector field defined around $w([0,1])$. By (2.39), the vector field $\zeta(t) = V(w(t)) - \mu(t)\cdot Y(w(t))$ is in $T_w\Omega^{(1)}_{p,\gamma}(\Delta)$ if and only if:

$$\mu(t) = \int_0^t \frac{\langle \nabla_{\dot w}V, Y\rangle - \langle V, \nabla_{\dot w}Y\rangle}{\langle Y, Y\rangle} \, \mathrm{d}r. \tag{5.7}$$

Setting

$$C_w = \sqrt{\int_0^1 \Theta_k(w)\langle \dot w, \dot w\rangle \, \mathrm{d}t},$$

we have:

(5.8)

$$\mathrm{d}G_k(w)[V - \mu\cdot Y] = \\ = -\mu(1) + \frac{k}{C_w}\int_0^1 \frac{1}{2}\langle \dot w, \dot w\rangle\Big(\langle \nabla\Theta_k(w), V\rangle - \mu\cdot\langle \nabla\Theta_k(w), Y\rangle\Big) \, \mathrm{d}t \\ + \frac{k}{C_w}\int_0^1 \Theta_k(w)\cdot\langle \dot w, \nabla_{\dot w}V - \mu' Y - \mu\nabla_{\dot w}Y\rangle \, \mathrm{d}t.$$

If we set $\lambda_k(s) = -(s^2 + sk^2)^{-1}$, we have $\Theta_k = \lambda_k(\langle Y, Y\rangle)$, and

$$\langle \nabla\Theta_k(w), \zeta\rangle = 2\lambda_k'(\langle Y, Y\rangle)\langle \nabla_\zeta Y, Y\rangle.$$

Moreover, since Y is Killing, we have:

$$\langle \nabla_Y Y, Y\rangle = \langle \nabla_{\dot w}Y, \dot w\rangle = 0;$$

and, since $w \in \Omega^{(1)}_{p,\gamma}(\Delta)$, it is $\langle \dot w, Y\rangle = 0$. Then, we get:

(5.9)

$$\mathrm{d}G_k(w)[V-\mu\cdot Y] = -\int_0^1 \frac{\langle \nabla_{\dot w} V, Y\rangle - \langle V, \nabla_{\dot w} Y\rangle}{\langle Y, Y\rangle}\,\mathrm{d}t + \\ + \frac{k}{C_w}\int_0^1 \left(\frac{1}{2}\langle \dot w,\dot w\rangle\langle \nabla\Theta_k(w), V\rangle + \Theta_k(w)\langle \dot w, \nabla_{\dot w} V\rangle\right)\mathrm{d}t.$$

Since w is a critical point for G_k, then the above expression vanishes for all smooth vector field V such that $V(0) = V(1) = 0$.

Now, we use covariant integration along w (see formulas (2.9) and (2.10)), and, by the Fundamental Theorem of Calculus of Variations, it is easy to show that the vanishing of (5.9) for all V implies the existence of a *parallel* vector field Z of class H^1 along w, i.e., $\nabla_{\dot w} Z = 0$, such that:

(5.10)

$$-\frac{Y}{\langle Y,Y\rangle} - \int_0^t {}^{w} \frac{\nabla_{\dot w} Y}{\langle Y, Y\rangle} - \frac{k}{2C_w}\left(\int_0^t {}^{w} \langle \dot w, \dot w\rangle \nabla\Theta_k(w)\right) + \frac{k}{C_w}\Theta_k(w)\,\dot w = Z.$$

From (5.10), we obtain immediately that $\Theta_k(w)\cdot \dot w$ is a continuous vector field along w; moreover, repeating the argument, since $\Theta_k(w) \neq 0$, we prove that $\dot w$ is of class C^1.

A straightforward integration by parts of (5.9) and a repeated application of the Fundamental Theorem of Calculus of Variations shows that, if w is a critical point of G_k, then it satisfies the differential equation:

$$\nabla_{\dot w}\left(\frac{Y}{\langle Y,Y\rangle}\right) + \frac{\nabla_{\dot w} Y}{\langle Y,Y\rangle} + \frac{k}{C_w}\left(\frac{1}{2}\langle \dot w,\dot w\rangle\nabla\Theta_k(w) - \nabla_{\dot w}\left(\Theta_k(w)\,\dot w\right)\right) = 0,$$

which is:

(5.11)

$$\frac{2\,\nabla_{\dot w} Y}{\langle Y,Y\rangle} - \frac{2\,\langle \nabla_{\dot w} Y, Y\rangle\cdot Y}{\langle Y,Y\rangle^2} + \\ + \frac{k}{C_w}\left(\frac{1}{2}\langle \dot w,\dot w\rangle\nabla\Theta_k(w) - \langle \nabla\Theta_k(w), \dot w\rangle\,\dot w - \Theta_k(w)\,\nabla_{\dot w}\dot w\right) = 0.$$

Since $\langle \nabla_{\dot w} Y, \dot w\rangle = \langle Y, \dot w\rangle \equiv 0$, multiplying (5.11) by $\dot w$ yields:

(5.12)

$$-\frac{1}{2}\langle \nabla\Theta_k(w), \dot w\rangle\langle \dot w,\dot w\rangle - \Theta_k(w)\,\langle \nabla_{\dot w}\dot w, \dot w\rangle = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\Theta_k(w)\langle \dot w,\dot w\rangle\right) = 0.$$

Formula (5.4) follows immediately from (5.12), and then (5.5) follows from (5.11), which concludes the first part of the proof.

Conversely, assume that $w \in \Omega^{(1)}_{p,\gamma}(\Delta)$ is of class C^2 and it satisfies (5.4) and (5.5). Then, w satisfies (5.11) and, arguing backwards, it follows that $\mathrm{d}G_k(w)[\zeta] = 0$ for all $\zeta \in T_w\Omega^{(1)}_{p,\gamma}(\Delta)$ of the form

$$\zeta = V - \mu \cdot Y, \tag{5.13}$$

with V a vector field of class H^1 along w such that $V(0) = V(1) = 0$. The conclusion follows easily from the fact that every vector $\zeta \in T_w\Omega^{(1)}_{p,\gamma}(\Delta)$ can be written in the form (5.13). Namely, if $\zeta \in T_w\Omega^{(1)}_{p,\gamma}(\Delta)$ is given and μ is any H^1-function such that $\mu(0) = 0$ and $\mu(1) = \big\langle \zeta(1), Y(w(1))\big\rangle \cdot \big\langle Y(w(1)), Y(w(1))\big\rangle^{-1}$, then $V = \zeta - \mu \cdot Y$ is the desired vector field along w that vanishes at the endpoints. ☐

Remark 5.3. For the study of the arrival time brachistochrones, we need a way to *pass* from curves w satisfying:

$$\phi_k(w)\langle \dot w, \dot w\rangle = E_w > 0 \text{ (constant)} \tag{5.14}$$

to curves $\tilde w$ satisfying the condition:

$$\Theta_k(\tilde w)\langle \dot{\tilde w}, \dot{\tilde w}\rangle = \tilde E_{\tilde w} > 0 \text{ (constant)}. \tag{5.15}$$

This can be done by taking a reparameterization of w of the form:

$$\tilde w(t) = w(\lambda(t)), \tag{5.16}$$

where $\lambda(0) = 0$, $\lambda(1) = 1$ and:

$$\tilde E_{\tilde w} = \Theta_k(\tilde w)\langle \dot{\tilde w}, \dot{\tilde w}\rangle = \frac{\phi_k(w(\lambda))}{\big\langle Y(w(\lambda)), Y(w(\lambda))\big\rangle}(\lambda')^2\langle \dot w(\lambda), \dot w(\lambda)\rangle =$$
$$= \frac{E_w}{\big\langle Y(w(\lambda)), Y(w(\lambda))\big\rangle}(\lambda')^2.$$

Then, λ must satisfy the Cauchy problem:

$$\lambda' = -\frac{\tilde E_{\tilde w}}{E_w}\langle Y(w(\lambda)), Y(w(\lambda))\rangle, \quad \lambda(0) = 0,$$

where $\tilde E_{\tilde w}$ has to be chosen in such a way that $\lambda(1) = 1$. Hence, we get:

$$\tilde E_{\tilde w} = -E_w \frac{\lambda'}{\big\langle Y(w(\lambda)), Y(w(\lambda))\big\rangle} = -E_w \int_0^1 \frac{\lambda'}{\big\langle Y(w(\lambda)), Y(w(\lambda))\big\rangle}\,\mathrm{d}t =$$
$$= -E_w \int_0^1 \frac{\mathrm{d}r}{\big\langle Y(w(r)), Y(w(r))\big\rangle}.$$

Thus, in order to (5.15) be satisfied, the function $\lambda = \lambda_w$ needs to satisfy the Cauchy problem:

$$\lambda' = \left(\int_0^1 \frac{dr}{\langle Y(w(r)), Y(w(r)) \rangle} \right) \langle Y(w(\lambda)), Y(w(\lambda)) \rangle, \quad \lambda(0) = 0. \tag{5.17}$$

Observe that the unique solution of (5.17) satisfies $\lambda(1) = 1$ and $\lambda' > 0$ on $[0,1]$; notice also that the map from $\mathcal{B}^{(1)}_{p,\gamma}(k)$ to $\{w \in \Omega^{(1)}_{p,\gamma}(\Delta) : w \text{ satisfies } (5.15)\}$ sending σ to $\widetilde{\mathcal{D}(\sigma)}$ is bijective.

The following theorem relates the brachistochrone curves to the critical points of G_k. Let's consider the map $\mathcal{D}$ defined by (4.2), and, for all $w \in \Omega^{(1)}_{p,\gamma}$ satisfying (5.14), let $\tilde{w}$ be given by (5.16), where $\lambda = \lambda_w$ satisfies (5.17).

Theorem 5.4. *Let $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ be fixed. The following statements are equivalent:*

(1) *σ is a localized minimizer for τ on $\mathcal{B}^{(1)}_{p,\gamma}(k)$;*
(2) *$\mathcal{D}(\sigma)$ is a localized minimizer for τ_k on $\Omega^{(1)}_{p,\gamma}(\Delta)$;*
(3) *$\widetilde{\mathcal{D}(\sigma)}$ is a localized minimizer for G_k on $\Omega^{(1)}_{p,\gamma}(\Delta)$;*
(4) *$\widetilde{\mathcal{D}(\sigma)}$ is a critical point for G_k on $\Omega^{(1)}_{p,\gamma}(\Delta)$;*
(5) *σ is an arrival time brachistochrone of energy k (namely, a C^2-curve joining p and γ and satisfying* (3.1) *and* (3.2)*).*

Moreover, in the above situation, it is $\tau(\sigma) = \tau_k(\mathcal{D}(\sigma)) = G_k(\widetilde{\mathcal{D}(\sigma)})$.

To prove Theorem 5.4 some preliminary results are given.

Lemma 5.5. *If w_0 is a localized minimizer for τ_k on $\Omega^{(1)}_{p,\gamma}(\Delta)$, then there exists $\hat{w} \in \Omega^{(1)}_{p,\gamma}(\Delta)$ localized minimizer for τ_k satisfying* (5.14) *(and $\tau_k(\hat{w}) = \tau(w_0)$).*

Proof. We prove first that, if p and γ are sufficiently close, with $p \notin \gamma(\mathbb{R})$, $I = [a,b] \subset [0,1]$ is sufficiently small and w_0 is a minimizer for τ_k on $\Omega^{(1)}_{p,\gamma}(\Delta, I)$, then there exists $\hat{w} \in \Omega^{(1)}_{p,\gamma}(\Delta, I)$ satisfying (5.14) and such that $\tau(\hat{w}) = \tau_k(w_0)$. Towards this goal, we can use an open and relatively compact neighborhood $U = V \times]\alpha, \beta[$ of $w_0([a,b])$, as in the proof of Proposition 4.9, where the metric g has the form $g[(\Xi,\Theta),(\Xi,\Theta)] = \langle \Xi,\Xi \rangle + 2\langle \delta(\mathrm{x}^i),\Xi \rangle \Theta - \beta(\mathrm{x}^i)\Theta^2$, for some smooth vector field δ on the closure $\overline{V}$ of V and β is a smooth positive scalar field on $\overline{V}$ such that $k^2 - \beta(\mathrm{x}) > 0$ in $\overline{V}$. For each $z \in U$, we write $z = (\mathrm{x},\theta)$, where $\mathrm{x} \in V$ and $\theta \in]\alpha,\beta[$. The horizontality of a curve $z(t) = (\mathrm{x}(t),\theta(t))$ ($\langle \dot{z}, Y \rangle = 0$) is written as in (4.16).

