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On the topology of the Milnor boundary for real analytic singularities

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Abstract

We study the topology of the boundaries ∂F_f and ∂F_I of the Milnor fibers F_f and F_I, respectively, of real analytic map-germs $f\colon(\mathbb{R}^M,0)\to(\mathbb{R}^K,0)$ and $f_I := \Pi_I \circ f\colon(\mathbb{R}^M,0)\to(\mathbb{R}^I,0)$ that admit Milnor's tube fibrations, where $\Pi_I\colon(\mathbb{R}^K,0)\to(\mathbb{R}^I,0)$ is the canonical projection for $1 \le I < K$. For each I we prove that the Milnor boundary ∂F_I is given by the double of the Milnor tube fiber F_{I+1}. Beside that, if $K - I \ge 2$, we prove that the pair $(\partial F_I, \partial F_f)$ is a generalized $(K-I-1)$-open-book decomposition with binding ∂F_f and page $F_f \backslash \partial F_f$—the interior of the Milnor fibre F_f. This allows us to prove several new Euler characteristic formulae connecting the Milnor boundaries ∂F_f, ∂F_I, with the respective links $\mathcal{L}_f$, $\mathcal{L}_I$, for each $1 \le I < K$, and a Lê–Greuel type formula for the Milnor boundary.

1 Introduction

One of the most active and challenging areas in singularity theory is the study of non-isolated singularities of complex spaces. For instance, if $f\colon(\mathbb{C}^n,0)\to(\mathbb{C},0)$ is a holomorphic germ of function with non-isolated critical point, the degeneration process of the non-critical levels to the non-isolated singularity hypersurface defined by f is still not well-understood, unlike the isolated singularity case.

One approach to this problem is to study such degeneration over a small sphere around the origin. In other words, one tries to understand the topology of the boundary of the Milnor fiber and how it degenerates to the link of f. This problem has been attacked by several authors like Siersma [28, 29], Nemethi and Szilard [25], Michel and Pichon [21–23], Bobadilla and Menegon [11], Menegon and Seade [20] and Aguilar et al. [1].

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The corresponding understanding for real analytic singularities is still very poor. Although one can define a Milnor fibration for many classes of real analytic germs of mapping $f\colon(\mathbb{R}^M,0)\to(\mathbb{R}^K,0)$, not much is known about the topology of the corresponding Milnor fiber or the link of f (see [15, 18, 20, 26] for some results), and even less about the boundary of such objects.

The first part of this paper aims to introduce a new perspective to deal with such problem, inspired mainly by [3, 9, 19]. The idea is to relate the topology of the boundary of the Milnor fiber of f, denoted by ∂F_f, with the boundary of the Milnor fiber of the composition f_I of f with some projection $(\mathbb{R}^K,0)\to(\mathbb{R}^I,0)$, which we denote by ∂F_I. As a result, in Sect. 3 we prove that for $K-I\geq 2$ there is a generalized open-book decomposition

$$\frac{f^{K-I}}{\|f^{K-I}\|}\colon \partial F_I\backslash\partial F_f \to S^{K-I-1}, \tag{1}$$

where f^{K-I} is the composition of f with the projection $(\mathbb{R}^K,0)\to(\mathbb{R}^{K-I},0)$. The particular case $0\leq K-I\leq 1$ is analyzed.

On the other hand, the understanding of the topology of the boundary of the Milnor fiber of the function-germ f_I also provides a tool to better understanding the topology of the Milnor fiber of the map-germ f itself. In fact, in Sect. 4 we use the aforementioned open-book decomposition to obtain some formulae relating the Euler characteristics of ∂F_I, F_f and the link $\mathcal{L}_I$ of f_I, for $I=1,\ldots,K$.

Finally, in the last section of the article we use those Euler characteristic formulae to get a hint on the possible topological behaviour of real analytic map-germs on an odd number of variables and how similar or different it can be when compared with the complex setting.

2 Notations and basics definitions

Let $f\colon(\mathbb{R}^M,0)\to(\mathbb{R}^K,0)$, $f=(f_1,\ldots,f_K)$ be an analytic map germ and consider the following diagram

$$\begin{array}{ccccc} & & (\mathbb{R}^M,0) & & \\ & \overset{f_I}{\swarrow} & \downarrow f & \overset{f^{K-I}}{\searrow} & \\ & & (\mathbb{R}^K,0) & & \\ & \overset{\Pi_I}{\swarrow} & & \overset{\Pi^{K-I}}{\searrow} & \\ (\mathbb{R}^I,0) & & & & (\mathbb{R}^{K-I},0) \end{array} \tag{2}$$

where the projections $\Pi_I(y_1,\ldots,y_K)=(y_1,\ldots,y_I)$ and $\Pi^{K-I}(y_1,\ldots,y_K):=(y_{I+1},\ldots,y_K)$, $f_I=\Pi_I\circ f$ and $f^{K-I}=\Pi^{K-I}\circ f$.

