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A characterization of the natural grading of the Grassmann algebra and its non-homogeneous $\mathbb{Z}_2$-gradings

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Let F be any field of characteristic different from two and let E be the Grassmann algebra of an infinite dimensional F-vector space L. In this paper we will provide a condition for a $\mathbb{Z}_2$-grading on E to behave like the natural $\mathbb{Z}_2$-grading E_{can}. More specifically, our aim is to prove the validity of a weak version of a conjecture presented in [10]. The conjecture poses that every $\mathbb{Z}_2$-grading on E has at least one non-zero homogeneous element of L. As a consequence, we obtain a characterization of E_{can} by means of its $\mathbb{Z}_2$-graded polynomial identities. Furthermore we construct a $\mathbb{Z}_2$-grading on E that gives a negative answer to the conjecture.

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Superalgebras
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1. Introduction

Let F be any field of characteristic different from two and let L be a vector space over F with basis $\{e_1, e_2, \dots\}$. The infinite dimensional Grassmann algebra E of L over F is the vector space with a basis consisting of 1 and all products $e_{i_1}e_{i_2}\cdots e_{i_k}$, where $i_1 < i_2 < \dots < i_k$, $k \geq 1$. The length of $e_{i_1}e_{i_2}\cdots e_{i_k}$ is the number k that is denoted by $|e_{i_1}e_{i_2}\cdots e_{i_k}|$. The multiplication in E is induced by $e_ie_j = -e_je_i$ for all i and j. We shall denote the above canonical basis of E by B_E. The Grassmann algebra has a natural $\mathbb{Z}_2$-grading $E_{can} = E_{(0)} \oplus E_{(1)}$, where $E_{(0)}$ is the vector space spanned by 1 and all products $e_{i_1}\cdots e_{i_k}$ with even k while $E_{(1)}$ is the vector space spanned by the products with odd k. It is well known that $E_{(0)} = Z(E)$ (here $Z(E)$ denotes the center of E), and $E_{(1)}$ is the "anticommuting" part of E.

The study of the Grassmann algebra E with its natural grading by the cyclic group $\mathbb{Z}_2$ is an important part of the theory of algebras with polynomial identities. In [11,12] Kemer developed a deep and far-reaching theory of varieties of associative algebras with polynomial identities. More precisely, the algebra E is the most powerful tool (and, at the moment, the only one) to construct a general T-ideal from the one of a finite dimensional one via the Grassmann envelope. Thus, in Kemer's theory the natural $\mathbb{Z}_2$-grading on E and the corresponding $\mathbb{Z}_2$-graded polynomial identities satisfied by E were of crucial importance. Consequently, gradings of the Grassmann algebra may shed light on what kind of mathematical construction should be investigated in order to find out how to cover identities from weaker ones. When the field is infinite and of positive characteristic, they turned out to be crucial too (see, for example, [13–15]). Apart from it, the Grassmann algebra arises naturally in many fields of physical and mathematical sciences, and the interested reader can consult [1] for a treatment of this topic. This highlights the importance of the algebra E in the science.

A natural question arises: Describe all possible $\mathbb{Z}_2$-gradings on E. The first studies on this direction were conducted by Anisimov, in 2001–2002, see [4,5]. When the field F is of characteristic 0 this was done by Di Vincenzo and Da Silva [6], and if F is infinite and of characteristic different from 2, by Centrone [2]. In those two last papers the authors assumed that the underlying vector space L of E is homogeneous in the grading. The way to define a homogeneous $\mathbb{Z}_2$-grading on E is relatively simple. For this, it is enough to choose the degrees of a basis of L. In [2,6,9] the authors studied the graded identities for such $\mathbb{Z}_2$-gradings on E. Furthermore, in [3,7], some cases of gradings on E by finite abelian groups of order greater than 2 were also investigated.

The construction of non-homogeneous $\mathbb{Z}_2$-gradings is more complicated, see [10]. Actually there is not a complete classification for such gradings of E, but it can be done via duality with automorphisms of order at most 2 on E. This duality is well known. It relies on the fact that if G is a finite abelian group then G is isomorphic to its dual group assuming that the field is large enough. As we are interested in gradings by the group $\mathbb{Z}_2$ we need no further assumptions on the base field (apart from its characteristic being different from 2). Thus if $\varphi \in Aut(A)$ is an automorphism of an algebra A of order at most two, then one has a $\mathbb{Z}_2$-grading on A given by $A = A_{0,\varphi} \oplus A_{1,\varphi}$. Here $A_{0,\varphi}$ and $A_{1,\varphi}$ are the eigenspaces in A associated to eigenvalues 1 and -1 of the linear transformation φ. Reciprocally to each $\mathbb{Z}_2$-grading on A one associates an automorphism of A of order ≤ 2 as follows. If $A = A_0 \oplus A_1$ is the $\mathbb{Z}_2$-grading the automorphism φ is defined by $\varphi(a_0 + a_1) = a_0 - a_1$ for every $a_i \in A_i$, $i = 0, 1$. We shall need this duality in the form of a duality between group gradings and group actions, see for example [8] for a discussion in the general case.