If we set $w_0 = (\mathrm{x}_0, \theta_0)$ and (cf. (4.15) and (4.16))

$$\tau_k(\mathrm{x}) = \int_a^b \left[\frac{\langle \delta(\mathrm{x}), \dot{\mathrm{x}} \rangle}{\beta(\mathrm{x})} + \frac{k}{\beta(\mathrm{x})} \sqrt{\frac{\beta(\mathrm{x})}{k^2 - \beta(\mathrm{x})} \langle \dot{\mathrm{x}}, \dot{\mathrm{x}} \rangle + \frac{\langle \delta(\mathrm{x}), \dot{\mathrm{x}} \rangle^2}{k^2 - \beta(\mathrm{x})}} \right] dt,$$

we only need to prove that, if x_0 is a minimizer for τ_k on the manifold:

$$\Omega^1(V;x_a,x_b)=\left\{x\in H^1([a,b],V):x(a)=x_a,\ x(b)=x_b\right\}$$

where x_a and x_b are given fixed points in V, then there exists $\hat{x}\in\Omega^1(V;x_a,x_b)$ and $\hat{c}>0$ such that

$$\frac{1}{k^2-\beta(\hat{x})}\left(\beta(\hat{x})\langle\dot{\hat{x}},\dot{\hat{x}}\rangle+\langle\delta(\hat{x}),\dot{\hat{x}}\rangle^2\right)=\hat{c}\text{ a.e., and }\tau_k(\hat{x})=\tau_k(x_0). \tag{5.18}$$

Denote by $\langle\cdot,\cdot\rangle_1$ the Riemannian metric in V given by:

$$\langle\Xi,\Xi\rangle_1=\frac{1}{k^2-\beta}\left(\beta\langle\Xi,\Xi\rangle+\langle\delta,\Xi\rangle^2\right),$$

and, for each $n\in\mathbb{N}$, let $\lambda_n:[a,b]\mapsto\mathbb{R}^+$ be the unique solution of the following Cauchy problem:

$$\begin{cases}\lambda_n'=\dfrac{1}{b-a}\left(\displaystyle\int_a^b\sqrt{\langle\dot{x}_0,\dot{x}_0\rangle_1+\frac{1}{n}}\,dt\right)\dfrac{1}{\sqrt{\langle\dot{x}_0(\lambda_n),\dot{x}_0(\lambda_n)\rangle_1+\frac{1}{n}}}\\ \lambda_n(a)=0.\end{cases}$$

Observe that λ_n is strictly increasing on $[a,b]$ and $\lambda_n(b)\equiv 1$.

We set $y_n=x_0(\lambda_n)$. Since:

$$\langle\dot{y}_n,\dot{y}_n\rangle_1=\frac{1}{(b-a)^2}\left(\int_a^b\sqrt{\langle\dot{x}_0,\dot{x}_0\rangle_1+\frac{1}{n}}\,dt\right)^2\frac{\langle\dot{x}_0(\lambda_n),\dot{x}_0(\lambda_n)\rangle_1}{\langle\dot{x}_0(\lambda_n),\dot{x}_0(\lambda_n)\rangle_1+\frac{1}{n}}, \tag{5.19}$$

the sequence y_n is bounded in H^1, and so there exists $y\in\Omega^1(V;x_a,x_b)$ such that

$$y_n\longmapsto y\ \text{ uniformly in }\ [a,b], \tag{5.20}$$

$$\int_a^b\langle\dot{y}_n,\varphi\rangle\,dt\longmapsto\int_a^b\langle\dot{y},\varphi\rangle\,dt,\quad\forall\varphi\in L^1([a,b],\mathbb{R}^{m-1}) \tag{5.21}$$

(recall that $m=\dim(\mathcal{M})$).

Note that, by the uniform convergence, we also have:

$$\int_a^b\langle\dot{y}_n,\varphi\rangle_1\,dt\longmapsto\int_a^b\langle\dot{y},\varphi\rangle_1\,dt,\quad\forall\varphi\in L^1([a,b],\mathbb{R}^{m-1}),$$

namely, $\dot{y}_n$ is weakly convergent to $\dot{y}$ in L^2 also with respect to the metric $\langle\cdot,\cdot\rangle_1$.

In particular, the above property is satisfied for all $\varphi\in L^\infty([a,b],\mathbb{R}^{m-1})$, hence (see [2]), we obtain:

$$\int_\alpha^\beta\sqrt{\langle\dot{y},\dot{y}\rangle_1}\,dt\le\liminf_{n\to\infty}\int_\alpha^b\sqrt{\langle\dot{y}_n,\dot{y}_n\rangle_1}\,dt,\quad\forall[\alpha,\beta]\subset[a,b]. \tag{5.22}$$

Since $k\beta(\mathbf{y}_n)^{-1} \mapsto k\beta(\mathbf{y})^{-1}$ uniformly and

$$\int_a^b \frac{\langle\delta(\mathbf{y}_n),\dot{\mathbf{y}}_n\rangle}{\beta(\mathbf{y}_n)}\,dt \longmapsto \int_a^b \frac{\langle\delta(\mathbf{y}),\dot{\mathbf{y}}\rangle}{\beta(\mathbf{y})}\,dt,$$

we get:

$$\tau_k(\mathbf{y}) \le \liminf_{n\to\infty} \tau_k(\mathbf{y}_n).$$

But τ_k is invariant by reparameterizations, therefore $\tau_k(\mathbf{y}_n) = \tau_k(\mathbf{x}_0)$. Since $\mathbf{x}_0$ is a minimizer, it must be $\tau_k(\mathbf{y}) = \tau_k(\mathbf{x}_0)$.

Now, by (5.19), we have:

(5.23)

$$\sqrt{\langle\dot{\mathbf{y}}_n(t),\dot{\mathbf{y}}_n(t)\rangle_1} \le \frac{1}{b-a}\int_a^b \sqrt{\langle\dot{\mathbf{x}}_0,\dot{\mathbf{x}}_0\rangle_1 + \frac{1}{n}}\,dt \text{ a.e. on } [a,b],\ \forall n \in \mathbb{N}.$$

Combining (5.22) and (5.23), a simple contradiction argument shows that

$$(\langle\dot{\mathbf{y}},\dot{\mathbf{y}}\rangle_1)^{\frac{1}{2}} \le \frac{1}{b-a}\int_a^b \sqrt{\langle\dot{\mathbf{x}}_0,\dot{\mathbf{x}}_0\rangle_1}\,dt \text{ a.e. on } [a,b]. \tag{5.24}$$

If it were

$$\sqrt{\langle\dot{\mathbf{y}},\dot{\mathbf{y}}\rangle_1} < \frac{1}{b-a}\int_a^b \sqrt{\langle\dot{\mathbf{x}}_0,\dot{\mathbf{x}}_0\rangle_1}\,dt = \frac{1}{b-a}\int_a^b \sqrt{\langle\dot{\mathbf{y}}_n,\dot{\mathbf{y}}_n\rangle_1}\,dt$$

on a set of positive Lebesgue measure (recall that $\mathbf{y}_n = \mathbf{x}_0(\lambda_n)$), then we would have:

$$\int_a^b \sqrt{\langle\dot{\mathbf{y}},\dot{\mathbf{y}}\rangle_1}\,dt < \liminf_{n\to\infty}\int_a^b \sqrt{\langle\dot{\mathbf{y}}_n,\dot{\mathbf{y}}_n\rangle_1}\,dt,$$

and therefore it should be:

$$\tau_k(\mathbf{y}) < \liminf_{n\to\infty} \tau_k(\mathbf{y}_n) = \tau_k(\mathbf{x}_0), \tag{5.25}$$

because $\int_a^b \langle\delta(\mathbf{y}_n),\dot{\mathbf{y}}_n\rangle\beta(\mathbf{y}_n)^{-1}\,dt \mapsto \int_a^b \langle\delta(\mathbf{y}),\dot{\mathbf{y}}\rangle_1\,dt$ and $k\beta(\mathbf{y}_n)^{-1} \mapsto k\beta(\mathbf{y})$ uniformly.

But $\mathbf{x}_0$ is a minimizer for τ_k, hence (5.25) is impossible, which implies:

$$\langle\dot{\mathbf{y}},\dot{\mathbf{y}}\rangle_1 = \left(\frac{1}{b-a}\int_a^b \sqrt{\langle\dot{\mathbf{x}}_0,\dot{\mathbf{x}}_0\rangle_1}\,dt\right)^2, \quad \text{a.e. on } [a,b],$$

and (5.18) is obtained by taking $\hat{\mathbf{x}} = \mathbf{y}$ and $\hat{c} = \left(\frac{1}{b-a}\int_a^b \sqrt{\langle\dot{\mathbf{x}}_0,\dot{\mathbf{x}}_0\rangle_1}\,dt\right)^2$.

Now, the above argument can be repeated on a finite covering of the interval $[0,1]$ consisting of closed intervals $[a_i,b_i]$, $i = 1,\ldots,r$, (whose interiors cover $[0,1]$); in this way, since τ_k is invariant by reparameterizations, we obtain a curve $w \in$

$\Omega^{(1)}_{p,\gamma}(\Delta)$ which is a *localized* minimizer for τ_k, a finite sequence of constants $c_i \geq 0$, $i = 1, \ldots, r$, such that:

$$\phi_k(w)\langle \dot{w}, \dot{w} \rangle \equiv c_i, \quad \text{a.e. on } [a_i, b_i], \quad i = 1, \ldots, r.$$

Moreover, $c_i = 0$ if and only if w is constant on $[a_i, b_i]$; since $p \notin \gamma(\mathbb{R})$ the curve w_0 is not constant, which implies that at least one of the c_i's is strictly positive. Hence, possibly by changing the intervals $[a_i, b_i]$, we can assume that $c_i > 0$ for all $i = 1, \ldots, r$. Therefore, we obtain a localized minimizer for τ_k and a positive constant c, defined as the minimum of the c_i's, such that:

$$\phi_k(w)\langle \dot{w}, \dot{w} \rangle \geq c > 0, \quad \text{a.e. on } [0, 1]. \tag{5.26}$$

Finally, we consider the solution λ of the following Cauchy problem:

$$\begin{cases} \lambda' = \left(\int_0^1 \sqrt{\phi_k(w)\langle \dot{w}, \dot{w} \rangle} \, \mathrm{d}t \right) \dfrac{1}{\phi_k(w)\langle \dot{w}, \dot{w} \rangle}, \\ \lambda(0) = 0, \end{cases} \tag{5.27}$$

which is well defined by (5.26). The desired curve $\hat{w}$ is then given by $w(\lambda)$; observe indeed that $\hat{w}$ is a localized minimizer for τ_k, because τ_k is invariant by reparameterizations, and

$$\phi_k(\hat{w})\langle \dot{\hat{w}}, \dot{\hat{w}} \rangle = \left(\int_0^1 \sqrt{\phi_k(w)\langle \dot{w}, \dot{w} \rangle} \, \mathrm{d}t \right)^2,$$

by (5.27). This concludes the proof. □

We now consider the function $\mu : [0, 1] \mapsto [0, 1]$ given by the inverse of the function λ defined by (5.17). The function μ can be obtained as the unique solution of the Cauchy problem:

$$\begin{cases} \mu' = \frac{1}{\langle Y(w(\mu)), Y(w(\mu)) \rangle} \left(\int_0^1 \langle Y(w), Y(w) \rangle \, \mathrm{d}t \right)^{-1} \\ \mu(0) = 0. \end{cases} \tag{5.28}$$

We have the following:

Lemma 5.6. *Let $w \in \Omega^{(1)}_{p,\gamma}(\Delta)$ be a curve satisfying (5.4) and (5.5), and μ be the solution of (5.28). Then, the curve $u = w \circ \mu$ satisfies the differential equation:*

$$\begin{aligned} & 2\nabla_{\dot{u}} Y - 2 \frac{\langle Y, \nabla_{\dot{u}} Y \rangle}{\langle Y, Y \rangle} Y + \\ & \quad + \frac{k}{\alpha_u} \left[\phi_k(u) \nabla_{\dot{u}} \dot{u} - 2\phi_k(u) \frac{\langle Y, \nabla_{\dot{u}} Y \rangle}{\langle Y, Y \rangle} \dot{u} - 2 \frac{\alpha_u^2}{\langle Y, Y \rangle} \nabla_Y Y \right] + \\ & \quad + \frac{k}{\alpha_u} \left[\langle \nabla \phi_k(u), \dot{u} \rangle \dot{u} - \frac{1}{2} \langle \dot{u}, \dot{u} \rangle \nabla \phi_k(u) \right] = 0, \end{aligned} \tag{5.29}$$

where α_u is the positive constant $\sqrt{\phi_k(u)\langle\dot u,\dot u\rangle}$.