Basic notations and definitions: The *zero locus* of f is defined and denoted by $V(f):=\{f=0\}$, respectively, $V(f_I)=\{f_I=0\}$ and $V(f^{K-I})=\{f^{K-I}=0\}$. Hence,

$$V(f_I)\supseteq V(f)\subseteq V(f^{K-I}).$$

The *singular set* of f, denoted by Sing f, is defined to be the set of points $x\in(\mathbb{R}^M,0)$ such that the rank of the Jacobian matrix $df(x)$ is lower than K. Analogously, we define the singular sets Sing f_I and Sing f^{K-I} of f_I and f^{K-I}, respectively. The *discriminant set* of

f is then defined by

$$\mathrm{Disc}\, f := f(\mathrm{Sing}\, f).$$

The *polar set* of f relative to $g(x) := \|x\|^2$ is defined and denoted by Sing (f, g). Analogously, we define Sing (f_I, g) and Sing (f^{K-I}, g).

The next diagram relates the singular and the polar sets:

$$\begin{array}{ccccc} \mathrm{Sing}\,(f_I) & \hookrightarrow & \mathrm{Sing}\,(f_I, g) & \hookleftarrow & \mathrm{Sing}\left(\frac{f_I}{\|f_I\|}, g\right) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Sing}\,(f) & \hookrightarrow & \mathrm{Sing}\,(f, g) & \hookleftarrow & \mathrm{Sing}\left(\frac{f}{\|f\|}, g\right) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Sing}\,(f^{K-I}) & \hookrightarrow & \mathrm{Sing}\,(f^{K-I}, g) & \hookleftarrow & \mathrm{Sing}\left(\frac{f^{K-I}}{\|f^{K-I}\|}, g\right) \end{array} \tag{3}$$

Remark 2.1 Some brief comments concerning diagram (3):

(a) each map on the right side of the diagram (3) should be read in their respective domain; for instance, $\frac{f}{\|f\|}\colon U\backslash V(f) \to S^{K-1}$, where U is an open neighborhood of 0 in $\mathbb{R}^M$,

(b) let us point out that the column on the right will be used in the proof of Corollary 3.3.

Definition 2.2 We say that a map germ $f\colon (\mathbb{R}^M, 0) \to (\mathbb{R}^K, 0)$, $f = (f_1, \ldots, f_K)$ is *tame*, or satisfies the transversality condition at the origin if

$$\overline{\mathrm{Sing}\,(f, g)\backslash V(f)} \cap \mathrm{Sing}\, f \subseteq \{0\}$$

as a germ of set at the origin.

The next lemma has a more general version as proved in [4]; however, the version below is sufficient.

Lemma 2.3 [9, Lemma 4.1] *Let* $1 \leq I \leq K-1$. *If* f *is tame and* Disc $f = \{0\}$, *then* f_I *and* f^{K-I} *are also tame, and* Disc $f_I = \{0\}$ *and* Disc $f^{K-I} = \{0\}$.

It follows from the proof of Lemma 4.2 in [16] that the tameness conditions for f, f_I and f^{K-I} induce the following fibrations on the boundary of the closed ball $S_\epsilon^{M-1} := \partial B_\epsilon^M$:

$$f_|\colon S_\epsilon^{M-1} \cap f^{-1}(B_{\eta_1}^K\backslash\{0\}) \to B_{\eta_1}^K\backslash\{0\} \tag{4}$$

$$f_{I|}\colon S_\epsilon^{M-1} \cap f_I^{-1}(B_{\eta_2}^I\backslash\{0\}) \to B_{\eta_2}^I\backslash\{0\} \tag{5}$$

$$f^{K-I}{}_|\colon S_\epsilon^{M-1} \cap (f^{K-I})^{-1}(B_{\eta_3}^{K-I}\backslash\{0\}) \to B_{\eta_3}^{K-I}\backslash\{0\} \tag{6}$$

Moreover, under the extra conditions Disc $f = \{0\}$ the [16, Theorem 4.4] ensure the existence of the *Milnor tube fibration* in the following sense: there exists $\epsilon_0 > 0$ small enough such that for all $0 < \epsilon \leq \epsilon_0$ there exists $0 < \eta_1 \ll \epsilon$ such that the restriction map

$$f_|\colon B_\epsilon^M \cap f^{-1}(B_{\eta_1}^K\backslash\{0\}) \to B_{\eta_1}^K\backslash\{0\} \tag{7}$$

is a locally trivial smooth fibration, where B_ϵ^M and B_ϵ^K stand for the closed ball in $\mathbb{R}^M$ with radius ϵ, centered at the origin, and in $\mathbb{R}^K$ with radius η_1, respectively.

Hence, we conclude the existence of Milnor tube fibrations for f_I and f^{K-I} :

$$f_{I_|}: B_\epsilon^M \cap f_I^{-1}(B_{\eta_2}^I \backslash \{0\}) \to B_{\eta_2}^I \backslash \{0\} \tag{8}$$

$$f^{K-I}{}_{|}: B_\epsilon^M \cap (f^{K-I})^{-1}(B_{\eta_3}^{K-I} \backslash \{0\}) \to B_{\eta_3}^{K-I} \backslash \{0\} \tag{9}$$

From now on, denote by F_f, F_I and F^{K-I} the Milnor fibers of the fibrations (7)–(9), respectively, by ∂F_f, ∂F_I and ∂F^{K-I} the fibers of (4)–(6).