In this paper we investigate the problem whether in every $\mathbb{Z}_2$-grading of E there is at least one element of the underlying vector space L that is homogeneous in the grading. This was posed in [10] as a conjecture. To this end our paper is organized as follows. In Section 2 we give the necessary background concerning $\mathbb{Z}_2$-gradings on the Grassmann algebra and their graded identities. In Section 3 we prove a weak version of the conjecture. As a consequence, we will provide a condition for a $\mathbb{Z}_2$-grading on E to behave as E_{can}. In other words, we obtain a characterization of E_{can} by means of its $\mathbb{Z}_2$-graded polynomial identities. Furthermore, in Section 4 we construct a $\mathbb{Z}_2$-grading on E that gives a negative answer to this conjecture.

We hope that our results about the natural grading of E may shed additional light on the construction of T-ideals, and consequently on the polynomial identities of PI-algebras.

2. Preliminaries

Let A be an unitary associative F-algebra. We say that A is a $\mathbb{Z}_2$-graded algebra (or superalgebra) whenever $A = A_0 \oplus A_1$ where A_0, A_1 are F-subspaces of A satisfying $A_iA_j \subset A_{i+j}$ for i, $j \in \mathbb{Z}_2$. For each $\mathbb{Z}_2$-grading on A, we will denote it by a specific symbol, for example Γ. The vector subspace A_i is called *homogeneous component* of degree i and a non-zero element a in it is homogeneous; we denote it by $\|a\| = i$. A vector subspace (subalgebra, ideal) $W \subset A$ is graded if $W = (W \cap A_0) \oplus (W \cap A_1)$.

We point out that we use freely the terms *superalgebra* and $\mathbb{Z}_2$-*graded algebra* as synonymous although this is an abuse of terminology. In the associative case they are indeed synonymous while in the nonassociative setting they are not. Indeed, a Lie or a Jordan superalgebra is not, as a rule, a Lie or a Jordan algebra. The correct setting in the general case should be as follows. Let $A = A_0 \oplus A_1$ be a $\mathbb{Z}_2$-graded algebra and let $\mathfrak{V}$ be a variety of algebras (not necessarily associative). Then A is a $\mathfrak{V}$-superalgebra whenever $A_0 \otimes E_{(0)} \oplus A_1 \otimes E_{(1)}$ is an algebra belonging to $\mathfrak{V}$. Since we shall deal with associative algebras only such a distinction is not relevant for our purposes, and we are not going to make any difference between superalgebras and $\mathbb{Z}_2$-graded algebras.

In what follows we assume that the reader knows the definitions of homomorphism, endomorphism and automorphism of algebras. Let A and B be superalgebras, a homomorphism $f\colon A \to B$ is a $\mathbb{Z}_2$-*graded homomorphism* if $f(A_i) \subset B_i$ for all $i \in \mathbb{Z}_2$. When there exists a $\mathbb{Z}_2$-graded isomorphism between A and B we say that A and B are isomorphic superalgebras and we denote it by $A \simeq B$.

One defines a free object in the class of superalgebras by considering the free F-algebra over the disjoint union of two countable sets of variables, denoted by Y and Z. We assume further that the elements of Y are of degree zero and the elements of Z are of degree 1. This algebra is denoted by $F\langle Y \cup Z\rangle$. Its even part is the vector space spanned by all monomials whose degree counting only the elements of Z, is an even integer. The remaining monomials span the odd component. The elements of $F\langle Y \cup Z\rangle$ are called $\mathbb{Z}_2$-graded polynomials (or simply polynomials). It is straightforward that $F\langle Y \cup Z\rangle$ is a free algebra in the sense that for every superalgebra A and for every map $\varphi\colon Y \cup Z \to A$ such that $\varphi(Y) \subseteq A_0$ and $\varphi(Z) \subseteq A_1$ there exists unique homomorphism of superalgebras $F\langle Y \cup Z\rangle \to A$ that extends φ.

Let $\Gamma\colon A = A_0 \oplus A_1$ be a superalgebra. We say that the polynomial $f(y_1, \dots, y_l, z_1, \dots, z_m) \in F\langle Y \cup Z\rangle$ is a $\mathbb{Z}_2$-*graded polynomial identity* for Γ if $f(a_1, \dots, a_l, b_1, \dots, b_m) = 0$ for all admissible substitution $a_1, \dots, a_l \in A_0$ and $b_1, \dots, b_m \in A_1$. The set of all $\mathbb{Z}_2$-graded polynomial identities of A is a graded ideal of $F\langle Y \cup Z\rangle$. It is called T_2-ideal, and denoted by $T_2(\Gamma)$. Given a superalgebra $\Gamma'\colon B = B_0 \oplus B_1$, we say that A and B are *PI-equivalent* as superalgebras if $T_2(\Gamma) = T_2(\Gamma')$.