Proof. If w satisfies (5.4) and (5.5), and $u = w\circ\mu$, then u satisfies:

(5.30)

$$-\frac{2}{\mu'\langle Y,Y\rangle}\nabla_{\dot u}Y+\frac{2}{\mu'\langle Y,Y\rangle^2}\langle Y,\nabla_{\dot u}Y\rangle Y+ \\ +\frac{k}{(\mu')^2C_w}\Big[\Theta_k(u)\left(\nabla_{\dot u}\dot u-\frac{\mu''}{\mu'}\dot u\right)+\langle\nabla\Theta_k(u),\dot u\rangle\,\dot u-\frac12\langle\dot u,\dot u\rangle\nabla\Theta_k(u)\Big]=0.$$

Now, $\dot u(t)=\dot w(\mu(t))\mu'(t)$, and

$$C_w=\sqrt{\Theta_k(w)\langle\dot w,\dot w\rangle}=\sqrt{\frac{\phi_k(u)\langle\dot u,\dot u\rangle}{(\mu')^2\langle Y,Y\rangle^2}}= \\ =-\frac{1}{\mu'\langle Y,Y\rangle}\sqrt{\phi_k(u)\langle\dot u,\dot u\rangle}=-\frac{\alpha_u}{\mu'\langle Y,Y\rangle}. \tag{5.31}$$

Observe that, by (5.28), α_u is a real constant. Moreover, by (5.31), setting $d_u = -\left(\int_0^1\langle Y(u(t)),Y(u(t))\rangle\,\mathrm{d}t\right)^{-1}$, we get

$$\mu'=-\frac{d_u}{\langle Y,Y\rangle},\quad\text{and}\quad\mu''=\frac{2d_u\langle Y,\nabla_{\dot u}Y\rangle}{\langle Y,Y\rangle^2}.$$

Therefore, after multiplying (5.30) by μ', using (5.31) to evaluate C_w, we get:

(5.32)

$$-\frac{2}{\langle Y,Y\rangle}\nabla_{\dot u}Y+\frac{2\,\langle Y,\nabla_{\dot u}Y\rangle}{\langle Y,Y\rangle^2}\,Y-\frac{k\langle Y,Y\rangle\Theta_k(u)}{\alpha_u}\Big[\nabla_{\dot u}\dot u+\frac{2}{\langle Y,Y\rangle}\langle Y,\nabla_{\dot u}Y\rangle\,\dot u\Big] \\ -\frac{k\langle Y,Y\rangle}{\alpha_u}\Big[\langle\nabla\Theta_k(u),\dot u\rangle-\frac12\langle\dot u,\dot u\rangle\nabla\Theta_k(u)\Big]=0.$$

Since $\langle Y,\nabla_\zeta Y\rangle=-\langle\zeta,\nabla_Y Y\rangle$, we have:

$$\langle\nabla\Theta_k(u),\zeta\rangle=\frac{\langle\nabla\phi_k(u),\zeta\rangle}{\langle Y,Y\rangle^2}-\frac{4\phi_k(u)}{\langle Y,Y\rangle^3}\langle Y,\nabla_\zeta Y\rangle= \\ =\langle\frac{\langle Y,Y\rangle\nabla\phi_k(u)+4\phi_k(u)\nabla_Y Y}{\langle Y,Y\rangle^3},\zeta\rangle,$$

and so:

$$\nabla\Theta_k(u)=\frac{\langle Y,Y\rangle\nabla\phi_k(u)+4\phi_k(u)\nabla_Y Y}{\langle Y,Y\rangle^3}. \tag{5.33}$$

Finally, substituting (5.33) in (5.32), and recalling that $\langle \nabla_Y Y, \dot{u} \rangle = -\langle \nabla_{\dot{u}} Y, Y \rangle$ and that $\phi_k(u)\langle \dot{u}, \dot{u} \rangle = \alpha_u^2$, we obtain (5.29). ☐

Remark 5.7. If $u =$ satisfies (5.29), by (4.8) we have:

$$\alpha_u = \sqrt{\phi_k(u)\langle \dot{u}, \dot{u} \rangle} = \mathcal{T}_\sigma,$$

where σ is the unique curve in $\mathcal{B}^{(1)}_{p,\gamma}(k)$ such that $u = \mathcal{D}(\sigma)$ and $\mathcal{T}_\sigma$ is its travel time.

Remark 5.8. From (4.1) we compute easily:

$$\langle \nabla \phi_k(u), \zeta \rangle = -\frac{2k^2}{(k^2 + \langle Y, Y \rangle)^2} \langle Y, \nabla_\zeta Y \rangle = \frac{2k^2}{(k^2 + \langle Y, Y \rangle)^2} \langle \zeta, \nabla_Y Y \rangle,$$

hence:

$$\nabla \phi_k(u) = \frac{2k^2}{(k^2 + \langle Y, Y \rangle)^2} \nabla_Y Y.$$

Then, since $\langle Y, \nabla_{\dot{u}} Y \rangle = -\langle \nabla_Y Y, \dot{u} \rangle$, if u satisfies (5.29) a straightforward computation (see also Remark 5.7) shows that the following differential equation is satisfied:

(5.34)

$$2\nabla_{\dot{u}} Y - 2\frac{\langle Y, \nabla_{\dot{u}} Y \rangle}{\langle Y, Y \rangle} Y + \frac{k}{\mathcal{T}_\sigma} \left[\phi_k(u) \nabla_{\dot{u}} \dot{u} + \frac{2\langle Y, Y \rangle + k^2}{(k^2 + \langle Y, Y \rangle)^2} \langle \dot{u}, \dot{u} \rangle \nabla_Y Y \right] + \\ - \frac{2k}{\mathcal{T}_\sigma} \frac{\langle Y, Y \rangle}{(k^2 + \langle Y, Y \rangle)^2} \langle \nabla_Y Y, \dot{u} \rangle \, \dot{u} = 0.$$

Now we are ready to prove the following variational principle:

Proposition 5.9. *Let $k > 0$ be fixed. Then, a curve $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ is an arrival time brachistochrone if and only if the curve $\widetilde{\mathcal{D}(\sigma)}$ is a critical point of the functional G_k in $\Omega^{(1)}_{p,\gamma}(\Delta)$.*

Proof. Let $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ be fixed, define $w = \mathcal{D}(\sigma)$ and $\tilde{w} = \widetilde{\mathcal{D}(\sigma)}$; since the critical points of G_k are curves of class C^2, we have that σ, w and $\tilde{w}$ are curves of class C^2. We consider the map $F : [0,1] \times I\!R \longmapsto \mathcal{M}$ given by:

$$F(t,s) = \psi(\sigma(t), s),$$

where ψ is the flow of Y. Denoting by $T(t,s)$ the vector field along F given by:

$$T = \frac{\partial F}{\partial t},$$

since $Y = \frac{\partial F}{\partial s}$, a standard argument in calculus of connections (see for instance [24, Proposition 6.9]) shows that:

$$\nabla_Y T - \nabla_T Y = 0. \tag{5.35}$$

By (4.2), $w(t) = F(t, \mathbf{r}_\sigma(t))$, and so:

$$\dot{w} = T(w) + \dot{\mathbf{r}}_\sigma Y(w);$$

thus, using (5.35), we compute:

$$\begin{aligned} \nabla_{\dot{w}}\dot{w} &= \nabla_{\dot{w}}(T + \dot{\mathbf{r}}_\sigma Y) = \nabla_{\dot{w}}T + \ddot{\mathbf{r}}_\sigma Y + \dot{\mathbf{r}}_\sigma \nabla_{\dot{w}}Y = \\ &= \nabla_{T+\dot{\mathbf{r}}_\sigma Y}T + \ddot{\mathbf{r}}_\sigma Y + \dot{\mathbf{r}}_\sigma \nabla_{T+\dot{\mathbf{r}}_\sigma Y}Y = \\ &= \nabla_T T + 2\dot{\mathbf{r}}_\sigma \nabla_T Y + \dot{\mathbf{r}}_\sigma^2 \nabla_Y Y + \ddot{\mathbf{r}}_\sigma Y. \end{aligned} \tag{5.36}$$

It is:

$$T(t,s) = \mathrm{d}_x\psi(\sigma(t),s)[\dot{\sigma}(t)], \quad \text{and} \quad Y(w(t)) = \mathrm{d}_x\psi(\sigma(t),s)[Y(\sigma(t))]; \tag{5.37}$$

and since $\langle Y, Y\rangle$ is constant along each flow line of Y, for all t it is $\phi_k(w(t)) = \phi_k(\sigma(t))$.

Considering that $\mathrm{d}_x\psi$ is an isometry, for all pair of smooth vector fields v_1 and v_2 in $\mathcal{M}$ we have:

$$\nabla_{\mathrm{d}_x\psi[v_1]}\left(\mathrm{d}_x\psi[v_2]\right) = \mathrm{d}_x\psi\left[\nabla_{v_1}v_2\right]. \tag{5.38}$$

Putting together (5.36), (5.37) and using (5.38), we get:

$$\nabla_{\dot{w}}\dot{w} = \mathrm{d}_x\psi\Big[\nabla_{\dot{\sigma}}\dot{\sigma} + 2\dot{\mathbf{r}}_\sigma\nabla_{\dot{\sigma}}Y(\sigma) + \dot{\mathbf{r}}_\sigma^2\nabla_{Y(\sigma)}Y(\sigma) + \ddot{\mathbf{r}}_\sigma Y(\sigma)\Big]. \tag{5.39}$$

Now, by (4.8), we have:

$$\phi_k(w)\langle\dot{w},\dot{w}\rangle = {T_\sigma}^2; \tag{5.40}$$

moreover, the following are easily computed:

$$\nabla_{Y(w)}Y(w) = \mathrm{d}_x\psi\big[\nabla_{Y(\sigma)}Y(\sigma)\big]; \tag{5.41}$$

$$\dot{w} = \mathrm{d}_x\psi[\dot{\sigma}] + \dot{\mathbf{r}}_\sigma Y(w) = \mathrm{d}_x\psi\big[\dot{\sigma} + \dot{\mathbf{r}}_\sigma Y(\sigma)\big]; \tag{5.42}$$

$$\nabla_{\dot{w}}Y = \nabla_{\mathrm{d}_x\psi[\dot{\sigma}+\dot{\mathbf{r}}_\sigma Y]}\Big(\mathrm{d}_x\psi[Y(\sigma)]\Big) = \mathrm{d}_x\psi\big[\nabla_{\dot{\sigma}}Y(\sigma) + \dot{\mathbf{r}}_\sigma\nabla_Y Y\big]; \tag{5.43}$$

$$\dot{\mathbf{r}}_\sigma = \frac{k\,T_\sigma}{\langle Y,Y\rangle}, \quad \ddot{\mathbf{r}}_\sigma = -2kT_\sigma\frac{\langle\nabla_{\dot{\sigma}}Y, Y\rangle}{\langle Y,Y\rangle^2}. \tag{5.44}$$

Patiently substituting (5.39)—(5.44) into (5.34) (that gives the differential equation satisfied by $w = \mathcal{D}(\sigma)$) yields the following differential equation for σ:

(5.45)

$$\nabla_{\dot\sigma}\dot\sigma - \frac{2\mathcal{T}_\sigma}{k}\nabla_{\dot\sigma}Y + 2\frac{\langle\nabla_Y Y,\dot\sigma\rangle}{\langle Y,Y\rangle + k^2}\dot\sigma + \frac{2\mathcal{T}_\sigma}{k}\frac{\langle\nabla_{\dot\sigma}Y,Y\rangle}{\langle Y,Y\rangle + k^2}Y - \mathcal{T}_\sigma{}^2\frac{2\langle Y,Y\rangle + k^2}{\langle Y,Y\rangle^2}\nabla_Y Y - \frac{\mathcal{T}_\sigma{}^2\,(k^2 + 2\langle Y,Y\rangle)}{\langle Y,Y\rangle(k^2 + \langle Y,Y\rangle)}\left(\frac{\langle\dot\sigma,\dot\sigma\rangle}{\mathcal{T}_\sigma{}^2} + \frac{k^2}{\langle Y,Y\rangle} + \frac{2k}{\mathcal{T}_\sigma\langle Y,Y\rangle}\langle\dot\sigma,Y\rangle\right)\nabla_Y Y = 0.$$

Finally, substituting $\langle\dot\sigma,\dot\sigma\rangle = -\mathcal{T}_\sigma{}^2$ and $\langle\dot\sigma,Y\rangle = -k\mathcal{T}_\sigma$ into (5.45) gives (3.1).