Consider the Milnor tube fibration $f_{I_|} : B_\epsilon^M \cap f_I^{-1}(B_{\eta_2}^I \backslash \{0\}) \to B_{\eta_2}^I \backslash \{0\}$, $f_I = \Pi_I \circ f$, and $z \in B_{\eta_2}^I \backslash \{0\}$. Thus the fiber $F_I = f^{-1}(\Pi_I^{-1}(z))$.

It was proved in [9, Theorem 6.3] that

$$F_I \approx F_f \times D^{K-I}, \tag{10}$$

where D^{K-I} is the $(K - I)$-dimensional closed disc. Hence, we get

$$\partial F_I \approx (\partial F_f \times D^{K-I}) \cup (F_f \times S^{K-I-1}). \tag{11}$$

Proposition 2.4 *Let $f : (\mathbb{R}^M, 0) \to (\mathbb{R}^K, 0)$, $M > K \geq 2$, be a tame map germ with* $\operatorname{Disc} f = \{0\}$ *and for $1 \leq I < K$ consider the composition map $f_I = \Pi_I \circ f$ where $\Pi_I : \mathbb{R}^K \to \mathbb{R}^I$ is the projection map. Then the boundary ∂F_I of the Milnor fiber F_I is obtained (up to homeomorphism) by the gluing together two disjoint copies of the Milnor fiber F_{I+1} along the common boundary ∂F_{I+1}.*

Proof The proof follows from the composition

$$\begin{array}{ccc} (\mathbb{R}^M, 0) & \xrightarrow{f_{I+1}} & (\mathbb{R}^{I+1}, 0) \\ & \searrow_{f_I = \widehat{\Pi}_I \circ f_{I+1}} & \downarrow \widehat{\Pi}_I \\ & & (\mathbb{R}^I, 0) \end{array}$$

where $\widehat{\Pi}_I(y_1, \ldots, y_{I+1}) = (y_1, \ldots, y_I)$ and the fact that f being tame implies the same to f_{I+1} and f_I. In the same manner $\operatorname{Disc} f = \{0\}$ implies $\operatorname{Disc} f_{I+1} = \{0\}$, $\operatorname{Disc} f_I = \{0\}$.

Hence, $F_I \approx F_{I+1} \times [-1, 1]$ and the boundary

$$\partial F_I \approx (\partial F_{I+1} \times [-1, 1]) \cup (F_{I+1} \times \{-1, 1\}) \tag{12}$$

Now it is easy to see that the closed manifold ∂F_I is obtained by the gluing the two disjoint copies of F_{I+1} given by $F_{I+1} \times \{-1\} \cup F_{I+1} \times \{1\}$ along the boundaries of the cylinder $\partial F_{I+1} \times [-1, 1]$. Therefore the result follows (Fig. 1). □

3 Fibration structure on the boundary of the milnor fiber

From now on we will consider f tame, $\operatorname{Disc} f = \{0\}$, and $V(f) \neq \{0\}$. Consider the map $f^{K-I}_{|\partial F_I} : \partial F_I \to \mathbb{R}^{K-I}$.

Lemma 3.1 *If f is tame, then $0 \in \mathbb{R}^{K-I}$ is a regular value of $f^{K-I}_{|\partial F_I}$.*

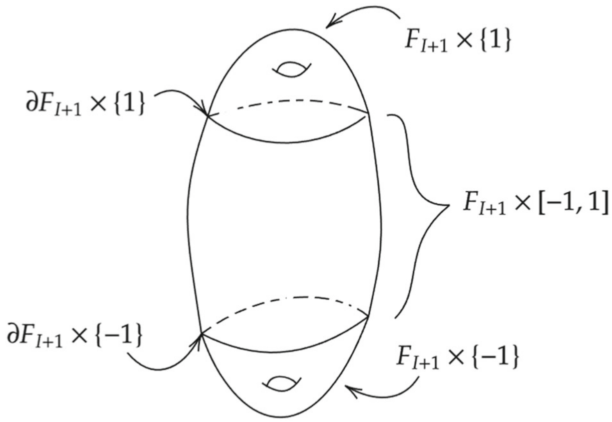

Fig. 1 Gluing two copies of (F_{I+1}) along (∂F_{I+1})

Proof Two different proofs concerning this statement may be found: one in [9, Lemma 4.1, p. 4855], and another more recently in [4]. But we will scratch one here for the sake of convenience.

