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Isomorphisms of Integral Group Rings of Infinite Groups

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Abstract

This paper deals with the isomorphism problem for integral group rings of infinite groups. In the first part we answer a question of Mazur by giving conditions for the isomorphism problem to be true for integral group rings of groups that are a direct product of a finite group and a finitely generated free abelian group. Also it is shown that the isomorphism problem for infinite groups is strongly related to the normalizer conjecture. Next we show that the automorphism conjecture holds for infinite finitely generated abelian groups G if and only if $\mathbb{Z}G$ has only trivial units. In the second part we partially answer a problem of Sehgal. It is shown that the class of a finitely generated nilpotent group G is determined by its integral group ring provided G has only odd torsion. When G has nilpotency class two then the finitely generated restriction is not needed. This, together with a result of Ritter and Sehgal, settles the isomorphism problem for finitely generated nilpotency class two groups. A link is pointed out between this problem and the dimension subgroup problem.

1 Introduction

Although recently Hertweck [4] gave a counterexample to the isomorphism problem for integral group rings of finite groups, it still remains a challenge to determine which groups do satisfy the conjecture. It is well known that a positive solution to this conjecture has been given by Roggenkamp and Scott for finite nilpotent groups and by Withcomb for finite metabelian

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groups. In the case of infinite groups very little is known. It is even unknown whether the nilpotency class is preserved for integral group rings of (infinite) nilpotent groups (see Sehgal's Problem 46 in [18]). Even the nilpotency two class is unresolved. Actually their is a satisfactory answer to the conjecture only for a special class of polycyclic-by-finite groups; namely for groups of the form $N \times \mathbf{Z}$, a direct product of a finite group N with the infinite cyclic group. In [9] Mazur proves that if H is another group such that $\mathbf{Z}(N \times \mathbf{Z}) \cong \mathbf{Z}H$ then $H = N \rtimes \mathbf{Z}$, a semidirect product, where the action of $\mathbf{Z}$ on the finite group N is given by a unit of $\mathbf{Z}N$. Hence, it follows that the isomorphism conjecture holds for $N \times \mathbf{Z}$ if and only if both the isomorphism conjecture and the normalizer conjecture hold for N (see also [10]). For the terminology and background on group rings we refer the reader to [18].

Mazur in [9] raised the question if his result can be extended to include direct products of finite groups N with finitely generated free abelian groups A. After some preliminary work in Section 2 on a result of Jespers, Parmenter and Sehgal [6] on central units of some integral group rings, we are able in Section 3 to answer Mazur's question. The techniques used in the proofs and the counterexample of Hertweck also allow us to construct many non isomorphic group bases for $\mathbf{Z}(N \times A)$; and thus we obtain infinite counterexamples to the isomorphism problem. Neverthless, for some semidirect products of finite groups and abelian groups we show that group bases have to be subisomorphic.

In Section 4 necessary and sufficient conditions are given for the automorphism conjecture to hold for $\mathbf{Z}(N \times A)$. It follows that this conjecture holds for the integral group ring of a finitely generated abelian group G if and only if either G is finite or all units of $\mathbf{Z}G$ are trivial.

The remainder of the paper concerns Sehgal's problem. In Section 5 we first prove that the isomorphism problem holds for finitely generated nilpotent class two groups. Note that in [11] Ritter and Sehgal proved the following result: if G and H are both finitely generated nilpotent class two groups such that $\mathbf{Z}G \cong \mathbf{Z}H$, then $G \cong H$. Second we show the nilpotency class of a finitely generated nilpotent group G is determined by its integral group ring, provided G has odd torsion or if it satisfies some other condition. We point out that there is a connection between this question and the dimension subgroup problem. If there exists a counterexample to the problem then it probably can be constructed using Rips' counterexample (see [3]) to the dimension subgroup problem.

2 Preliminaries

The following theorem is proved in [6] for nilpotent groups, but its proof remains valid under the weaker assumptions.

The centre of a group G is denoted $\mathcal{Z}(G)$ and the group of units (respectively augmented units) is denoted $\mathcal{U}(\mathbb{Z}G)$ (respectively $\mathcal{U}_1(\mathbb{Z}G)$).