For the converse we observe that the above steps can be done backwards, to prove that if σ satisfies (3.1) and (3.2), then $\widetilde{\mathcal{D}(\sigma)}$ satisfies (5.4) and (5.5). This concludes the proof. □

We are finally ready to prove Theorem 5.4:

Proof of Theorem 5.4. The equivalence between the statements (1) and (2) is an immediate consequence of Lemma 5.5 and the construction of the map $\mathcal{D}$. Observe that, by the regularity of G_k, a local minimizer for G_k is a critical point of G_k, and in particular it satisfies (5.4). Then, the equivalence between the statements (2) and (3) follows by Lemma 5.5, the invariance by reparameterization of τ_k, the construction of $\widetilde{\mathcal{D}(\sigma)}$ (see Remark 5.3), and the Hölder inequality applied to the second summand of τ_k in formula (4.5). Moreover, the construction of $\widetilde{\mathcal{D}(\sigma)}$, formula (5.4) and the invariance by reparameterization shows that $\tau_k(\mathcal{D}(\sigma)) = G_k(\widetilde{\mathcal{D}(\sigma)})$.

We have already pointed out that the statement (3) implies (4). For the converse, we use the Cauchy problem satisfied by the critical points of G_k (see (5.5)) to prove the local invertibility of the map $v \mapsto w(1)$ from $T_q\mathcal{M}$ to M, where w is the unique solution of (5.5) satisfying $w(0) = q$ and $\dot w(0) = v$. This fact allows to deduce, in analogy with the Riemannian geodesic problem, that a critical point of G_k must be a local minimizer.

Finally, the equivalence of the statements (4) and (5) is given by Proposition 5.9. Observe that the equality $\tau(\sigma) = \tau_k(\mathcal{D}(\sigma))$ is given by Lemma 4.3. □

6. The Penalized Functional $G_{k,\varepsilon}$ and the Palais–Smale Condition

The presence of the boundary ∂U_k implies that the functional G_k does not satisfy good compactness properties on $\Omega^{(1)}_{p,\gamma}(\Delta)$; moreover, for the same reason its sublevels fails to be complete subspaces of $\Omega^{(1)}_{p,\gamma}(\Delta)$.

To overcome this problem, we now introduce a *penalization* argument, which has been used systematically in the study of unidimensional variational problem in subsets with convex boundary.

Recalling the definition (4.1) of the function Ψ_k, for all $k > 0$ fixed we introduce a family $G_{k,\varepsilon}$ of functionals, depending on the parameter $\varepsilon \in [0,1]$, defined by:

$$G_{k,\varepsilon}(w) = G_k(w) + \varepsilon \int_0^1 \frac{\mathrm{d}t}{\Psi_k(w)^2}. \tag{6.1}$$

Observe that $G_{k,0} = G_k$.

Since G_k and Ψ_k are smooth, then, for all ε, $G_{k,\varepsilon}$ is a smooth functional on $\Omega^{(1)}_{p,\gamma}(\Delta)$, and its Gateaux derivative is easily computed as:

$$\mathrm{d}G_{k,\varepsilon}(w)[\zeta] = \mathrm{d}G_k(w)[\zeta] - 2\varepsilon \int_0^1 \frac{\langle \nabla\Psi_k(w), \zeta\rangle}{\Psi_k(w)^3}\,\mathrm{d}t. \tag{6.2}$$

For all $\varepsilon > 0$, the sublevel of the penalized functional $G_{k,\varepsilon}$ are complete subsets of $\Omega^{(1)}_{p,\gamma}(\Delta)$. In order to prove this, we use the following result, which is known as the *Gordon's Lemma*, whose proof can be found, for instance, in Reference [16].

We denote by $\mathrm{dist}(\cdot,\cdot)$ the distance function on $\mathcal{M}$ induced by the Riemannian metric g_{R} defined in (1.9).

Lemma 6.1. *Let $\{w_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $\Omega^{(1)}_{p,\gamma}(\Delta)$ such that:*

$$\sup_n \int_0^1 \frac{\mathrm{d}t}{\Psi_k(w_n)^2} < +\infty.$$

Then, w_n stays uniformly far *from ∂U_k, i.e., there exists $\rho > 0$ such that*

$$\mathrm{dist}(w_n(t), \partial U_k) \geq \rho$$

for all $t \in [0,1]$ and all $n \in \mathbb{N}$. □

From Lemma 6.1 we obtain the following:

Proposition 6.2. *For all $\varepsilon \in]0,1]$ and all $c \in \mathbb{R}$, the sublevel*

$$G^c_{k,\varepsilon} = \left\{ w \in \Omega^{(1)}_{p,\gamma}(\Delta) : G_{k,\varepsilon}(w) \leq c \right\}$$

is a possibly empty complete metric subspace of $\Omega^{(1)}_{p,\gamma}(\Delta)$.

Proof. Let $\{w_n\}_n$ be a Cauchy sequence in $G^c_{k,\varepsilon}$; since $\overline{U}_k = U_k \bigcup \partial U_k$ is complete, then w_n is convergent in $H^1([0,1], \overline{U}_k)$ to a curve w with image in $\overline{U}_k$. Since $G_{k,\varepsilon}(w_n)$ is bounded, then, by Lemma 6.1, w_n stays uniformly far from ∂U_k, so that w has image in U_k and w_n converges to w in $\Omega^{(1)}_{p,\gamma}(\Delta)$. Moreover, by the continuity of $G_{k,\varepsilon}$, it is $G_{k,\varepsilon}(w) \leq c$, and we are done. □

We recall the definition of the Palais–Smale condition for a C^1-functional on a Hilbert manifold:

Definition 6.3. Let (X, h) be a Hilbertian manifold and $L : X \longmapsto \mathbb{R}$ be a functional of class C^1 on X. For all $x \in X$, we denote by $\| \cdot \|$ the norm on the dual Hilbert space $(T_xX)^*$. A *Palais–Smale sequence* for L at the level $c \in \mathbb{R}$ is a sequence $\{x_n\}_n$ in X such that the following two conditions are satisfied:

$(\mathrm{PS1})_c \quad \lim_{n\to\infty} L(x_n) = c;$

$(\mathrm{PS2})_c \quad \lim_{n\to\infty} \|\mathrm{d}L(x_n)\| = 0.$

The functional L is said to satisfy the *Palais–Smale condition* at the level c if every Palais–Smale sequence at the level c for L are convergent in X.

Proposition 6.4. *For all $\varepsilon \in]0,1]$ and all $c \in \mathbb{R}$, the penalized $G_{k,\varepsilon}$ satisfies the Palais–Smale condition at the level c on $\Omega^{(1)}_{p,\gamma}(\Delta)$.*

Proof. Let $c \in \mathbb{R}$ and $\varepsilon \in]0,1]$ be fixed; let w_n be a Palais–Smale sequence for $G_{k,\varepsilon}$ at the level c in $\Omega^{(1)}_{p,\gamma}(\Delta)$, with respect to the Hilbertian structure (2.6).

Since $G_{k,\varepsilon}(w_n)$ is bounded with respect to n, we have the existence of a constant $D = D(c) > 0$ (see Remark 5.1) such that:

$$\int_0^1 \langle \dot{w}_n, \dot{w}_n \rangle_{(\mathrm{R})} \,\mathrm{d}t \leq D, \quad \forall n \in \mathbb{N}. \tag{6.3}$$

Hence, up to passing to subsequences, by Lemma 6.1 it follows that there exists a curve $w : [0,1] \mapsto U_k$, $w \in \Omega^{(1)}_{p,\gamma}(\Delta)$, such that:

$$w_n \longmapsto w \text{ uniformly} \quad \text{and} \quad \dot{w}_n \longmapsto \dot{w} \quad \textit{weakly} \text{ in } L^2([0,1], T\mathcal{M}). \tag{6.4}$$

To prove the proposition, we need to show that the convergence of $\dot{w}_n$ to $\dot{w}$ is *strong* in L^2.

Since w_n is a Palais–Smale sequence, for all $n \in \mathbb{N}$ there exists a vector field a_n along w_n such that:

$$\mathrm{d}G_{k,\varepsilon}(w_n)[\zeta] = \int_0^1 \langle a_n, \nabla^{(\mathrm{R})}_{\dot{w}_n} \zeta \rangle_{(\mathrm{R})} \,\mathrm{d}t, \quad \forall \zeta \in T_{w_n}\Omega^{(1)}_{p,\gamma}(\Delta), \tag{6.5}$$

and

$$a_n \longmapsto 0 \quad \text{in } L^2([0,1], T\mathcal{M}). \tag{6.6}$$

Now, we can write the Riemannian covariant derivative $\nabla^{(\mathrm{R})}$ in terms of the Lorentzian covariant derivative ∇ by a formula of the type:

$$\nabla^{(\mathrm{R})}_{\dot{w}_n} \zeta = \nabla_{\dot{w}_n} \zeta + \Gamma(w_n)[\dot{w}_n, \zeta],$$

where $\Gamma(w_n)[\cdot, \cdot]$ is bilinear.

Since w_n is bounded in L^∞ and $\overline{U}_k$ is complete with respect to g_{R}, by (6.3) and (6.6) for all $n \in I\!N$ we have the existence of a vector field b_n along w_n such that:

$$\int_0^1 \langle a_n, \nabla^{(\mathrm{R})}_{\dot w_n}\zeta\rangle_{(\mathrm{R})}\,\mathrm{d}t = \int_0^1 \Big(\langle a_n, \nabla_{\dot w_n}\zeta\rangle_{(\mathrm{R})} + \langle b_n, \zeta\rangle_{(\mathrm{R})}\Big)\,\mathrm{d}t,$$

and

$$b_n \longmapsto 0 \quad \text{in } L^1([0,1], T\mathcal{M}).$$

Then from (1.9) and (6.4), for all $n \in I\!N$ we have the existence of vector fields A_n and B_n along w_n such that:

$$\mathrm{d}G_{k,\varepsilon}(w_n)[\zeta] = \mathrm{d}G_k(w_n)[\zeta] - 2\varepsilon \int_0^1 \frac{\langle \nabla\Psi_k(w_n), \zeta\rangle}{\Psi_k(w_n)^3}\,\mathrm{d}t =$$

$$= \int_0^1 \left(\langle A_n, \nabla_{\dot w_n}\zeta\rangle + \langle B_n, \zeta\rangle\right)\,\mathrm{d}t,$$

with

$$A_n \longmapsto 0 \quad \text{in } L^2([0,1], T\mathcal{M}), \qquad B_n \longmapsto 0 \quad \text{in } L^1([0,1], T\mathcal{M}).$$

Let V be any vector field along w_n in $H^1([0,1], \mathcal{M})$ such that $V(0) = V(1) = 0$; we set:

$$\mu(t) = \int_0^t \frac{\langle \nabla_{\dot w_n} V, Y\rangle - \langle V, \nabla_{\dot w_n} Y\rangle}{\langle Y, Y\rangle}\,\mathrm{d}r. \tag{6.7}$$

Arguing as in the proof of Lemma 5.2, every vector field in $T_{w_n}\Omega^{(1)}_{p,\gamma}(\Delta)$ can be written in the form $V - \mu\cdot Y$, where V is as above. Hence, recalling that $\langle \nabla\Psi_k, Y\rangle \equiv 0$, we get (see (5.9)):

$$\begin{aligned}
&-\int_0^1 \frac{\langle \nabla_{\dot w_n} V, Y\rangle - \langle V, \nabla_{\dot w_n} Y\rangle}{\langle Y, Y\rangle}\,\mathrm{d}t +\\
&+\frac{k}{C_{w_n}}\int_0^1 \left[\frac{1}{2}\langle \dot w_n, \dot w_n\rangle\langle \nabla\Theta_k(w_n), V\rangle + \Theta_k(w_n)\langle \dot w_n, \nabla_{\dot w_n} V\rangle\right]\,\mathrm{d}t +\\
&-2\varepsilon\int_0^1 \frac{\langle \nabla\Psi_k(w_n), V\rangle}{\Psi_k(w_n)^3}\,\mathrm{d}t =\\
&= \int_0^1 \left[\langle A_n, \nabla_{\dot w_n} V - \mu' Y - \mu\nabla_{\dot w_n} Y\rangle + \langle B_n, V - \mu Y\rangle\right]\,\mathrm{d}t,
\end{aligned} \tag{6.8}$$

where $C_{w_n} = \left(\int_0^1 \Theta_k(w_n)\langle \dot w_n, \dot w_n\rangle\,\mathrm{d}t\right)^{\frac{1}{2}}$. Observe that, from (6.3) and (6.4) it follows that C_{w_n} is bounded.