As already mentioned in the introduction, to define a homogeneous $\mathbb{Z}_2$-grading on E is enough to choose the degrees of a basis of L. More specifically, according to [6], given $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, there are the following possibilities:

$$\|e_i\|_k = \begin{cases} 0, & \text{if } i = 1, \dots, k \\ 1, & \text{otherwise} \end{cases},$$

$$\|e_i\|_{k^*} = \begin{cases} 1, & \text{if } i = 1, \dots, k \\ 0, & \text{otherwise} \end{cases},$$

and

$$\|e_i\|_\infty = \begin{cases} 0, & \text{if } i \text{ is even} \\ 1, & \text{otherwise} \end{cases}.$$

The degree of a monomial $e_{i_1} e_{i_2} \cdots e_{i_t}$ is $\|e_{i_1} e_{i_2} \cdots e_{i_t}\| = \|e_{i_1}\| + \|e_{i_2}\| + \cdots + \|e_{i_t}\|$, where the latter is taken in $\mathbb{Z}_2$. These gradings are denoted by E_k, E_{k^*}, and E_∞, respectively. When $\|e_i\| = 1$ for all i, we recover E_{can}. Thus, we have three structures of homogeneous superalgebra on E, up to an isomorphism. Notice that all homogeneous $\mathbb{Z}_2$-gradings on E correspond to the automorphisms φ on E satisfying, for an appropriate basis of E, the conditions

$$\varphi(e_i) = \pm e_i.$$

In general, for each automorphism φ on E of order at most 2, we have a structure of $\mathbb{Z}_2$-grading on E, denoted by $E_\varphi = E_{0,\varphi} \oplus E_{1,\varphi}$.

3. A characterization of E_{can}

The goal of this section is to prove a weak version of the conjecture formulated in [10]. As a consequence, we obtain a characterization of E_{can} by means of its $\mathbb{Z}_2$-graded polynomial identities.

Let φ be an automorphism on E of order at most 2, i.e., $\varphi^2 = id$. Here the automorphism identity of E is denoted by id. As in the above section, $E_\varphi = E_{0,\varphi} \oplus E_{1,\varphi}$ denotes the $\mathbb{Z}_2$-grading on E induced by φ. We observe that

$$e_i = (e_i + \varphi(e_i))/2 + (e_i - \varphi(e_i))/2, \text{ for } i \in \mathbb{N}.$$

If we set $V_0 = \{(e_i + \varphi(e_i))/2 \mid i \in \mathbb{N}\}$ and $V_1 = \{(e_i - \varphi(e_i))/2 \mid i \in \mathbb{N}\}$, then each element of B_E is written as a sum of products of elements in $V_0 \cup V_1$.

The next lemma is of easy deduction, and for this reason we omit its proof.

Lemma 1. *The component $E_{0,\varphi}$ is spanned by all products of elements in V_0 and V_1 with an even number of factors in V_1, and $E_{1,\varphi}$ is spanned by all products of elements in V_0 and V_1 with an odd number of factors in V_1.*

Now, we define $I_\beta = \{n \in \mathbb{N} \mid \varphi(e_n) = \pm e_n\}$. We distinguish the following possibilities:

(1) $I_\beta = \mathbb{N}$.
(2) $I_\beta \neq \mathbb{N}$ is infinite.
(3) I_β is finite and non-empty.

Pay attention there might be a basis β of L such that $I_\beta = \emptyset$ but $I_{\beta'} \neq \emptyset$ for some other basis β' of L. Hence the fourth possibility is

(4) $I_\gamma = \emptyset$ for every basis γ of the vector space L.

We shall call these automorphisms (and also the corresponding $\mathbb{Z}_2$-gradings), automorphisms ($\mathbb{Z}_2$-grading) of type (1), (2), (3), and (4), respectively.

Now we obtain the following lemma.

Lemma 2. *Let φ be an automorphism on E of order at most 2. If $T_2(E_\varphi) = T_2(E_{can})$, then $E_{0,\varphi} \subset Z(E)$. In particular, $e_i + \varphi(e_i) \in Z(E)$, for all i.*

Proof. It is known that

$$[y,x] \in T_2(E_{can}) = T_2(E_\varphi),$$

where $x \in Y \cup Z$. It implies immediately that $E_{0,\varphi} \subset Z(E)$. From $e_i + \varphi(e_i) \in E_{0,\varphi}$, the latter part follows. ■

The next steps to obtain the main result of this section are to analyze the automorphisms φ of E according to its type. To this end we need a series of results.