Theorem 2.1 *Let G be a group such that $T(G)$, the set of torsion elements of G, forms a finite subgroup. Suppose that $G/T(G)$ is an ordered group. Then, given $u \in \mathcal{Z}(\mathcal{U}_1(\mathbb{Z}G))$ there exist $v \in \mathbb{Z}T(G)$ and $g \in G$ such that $u = vg$. Moreover, there exists a fixed positive integer n such that $g^n, v^n \in \mathcal{Z}(\mathcal{U}_1(\mathbb{Z}G))$.*

Corollary 2.2 *Let G be as in Theorem 2.1. Then, the central units of $\mathbb{Z}G$ are trivial if and only if the only units of $\mathbb{Z}T(G)$ that are central in $\mathbb{Z}G$ are trivial.*

Proof. Of course one implication is trivial. For the converse, assume that the only units of $\mathbb{Z}T(G)$ that are central in $\mathbb{Z}G$ are trivial. We have to prove that any augmented central unit x of $\mathbb{Z}G$ is trivial. Since central units of finite order are trivial [17, Corollary I.1.7], it is sufficient to deal with the case that x has infinite order. So let x be such a unit. Use Theorem 2.1 to write $x = vg$ with $g \in G$ and $v \in \mathbb{Z}T(G)$. Since there exists an integer n such that v^n is central it follows from the assumption that v is a torsion unit. Hence if we set $m = o(v)$, the order of v, then $x^m = g^m$. Let $A = \langle x^m \rangle$. Then x and g project to torsion units in $\mathbb{Z}(G/A)$ and since x is central, it follows that x and thus also v projects to a trivial unit. But A is torsion free and $T(G)$ forms a subgroup and thus this projection is injective on the support of v and so v is a trivial unit. It follows that $x \in G$. $\square$

The case when G is nilpotent is proved in [6] and finite groups with trivial centre were characterized by Ritter-Sehgal (see [18]). In [6] an example is given to show that $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ being trivial does not imply that $\mathcal{Z}(\mathcal{U}(\mathbb{Z}T(G)))$ is trivial.

The normalizer of a subgroup H in a group G is denoted $\mathcal{N}_G(H)$, and the centralizer of H in G is denoted $\mathcal{C}_G(H)$.

Corollary 2.3 *Let G and $u = vg$ be as in Theorem 2.1. The element $v \in \mathcal{N}_{\mathcal{U}_1(\mathbb{Z}T(G))}(T(G))$ and, if $\mathcal{N}_{\mathcal{U}_1(\mathbb{Z}T(G))}(T(G)) = \mathcal{Z}(\mathcal{U}_1(\mathbb{Z}T(G)))T(G)$ then we may take $v \in \mathcal{Z}(\mathcal{U}_1(\mathbb{Z}T(G)))$ and in this case $g \in \mathcal{Z}(\mathcal{C}_G(T(G)))$.*

In particular this is the case when $T(G)$ is nilpotent or of odd order.

Proof. Since $u = vg \in \mathcal{Z}(\mathcal{U}_1(\mathbf{Z}G))$ one gets that $(vg)^{-1}T(G)(vg) = T(G)$ and thus $v^{-1}T(G)v = gT(G)g^{-1} = T(G)$. Hence $v \in \mathcal{N}_{\mathcal{U}_1(\mathbf{Z}T(G))}(T(G))$. If $\mathcal{N}_{\mathcal{U}_1(\mathbf{Z}T(G))}(T(G)) = \mathcal{Z}(\mathcal{U}_1(\mathbf{Z}T(G)))T(G)$ then we may write $v = v_1t$ with $t \in T(G)$ and $v_1 \in \mathcal{Z}(\mathcal{U}_1(\mathbf{Z}T(G)))$. This proves that in this case we may take $v \in \mathcal{Z}(\mathcal{U}_1(\mathbf{Z}T(G)))$. But then $g = uv^{-1}$ centralizes $T(G)$. So, if $g_1 \in C_G(T(G))$, then

$$vg_1g = g_1vg = g_1u = ug_1 = vgg_1$$

and thus $g_1g = gg_1$. Hence $g \in \mathcal{Z}(C_G(T(G)))$, as desired.