Combining (6.7) and (6.8) and using Fubini's theorem, we obtain:

$$
\begin{aligned}
&\int_0^1 \left[(-C_{w_n}\langle Y, \nabla_{\dot w_n} Y\rangle + C_{w_n}\langle \nabla_{\dot w_n} Y, V\rangle)\langle Y, Y\rangle^{-1}\right]\,dt+\\
&+k\int_0^1 \left[\frac{1}{2}\langle \dot w_n, \dot w_n\rangle\langle \nabla\Theta_k(w_n), V\rangle + \langle \Theta_k(w_n)\,\dot w_n, \nabla_{\dot w_n} V\rangle\right]\,dt+\\
&-2\varepsilon\, C_{w_n}\int_0^1 \frac{\langle \nabla\Psi_k(w_n), V\rangle}{\Psi_k(w_n)^3}\,dt =\\
&=\int_0^1 \left[\langle \alpha_n, \nabla_{\dot w_n} V\rangle + \langle \beta_n, V\rangle\right]\,dt,
\end{aligned}
\tag{6.9}
$$

where $\alpha_n \mapsto 0$ in $L^2([0,1], T\mathcal{M})$ and $\beta_n \mapsto 0$ in $L^1([0,1], T\mathcal{M})$.

Integration by parts in (6.9) gives the existence of vector fields H_n and W_n along w_n such that $\nabla_{\dot w_n} W_n \equiv 0$ and such that $\nabla_{\dot w_n} H_n$ is bounded in $L^1([0,1], T\mathcal{M})$, and satisfying:

$$
\Theta_k(w_n)\,\dot w_n + H_n = W_n, \quad \forall n \in I\!N. \tag{6.10}
$$

Integrating (6.10) on $[0,1]$ we see that W_n is uniformly bounded, and therefore W_n is bounded in $H^1([0,1], T\mathcal{M})$. By (6.4), $\Theta_k(w_n)$ is uniformly bounded away from zero, and thus from (6.10) we see that $\nabla_{\dot w_n} \dot w_n$ is bounded in $L^1([0,1], T\mathcal{M})$. Then, up to subsequences, $\dot w_n$ is convergent in $L^2([0,1], T\mathcal{M})$ (see [2, Theorem VII.7]), which concludes the proof. □

We can now use standard techniques from Critical Point Theory to prove the existence of minima for the penalized functional $G_{k,\varepsilon}$:

Corollary 6.5. *For all $\varepsilon \in]0,1]$, the penalized functional $G_{k,\varepsilon}$ attains its minimum in $\Omega^{(1)}_{p,\gamma}(\Delta)$.*

Proof. Since G_k is bounded from below, then also $G_{k,\varepsilon}$ is bounded from below. The existence of a minimizer is a classical argument in Critical Point Theory. Let's fix $\varepsilon \in]0,1]$. Thanks to the Palais–Smale condition and the completeness of the sublevels of the functionals $G_{k,\varepsilon}$, if the infimum i_ε of $G_{k,\varepsilon}$ on $\Omega^{(1)}_{p,\gamma}(\Delta)$ weren't a critical value, then it would be possible to find a homotopy between the sublevels $G^{i-\eta}_{k,\varepsilon}$ and $G^{i+\eta}_{k,\varepsilon}$, where $\eta > 0$ is sufficiently small. This is clearly impossible, because, for every $\eta > 0$, $G^{i-\eta}_{k,\varepsilon} = \emptyset$ while $G^{i+\eta}_{k,\varepsilon} \neq \emptyset$. Hence, $G_{k,\varepsilon}$ attains its minimum on $\Omega^{(1)}_{p,\gamma}(\Delta)$. □

7. A Priori Estimates for the Penalized Functional

AND THE PROOF OF THEOREM 1.1

The goal of this section is to prove that, given a family w_ε of stationary points for the penalized functionals $G_{k,\varepsilon}$, with $\varepsilon \in]0,1]$, there exists a sequence $\varepsilon_n \downarrow 0$ such that the corresponding sequence w_{ε_n} tends to a curve w which is a critical point for G_k in $\Omega^{(1)}_{p,\gamma}(\Delta)$.

Since $\langle \nabla\Psi_k, Y\rangle \equiv 0$, using the Euler–Lagrange equation for the penalized functional $G_{k,\varepsilon}$ and the same technique of Lemma 5.2, it is easy to prove the following:

Proposition 7.1. *Let $\varepsilon \in]0,1]$ and $k > 0$ be fixed. If curve $w_\varepsilon \in \Omega^{(1)}_{p,\gamma}(\Delta)$ is a critical point for the functional $G_{k,\varepsilon}$ then it is a smooth curve that satisfies the differential equation:*

(7.1)
$$\frac{k}{C_{w_\varepsilon}}\left(\Theta_k(w_\varepsilon)\,\nabla_{\dot w_\varepsilon}\dot w_\varepsilon + \langle \nabla\Theta_k(w_\varepsilon), \dot w_\varepsilon\rangle\,\dot w_\varepsilon - \frac{1}{2}\langle \dot w_\varepsilon, \dot w_\varepsilon\rangle \nabla\Theta_k(w_\varepsilon)\right)+$$
$$-\frac{2}{\langle Y,Y\rangle}\nabla_{\dot w_\varepsilon} Y + 2\frac{\langle Y, \nabla_{\dot w_\varepsilon} Y\rangle}{\langle Y,Y\rangle^2}\,Y + \frac{2\varepsilon}{\Psi_k(w_\varepsilon)^3}\nabla\Psi_k(w_\varepsilon) = 0,$$

where $C_{w_\varepsilon} = \left(\int_0^1 \Theta_k(w_\varepsilon)\langle \dot w_\varepsilon, \dot w_\varepsilon\rangle\, \mathrm{d}t\right)^{\frac{1}{2}}$. □

Lemma 7.2. *Let $k > 0$ be fixed. Suppose that $\{w_\varepsilon\}_{\varepsilon\in]0,1]}$ is a family of critical points of $G_{k,\varepsilon}$ in $\Omega^{(1)}_{p,\gamma}(\Delta)$ and c is a positive constant such that:*

(7.2)
$$G_{k,\varepsilon}(w_\varepsilon) \le c < +\infty.$$

Then, there exists a positive constant $\delta = \delta(c) > 0$ such that $\Psi_k(w_\varepsilon(t)) \ge \delta$ for all $\varepsilon \in]0,1]$ and all $t \in [0,1]$.

Proof. Since $G_{k,\varepsilon} \le G_k$, from Remark 5.1 we deduce the existence of a positive constant $D = D(c)$ such that:

$$\int_0^1 \langle \dot w_\varepsilon, \dot w_\varepsilon\rangle_{(\mathrm{R})}\, \mathrm{d}t \le D(c),$$

for all $\varepsilon \in]0,1]$ (recall that w_ε is horizontal). The completeness of $\overline{U}_k$ imply then the existence of a compact subset K of $\overline{U}_k$ that contains the image of all the curves w_ε.

For all $\varepsilon \in]0,1]$, we consider the smooth function $\rho_\varepsilon(t) = \Psi_k(w_\varepsilon(t))$ on the interval $[0,1]$; let t_ε be a minimum point for ρ_ε.

By contradiction, assume that:

(7.3)
$$\liminf_{\varepsilon\to 0} \rho_\varepsilon(t_\varepsilon) = 0,$$

and since Ψ_k is bounded away from 0 on $\gamma(\mathbb{R})$ (recall (4.1) and the fact that γ is an integral curve of Y), for ε small enough it is $t_\varepsilon \in]0,1[$. Hence, we have:

$$\rho'_\varepsilon(t_\varepsilon) = \langle \nabla\Psi_k(w_\varepsilon(t_\varepsilon)), \dot{w}_\varepsilon(t_\varepsilon)\rangle = 0, \tag{7.4}$$

and

(7.5)
$$0 \le \rho''_\varepsilon(t_\varepsilon) = \langle H^{\Psi_k}(w_\varepsilon(t_\varepsilon))[\dot{w}_\varepsilon(t_\varepsilon)], \dot{w}_\varepsilon(t_\varepsilon)\rangle + \langle \nabla\Psi_k(w_\varepsilon(t_\varepsilon)), \nabla_{\dot{w}_\varepsilon}\dot{w}_\varepsilon(t_\varepsilon)\rangle,$$

where H^{Ψ_k} is the Hessian of Ψ_k. To ease the notation, in all the calculations that follow we will omit the argument t_ε.

Since $\Theta_k = -\left[\langle Y,Y\rangle(k^2+\langle Y,Y\rangle)\right]^{-1}$, the gradient of the functions Ψ_k and Θ_k are easily computed as:

$$\nabla\Psi_k = \nabla_Y Y, \quad \nabla\Theta_k = \frac{k^2 + 2\langle Y,Y\rangle}{(\langle Y,Y\rangle^2 + k^2\langle Y,Y\rangle)^2}\nabla_Y Y; \tag{7.6}$$

thus, by (7.4), we have:

(7.7)
$$\langle \nabla\Theta_k(w_\varepsilon), \dot{w}_\varepsilon\rangle = 0, \quad \langle \nabla_{\dot{w}_\varepsilon} Y(w_\varepsilon), Y(w_\varepsilon)\rangle = -\langle \nabla_{Y(w_\varepsilon)} Y(w_\varepsilon), \dot{w}_\varepsilon\rangle = 0.$$

Combining (7.1), (7.5) and (7.7) gives:

(7.8)
$$\begin{aligned}
0 \le \rho''_\varepsilon &= \\
&= \langle H^{\Psi_k}(w_\varepsilon)[\dot{w}_\varepsilon], \dot{w}_\varepsilon\rangle + \frac{2\,C_{w_\varepsilon}}{k\Theta_k(w_\varepsilon)\langle Y(w_\varepsilon), Y(w_\varepsilon)\rangle}\langle \nabla\Psi_k(w_\varepsilon), \nabla_{\dot{w}_\varepsilon} Y(w_\varepsilon)\rangle \\
&+ \frac{\langle \nabla\Psi_k(w_\varepsilon), \nabla\Theta_k(w_\varepsilon)\rangle}{2\,\Theta_k(w_\varepsilon)}\langle \dot{w}_\varepsilon, \dot{w}_\varepsilon\rangle - \frac{2\,\varepsilon\, C_{w_\varepsilon}}{k\,\Psi_k(w_\varepsilon)^3}\langle \nabla\Psi_k(w_\varepsilon), \nabla\Psi_k(w_\varepsilon)\rangle,
\end{aligned}$$

where $C_{w_\varepsilon} = \left(\int_0^1 \Theta_k(w_\varepsilon)\langle \dot{w}_\varepsilon, \dot{w}_\varepsilon\rangle\, \mathrm{d}t\right)^{\frac{1}{2}}$.

Now, by the compactness of K, there exist positive constants d_1 and d_2 such that the following inequalities are satisfied:

$$\left|\langle H^{\Psi_k}(w_\varepsilon)[\dot{w}_\varepsilon], \dot{w}_\varepsilon\rangle\right| \le d_1 \cdot \langle \dot{w}_\varepsilon, \dot{w}_\varepsilon\rangle_{(\mathrm{R})} = d_1 \cdot \langle \dot{w}_\varepsilon, \dot{w}_\varepsilon\rangle,$$

$$2\left|\langle \nabla\Psi_k(w_\varepsilon), \nabla_{\dot{w}_\varepsilon} Y(w_\varepsilon)\rangle\right| \le d_2 \cdot \sqrt{\langle \dot{w}_\varepsilon, \dot{w}_\varepsilon\rangle}.$$

Therefore, since $\Theta_k = (-\langle Y, Y\rangle \Psi_k)^{-1}$, (7.6) and (7.8) give:

(7.9)

$$0 \le d_1\langle \dot w_\varepsilon, \dot w_\varepsilon\rangle + d_2\langle \dot w_\varepsilon, \dot w_\varepsilon\rangle^{\frac{1}{2}} C_{w_\varepsilon}\Psi_k(w_\varepsilon) +$$

$$-\frac{\Psi_k(w_\varepsilon)\,\langle Y(w_\varepsilon), Y(w_\varepsilon)\rangle\,\langle \nabla\Psi_k(w_\varepsilon), \nabla\Psi_k(w_\varepsilon)\rangle\langle \dot w_\varepsilon, \dot w_\varepsilon\rangle\Big(k^2 + 2\langle Y(w_\varepsilon), Y(w_\varepsilon)\rangle\Big)}{2\Big(\langle Y(w_\varepsilon), Y(w_\varepsilon)\rangle^2 + k^2\langle Y(w_\varepsilon), Y(w_\varepsilon)\rangle\Big)^2}$$

$$-\frac{2\,\varepsilon\, C_{w_\varepsilon}}{k\,\Psi_k(w_\varepsilon)^3}\langle \nabla\Psi_k(w_\varepsilon), \nabla\Psi_k(w_\varepsilon)\rangle.$$

Now, if we multiply both sides of (7.1) by $\dot w_\varepsilon$, we obtain the existence of a constant E_ε such that:

$$\frac{k}{2\,C_{w_\varepsilon}}\,\Theta_k(w_\varepsilon)\langle \dot w_\varepsilon, \dot w_\varepsilon\rangle - \frac{\varepsilon}{\Psi_k(w_\varepsilon)^2} \equiv E_\varepsilon, \quad \text{on } [0,1],\ \forall \varepsilon \in\,]0,1].$$