By definition, we have that $V(f) \subset V(f^{K-I})$, hence $V(f^{K-I})^c \subset V(f)^c$, where $V(f)^c$ means the complement of $V(f)$ inside a closed small ball B_ϵ^M, respectively for $V(f^{K-I})^c$. By the diagram (3), we get that

$$\text{Sing}\,(f^{K-I}, g) \subset \text{Sing}\,(f, g) \;\; \text{and} \;\; \text{Sing}\,(f^{K-I}) \subset \text{Sing}\,(f), \tag{13}$$

then $\text{Sing}\,(f^{K-I}, g) \cap V(f^{K-I})^c \subset \text{Sing}\,(f, g) \cap V(f)^c$ and

$$\overline{\text{Sing}\,(f^{K-I}, g) \backslash V(f^{K-I})} \subset \overline{\text{Sing}\,(f, g) \backslash V(f)}. \tag{14}$$

From (13) and (14) together, we get

$$\overline{\text{Sing}\,(f^{K-I}, g) \backslash V(f^{K-I})} \cap \text{Sing}\,(f^{K-I}) \subset \overline{\text{Sing}\,(f, g) \backslash V(f)} \cap \text{Sing}\,(f).$$

Therefore, clearly the statement follows by the tameness of f. □

By compactness of ∂F_I one may choose $\tau > 0$ small enough and a closed disc $D_\tau^{K-I} \subset \mathbb{R}^{K-I}$ centered at the origin $0 \in \mathbb{R}^{K-I}$, such that all $y \in D_\tau^{K-I}$ is a regular value of the restriction $f^{K-I}_{|\partial F_I}$. Hence the restriction map

$$f^{K-I}_{|\partial F_I} : \partial F_I \cap (f^{K-I}_{|\partial F_I})^{-1}(D_\tau^{K-I}) \to D_\tau^{K-I} \tag{15}$$

is a smooth, proper and onto submersion, then a trivial fibration with the fiber diffeomorphic to $\partial F_f = \partial F_I \cap (f^{K-I})^{-1}(0)$. Therefore,

$$\partial F_I \cap (f^{K-I})^{-1}(D_\tau^{K-I}) \approx \partial F_f \times D_\tau^{K-I}. \tag{16}$$

For the sake of simplicity, we denote $f^{K-I}_{|\partial F_I}$ by f^{K-I} if no confusion is allowed.

Denote by $T_\tau(\partial F_f) := \partial F_f \times D_\tau^{K-I}$ the closed tubular neighbourhood of the embedded submanifold $\partial F_f \hookrightarrow \partial F_I$ (Fig. 2).

Then, by (11) it follows that the complement

$$\partial F_I \backslash int(T_\tau(\partial F_f)) \approx F_f \times S^{K-I-1} \tag{17}$$

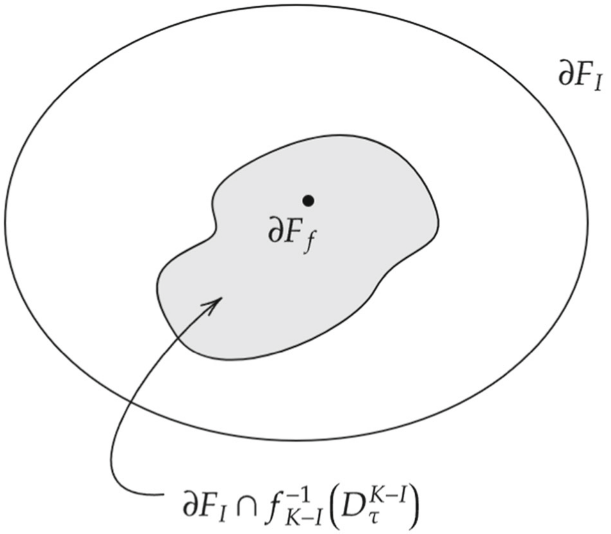

Fig. 2 A tubular neighborhood ($T(N)$), the radial projection, and a fibration over (S^{k-1})

In this section we will prove that the embedded submanifold $\partial F_f \hookrightarrow \partial F_I$ yields on the boundary ∂F_I an interesting structure. For that, let $f\colon (\mathbb{R}^M, 0) \to (\mathbb{R}^K, 0)$, $M > K \geq 2$ and take $1 \leq I \leq K-2$ as in the beginning of Sect. 2.

We are now ready to introduce the main result of this section. Before that, we introduce an appropriate definition which fits with the type of fibration structure we are able to prove on the boundary of the Milnor fiber.

Following Winkelnkemper [30], Ranicki [27],[1] Looijenga [13], see also [10] and [5, section 3]: given M a smooth manifold M and $N \subset M$ a submanifold of codimension $k \geq 2$ in M, suppose that for some trivialization $t\colon T(N) \to N \times B^k$ of a tubular neighbourhood $T(N)$ of N in M, the fiber bundle defined by the composition $\pi \circ t$ in the diagram below

$$\begin{array}{ccc} T(N)\backslash N & \xrightarrow{t} & N \times (B^k\backslash\{0\}) \\ & \searrow^{\pi\circ t} & \downarrow \pi \\ & & S^{k-1} \end{array}$$

where $\pi(x, y) := \dfrac{y}{\|y\|}$, extends to a smooth locally trivial fiber bundle $p : M\backslash N \to S^{k-1}$; i.e., $p_{|T(N)\backslash N} = \pi \circ t$.

In such a case the pair (M, N) above will be called a *generalized* $(k-1)$*-open-book decomposition on M with binding N* and *page of the book* the fiber $p^{-1}(y)$, $y \in S^{k-1}$.