Proposition 3. *Let E_d be a homogeneous $\mathbb{Z}_2$-grading on the Grassmann algebra E and let φ be an automorphism on E of order at most 2. If φ is of type* (1)*, the following statements hold:*

(*a*) *$T_2(E_d) = T_2(E_\varphi)$ if and only if $E_d \simeq E_\varphi$.*
(*b*) *If $T_2(E_\varphi) \supset T_2(E_{can})$, then $E_\varphi \simeq E_{can}$.*

Proof. The fact that φ is an automorphism of type (1) implies that E_φ is a homogeneous $\mathbb{Z}_2$-grading on E. Item (a) follows as a straightforward consequence of [2,6,9]. For the statement in (b), notice that $T_2(E_\varphi) \supset T_2(E_{can})$ implies $[y,x]$ lies in $T_2(E_\varphi)$, for any $x \in Y \cup Z$. Since the gradings E_{k^*} and E_∞ do not satisfy such identity, it follows that $E_\varphi \simeq E_k$, for some $k \in \mathbb{N}_0$. Furthermore there exists a homogeneous subalgebra of E_k which is isomorphic to E_{can}. Hence,

$$T_2(E_\varphi) \subset T_2(E_{can}),$$

so we conclude $T_2(E_\varphi) = T_2(E_{can})$. Applying statement ($a$), we obtain $E_\varphi \simeq E_{can}$. ■

Proposition 4. *There does not exist $\varphi \in Aut(E)$ of type* (2) *such that $T_2(E_\varphi) = T_2(E_{can})$.*

Proof. Due to Lemma 2, $e_i + \varphi(e_i) \in E_{(0)}$. On the other hand, if φ is of type (2), according to [10, Proposition 1], φ satisfies $\varphi(e_i) \in E_{(1)}$, for all i. Hence $e_i + \varphi(e_i) \in E_{(1)}$, which gives us

$$e_i + \varphi(e_i) \in E_{(0)} \cap E_{(1)},$$

and then $\varphi(e_i) = -e_i$, for all $i \in \mathbb{N}$. The latter statement implies φ of type (1) which is a contradiction. This completes the proof. ■

Lemma 5. *Let φ be an automorphism on E of type* (3)*. If $T_2(E_\varphi) = T_2(E_{can})$, then:*

(*a*) *there exists a natural number k such that φ is defined by*

$$\varphi(e_n) = \begin{cases} -e_n, & \text{if } n \le k \\ -e_n + e_n e_1 \cdots e_k W_n, & \text{if } n > k \end{cases}.$$

Moreover, k even implies $W_n \in E_{(1)}$ and k odd implies $W_n \in Z(E)$, for all $n > k$.
(*b*) $E_{0,\varphi} = Z(E)$.

Proof. Statement (*a*): As φ is of type (3), there exists a finite number of homogeneous generators in the grading, namely $e_1, \ldots, e_k$, for some k.

From Lemma 2 it follows that $e_1, \ldots, e_k \in E_{1,\varphi}$, which implies $\varphi(e_n) = -e_n$, for $n = 1, \ldots, k$. Given $n > k$, using again Lemma 2, we have that $Z_n = \varphi(e_n) + e_n \in Z(E)$ is non-zero. Since $\varphi(e_n)^2 = 0$, we obtain that $-2e_n Z_n + Z_n^2 = 0$. Comparing the lengths of each parcel, we conclude $Z_n = e_n Z'_n$, where $Z'_n \in E_{(1)}$. Now, given $j = 1, \ldots, k$, due to $\varphi(e_n)\varphi(e_j) + \varphi(e_j)\varphi(e_n) = 0$, it follows $Z'_n = e_1 \cdots e_k W_n$. The last part is immediate by the previous equality.

Statement (*b*): Lemma 2 implies the inclusion $E_{0,\varphi} \subset Z(E)$. To prove the reverse inclusion, it is enough to show that $e_i e_j \in E_{0,\varphi}$, for all i, j. When $i, j \in \{1, \ldots, k\}$, $e_i e_j \in E_{0,\varphi}$ is immediate. If $e_i \in E_{1,\varphi}$ and

$$\varphi(e_j) - e_j = -2e_j + e_j e_1 \cdots e_k W_j \in E_{1,\varphi},$$

for every $j \notin \{1, \ldots, k\}$, then $e_i e_j \in E_{0,\varphi}$. Finally, when $i, j \notin \{1, \ldots, k\}$, notice that

$$(\varphi(e_i) - e_i)(\varphi(e_j) - e_j) \in E_{0,\varphi},$$

i.e.,

$$(-2e_i + e_i e_1 \cdots e_k W_i)(-2e_j + e_j e_1 \cdots e_k W_j) \in E_{0,\varphi} \subset Z(E).$$

Therefore, $e_i e_j \in E_{0,\varphi}$, since $e_j e_i e_1 \cdots e_k W_i + e_i e_j e_1 \cdots e_k W_j \in E_{(1)}$, and we are done. ■

The following proposition is an important tool to prove the main result of this section.