Integrating the above formula on $[0, 1]$ gives:

$$\frac{k\,C_{w_\varepsilon}}{2} - \varepsilon\int_0^1 \frac{\mathrm{d}t}{\Psi_k(w_\varepsilon)^2} = E_\varepsilon,$$

therefore, for all $t \in [0,1]$, and in particular for $t = t_\varepsilon$, it is:

$$\text{(7.10)}\qquad \frac{k}{2\,C_{w_\varepsilon}}\Theta_k(w_\varepsilon)\langle \dot w_\varepsilon, \dot w_\varepsilon\rangle - \frac{\varepsilon}{\Psi_k(w_\varepsilon)^2} \equiv \frac{k\,C_{w_\varepsilon}}{2} - \varepsilon\int_0^1 \frac{\mathrm{d}t}{\Psi_k(w_\varepsilon)^2}.$$

Now, since $\Psi_k(w_\varepsilon(t_\varepsilon)) = \min \Psi_k(w_\varepsilon)$, it is

$$\text{(7.11)}\qquad \frac{1}{\Psi_k(w_\varepsilon(t_\varepsilon))} \ge \int_0^1 \frac{\mathrm{d}t}{\Psi_k(w_\varepsilon)^2};$$

solving (7.10) for C_{w_ε}, we obtain:

$$C_{w_\varepsilon} = -\frac{1}{k}\left(\frac{\varepsilon}{\Psi_k(w_\varepsilon)^2} - \varepsilon\int_0^1 \frac{\mathrm{d}t}{\Psi_k(w_\varepsilon)^2}\right) + \\ + \frac{1}{k}\sqrt{\frac{\varepsilon}{\Psi_k(w_\varepsilon)^2} - \varepsilon\int_0^1 \frac{\mathrm{d}t}{\Psi_k(w_\varepsilon)^2} + k^2\Theta_k(w_\varepsilon)\langle \dot w_\varepsilon, \dot w_\varepsilon\rangle},$$

and so, by (7.11), we get:

$$C_{w_\varepsilon} \le \frac{1}{k}\sqrt{\Theta_k(w_\varepsilon(t_\varepsilon))\langle \dot w_\varepsilon(t_\varepsilon), \dot w_\varepsilon(t_\varepsilon)\rangle}.$$

Recalling the definition of Θ_k and Ψ_k, by (7.9) we have:

(7.12)

$$
\begin{aligned}
0 \leq\ & d_1\langle \dot w_\varepsilon, \dot w_\varepsilon\rangle + d_2\langle \dot w_\varepsilon, \dot w_\varepsilon\rangle \sqrt{\frac{k^2 + \langle Y(w_\varepsilon), Y(w_\varepsilon)\rangle}{-\langle Y(w_\varepsilon), Y(w_\varepsilon)\rangle}} + \\
& - \frac{2\varepsilon\, C_{w_\varepsilon}}{k\,\Psi_k(w_\varepsilon)^3}\langle \nabla\Psi_k(w_\varepsilon), \nabla\Psi_k(w_\varepsilon)\rangle + \\
& - \frac{\langle Y(w_\varepsilon), Y(w_\varepsilon)\rangle(k^2 + 2\langle Y(w_\varepsilon), Y(w_\varepsilon)\rangle)}{2\langle Y(w_\varepsilon), Y(w_\varepsilon)\rangle^2 (k^2 + \langle Y(w_\varepsilon), Y(w_\varepsilon)\rangle)}\langle \nabla\Psi_k(w_\varepsilon), \nabla\Psi_k(w_\varepsilon)\rangle\langle \dot w_\varepsilon, \dot w_\varepsilon\rangle,
\end{aligned}
$$

where the expression above is computed at $t = t_\varepsilon$.

But $-\langle Y, Y\rangle^{-1} \leq \frac{1}{\nu}$ and $\langle \nabla\Psi_k, \nabla\Psi_k\rangle \geq \nu_0 > 0$ for some ν_0 independent of ε; moreover

$$
\frac{\langle Y, Y\rangle(k^2 + 2\langle Y, Y\rangle)}{2\langle Y, Y\rangle^2(k^2 + \langle Y, Y\rangle)} \longmapsto +\infty \quad \text{as} \quad k^2 + \langle Y, Y\rangle \mapsto 0.
$$

Therefore, if $\Psi_k(w_\varepsilon(t_\varepsilon)) \mapsto 0$, by (7.12) there exists a positive constant $a_0 > 0$ such that:

$$
0 \leq -a_0\langle \dot w_\varepsilon, \dot w_\varepsilon\rangle - \frac{2\varepsilon\, C_{w_\varepsilon}}{k\,\Psi_k(w_\varepsilon)^3}\langle \nabla\Psi_k(w_\varepsilon), \nabla\Psi_k(w_\varepsilon)\rangle.
$$

This is a contradiction, because $\langle \dot w_\varepsilon, \dot w_\varepsilon\rangle \geq 0$ and $C_{w_\varepsilon}\langle \nabla\Psi_k(w_\varepsilon), \nabla\Psi_k(w_\varepsilon)\rangle$ is strictly positive. So the proof is concluded. □

Let's assume now that, for all $\varepsilon \in]0, 1]$, w_ε is a critical point for $G_{k,\varepsilon}$ in $\Omega^{(1)}_{p,\gamma}(\Delta)$ such that:

(7.13)
$$
\sup_\varepsilon G_{k,\varepsilon}(w_\varepsilon) < +\infty.
$$

Such a family is given for instance, by a family of minimum points for $G_{k,\varepsilon}$; observe indeed that, for all $\varepsilon \in]0, 1]$, we have

$$
\min_{\Omega^{(1)}_{p,\gamma}(\Delta)} G_{k,\varepsilon} \leq \min_{\Omega^{(1)}_{p,\gamma}(\Delta)} G_{k,1} < +\infty.
$$

Given such a family w_ε, we now prove that we can pass to the limit as ε tends to 0, obtaining a critical point for the functional G_k:

Proposition 7.3. *There exists a sequence $\{\varepsilon_n\}_n \subset]0, 1]$ tending to zero as $n \to \infty$ and a smooth curve $w \in \Omega^{(1)}_{p,\gamma}(\Delta)$ such that w_{ε_n} tends to w_0 in $C^2([0, 1], \mathcal{M})$. Moreover, such a curve w_0 is a critical point for the functional G_k in $\Omega^{(1)}_{p,\gamma}(\Delta)$.*

Proof. By Remark 5.1, since $G_k(w_\varepsilon) \le G_{k,\varepsilon}(w_\varepsilon) \le \sup_\varepsilon G_{k,\varepsilon}(w_\varepsilon) < +\infty$, then there exists a constant $d_0 \in \mathbb{R}^+$ such that:

$$\int_0^1 \langle \dot{w}_\varepsilon, \dot{w}_\varepsilon \rangle_{(\mathbb{R})} \, \mathrm{d}t \le d_0 < +\infty. \tag{7.14}$$

Then, by the completeness of $\overline{U}_k$, there exists a compact set $K \subset \overline{U}_k$ such that $w_\varepsilon([0,1]) \subset K$ for all ε. Moreover, by Lemma 7.2, since

$$\Psi_k(w_\varepsilon(t)) \ge \delta > 0, \tag{7.15}$$

then $K \subset U_k$.

By the Ascoli–Arzelá's Theorem, from (7.14) and the compactness of K it follows that there exists a sequence $\{\varepsilon_n\}_n \subset]0,1]$ such that w_{ε_n} is uniformly convergent to an absolutely continuous curve w_0 joining p and γ in $K \subset U_k$; by continuity, $\Psi_k(w_0(t)) \ge \delta$ for all t.

We consider the following facts:

(1) $\Theta_k(w_{\varepsilon_n})$ and $\Psi_k(w_{\varepsilon_n})$ are bounded away from 0 by (7.15), and $\Theta_k(w_{\varepsilon_n})^{-1}$ and $\Psi_k(w_{\varepsilon_n})^{-1}$ are bounded in $L^\infty([0,1],\mathbb{R})$;

(2) $\dot{w}_{\varepsilon_n}$ is bounded in $L^2([0,1],T\mathcal{M})$ by (7.14);

(3) $\langle Y(w_{\varepsilon_n}), Y(w_{\varepsilon_n})\rangle^{-1}$ is bounded in $L^\infty([0,1],\mathbb{R})$, while the vector fields $\nabla\Theta_k(w_{\varepsilon_n})$, $\nabla\Psi_k(w_{\varepsilon_n})$ and $Y(w_{\varepsilon_n})$ are bounded in $L^\infty([0,1],T\mathcal{M})$ by (1.11) and the compactness of K; in particular, we have:

$$\lim_{n\to\infty} \frac{2\,\varepsilon_n}{\Psi_k(w_{\varepsilon_n})^3} \nabla\Psi_k(w_{\varepsilon_n}) = 0 \quad \text{in } L^\infty([0,1],T\mathcal{M}). \tag{7.16}$$

(4) by the previous two facts, $\nabla_{\dot{w}_{\varepsilon_n}} Y(w_{\varepsilon_n})$ is bounded in $L^2([0,1],T\mathcal{M})$;

(5) $C_{w_{\varepsilon_n}} = \left(\int_0^1 \Theta_k(w_{\varepsilon_n})\langle \dot{w}_{\varepsilon_n}, \dot{w}_{\varepsilon_n}\rangle_{(\mathbb{R})} \mathrm{d}t\right)^{\frac{1}{2}}$ is bounded by (7.14) and the boundedness of $\Theta_k(w_{\varepsilon_n})$.

Using the facts above, by analyzing the differential equation (7.1) satisfied by the w_{ε_n}, we obtain that the second derivative $\nabla_{\dot{w}_{\varepsilon_n}} \dot{w}_{\varepsilon_n}$ is bounded in $L^1([0,1],T\mathcal{M})$. Then, $\langle \dot{w}_{\varepsilon_n}, \dot{w}_{\varepsilon_n}\rangle_{(\mathbb{R})}$ is bounded in $L^\infty([0,1],\mathbb{R})$ and, again by (7.1), we obtain that w_{ε_n} is bounded in $W^{1,\infty}([0,1],\mathcal{M})$.

Using this new information and arguing similarly, we have that w_{ε_n} is bounded in $W^{2,2}([0,1],\mathcal{M})$, which implies that, up to subsequences, w_{ε_n} converges to w_0 in $C^1([0,1],\mathcal{M})$. Then, using again the differential equation (7.1), we finally conclude that w_{ε_n} converges in $C^2([0,1],\mathcal{M})$, and we can pass to the limit as $\varepsilon \to 0$ in (7.1) using (7.16), obtaining that w_0 satisfies the differential equation (5.5). Arguing as in Proposition 7.3, we see that w_0 also satisfies (5.4); this can be easily checked by multiplying equation (5.5) by $\dot{w}_0$. Hence, by Lemma 5.2, w_0 is a critical point of G_k in $\Omega^{(1)}_{p,\gamma}(\Delta)$, and we are done. □

We can now prove Theorem 1.1:

Proof of Theorem 1.1. For all $\varepsilon \in]0,1]$, let w_ε be a minimal point for the functional $G_{k,\varepsilon}$, which exists by Corollary 6.5. The family $\{w_\varepsilon\}_\varepsilon$ satisfies (7.13) . Observe indeed that, for all $w \in \Omega^{(1)}_{p,\gamma}(\Delta)$ and for all $\varepsilon \in]0,1]$, it is $G_{k,\varepsilon}(w) \le G_{k,1}(w)$. By Proposition 7.3, we can find a sequence w_{ε_n} tending to a smooth curve w_0 that is a critical point for G_k in $\Omega^{(1)}_{p,\gamma}(\Delta)$. Let σ_0 be the unique curve in $\mathcal{B}^{(1)}_{p,\gamma}(k)$ such that $\mathcal{D}(\sigma_0) = w_0$; by Theorem 5.4, such a σ_0 gives an arrival time brachistochrone of energy k between p and γ, which concludes the proof. □

8. Ljusternik–Schnirelman Theory and the Proof of Theorem 1.2

The goal of this section is to give a proof of Theorem 1.2 by means of the Ljusternik–Schnirelman theory for functionals satisfying the Palais–Smale condition. We use well known techniques from Critical Point Theory, which are repeated here for the reader's convenience.

We recall the following definition:

Definition 8.1. If X is a topological space and B any subset of X, the Ljusternik–Schnirelman category $\mathrm{cat}_X(B)$ of B in X is the minimal number (possibly infinite) of closed, contractible subsets of X that cover B. We denote by $\mathrm{cat}(X) = \mathrm{cat}_X(X)$.