The main result of this section is:

Theorem 3.2 *Let* $f\colon (\mathbb{R}^M, 0) \to (\mathbb{R}^K, 0)$, $M > K$, *be a tame map germ with* $\operatorname{Disc} f = \{0\}$. *Consider* $1 \leq I < K$, $K - I \geq 2$, *such that* $\operatorname{Sing}\left(\dfrac{f^{K-I}}{\|f^{K-I}\|}, g\right) = \emptyset$. *Then the pair* $(\partial F_I, \partial F_f)$ *is a generalized* $(K-I-1)$*-open-book decomposition, with binding* ∂F_f *and page* $F_f\backslash\partial F_f$ *the interior of the Milnor fiber* F_f.

[1] Including the appendix "The history and applications of open books", by H. E. Winkelnkemper.

Proof The diagram below follows from Eq. (15)

$$\begin{array}{ccc} \partial F_I \cap (f^{K-I})^{-1}(D_\tau^{K-I} - \{0\}) & \xrightarrow{f^{K-I}} & D_\tau^{K-I} - \{0\} \\ & \searrow^{\frac{f^{K-I}}{\|f^{K-I}\|}} & \downarrow \Pi_R(z)=\frac{z}{\|z\|} \\ & & S^{K-I-1} \end{array} \tag{18}$$

where we get that the projection $\frac{f^{K-I}}{\|f^{K-I}\|}$ is a (trivial) fiber bundle, where Π_R is the radial projection. □

It induces the trivial fibration on the diagonal projection

$$\begin{array}{ccc} \partial F_I \cap (f^{K-I})^{-1}(S_\tau^{K-I-1}) & \xrightarrow{f^{K-I}} & S_\tau^{K-I-1} \\ & \searrow^{\frac{f^{K-I}}{\|f^{K-I}\|}} & \downarrow \Pi_R(z)=\frac{z}{\|z\|} \\ & & S^{K-I-1} \end{array} \tag{19}$$

The condition Sing $\left(\frac{f^{K-I}}{\|f^{K-I}\|}, g\right) = \emptyset$, assure that

$$\frac{f^{K-I}}{\|f^{K-I}\|}: \partial F_I \backslash (f^{K-I})^{-1}(int(D_\tau^{K-I})) \to S^{K-I-1}. \tag{20}$$

is a smooth submersion which is also proper and onto by (19). Consequently, it is a smooth locally trivial fibration.

Now we may glue together the fibrations (18), (20) and (19) to get the smooth locally trivial fiber bundle

$$\frac{f^{K-I}}{\|f^{K-I}\|}: \partial F_I \backslash \partial F_f \to S^{K-I-1}. \tag{21}$$

Now see that the diffeomorphism of (16) says that

$$\partial F_I \cap (f^{K-I})^{-1}(D_\tau^{K-I} - \{0\}) \approx \partial F_f \times (S_\tau^{K-I-1} \times (0, \tau]).$$

Hence the fiber of the diagonal fibration in the diagram (18) should be diffeomorphic to $\partial F_f \times (0, \tau]$.

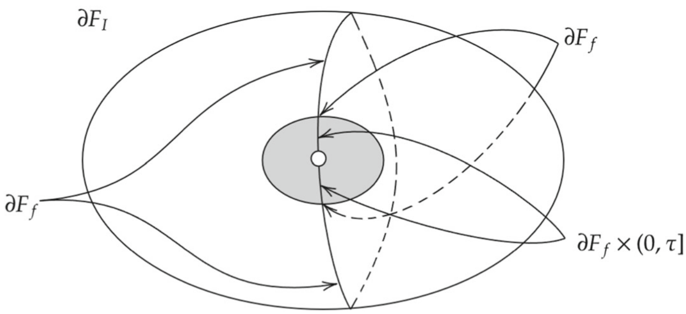

Fig. 3 An open book structure on (∂F_I), with binding (∂F_f) and pages given by the interior $(F_f \setminus \partial F_f)$

On the other hand, the diffeomorphism (17) assures that the fiber of the diagram (20) is diffeomorphic to F_f. The fiber of the fibration (19) is clearly diffeomorphic to ∂F_f. Therefore, we conclude that the fiber of fibration (21) must be diffeomorphic to the gluing (using the identity diffeomorphism on the boundary) $F_f \cup_{\partial F_f} (\partial F_f \times (0, \tau]) = F_f \backslash \partial F_f$. See Fig. 3 and the proof is finished.

Corollary 3.3 *Let* $f\colon (\mathbb{R}^M, 0) \to (\mathbb{R}^K, 0)$*,* $M > K$*, be a tame map germ with* Disc $f = \{0\}$ *and* Sing $\left(\frac{f}{\|f\|}, g\right) = \emptyset$*. Then, for any* I*, with* $1 \leq I < K$*,* $K - I \geq 2$*, the pair* $(\partial F_I, \partial F_f)$ *is a generalized* $(K - I - 1)$*-open-book decomposition, with binding* ∂F_f *and page* $F_f \backslash \partial F_f$ *the interior of the Milnor fiber* F_f*.*

Proof It follows directly from the right column of the diagram (3) and Theorem 3.2. □

Remark 3.4 Two important remarks about the Theorem 3.2.

