EXPONENTIAL SOLUTION FOR INFINITE DIMENSIONAL VOLTERRA-STIELTJES LINEAR INTEGRAL EQUATION OF TYPE (K)

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Abstract: Here we are dealing with the linear Volterra-Stieltjes integral equation of type (K) with kernel K and resolvent R. A bound on the semivariation of K is done to get for some operator B, $R = e^B$. Moreover, we will show that if K satisfies a special condition we can take $B = K$.

Introduction

It is exhaustively well known the importance in to get the operator-solution of an evolutive system having the exponential form.

In the frame of the linear integral equations of type (K) we have, until now, results yielding the resolvent being an exponential operator only for special kernels - the ones for which the equation (K) is a Stieltjes equation. (See remark 2.10, below). Here we get conditions for general kernels K, enlarging our options.

In the following section 1 we point general results in the theory of the linear integral equations of type (K), and in section 2, we give the results concerning the exponential expression of the resolvent R.

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1. Volterra-Stieltjes integral equations of type (K).

The Volterra-Stieltjes linear integral equation of type (K) deals with the forcing and state terms being regulated mappings and is considered in many works. The main events in the development of the theory are due to D. B. Hinton, who originated it in 1966, and C. S. Hõnig and Š. Schwabik (1974), and others. For historical remarks and references, see Hõnig [4]. Here we work in the context by Hõnig.

This type of equation encompasses very general classes of evolutive systems, as the linear ODE, PDE, Neutral Functional equations, equations with impulsive action and the Stieltjes equations

$$(L) \qquad y(t) - x + \int_a^t dA(s)y(s) = f(t)$$

(see [3, pp. 82-94], [2], and in [5], [9]).

Given $[a,b] \subset R$ and a Banach space X, we define the *semi-variation* of $g : [a,b] \to L(X)$ as

$$SV[g] = \sup_{d\in D} \sup \left\{ \sum_{i=1}^{|d|} ||[g(t_i) - g(t_{i-1})]x_i|| \in X, ||x_i|| \leq 1 \right\}$$

where D is the set of all finite partitions of the interval $[a,b]$, $d : t_0 = a < t_1 < \ldots < t_n = b$, and $|d| = n$. Sometimes we will denote $SV[g]$ as $SV_{[a,b]}[g]$. If $SV[g] < \infty$, we say that g is of bounded semivariation, and we write $g \in SV([a,b], L(X))$. Note that SV is a seminorm.

If $\alpha \in SV([a,b], L(X))$ and $[c,d] \subset [a,b]$ we have: $\alpha \in SV([c,d], L(X))$, $SV_{[c,d]}[\alpha] \leq SV_{[a,b]}[\alpha]$ and

$$SV_{(c,d]}[\alpha] = \lim_{\tau \downarrow c} SV_{[\tau,d]}[\alpha]$$

In the same way we define $SV_{[c,d)}[\alpha]$. The following properties on the semi-variation of α will be useful:

Proposition 1.1 [1; 1.15] If $\alpha \in SV([a,b], L(X))$, then:

(i) the function $t \in [a,b] \to SV_{[a,t]}[\alpha]$ is increasing,

(ii) if $c \in (a,b)$ then $SV_{[a,b]}[\alpha] \leq SV_{[a,c]}[\alpha] + SV_{[c,b]}[\alpha]$,

(iii) α is bounded and $||\alpha(t)|| \leq ||\alpha(a)|| + SV_{[a,t]}[\alpha]$.

We say that $f : [a,b] \to X$ is *regulated*, and write $f \in G([a,b],X)$ if f has discontinuities only of the first kind. $G([a,b],X)$ is a Banach space when endowed with the *sup* norm.

For $\alpha \in SV([a,b],L(X))$ and $f \in G([a,b],X)$, there exists the *interior* (or Dushnik-type) *integral*

$$F_\alpha(f) = \int_a^b \cdot d\alpha(t).f(t) = \lim_{d\in D} \sum_{i=1}^{|d|} [\alpha(t_i) - \alpha(t_{i-1})]f(\dot{s}_i) \in X$$

where $\dot{s}_i \in (t_{i-1}, t_i)$ (see [4, Theorem 1.11]).

The interior integral is an extension of the usual Riemann-Stieltjes integral as seen in:

Proposition 1.2 ([3; Th. I.1.1]) Let be $\alpha : [a,b] \to L(X)$ and $f : [a,b] \to X$:

(i) if $\int_a^b d\alpha(t).f(t)$ exists then $\int_a^b \cdot d\alpha(t).f(t)$ exists and has the same value,

(ii) if α and f have no common points of discontinuities and there exists $\int_a^b \cdot d\alpha(t).f(t)$, then $\int_a^b \cdot d\alpha(t) \cdot f(t) = \int_a^b d\alpha(t).f(t)$.