Clearly, the Ljusternik–Schnirelman category is increasing with respect to the inclusion, i.e., if A and B are subsets of X, then:

$$A \subseteq B \quad \Longrightarrow \quad \mathrm{cat}_X(A) \le \mathrm{cat}_X(B). \tag{8.1}$$

Moreover, the Ljusternik–Schnirelman category is a homotopical invariant, i.e., if A and B are homotopical subsets of X, then $\mathrm{cat}_X(A) = \mathrm{cat}_X(B)$.

It is not difficult to see that, since Y is complete and γ is contractible in $\mathcal{M}$, then, for any fixed point $q \in \gamma$, the three spaces $\Omega_{p,q}$, $\Omega^{(1)}_{p,\gamma}$ and $\Omega^{(1)}_{p,\gamma}(\Delta)$ have the same homotopy type. Namely, we have already observed in Remark 4.2 that $\Omega^{(1)}_{p,\gamma}$ and $\Omega^{(1)}_{p,\gamma}(\Delta)$ have the same homotopy type.

In particular, it follows that, for every subset $B \subseteq \Omega^{(1)}_{p,\gamma}(\Delta)$, we have:

$$\mathrm{cat}_{\Omega^{(1)}_{p,\gamma}(\Delta)}(B) = \mathrm{cat}_{\Omega^{(1)}_{p,\gamma}}(B).$$

Moreover, given any curve $z \in \Omega^{(1)}_{p,\gamma}$, one can consider the curve $\tilde z \in \Omega_{p,q}$ given by:

$$\tilde z(t) = \psi(z(t), \phi_z(t)),$$

where

$$\phi_z(t) = \left(\gamma^{-1}(q) - \gamma^{-1}(z(1))\right) \cdot t.$$

Observe that the map γ^{-1} is well defined and continuous on $\gamma(\mathbb{R})$ because γ is injective. Moreover, the evaluation map $z \longmapsto z(1)$ is continuous in $\Omega^{(1)}_{p,\gamma}$, so the map $z \longmapsto \phi_z \in C^1([0,1],\mathbb{R})$ is continuous in $\Omega^{(1)}_{p,\gamma}$.

The map $\mathcal{H} : \Omega^{(1)}_{p,\gamma} \times [0,1] \longmapsto \Omega^{(1)}_{p,\gamma}$ given by:

$$\mathcal{H}(z,r)(t) = \psi(z(t), \phi_z(t)\cdot r),$$

is clearly continuous; moreover it satisfies $\mathcal{H}(z,0) = z, \mathcal{H}(z,r) = z$ for all $z \in \Omega_{p,q}$ and all $r \in [0,1]$, and $\mathcal{H}(z,1) = \tilde{z}$, and so it is a *strong deformation retract* of $\Omega^{(1)}_{p,\gamma}$ onto $\Omega_{p,q}$. In particular, the map $\mathcal{H}(\cdot,1)$ is a homotopy equivalence between $\Omega^{(1)}_{p,\gamma}$ and $\Omega_{p,q}$ and the two spaces have the same homotopy type.

A well known result by Fadell and Husseini (see [3]) states that, if U_k is not contractible, then the category of the space $\Omega_{p,q}$, and so also the category of $\Omega^{(1)}_{p,\gamma}$, is infinite:

$$\mathrm{cat}(\Omega^{(1)}_{p,\gamma}) = \mathrm{cat}(\Omega_{p,q}) = +\infty;$$

moreover, there exists a sequence K_n of compact subsets of $\Omega^{(1)}_{p,\gamma}$ such that:

$$\lim_{n\to\infty} \mathrm{cat}_{\Omega^{(1)}_{p,\gamma}}(K_n) = +\infty. \tag{8.2}$$

By Remark 4.2, we can assume that $K_n \subset \Omega^{(1)}_{p,\gamma}(\Delta)$ for all $n \in \mathbb{N}$.

We denote by Γ_n, $n \in \mathbb{N}$, the collection of all compact subsets of $\Omega^{(1)}_{p,\gamma}(\Delta)$ having category strictly larger than to n:

$$\Gamma_n = \left\{B \text{ compact subset of } \Omega^{(1)}_{p,\gamma}(\Delta) : \mathrm{cat}_{\Omega^{(1)}_{p,\gamma}(\Delta)}(B) \geq n+1\right\}. \tag{8.3}$$

Observe that, by (8.2), $\Gamma_n \neq \emptyset$ for all $n \in \mathbb{N}$.

For all $c \in \mathbb{R}$ and $\varepsilon \in [0,1]$, we denote by $G^c_{k,\varepsilon}$ the closed c-sublevel of the functional $G_{k,\varepsilon}$ in $\Omega^{(1)}_{p,\gamma}(\Delta)$:

$$G^c_{k,\varepsilon} = \left\{w \in \Omega^{(1)}_{p,\gamma}(\Delta) : G_{k,\varepsilon}(w) \leq c\right\};$$

clearly

$$G^c_{k,\varepsilon} \subseteq G^c_k, \qquad \forall c \in \mathbb{R}, \forall \varepsilon \in [0,1].$$

A well known argument in Critical Point Theory shows that, if $L : X \mapsto \mathbb{R}$ is a C^1-functional defined on the Banach manifold X, satisfying the Palais–Smale condition at the level $c' \in \mathbb{R}$ and such that the sublevel $L^{c'}$ is complete, then, the category $\mathrm{cat}_X(L^{c'})$ is finite.

In the following preliminary Lemma we show that, for all $c \in \mathbb{R}$, the category $\mathrm{cat}_{\Omega^{(1)}_{p,\gamma}}(G^c_{k,\varepsilon}) = \mathrm{cat}_{\Omega^{(1)}_{p,\gamma}(\Delta)}(G^c_{k,\varepsilon})$ is bounded *uniformly* with respect to ε.

Lemma 8.2. *For all $c \in \mathbb{R}$ there exists $N = N(c) \in \mathbb{N}$ such that:*

$$\mathrm{cat}_{\Omega^{(1)}_{p,\gamma}}(G^c_{k,\varepsilon}) \leq N(c), \tag{8.4}$$

for all $\varepsilon \in [0,1]$.

Proof. We will show that, for all $c \in \mathbb{R}$, there exists $c' \in \mathbb{R}$ and, for all $\varepsilon \in [0,1]$, a homotopy between the sublevel $G^c_{k,\varepsilon}$ and a subset of the sublevel $L^{c'}$, where L is a suitable smooth functional satisfying the properties described above.

By the completeness of $\overline{U}_k$, for all $c \in \mathbb{R}$ there exists a compact subset $K_c \subset \overline{U}_k$ containing the images of all the curves in $G^c_{k,\varepsilon}$, for all $\varepsilon \in [0,1]$. Indeed, as we have already observed at the beginning of the proof of Proposition 7.3, all such curves w have bounded energy:

$$\int_0^1 \langle \dot{w}, \dot{w} \rangle_{(R)} \, dt \leq d_0 < +\infty.$$

For $\delta > 0$, we denote by V_δ the closed subset of U_k given by:

$$V_\delta = \Big\{ x \in U_k : \Psi_k(x) \geq \delta \Big\}. \tag{8.5}$$

Let now $\delta > 0$ be small enough and $H : [0,1] \times (\overline{U}_k \cap K_c) \longmapsto K_c$ be a map of class C^1 satisfying:

(1) $H(0,x) = x$, for all $x \in \overline{U}_k \cap K_c$;
(2) $H(r, V_\delta) \subset V_\delta$, for all $r \in [0,1]$;
(3) $H(1, \overline{U}_k \cap K_c) \subset V_\delta$.

Such a map is built (up to considering a *larger* K_c) by using the flow of the vector field $\nabla\Psi_k$, which is bounded away from 0 in the compact set $K_c \cap \partial U_k$.

Finally, we consider the map $\mathcal{H} : [0,1] \times G^c_k \longmapsto \Omega^{(1)}_{p,\gamma}$ defined by:

$$\mathcal{H}(r,w)(t) = H(r \cdot t, w(t)), \quad \forall r,t \in [0,1]. \tag{8.6}$$

Clearly, $\mathcal{H}$ is continuous; moreover, by the construction of H, the map $\mathcal{H}(0,\cdot)$ is the identity on G^c_k. By construction, the curves in $\mathcal{H}(1, G^c_k)$ have image in the set V_δ, and so they stay uniformly far from ∂U_k.

Since the curves in G^c_k have image in a fixed compact set, it is easy to see by a direct computation that the map H has bounded partial derivatives, and so the map $\mathcal{H}(1,\cdot)$ is Lipschitz continuous on G^c_k. It follows that there exists a real number c' such that:

$$\sup_{w \in \mathcal{H}(1, G^c_k)} G_k(w) \leq c'.$$

We denote by $A_{k,\delta,c'}$ the set of curves in $G^{c'}_k$ having image in V_δ:

$$A_{k,\delta,c'} = \Big\{ w \in G^{c'}_k : w(t) \in V_\delta, \ \forall t \Big\};$$

we have just proven that:

$$\mathcal{H}(1, G^c_k) \subseteq A_{k,\delta,c'}.$$

Let's consider the functional $L : \Omega^{(1)}_{p,\gamma} \longmapsto I\!R$ given by:

$$L(w) = G_k(w) + \int_0^1 \frac{r(t)}{\Psi_k(w(t))^2}\,\mathrm{d}t, \tag{8.7}$$

where $r : [0, +\infty[\longmapsto I\!R^+$ is a smooth function satisfying:

$$r(0) = 1, \quad r'(0) > 0, \quad \text{and} \quad r(s) \equiv 0 \quad \text{if} \quad s \in [\frac{\delta}{2}, +\infty[.$$

Now, the same proofs of Propositions 6.2 and 6.4 can be repeated *verbatim* to show that L is a smooth functional on $\Omega^{(1)}_{p,\gamma}(\Delta)$ that satisfies the Palais–Smale condition at every level c, and whose sublevels are complete. It follows that, for all $c' \in I\!R$, there exists a natural number $N(c')$ such that:

$$\mathrm{cat}_{\Omega^{(1)}_{p,\gamma}}(L^{c'}) = N(c') < +\infty. \tag{8.8}$$

Since the function r vanishes identically on the images of the curves in $A_{k,\delta,c'}$, then $A_{k,\delta,c'} \subseteq L^{c'}$, and, in particular,

$$\mathrm{cat}_{\Omega^{(1)}_{p,\gamma}}(A_{k,\delta,c'}) \leq \mathrm{cat}_{\Omega^{(1)}_{p,\gamma}}(L^{c'}) = N(c'). \tag{8.9}$$

Thus, by the homotopical invariance and the monotonicity of the Ljusternik-Schnirelman category, we have:

$$\mathrm{cat}_{\Omega^{(1)}_{p,\gamma}}(G^c_{k,\varepsilon}) \leq \mathrm{cat}_{\Omega^{(1)}_{p,\gamma}}(G^c_k) = \mathrm{cat}_{\Omega^{(1)}_{p,\gamma}}(\mathcal{H}(1, G^c_k)) \leq \mathrm{cat}_{\Omega^{(1)}_{p,\gamma}}(A_{k,\delta,c'}), \tag{8.10}$$

which concludes the proof. □

A well known *minimax* argument in Critical Point Theory shows that the numbers:

$$c^\varepsilon_m = \inf_{B\in\Gamma_m} \left[\sup_{x\in B} G_{k,\varepsilon}(x)\right] \tag{8.11}$$

are critical values for the functional $G_{k,\varepsilon}$ for all $m \in I\!N$ and all $\varepsilon \in]0, 1]$. Observe that, by the definition (8.3) of Γ_n, each c^ε_m is well defined and finite. Also, since $\varepsilon \longmapsto G_{k,\varepsilon}(w)$ is increasing on $[0, 1]$ for all $w \in \Omega^{(1)}_{p,\gamma}$, it follows easily that:

$$0 \leq c^\varepsilon_m \leq c^1_m, \qquad \forall \varepsilon \in [0, 1],\ \forall m \in I\!N. \tag{8.12}$$

Since Γ_n is non empty for all n, from (8.11) and Lemma 8.2 it follows that $G_{k,\varepsilon}$ has arbitrarily large critical values for all $\varepsilon \in]0, 1]$; in order to prove Theorem 1.2, we need to prove that the same claim is true for the functional $G_k = G_{k,0}$.