(1) For $K - I = 2$ the generalized open-book decomposition is classically also called an *open book structure* on ∂F_I with binding ∂F_f and page $F_f \backslash \partial F_f$ (see [2]).
(2) In the Theorem 3.2 it was considered that $K - I \geq 2$. For completeness let us say some works regarding the reminder conditions, i.e., $0 \leq K - I \leq 1$. For $K = I$ one may use the convention that $f^{K-I} := f_0 \equiv 0$ and $\mathbb{R}^0 = \{0\}$. Hence, nothing else to be said. For $I = K - 1$, then the study of the restriction function $f^{K-I} : \partial F_I \to \mathbb{R}$ reduce to that of Proposition 2.4. Thus, in the view of equation (11) one can see that the construction above leads to a "fibration" over $S^0 = \{-1, 1\}$.

4 The euler characteristic formulae

In this section we prove some Euler characteristic formulae relating the Milnor fibers, the Milnor boundaries, and the link of the singularity along the projections. We should say that many of them were inspired by [9].

Let $f\colon (\mathbb{R}^M, 0) \to (\mathbb{R}^K, 0)$, $M > K \geq 2$, be an analytic map-germ. We will assume along the section that f is tame and Disc $f = \{0\}$. Thus, dim $F_f = M - K$ and dim $\partial F_f = M - K - 1$.

Denote by $\widehat{F_f} = F_f \cup_{\partial F_f} F_f$ the closed manifold built by gluing two copies of F_f along the boundary ∂F_f using the identity diffeomorphism on ∂F_f. By the additive property of the Euler characteristic we have that $\chi(\widehat{F_f}) = 2\chi(F_f) - \chi(\partial F_f)$. Since closed odd dimensional manifolds have Euler characteristic 0, one has

$$\chi(\partial F_f) = \begin{cases} 0, & \text{if M-K is even.} \\ 2\chi(F_f), & \text{if M-K is odd.} \end{cases} \tag{22}$$

Applying again the additive Euler characteristic to the diffeomorphism (11) we get

$$\begin{aligned} \chi(\partial F_I) &= \chi(\partial F_f \times D^{K-I}) + \chi(F_f \times S^{K-I-1}) - \chi(\partial F_f \times S^{K-I-1}) \\ &= \chi(\partial F_f) + \chi(F_f) \cdot \chi(S^{K-I-1}) - \chi(\partial F_f) \cdot \chi(S^{K-I-1}). \end{aligned}$$

But $2 - \chi(S^{K-I-1}) = \chi(S^{K-I})$. Thus, together with (22) one gets

$$\chi(\partial F_I) = \begin{cases} \chi(F_f) \cdot \chi(S^{K-I-1}), & \text{if M-K is even.} \\ \chi(F_f) \cdot \chi(S^{K-I}), & \text{if M-K is odd.} \end{cases} \tag{23}$$

We may consider the convention $\chi(S^{-1}) = \chi(\emptyset) = 0$ and the fact that for the 0-dimensional sphere $\chi(S^0) = \chi(\{-1, 1\}) = 2$. Then all the above discussion, including the special cases of $I = 1$ and $I = K$ may be summarized as below.

Theorem 4.1 *Let* $f : (\mathbb{R}^M, 0) \to (\mathbb{R}^K, 0)$, $M > K \geq 2$, *be an analytic map-germ, tame with* $\operatorname{Disc} f = \{0\}$. *Then:*

(1) $\chi(\partial F_I) = \chi(F_f) \cdot \chi(S^{M-I-1})$, *for any* $1 \leq I \leq K$.
(2) *Lê–Greuel's type formula:* $\chi(\partial F_{I+1}) - \chi(\partial F_I) = 2(-1)^{M-I}\chi(F_f)$, *for any* $1 \leq I < K$.
(3) $\chi(\partial F_I) = \chi(\partial F_{I+2})$, *for any* $1 \leq I < K - 1$.

Proof The item (1) follows from the identity (23). To prove the item (2) just exchange I by $I + 1$ in the item (1) and take the difference. The item (3) is immediate from item (1). □

4.1 Relating the euler characteristic of the links

Consider again $f : (\mathbb{R}^M, 0) \to (\mathbb{R}^K, 0)$, $M > K \geq 2$, a tame polynomial map-germ with $\operatorname{Disc} f = \{0\}$. For each $1 < I \leq K$ the map $f_I : (\mathbb{R}^M, 0) \to (\mathbb{R}^I, 0)$ admits the Milnor tube fibrations (5) and by restriction it induces the fibrations $f_I : B^M_\epsilon \cap f_I^{-1}(S^{I-1}_\eta) \to S^{I-1}_\eta$ with Milnor fibers F_I and the fibration $f_I : S^{M-1}_\epsilon \cap f_I^{-1}(S^{I-1}_\eta) \to S^{I-1}_\eta$ with fiber ∂F_I.

Denote by $T_\eta(f_I) := B^M_\epsilon \cap f_I^{-1}(S^{I-1}_\eta)$ the Milnor tube of f_I and by $\mathcal{L}_I := f_I^{-1}(0) \cap S^{M-1}_\epsilon$ the respective link.