Proposition 6. *Let φ be an automorphism on E of order at most 2. If φ is of type* (3) *and $T_2(E_\varphi) = T_2(E_{can})$, then $E_\varphi \simeq E_{can}$.*

Proof. Following the notation of the previous lemma, the set V_1, defined before of Lemma 1, is $\{v_n \mid v_n = e_n \text{ if } n \leq k, v_n = u_n \text{ if } n > k\}$, where $u_n = -2e_n + e_n e_1 \cdots e_k W_n$. According to item (b) of Lemma 5 and Lemma 1, we conclude that

$$E_{0,\varphi} = Z(E) = span_F\{v_{i_1} \cdots v_{i_t} \mid i_1 < \ldots < i_t\},$$

for t even, and

$$E_{1,\varphi} = span_F\{v_{i_1} \cdots v_{i_t} \mid i_1 < \ldots < i_t\},$$

for t odd. The map $f_\varphi : E_{can} \to E_\varphi$, defined by

$$f_\varphi(e_n) = \begin{cases} -2e_n, & \text{if } n \le k \\ v_n, & \text{if } n > k \end{cases},$$

can be extended to a $\mathbb{Z}_2$-graded homomorphism, since it satisfies $f_\varphi(e_i)f_\varphi(e_j) + f_\varphi(e_j)f_\varphi(e_i) = 0$, for all i, j. It is also clear that f_φ is surjective. Therefore,

$$E_\varphi \simeq \frac{E_{can}}{\mathcal{I}},$$

where $\mathcal{I} = \ker(f_\varphi)$ is a graded ideal of E_{can}.

We claim that $\mathcal{I} = \{0\}$. Indeed, given $e_{i_1} \cdots e_{i_t} \in B_E$, we have

$$f_\varphi(e_{i_1} \cdots e_{i_t}) = (-2)^t e_{i_1} \cdots e_{i_t} + \sum_{P_j \in B_E, |P_j| > t} \alpha_j P_j,$$

where $\alpha_j \in F$.

Let $w_1, \dots, w_s \in B_E$ be distinct monomials and $\lambda_1, \dots, \lambda_s \in F$, with each $\lambda_i \neq 0$, satisfying

$$\lambda_1 w_1 + \cdots + \lambda_s w_s \in \mathcal{I}.$$

We can assume $|w_1| \le \ldots \le |w_s|$. It follows that

$$\begin{aligned} 0 &= f_\varphi(\lambda_1 w_1 + \cdots + \lambda_s w_s) \\ &= (-2)^{|w_1|} \lambda_1 w_1 + \sum_{Q_j \in B_E, |Q_j| \ge |w_1|} \sigma_j Q_j, \end{aligned}$$

where each $Q_j \neq w_1$ and $\sigma_j \in F$. The last equality implies $\lambda_1 = 0$, which is a contradiction. Therefore, $\mathcal{I} = \{0\}$ and we are done. ■

Remark 7. We draw the reader's attention that there exists at least one automorphism φ on E of type (3) such that $T_2(E_\varphi) = T_2(E_{can})$, see [10, Proposition 11].

From now on, we will deal with automorphisms of type (4). As promised at the beginning of the section, the next result is a weak version of the conjecture presented in [10].

Theorem 8. *Let φ be an automorphism on the Grassmann algebra E of order at most 2. If $T_2(E_\varphi) = T_2(E_{can})$, then there exists a non-zero vector $v \in L$ homogeneous in the $\mathbb{Z}_2$-grading E_φ. Consequently, φ is not of type* (4).

Proof. By Lemma 2, it is known that $e_i + \varphi(e_i) \in E_{(0)}$. Following word by word the argument presented in item (a) of Lemma 5, for each natural number i, we may write

$$\varphi(e_i) = -e_i + e_i W_i, \tag{1}$$

where $W_i = \lambda_i^1 w_i^1 + \cdots + \lambda_i^{n_i} w_i^{n_i}$, $e_i w_i^{n_r} \neq 0$, for $1 \leq r \leq i$, and the set $\{w_i^j \mid j = 1, \ldots, n_i\} \subset E_{(1)}$ is linearly independent.

Next we consider the following notations. Given a basic element (or monomial) $w = e_{i_1} \cdots e_{i_k}$ in B_E, the set $supp(w) = \{e_{i_1}, \ldots, e_{i_k}\}$ is called *support* of w. For any monomial $w_1 = e_{j_1} \cdots e_{j_l}$ in B_E, we say that w and w_1 have *pairwise disjoint supports* if $supp(w) \cap supp(w_1) = \emptyset$ and, consequently, $ww_1 \neq 0$. And to finish, for each $W = \lambda_1 w_1 + \cdots + \lambda_n w_n$ in E, denote by $\mathcal{S}(W)$ the set formed by the union of $supp(w_i)$, with $1 \leq i \leq n$. Here each w_i lies in B_E and $\lambda_i \in F$. In particular, if $W = W_i$ given in (1), then $e_i \notin \mathcal{S}(W_i)$, for all i.