The following is true:

Proposition 1.3 [5; Th. 2.4] If $\alpha \in SV([a,b].L(X))$ and $f \in G([a,b],X)$, then

$$\|\int_a^b \cdot d\alpha(t).f(t)\| \leq SV[\alpha].\|f\|. \tag{1.1}$$

We will consider the *operator interior integral* $\int_a^b \cdot d\alpha(t) \circ \beta(t) \in L(X)$ with $\alpha, \beta : [a,b] \to L(X)$ being defined by:

$$\left[\int_a^b \cdot d\alpha(t) \circ \beta(t)\right] x = \int_a^b \cdot d\alpha(t).[\beta(t)x] \qquad (x \in X).$$

Given I_X the identity mapping of $L(X)$, and the set

$$Q = \{(t,s) \in [a,b] \times [a,b]; a \leq s \leq t \leq b\} \subset \mathbf{R}^2$$

and a mapping $T : Q \to L(X)$, with $T^t(s) = T_s(t) = T(t,s)$, we write $T \in G_0^\sigma.SV^u(Q,L(X))$ or shortly $T \in G_0^\sigma.SV^u$ if T satisfies:

$$T(t,t) = 0, \tag{Δ°}$$

$$T_s x \in G([a,b],L(X)) \tag{G^σ}$$

where $T_s x(t) = T(t,s)x$, for every $t \in [a,b]$ and $x \in X$, and

(SV^u)
$$SV^u[T] = \sup_{a \leq t \leq b} SV[T^t] < \infty.$$

If, instead of (Δ^0), T satisfies

(Δ^I)
$$T(t,t) = I_X$$

we write $T \in G_I^\sigma.SV^u(Q, L(X))$, or shortly $T \in G_I^\sigma.SV^u$. Note that SV^u is a seminorm.

The operators $T \in G_0^\sigma.SV^u$ represent in the sense of the classical Riesz representation theorem - by using now the interior integral - (see [4, Th. 2.10]) - the non-antecipative [or causal] operators acting on the left continuous elements of $G([a,b], X)$. The equation (K) which we will be dealing with is

(K)
$$x(t) - x(a) + \int_a^t \cdot d_s K(t,s).x(s) = u(t) - u(a) \qquad (a \leq t \leq b)$$
$$u \in G([a,b], X), \text{ and } K \in G_0^\sigma.SV^u.$$

The following proposition define the resolvent R, associated to K:

Proposition 1.4 ([4, Th. 3.4]) Suppose (K) and that there exists one and only one mapping $R \in G_I^\sigma.SV^u(Q, L(X))$ satisfying

(R^*)
$$R(t,s)x - x + \int_s^t \cdot d_\tau K(t,\tau).R(\tau,s)x = 0$$

for every $x \in X$ and $a \leq s \leq t \leq b$. Then the solution of (K) forced by u, is given by

$$x(t) = u(t) + R(t,s)[x(a) - u(a)] - \int_a^t \cdot d_s R(t,s).u(s).$$

In the next propositions we give a necessary and sufficient condition on K, for having the existence and unicity of such R. We need the following definitions before:

Definition 1.5 For $K \in G_0^\sigma.SV^u$ and $d \in D$ we denote:

$$c(K,d) = \sup_{1 \leq i \leq |d|} \sup\{SV_{[s_{i-1},t]}[K^t]; s_{i-1} \leq t \leq s_i\}$$

and

$$c(K,d^*) = \sup_{1\le i\le |d|} \sup\{SV_{(s_{i-1},t]}[K^t]; s_{i-1} \le t \le s_i\}.$$

Let be, for all $t \in [a,b)$ ($t \in (a,b]$, respectively), $K(t+,s) = \lim_{\tau\downarrow t} K(\tau,s)$ ($K(t-,s) = \lim_{\tau\uparrow t} K(\tau,s)$, respectively). The existence of such operators is released in a straightforward way by the Banach-Steinhaus theorem.

Definition 1.6 If $K \in G_0^\sigma \cdot SV^u$ we define $K^- \in G_0^\sigma \cdot SV^u$ by

$$K^-(t,s) = K(t-,s) \qquad (a < s < t)$$

A result allowing the existence and unicity of the resolvent R is done by

Proposition 1.7 ([5; Th. 3.4]) Let be $k \in L(G([a,b],X))$ the operator defined by $K \in G_0^\sigma \cdot SV^u$:

$$ky(t) = \int_a^t \cdot d_\sigma K(t,\sigma).y(\sigma),$$

being compact. Then the following properties are equivalent:

(P_1): for every $t \in [a,b]$, $[I_X - K(t+,t)]^{-1} \in L(X)$

(P_2): there exists an unique R fulfilling (R^*).

A strictly more general result in this frame is possible:

Proposition 1.8 [5; Th. 3.8] Let be $K \in G_0^\sigma \cdot SV^u$, satisfying:

(P_3): there exists a division $d \in D$ with $c(K^-,d^*) < 1$.

Then (P_1) is equivalent to (P_2).

After those preliminary results, the main section in this work can be done.