We may consider η small enough such that the sphere S^{M-1}_ϵ is homeomorphic to the gluing $T_\eta(f_I) \cup_{\partial T_\eta(f_I)} N_\eta(f_I)$, where $\mathcal{L}_I \subset N_\eta(f_I) := f_I^{-1}(B^I_\eta)$ is a semi-algebraic neighbourhood that retract to the link $\mathcal{L}_I$, as proved by Durfee [8].

Thus,

$$\begin{aligned}\chi(S^{M-1}_\epsilon) &= \chi(T_\eta(f_I)) + \chi(N_\eta(f_I)) - \chi(\partial T_\eta(f_I)) \\ &= \chi(F_I)\chi(S^{I-1}) + \chi(\mathcal{L}_I) - \chi(\partial F_I)\chi(S^{I-1}).\end{aligned}$$

Hence, it follows from homeomorphism in (10)

$$\chi(\mathcal{L}_I) = \chi(S^{M-1}) - \chi(F_f)\chi(S^{I-1}) + \chi(\partial F_I)\chi(S^{I-1}) \tag{24}$$

Lemma 4.2 *The following holds true:*

$$\chi(\mathcal{L}_I) = \chi(S^{M-1}) + (-1)^{M-I-1}\chi(F_f)\chi(S^{I-1}).$$

Proof The proof follows from item (1) of Theorem 4.1 and Eq. (24). □

The next result provides in particular a second proof of [9, Proposition 7.1, p. 4861].

Proposition 4.3 *Let* $f : (\mathbb{R}^M, 0) \to (\mathbb{R}^K, 0)$, $M > K \geq 2$, *be a tame polynomial map-germ with* $\operatorname{Disc} f = \{0\}$. *Then:*

(1) $\chi(\mathcal{L}_{I+1}) - \chi(\mathcal{L}_I) = 2(-1)^{M-I}\chi(F_f)$, *for each* $1 \leq I < K$.
(2) $\chi(\mathcal{L}_{I+2}) = \chi(\mathcal{L}_I)$, *for each* $1 \leq I < K - 1$.

Proof It follows directly from Lemma 4.2. □

Remark 4.4 (1) We point out that the Lê–Greuel type formula obtained in the Theorem 4.1, item (2), is somehow similar to that obtained in [7, Theorem 1, p. 3], but with the difference that in [7] the authors worked with the Euler number of the Milnor fibers, instead of its boundary. The extra term in the formula from [7] is related to the behavior inside the Milnor fiber, which, at least intuitively, explains why it is not present in this work, which only involves the boundary.

(2) In view of the Theorem 4.1 and the Proposition 4.3, we can see that for all $1 \leq I < K$ we have $\chi(\partial F_{I+1}) - \chi(\partial F_I) = \chi(\mathcal{L}_{I+1}) - \chi(\mathcal{L}_I)$. Thus, $\chi(\partial F_{I+1}) - \chi(\mathcal{L}_{I+1}) = \chi(\partial F_I) - \chi(\mathcal{L}_I) = \cdots = \chi(\partial F_2) - \chi(\mathcal{L}_2) = \chi(\partial F_1) - \chi(\mathcal{L}_1)$. Hence, it suggests the following definition.

Definition 4.5 Let $f: (\mathbb{R}^M, 0) \to (\mathbb{R}^K, 0)$, $M > K \geq 2$, be a tame polynomial map-germ with Disc $f = \{0\}$. The degree of *degeneracy on the Milnor boundary of* f is defined as the number

$$DB(f) := \chi(\partial F_1) - \chi(\mathcal{L}_1).$$

Clearly, if f has an isolated singularity at the origin one has that $DB(f) = 0$.

5 On the boundaries of the milnor fibers and the links on each stage I

In the real setting we do not expect to prove theorems regarding the degree of connectivity of the Milnor fibers, its boundaries nor the respective links of f_I, on each stage I. Notwithstanding, in the case where the dimension M of the source space is even, for all I, $1 \leq I \leq K$, we may write $\chi(\partial F_I) = \chi(S^{I+1})\chi(F_I)$ and as an application of Lemma 4.2 we conclude that $\chi(\partial F_I) = \chi(\mathcal{L}_I)$, and hence $DB(f) = 0$. However, if the source space M is odd-dimensional some interesting relations between the boundaries of the Milnor fiber, the links of the singularities and the Milnor fibers on the Milnor tubes come up on each stage I, and it provides a way to distinguish between the homotopy type of the Milnor boundary and the link of the singularities f_I for each $1 \leq I \leq K$, as described below.

We first remind that for odd dimension $M \geq 2$ the Eq. (24) becomes

$$(*):\ \chi(\mathcal{L}_I) = 2 - \chi(F_f)\chi(S^{I-1}) + \chi(\partial F_I)\chi(S^{I-1}).$$

This allows us to prove the below result whose proof we left as an exercise.

Lemma 5.1 *Let M be odd and I such that $M > K \geq I \geq 2$. Then, for each I the following conditions hold for the links $\mathcal{L}_I$ and the boundaries ∂F_I of the Milnor fibers F_I :*

(1) *if I is even then $\chi(\mathcal{L}_I) = 2$, by equation $(*)$. Moreover, since* dim $F_I = M - I$ *is odd then $\chi(\partial F_I) = 2\chi(F_I) = 2\chi(F_f)$;*

(2) *if I is odd then $\chi(\mathcal{L}_I) = 2 - 2\chi(F_f)$, by equation $(*)$. Moreover, since* dim $\partial F_I = M - I - 1$ *is odd then $\chi(\partial F_I) = 0$.*

Now we are ready to state the main result of this section.