Since

$$\varphi(e_k)\varphi(e_i) + \varphi(e_i)\varphi(e_k) = 0,$$

for all $i, k \in \mathbb{N}$, we have

$$(-e_k e_i W_i - e_k W_k e_i + e_k W_k e_i W_i) + (-e_i e_k W_k - e_i W_i e_k + e_i W_i e_k W_k) = 0.$$

It follows from the latter equality that

$$-e_k e_i W_i - e_i e_k W_k + e_k W_k e_i W_i = 0.$$

Due to both $e_k e_i W_i + e_i e_k W_k \in E_{(1)}$ and $e_k e_i W_k W_i \in E_{(0)}$, we conclude that $e_k e_i W_i + e_i e_k W_k = 0$ and $e_k e_i W_k W_i = 0$. Consequently,

$$e_k e_i W_i = e_k e_i W_k. \tag{2}$$

Moreover, for any i and j, we can write

$$W_i = e_j P_i + T_i, \tag{3}$$

where $P_i \in E_{(0)}$, $T_i \in E_{(1)}$ and $e_j \notin \mathcal{S}(T_i)$. By (2), if $e_i \notin \mathcal{S}(W_j)$, we then have

$$W_j = T_i \tag{4}$$

Recall that n_i is the number of parcels that occur in W_i. Let $n_k = \min\{n_i \mid i \in \mathbb{N}\}$. If $n_k = 0$, then $\varphi(e_k) = -e_k$ and the result follows. Hence we may assume that $n_k > 0$. We will prove that $W_i = W_k$, for all i. If $e_k \notin \mathcal{S}(W_i)$, for some i, then $T_k = W_i$. By the minimality of k, we obtain $W_k = W_i$. Thus, we suppose that there exists at least one j such that $e_k \in \mathcal{S}(W_j)$. As the set $\mathcal{S}(W_j) \cup \mathcal{S}(W_k)$ is finite, we take $e_i \notin \mathcal{S}(W_j) \cup \mathcal{S}(W_k)$. Due to (4), it follows that $W_j = T_i = W_k$. Therefore, there exists a non-zero $Q \in E_{(1)}$ such that $W_i = Q$, for all i.

So

$$\varphi(e_i) = -e_i + e_iQ,$$

where $Q = W_i$ and $e_i \notin \mathcal{S}(W_i)$. Assuming $e_t \in \mathcal{S}(Q)$, we obtain

$$\varphi(e_t) = -e_t + e_tQ,$$

where $Q = W_t$, which contradicts the fact that $e_t \notin \mathcal{S}(W_t) = \mathcal{S}(Q)$. We conclude that $n_k = 0$, and e_k is a homogeneous element in the $\mathbb{Z}_2$-grading E_φ. ■

Due to our results, we obtain a characterization of E_{can} via $T_2(E_{can})$. The next is one of the main results of this section.

Theorem 9. *Let φ be an automorphism on the Grassmann algebra E of order at most 2. If $T_2(E_\varphi) = T_2(E_{can})$, then E_φ and E_{can} are $\mathbb{Z}_2$-isomorphic.*

Proof. As E_φ is PI-equivalent to E_{can}, Proposition 4 and Theorem 8 imply that φ is of type (1) or (3). If φ is of type (1), we apply item (b) of Proposition 3. If φ is of type (3), we apply Proposition 6. So we are done. ■

The arguments applied in Proposition 4, Proposition 6 and Theorem 8 work with a weaker assumption. More specifically, assuming $\varphi \in Aut(E)$ such that $\varphi^2 = id$, the hypothesis $T_2(E_\varphi) = T_2(E_{can})$ can be replaced for

$$[y, x] \in T_2(E_\varphi),$$

where $x \in Y \cup Z$.

In the light of the last comment, we have a better characterization for the structure E_{can}.

Theorem 10. *Let φ be an automorphism on the Grassmann algebra E of order at most 2, and let $E_\varphi = E_{0,\varphi} \oplus E_{1,\varphi}$ be the $\mathbb{Z}_2$-grading on E induced by φ. The following statements are equivalent:*

(1) $T_2(E_\varphi) = T_2(E_{can})$.
(2) $[y, x] \in T_2(E_\varphi)$, *where* $x \in Y \cup Z$.
(3) $E_{0,\varphi} = Z(E)$.
(4) $E_\varphi \simeq E_{can}$.

Proof. Notice that (1) ⇒ (2) and (4) ⇒ (1) are immediate. Already for (2) ⇒ (3) and (3) ⇒ (4) we use similar arguments contained in the proof of Lemma 5. ■

4. Concrete $\mathbb{Z}_2$-gradings of type (4)

In the paper [10], the authors conjectured the non-existence of automorphisms of type (4), and it was strengthened by Theorem 8. The goal of this section is to construct a kind of automorphism that gives a negative answer to the conjecture.