Lemma 8.3. *In the notations of Lemma 8.2, for all $c \in I\!R$, if $m \geq N(c)$ then*

$$c^\varepsilon_m \geq c, \qquad \forall \varepsilon \in]0, 1]. \tag{8.13}$$

Proof. Let $\varepsilon \in]0,1]$ and $c \in \mathbb{R}$ be fixed. By contradiction, suppose that there exists $m > N(c)$ and $B \in \Gamma_m$ such that:

$$\sup_{x \in B} G_{k,\varepsilon}(x) \leq c.$$

Then, it would be:

$$B \subseteq G^c_{k,\varepsilon},$$

and, by the monotonicity of the Ljusternik–Schnirelman category and Lemma 8.2, it would be:

$$\text{cat}_{\Omega^{(1)}_{p,\gamma}(\Delta)}(B) \leq \text{cat}_{\Omega^{(1)}_{p,\gamma}(\Delta)}(G^c_{k,\varepsilon}) \leq N(c). \tag{8.14}$$

The inequality (8.14) contradicts the fact that $B \in \Gamma_m$ and concludes the proof. □

We are ready for the proof of Theorem 1.2:

Proof of Theorem 1.2. Let $\{\varepsilon_n\}_n \subset]0,1]$ be any decreasing sequence converging to 0 and let G_{k,ε_n} be the corresponding sequence of functionals on $\Omega^{(1)}_{p,\gamma}(\Delta)$.

For all $m \in \mathbb{N}$, we define:

$$\overline{c}_m = \liminf_{n \to \infty} c^{\varepsilon_n}_m. \tag{8.15}$$

By (8.12), the $\overline{c}_m$ are well defined (finite), and they form a non decreasing sequence.

By Proposition 7.3, for all $m \in \mathbb{N}$, $\overline{c}_m$ is a critical value for G_k.

Finally, by Lemma 8.3, the sequence $\overline{c}_m$ is unbounded, hence G_k has arbitrarily large critical values. Let w_m be a sequence of critical points of G_k in $\Omega^{(1)}_{p,\gamma}(\Delta)$ such that $G_k(w_m) = \overline{c}_m$ and, for each $m \in \mathbb{N}$, let σ_m be the unique curve in $\mathcal{B}^{(1)}_{p,\gamma}(k)$ such that $\mathcal{D}(\sigma_m) = w_m$. The conclusion follows then immediately from Theorem 5.4. □

Appendix A. The arrival time brachistochrones in the exterior Schwarzschild's spacetime

We now present an explicit calculation of the arrival time brachistochrones in the exterior Schwarzschild spacetime, that is the relativistic model for the gravitational field outside a star static and spherically symmetric (see [1, 12, 17]). We will prove that any event p and any observer γ can be joined by an arrival time brachistochrone of arbitrary (positive) energy. We emphasize that, by a result of [20], the same result does *not* hold for the travel time brachistochrones.

We consider polar coordinates (r, ϕ, θ) in $\mathbb{R}^3$. The exterior Schwarzschild spacetime is given by a product manifold $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$, where $\mathcal{M}_0 \subset \mathbb{R}^3$ is the open subset given by the region outside the sphere of radius $2m$:

$$\mathcal{M}_0 = \{(r, \phi, \theta) : r > 2m\},$$

and the metric g on $\mathcal{M}$ is:

$$g = \beta(r)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 - \beta(r)\,\mathrm{d}t^2, \tag{A.1}$$

where

$$\beta(r) = 1 - \frac{2m}{r}$$

and $\mathrm{d}\Omega^2 = \sin^2\theta\,\mathrm{d}\phi^2 + \mathrm{d}\theta^2$ is the standard Riemannian metric induced on the unit sphere of $\mathbb{R}^3$ by the Euclidean metric. The positive constant m represent the mass of the star.

We consider the timelike Killing vector field $Y = \frac{\partial}{\partial t}$; its orthogonal distribution Δ is completely integrable, and its integral submanifolds are the spacelike surfaces given by the *time slices* $\mathcal{M}_0 \times \{t_0\}$. Please note that, differently from the previous sections, we are now using the letter t to indicate a the global coordinate function on $\mathcal{M}$. The curve parameter will now be denoted by the letter s. Observe also that $\langle Y, Y\rangle = -\beta$, where $\langle\cdot,\cdot\rangle$ denotes the metric (A.1).

Fix any positive constant k; the *potential well* U_k defined in (1.10) is given by:

$$U_k = \Big\{(r,\phi,\theta,t) : 1 - \frac{2m}{r} < k^2\Big\}.$$

If $k^2 \geq 1$, then U_k coincides with the entire manifold $\mathcal{M}$; if $0 < k^2 < 1$, then

$$U_k = \Big\{(r,\phi,\theta,t) : 1 - k^2 < \frac{2m}{r} < 1\Big\}.$$

As we have observed, a curve $z = (\mathbf{x}, t)$ in $\mathcal{M}$, with $\mathbf{x} \in \mathcal{M}_0$ and $t \in \mathbb{R}$, is horizontal with respect to the distribution Δ if and only if its image stays inside a time slice $\mathcal{M}_0 \times \{t_0\}$, i.e., if and only if $\dot t = 0$. Consequently, by Theorem 5.4, the arrival time brachistochrones in the exterior Schwarzschild spacetime are curves $\sigma = (r,\phi,\theta,t)$ such that $\mathbf{x} = (r,\phi,\theta)$ is a critical point for the functional

$$G_k(r,\phi,\theta) = \int_0^1 \frac{1}{\beta(r)(k^2-\beta(r))}\left[\frac{1}{\beta(r)}\dot r^2 + r^2\left(\sin^2\theta\,\dot\phi^2 + \dot\theta^2\right)\right]\,\mathrm{d}s,$$

parameterized by:

$$\frac{\beta(r)}{k^2-\beta(r)}\left[\frac{1}{\beta(r)}\dot r^2 + r^2\left(\sin^2\theta\,\dot\phi^2 + \dot\theta^2\right)\right] \equiv C \text{ (constant)},$$

and t satisfies $\dot t = -\beta(r)^{-1} k\, \mathcal{T}_\sigma$.

By the radial symmetry of the Schwarzschild metric, we can restrict our attention to the equatorial plane $\theta = \frac{\pi}{2}$; in this case, the spatial part $\mathbf{x} = (r, \phi, \frac{\pi}{2})$ of the arrival time brachistochrones are characterized as the critical points for the functional:

$$G_k(r,\phi) = \int_0^1 \frac{1}{\beta(r)(k^2-\beta(r))}\left(\frac{1}{\beta(r)}\dot r^2 + r^2\,\dot\phi^2\right)\,\mathrm{d}s \tag{A.2}$$

that are parameterized by:

$$\frac{\beta(r)}{k^2-\beta(r)}\left(\frac{1}{\beta(r)}\dot{r}^2+r^2\dot{\phi}^2\right)\equiv C \text{ (constant)}.$$

The functions ϕ_k and Θ_k defined respectively in (4.1) and (5.1) are now given by:

$$\phi_k=\frac{\beta}{k^2-\beta},\quad \Theta_k=\frac{1}{\beta(k^2-\beta)}.$$

If (r,ϕ) is a critical point for the functional G_k of (A.2), then it satisfies:

$$\int_0^1 \frac{2\beta-k^2)\beta'}{\beta^2(k^2-\beta)^2}\left(\frac{\dot{r}^2}{\beta}+r^2\dot{\phi}^2\right)\rho\,\mathrm{d}s+$$
$$+\int_0^1\frac{1}{\beta\,(k^2-\beta)}\left(-\frac{1}{\beta^2}\beta'\rho\dot{r}^2+\frac{2\dot{r}\dot{\rho}}{\beta}+2r\rho\dot{\phi}^2+2r^2\dot{\phi}\,\dot{\Phi}\right)\,\mathrm{d}s=0,$$

for all ρ and Φ smooth functions on $[0,1]$ vanishing at 0 and 1. Then, integration by parts and the Fundamental Theorem of Calculus of Variations give the following system of differential equations satisfied by (r,ϕ):

$$\begin{cases}-\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{2\dot{r}}{\beta^2(k^2-\beta)}\right)+\left(\frac{2\beta-k^2-1}{\beta^3(k^2-\beta)^2}\right)\beta'\dot{r}^2+\left[\frac{(2\beta-k^2)\beta' r^2}{\beta^2(k^2-\beta)^2}-2r\right]\dot{\phi}^2=0,\\ \\ 2r^2\dot{\phi}\equiv L \text{ (constant)},\end{cases}$$

which is also written as:

(A.3)
$$\begin{cases}-\frac{2\ddot{r}}{\beta^2(k^2-\beta)}+\frac{2\dot{r}^2\beta'}{\beta^3(k^2-\beta)^2}(k^2-1-\beta)+\left[\frac{(2\beta-k^2)\beta' r^2}{\beta^2(k^2-\beta)^2}-2r\right]\dot{\phi}^2=0,\\ \\ 2r^2\dot{\phi}\equiv L \text{ (constant)}.\end{cases}$$

Now, as in (5.28), we define μ to be the unique solution of the Cauchy problem:

(A.4)
$$\begin{cases}\mu'=-\frac{1}{\beta(r(\mu))}\left(\int_0^1\frac{\mathrm{d}s}{\beta(r(s))}\right)^{-1},\\ \\ \mu(0)=0.\end{cases}$$

The brachistochrone differential equation is the equation satisfied by the pair

$$(r_1,\phi_1)=(r\circ\mu,\phi\circ\mu)$$

(see Lemma 5.6), which is given by:

(A.5)
$$\begin{cases} -2\left(\ddot{r}_1 - \dfrac{\mu''\dot{r}_1}{\mu'}\right)\dfrac{1}{\beta^2(k^2-\beta)} + \dfrac{2\dot{r}_1^2(k^2-1-\beta)}{\beta^3(k^2-\beta)^2} + \left[\dfrac{(2\beta-k^2)\beta' r_1^2}{\beta^2(k^2-\beta)^2} - 2r_1\right]\dot{\phi}_1^2 = 0, \\ 2r_1^2\dfrac{\dot{\phi}_1}{\mu'} \equiv L \text{ (constant)}. \end{cases}$$

By (A.4), we get:

$$\frac{\mu''}{\mu'} = -\beta(r_1)\,\frac{\beta'(r_1)\dot{r}_1}{\beta(r_1)^2},$$

and, by (A.4) and (A.5) we have:

$$2\,r_1^2\dot{\phi}_1\beta(r_1) \equiv \tilde{L} \text{ (constant)}.$$

Therefore, up to replacing (r_1, ϕ_1) by (r, ϕ) and $\tilde{L}$ by L, we see that the arrival time brachistochrones in the equatorial plane $\theta = \frac{\pi}{2}$ of the exterior Schwarzschild spacetime are characterized by the differential equations:[3]

(A.6)
$$\begin{cases} -\dfrac{2}{\beta^2(k^2-\beta)}\left(\ddot{r} + \dfrac{\beta'\dot{r}^2}{\beta}\right) + \dfrac{2\dot{r}^2(k^2-1-\beta)}{\beta^3(k^2-\beta)^2} + \left[\dfrac{(2\beta-k^2)\beta' r^2}{\beta^2(k^2-\beta)^2} - 2r\right]\dot{\phi} = 0, \\ 2\,r^2\beta\dot{\phi} \equiv L \text{ (constant)}. \end{cases}$$

We now prove that, for $\varepsilon > 0$ suitably fixed, the following subset of $\mathcal{M}$:

$$\mathcal{M}_\varepsilon = \left\{(r, \phi, \theta, t) \in \mathcal{M} : \beta(r) \geq \varepsilon\right\}$$

satisfies a *convexity* property, analogous to a similar condition employed in the proof of Lemma 7.2. More precisely, we prove that if (r, ϕ) satisfies (A.6), with $\beta(r(0)) = \varepsilon$ and $\dot{r}(0) = 0$, then the following inequality holds:

(A.7)
$$\left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right|_{s=0} \beta(r(s)) < 0.$$

Geometrically, the above condition means that every arrival time brachistochrone in the exterior Schwarzschild spacetime having image in $\mathcal{M}_\varepsilon$ and both endpoints in the interior of $\mathcal{M}_\varepsilon$ never reaches the boundary $\partial\mathcal{M}_\varepsilon$.

---

[3] Observe that in Reference [20] the author only writes the differential equations satisfied by the *travel time brachistochrones*.

Now, a simple computation shows that (A.7) is equivalent the following inequality:

$$\frac{(2\beta - k^2)\beta' r}{\beta^2(k^2 - \beta)} < 2,$$

at the points where $\beta = \varepsilon$. This is certainly true for ε sufficiently small, because $\beta' r$ tends to 1 as ε goes to 0, while

$$\lim_{\beta \to 0^+} \frac{2\beta - k^2}{\beta^2(k^2 - \beta)} = -\infty.$$

Thanks to the above convexity property it is possible to prove, essentially by the same arguments in the proof of Theorem 1.1, that any event and any observer of the exterior Schwarzschild spacetime can be joined by an arrival time brachistochrone of any fixed positive energy. Note that, as it was pointed out in [20], that this is not true for the travel time brachistochrones.

Finally, we observe that it is possible to extend the results of Theorem 1.1 and Theorem 1.2 to the case of a *potential well* with boundary, as long as the above convexity property is satisfied.

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