2. Exponential representation of the resolvent

Let be as in [1; p. 31] the operator

$$\tau : G_\Delta^\sigma \longrightarrow G_0^\sigma \cdot SV^u(Q, L(X))$$

defined by

$$(\tau U)(t,s) = \int_s^t \cdot d_\sigma K(t,\sigma) \circ U(\sigma,s)$$

The next result gives a condition for have R expressed as an exponential operator for a particular point $(t,s) \in Q$:

Theorem 2.1 Suppose $K \in G_0^\sigma \cdot SV^u$ satisfying (P_3) and R being the resolvent operator associated to K in (K).

If for $(t,s) \in Q$ we have

$$\sup_{\sigma \in [s,t]} ||R(\sigma,s)|| < \frac{1}{SV_{[s,t]}[K^t]}, \tag{2.1}$$

then there exists an operator $B(t,s) \in L(X)$ such that:

$$R(t,s) = e^{B(t,s)}. \tag{2.2}$$

Proof: According a result by Nagumo in Banach algebra ([7, Th. I.4.12]) using straightforward arguments - see for example [8; Lemma] - if we have, for all nonnegative real r,

$$(R(t,s) + rI)^{-1} \in L(X), \tag{2.3}$$

then we get (2.2).

According (R^*), the expression in (2.3) is true, provide

$$[\lambda I_X - \tau R]^{-1} \in L(X),$$

for all real $\lambda \geq 1$. If

$$|| \int_s^t \cdot d_\sigma K(t,\sigma) \circ R(\sigma,s)|| < 1 \tag{2.4}$$

we get so (see [6; Ex. I.4.5]).

The inequality (1.1) applied to the left term in (2.4) ends the proof. ∎

We get immediately as a consequence of this theorem:

Corollary 2.2 If $\sigma \in [s,t]$, then:

$$SV_{[s,t]}[K^t] < \frac{1}{||R(\sigma,s)||}.$$

Moreover, $R(s,s) = I_X$ gets:

$$SV_{[s,t]}[K^t] < 1.$$

If R is done by a Neumann series in the sense of [5; Def. 3.3], we get a sufficient condition to have R being of the form (2.2) in a point $(t,s) \in Q$, only by considerations directly on K.

Let us state,

Proposition 2.3 [5; Th. 3.9] Suppose $K \in G_0^\sigma \cdot SV^u$ satisfying the property

(P_4): there exists a $d \in D$ such that $c(K^-, d) < 1$.

Then R associated to K in (K), is done by the Neumann series.

$$R = I_X + \sum_{n=1}^{+\infty}(-1)^n \tau^n(I_X). \tag{2.5}$$

In the next theorem we state a result that allows

$$R = e^B \tag{2.6}$$

in Q.

Theorem 2.4 Suppose $K \in G_0^\sigma.SV^u$, and the operator $1.U = U$ for all $U \in G_\Delta^\sigma$.

If $SV^u[K] < \frac{1}{4}$ then there exists an operator $B \in G_0^\sigma.SV^u$ such that for all $(t,s) \in Q$:

$$R(t,s) = e^{B(t,s)}$$

Proof: According [1; 2.18 and 2.7] we have

$$c(K^-, d) < SV^u[K^-] < SV^u[K].$$

This implies (P_4).

The expression (2.5) implies

$$R = (1+\tau)^{-1} I_X,$$

and then, according [6; I.4.5]

$$||\tau R|| \leq \frac{||\tau||}{1 - ||\tau||}. \tag{2.7}$$

By other hand we have, [1; 2.23.c]:

$$||\tau|| \leq 2SV^u[K] \tag{2.8}$$

Finally, $SV^u[K] < \frac{1}{4}$, (2.8) and (2.7) imply (2.3) and so (2.2) for every $(t,s) \in Q$. ∎

If R has the exponential expression (2.6), we can get in some special cases $B(t,s)[(t,s) \in Q]$, done in terms of K, as shown in the next theorem. Before, we need a definition:

Definition 2.5 $K \in G_0^\sigma.SV^u$ satifies the property (P_s) if

$$(P_s) \qquad \sum_{n=2}^{\infty}(-1)^{n+1}\tau^n(I_X)(t,s) = 0$$

Theorem 2.6 Let be $K \in G_0^\sigma.SV^u$ satisfying (P_s) and with $SV^u[K] < \frac{1}{4}$.

Then $B : Q \to L(X)$ in (2.6) can be expressed by

$$(2.9) \qquad B(t,s) = K^1(t,s) - \frac{K^2(t,s)}{2} + \frac{K^3(t,s)}{3} - \dots$$

where $K^1(t,s) = K(t,s)$ and $K^{n+1}(t,s) = K(t,s) \circ K^n(t,s), n = 1, 2\dots$.

Proof: According the theorem 2.4 we get

$$(2.10) \qquad e^B = (1+\tau)^{-1}I_X.$$

K satisfying (P_s), and $\tau(I_X) = -K$ yield:

$$e^B = I_X - K$$

The definition of $\log T$, for a linear operator T in [6; X.1.4] implies (2.9) for every $(t,s) \in Q$.

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