First we will introduce some notations that will be useful.

Definition 11. Let $I = \{2n \in \mathbb{N}, n > 1 \mid \exists m \in \mathbb{N} \text{ such that } 2^m \leq n < 2^m + 2^{m-1}\}$. We define the sequence $\{\epsilon_i\}_{i\in\mathbb{N}}$ by

$$\begin{aligned}
&\epsilon_{2n-1} = -1, \text{ if } 2n \in I;\\
&\epsilon_{2n-1} = 1, \text{ if } 2n \in \mathbb{N} \setminus I;\\
&\epsilon_{2n} = 1, \text{ for all n.}
\end{aligned}$$

Remark 12. According to Definition 11, we have $\epsilon_1 = 1$, $\epsilon_2 = 1$, $\epsilon_3 = -1$, $\epsilon_4 = 1$, $\epsilon_5 = 1$, $\epsilon_6 = 1$, $\epsilon_7 = -1$, and so on.

Lemma 13. *Given a natural number n, we have $\epsilon_1 \cdots \epsilon_{2n+1} = -\epsilon_n$.*

Proof. We use induction on n. In the proof we omit the scalars with even indexes in the product. By Remark 12, we have that $\epsilon_1\epsilon_3 = -\epsilon_1$. So the result holds for $n = 1$. From now on, assume the validity of the result for $n > 1$. We take into consideration the following cases:

If n is even, there exists $k \in \mathbb{N}$ such that $n = 2k$. In this case, by induction hypothesis

$$\epsilon_1\epsilon_3 \cdots \epsilon_{4k-1}\epsilon_{4k+1} = -1,$$

and we need to show

$$\epsilon_1\epsilon_3 \cdots \epsilon_{4k+1}\epsilon_{4k+3} = -\epsilon_{2k+1},$$

which is equivalent to show that $\epsilon_{4k+3} = \epsilon_{2k+1}$. If $\epsilon_{2k+1} = -1$, then $2(k+1) \in I$. Hence, there exists a natural number m such that

$$2^m \leq k+1 < 2^m + 2^{m-1}.$$

Therefore,

$$2^{m+1} \leq 2k+2 < 2^{m+1} + 2^m.$$

This implies $2k+2 \in I$, hence $\epsilon_{4k+3} = -1$. Similarly $\epsilon_{4k+3} = -1$ implies $\epsilon_{2k+1} = -1$, since $k > 0$. Therefore, we proved that $\epsilon_{4k+3} = -1$ if and only if $\epsilon_{2k+1} = -1$ and the result follows for n even.

Now, we analyze the case n odd, in other words, there exists $k \in \mathbb{N}$ such that $n = 2k+1$. By induction, we assume

$$\epsilon_1 \cdots \epsilon_{4k+3} = -\epsilon_{2k+1}.$$

As it was done before

$$\epsilon_1 \cdots \epsilon_{4k+5} = -1$$

is equivalent to show that $\epsilon_{4k+5} = \epsilon_{2k+1}$. If $\epsilon_{2k+1} = -1$, then $2(k+1) \in I$. Hence, there exists natural number m such that

$$2^m \leq k+1 < 2^m + 2^{m-1}.$$

As $2^m + 2^{m-1} - 1 < 2^m + 2^{m-1}$, we have

$$2^m \leq k+1 \leq 2^m + 2^{m-1} - 1,$$

so

$$2^{m+1} \leq 2k+2 \leq 2^{m+1} + 2^m - 2.$$

Due to this last inequality, it follows that

$$2^{m+1} \leq 2^{m+1} + 1 \leq 2k+3 \leq 2^{m+1} + 2^m - 1 < 2^{m+1} + 2^m.$$

Hence $2k+3 \in I$ and it implies $\epsilon_{4k+5} = -1$. Analogously, if $\epsilon_{4k+5} = -1$, then $\epsilon_{2k+1} = -1$. The result is proved for n odd, and we are done. ■

Let $w_n = e_1 e_2 \cdots e_{2n+1}$. We define the linear transformation $\lambda : L \to L$ by

$$\lambda(e_i) = \epsilon_i e_i,$$

where the sequence $\{\epsilon_i\}_{i\in\mathbb{N}}$ was presented in Definition 11. We extend λ to an unique automorphism of E. According to the previous lemma, we have

$$\lambda(w_n) = \epsilon_1 \cdots \epsilon_{2n+1} w_n = -\epsilon_n w_n.$$

Next, we will construct an automorphism of type (4) of the Grassmann algebra E. For this, we define the linear transformation $\varphi : E \to E$ by

$$\varphi(e_i) = \epsilon_i e_i + w_i.$$

The following theorem is the main result of this section.