The final part follows from results of Jackowski and Marciniak [5] and Sehgal [18, Corollary 9.6]. □

Recall that a *group basis* of an integral group ring $\mathbf{Z}G$ is a subgroup H of augmented units such that the elements of H are independent over $\mathbf{Z}$ and $\mathbf{Z}H = \mathbf{Z}G$.

Corollary 2.4 *Let G be a group as in Theorem 2.1 and suppose that $\mathcal{Z}(\mathcal{U}_1(\mathbf{Z}(G))) = \mathcal{Z}(G)$. If $H \subseteq \mathcal{U}_1(\mathbf{Z}G)$ is another group basis for $\mathbf{Z}G$, then $\mathcal{Z}(H) = \mathcal{Z}(G)$.*

Proof. ¿From the correspondence theorem between finite normal subgroups of G and H ([17, Theorem III.4.17]) it follows that $\Delta(G, T(G)) = \Delta(H, T)$ for some finite normal subgroup T of H. Since

$$\mathbf{Z}(G/T(G)) \cong \mathbf{Z}G/\Delta(G, T(G)) = \mathbf{Z}H/\Delta(H, T) \cong \mathbf{Z}(H/T)$$

is a domain (as $G/T(G)$ is ordered) it follows that H/T is torsion free and thus $T = T(H)$ is a finite subgroup. Since units of $\mathbf{Z}(G/T(G))$ are trivial, it also follows that $G/T(G) \cong H/T(H)$, in particular $H/T(H)$ is ordered as well.

We now prove the result. Clearly the assumptions imply $\mathcal{Z}(H) \subseteq \mathcal{Z}(G)$. For the converse inclusion, let $x \in \mathcal{Z}(G)$ and use Theorem 2.1 to write $x = vh$ with $v \in \mathcal{U}_1(\mathbf{Z}T(H))$ and $h \in H$. By Theorem 2.1, $v^n \in \mathcal{Z}(\mathcal{U}_1(\mathbf{Z}H)) \subseteq \mathcal{Z}(\mathcal{U}_1(\mathbf{Z}G)) = \mathcal{Z}(G)$ for some positive integer n. Since

$$v^n = 1 \ mod \ \Delta(G, T(G)),$$

we thus get that $v^n \in T(G)$. Hence v has finite order, say m. It follows that $x^m = h^m$. Set $A = \langle h^m \rangle$ and note that $\Delta(G, A) = \Delta(H, A)$. So $\overline{H}$, the image of H in $\mathbf{Z}(G/A)$, is a group basis. But since the image of x is a central torsion unit it must belong to $\overline{H}$. We conclude that $\overline{v}$, the image of v, is in $\overline{H}$ and so its support has cardinality 1. The rest of the proof is as in Corollary 2.2. □

In the remainder of this section we prove some technical results needed in the next section.

Let $N \xrightarrow{i_1} E_1 \xrightarrow{\epsilon_1} G$ be a split exact sequence and let $s_1 : G \to E_1$ be a transversal homomorphism. If $A \lhd G$ is such that $B = s_1(A)$ is normal in E_1 then there exists a split exact sequence $N \xrightarrow{i_2} E_2 \xrightarrow{\epsilon_2} G_1$ and a transversal homomorphism $s_2 : G_1 \to E_2$. Here $E_2 = E_1/B$ and $G_1 = G/A$. To see this define $i_2(n) = i_1(n)B$ and $\epsilon_2(eB) = \epsilon_1(e)A$ and use diagram chasing to prove the exactness of the sequence. The transversal is defined by $s_2(gA) = s_1(g)B$. Let $\pi_1 : E_1 \to E_2$ and $r_1 : G \to G_1$ be the natural projections. Then it is easy to see that $(1, \pi_1, s_1)$ defines a morphism between the two sequences such that $\pi_1 \circ s_1 = s_2 \circ r_1$.