Theorem 5.2 *Consider M odd, $M > K \geq I \geq 2$. Let $f: (\mathbb{R}^M, 0) \to (\mathbb{R}^K, 0)$ and $f_I: (\mathbb{R}^M, 0) \to (\mathbb{R}^I, 0)$ be real analytic map germs as in the diagram (2). Then, $\chi(F_f) = 1$ if and only if $\chi(\partial F_I) = \chi(\mathcal{L}_I)$ in some stage I. Moreover, if the last equality holds true on any stage I, it also will holds true on all stages I, $2 \leq I \leq K < M$.*

Proof The proof follows from Lemma 5.1.

For the "if" case, we can see that $\chi(F_I) = \chi(F_f) = 1$ implies that the two quantities $\chi(\partial F_I) = \chi(\mathcal{L}_I)$ in the either cases of Lemma 5.1. Moreover, the equality in some stage I clearly implies that on all I, $2 \leq I \leq K < M$.

For the "only if" case, if we suppose that in some stage (even or odd) I the equality $\chi(\partial F_I) = \chi(\mathcal{L}_I)$ holds true, then again by Lemma 5.1 we conclude that $2\chi(F_I) = 2$. Therefore, $\chi(F_f) = 1$. □

The next result provides a natural class of map germs where one of two conditions above holds true. Beside that, it also provides another proof of [3, Proposition 3, item ii), p. 71].

Corollary 5.3 *Let $f: (\mathbb{R}^M, 0) \to (\mathbb{R}^K, 0)$, with $M > K \geq 2$ and M odd, be a real analytic map-germ with an isolated critical point at the origin. Then, for each I, $M > K \geq I \geq 1$, we have that $\chi(F_I) = 1$.*

Proof The proof might be left as exercise, but we will sketch it below for the sake of convenience.

Since f has an isolated singular point at the origin, one may apply the diagram (3) to get that $\text{Sing}\,(f_I) \subseteq \{0\}$ for each fixed I. It is enough to consider the case $\text{Sing}\,(f_I) = \{0\}$, because the case $\text{Sing}\,(f_I) = \emptyset$ the result follows as an easy application of the Inverse Function Theorem version for map germs.

Now, if we assume further that M is odd then for each I the link $\mathcal{L}_I$ must be not empty, and it is in fact a smooth manifold diffeomorphic to ∂F_I and thus $\chi(\partial F_I) = \chi(\mathcal{L}_I)$. Therefore, one may apply the Theorem 5.2 and conclude that $\chi(F_I) = 1$, for each $1 \leq I \leq K < M$. □

Remark 5.4 For the existence of map germ $(\mathbb{R}^M, 0) \to (\mathbb{R}^K, 0)$, M odd, $M > K \geq 2$, with isolated critical point at the origin, the reader may consult [5, section 5.2, p. 101].

Corollary 5.5 *Let $M > K \geq I \geq 1$ and f be as in Theorem 5.2. If $\chi(F_f) \neq 1$ then at all stages I, the Milnor boundary ∂F_I and the respective link $\mathcal{L}_I$ of f_I cannot be homotopically equivalent.*

Proof It is now trivial because $\chi(F_I) = \chi(F_f) \neq 1$, on each stage I. Therefore, by Theorem 5.2 the Milnor boundary ∂F_I and the link $\mathcal{L}_I$ can not be homotopically equivalent. □

The next example shows that for an odd dimension M it is easy to construct a family of map germs where the Euler characteristic of the Milnor fiber is not equal to one.

Example 5.6 Let $f: (\mathbb{R}^M, 0) \to (\mathbb{R}^K, 0)$ be an analytic map germ $M > K \geq 2$, with $\text{Sing}\, f = \{0\}$, and $g: (\mathbb{R}^K, 0) \to (\mathbb{R}^K, 0)$ be an analytic ramified covering map branched along $\{0\}$ with t-sheets, $t \geq 2$. Then, for all fixed $z \in (\mathbb{R}^K, 0)$, $0 < \|z\| \ll 1$, and all $x \in g^{-1}(z)$ the map g is a local diffeomorphism and the fiber $g^{-1}(z)$ consists of a finite number of points, and we have $t := \#g^{-1}(z)$. Thus, $\text{Sing}\, g = \{0\}$ and the composition map germ $h = g \circ f: (\mathbb{R}^M, 0) \to (\mathbb{R}^K, 0)$ safisfies that $\text{Sing}\, h = f^{-1}(0) \subseteq V_h$. Since f is tame, it is not hard to see that h is tame as well. Then the map h admits a Milnor tube fibration with Milnor fiber $F_h = \sqcup_{i=1}^{t} F_f$ (t-disjoint copies of F_f). Therefore we have that $\chi(F_h) = t.\chi(F_f) = t \geq 2$, where we use that $\chi(F_f) = 1$ by Corollary 5.3.

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