Theorem 14. *The linear transformation φ defined above is an automorphism of type (4) of the Grassmann algebra.*

Proof. It is well known that any linear transformation ϕ on E satisfying

$$\phi(e_i)\phi(e_j) + \phi(e_j)\phi(e_i) = 0,$$

for any i, $j \in \mathbb{N}$, can be extended to an unique endomorphism of E. As $\epsilon_i e_i + w_i \in E_{(1)}$, it follows that φ can be extended to an endomorphism of the algebra E. Besides,

$$\varphi^2(e_i) = \epsilon_i(\epsilon_i e_i + w_i) + \varphi(w_i).$$

Notice that

$$\varphi(w_i) = \varphi(e_1)\cdots\varphi(e_{2i+1}) = \epsilon_1 \cdots \epsilon_{2i+1} w_i = -\epsilon_i w_i.$$

The last equality follows from Lemma 13. Thus, φ is an automorphism of order 2.

Now we claim that φ is of type (4). Assume $v \in L$ such that $\varphi(v) = \pm v$. There exist $\alpha_1, \ldots, \alpha_n \in F$ so that $v = \alpha_1 e_{i_1} + \cdots + \alpha_n e_{i_n}$. Thus,

$$\pm(\sum_{k=1}^{n} \alpha_k e_{i_k}) = \pm v = \varphi(v) = \sum_{k=1}^{n} \alpha_k(\epsilon_k e_{i_k} + w_{i_k}).$$

Comparing the lengths of the parcels in the last equality, we conclude that $\sum_{k=1}^{n} \alpha_k w_{i_k} = 0$. As the set $\{w_i \mid i \in \mathbb{N}\}$ is linearly independent, we have $\alpha_1 = \cdots = \alpha_n = 0$. Therefore, we have the result. ■

Let E_φ be the $\mathbb{Z}_2$-grading on E induced by the automorphism φ constructed above. As a consequence of Theorem 8, we conclude immediately that

$$T_2(E_\varphi) \neq T_2(E_{can}).$$

A natural question arises. What are the generators of the T_2-ideal of the $\mathbb{Z}_2$-graded polynomial identities of E_φ? The following result answers such question.

Proposition 15. *The $\mathbb{Z}_2$-graded algebras E_φ and E_∞ are isomorphic superalgebras.*

Proof. Let $\{\epsilon_i\}_{i\in\mathbb{N}}$ be the sequence given in Definition 11. We consider $A = \{i \in \mathbb{N} \mid \epsilon_i = 1\}$ and $B = \{i \in \mathbb{N} \mid \epsilon_i = -1\}$. Assume that E_h is the homogeneous $\mathbb{Z}_2$-grading on E given by

$$\|e_i\| = \begin{cases} 0, & \text{if } i \in A \\ 1, & \text{if } i \in B \end{cases}.$$

Since both A and B are infinite, we conclude that E_h and E_∞ are isomorphic superalgebras. Now we define the map $f : E_h \to E_\varphi$ by

$$f(e_i) = \begin{cases} e_i + w_i/2, & \text{if } i \in A \\ e_i - w_i/2, & \text{if } i \in B \end{cases} .$$

It follows that f can be extended to an endomorphism of E, once $w_i \in E_{(1)}$, for each $i \in \mathbb{N}$. Moreover, f is a $\mathbb{Z}_2$-graded endomorphism.

For each i, notice that

$$f(w_i) = f(e_1 \cdots e_{2i+1}) = (e_1 \pm \frac{w_1}{2}) \cdots (e_{2i+1} \pm \frac{w_{2i+1}}{2}) = w_i.$$

We claim that f is an isomorphism, for this it is enough to provide the inverse map for f. Let $g : E_\varphi \to E_h$ be the map given by

$$g(e_i) = \begin{cases} e_i - w_i/2, & \text{if } i \in A \\ e_i + w_i/2, & \text{if } i \in B \end{cases} .$$

As it was done for f, it follows that $g(w_i) = w_i$, for each $i \in \mathbb{N}$. When $i \in A$, we have

$$g \circ f(e_i) = g(e_i + w_i/2) = e_i - w_i/2 + w_i/2 = e_i.$$

Similarly, $g \circ f(e_i) = e_i$, if $i \in B$. Therefore, we have $g = f^{-1}$, and the result follows. ∎

In [10], the authors constructed $\mathbb{Z}_2$-gradings on E of types (2) and (3). In the present section, we provide a concrete $\mathbb{Z}_2$-grading of type (4). It turns out that all these structures are $\mathbb{Z}_2$-isomorphic to some homogeneous $\mathbb{Z}_2$-grading of E. These statements provide us with the ground to pose the following conjecture.

Conjecture 1. *Every $\mathbb{Z}_2$-grading of E is $\mathbb{Z}_2$-graded isomorphic to some homogeneous $\mathbb{Z}_2$-grading of E.*

Declaration of competing interest

None declared.

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