Lemma 2.5 *Let $f_1 : G \to Aut(N)$ and $f_2 : G_1 \to Aut(N)$ be the maps induced by the transversals s_1 and s_2 respectively. Then $f_1 = f_2 \circ r_1$.*

Proof. Since we have commuting diagrams, it follows easily that, for any $g \in G$,

$$s_2(gA)^{-1} i_2(n) s_2(gA) = \pi_1(s_1(g)^{-1} i_1(n) s_1(g)).$$

From this and the injectivity of i_2 the result follows. In fact

$$i_2(f_2(gA)(n)) = \pi_1(s_1(g)^{-1} i_1(n) s_1(g)) = \pi_1(i_1(f_1(g)(n))) = i_2(f_1(g)(n))$$

for all $n \in N$ and thus $f_1 = f_2 \circ r_1$. □

Lemma 2.6 *Suppose there exists a monomorphism $i_3 : M \to E_3$ and an epimorphism $\pi_2 : E_3 \to E_2$ such that $\pi_2(i_3(M)) = i_2(N)$ and the restriction of π_2 on $i_3(M)$ is injective. Then there exists a split exact sequence $M \xrightarrow{i_3} E_3 \xrightarrow{\epsilon_3} G_3$, a morphism (α, π_2, r_2) and a transversal $s_3 : G_3 \to E_3$ such that $\pi_2 \circ s_3 = r_2 \circ s_2$.*

Proof. Let $K = \pi_2^{-1}(s_2(G_1))$ and let E be the subgroup generated by $i_3(M)$ and K. Since $Ker(\pi_2) \subseteq K$ and (i_2, ϵ_2) splits, it follows that $E = E_3$ and $K \bigcap i_3(M) = 1$. Hence K is a complement of $i_3(M)$ in E_3. Set $G_3 = K$, $r_2 = \epsilon_2 \circ \pi_2$ restricted on M, and let $\alpha : M \to N$ be the morphism defined by $i_2\alpha(m) = \pi_2 i_3(M)$. It is now easy to see that we have a commuting diagram and if we set $s_3(k) = k$, for any $k \in K$, then s_3 is a transversal homomorphism such that $\pi_2 \circ s_3 = s_2 \circ r_2$. □

Lemma 2.7 *With notations as in the previous Lemmas and the proof of Lemma 2.6, let f_3 be the map induced by s_3 from G_3 to $Aut(M)$. Then $f_2 \cdot r_2(k) = \alpha \circ f_3(k) \circ \alpha^{-1}$, for any $k \in K$.*

Proof. Note that α is an isomorphism. The proof is now similar as that of Lemma 2.5. □

Lemma 2.8 *If there exists a monomorphism $\rho : G \to G_3$ such that $r_2 \circ \rho = r_1$, then E_1 is isomorphic to a subgroup of E_3.*

Proof. Define $\Psi : E_1 \to E_3$ by $\Psi(i_1(n)s_1(g)) = (i_3 \cdot \alpha^{-1}(n))s_3(\rho(g))$. It is readily verified that Ψ is a well defined injective map with image the group $i_3(M) \rtimes s_3\rho(G)$. So it is sufficient to show that Ψ is a group homomorphism. Since $\Psi(1) = 1$ it is sufficient to show that Ψ preserves products. For this let $a = i_1(n)s_1(g) \in E_1$ and $b = i_1(m)s_1(h) \in E_1$, where $n, m \in N$ and $g, h \in G$. Then

$$\begin{aligned}
\Psi(ab) &= \Psi\left(i_1(n)s_1(g)i_1(m)s_1(g)^{-1}s_1(g)s_1(h)\right) \\
&= \Psi\left(i_1(n)(i_1f_1(g^{-1}(m))s_1(gh)\right) \\
&= i_3\alpha^{-1}(nf_1(g^{-1})(m))s_3(gh) \\
&= i_3\left(\alpha^{-1}(n)\right)\left(i_3\alpha^{-1}f_1(g^{-1})(m)\right)s_3(gh).
\end{aligned}$$

Also

$$\begin{aligned}
\Psi(a)\Psi(b) &= \left(i_3\alpha^{-1}(n)s_3(\rho(g))\right)\left(i_3\alpha^{-1}(m)s_3(\rho(h))\right) \\
&= i_3\alpha^{-1}(n)s_3(\rho(g))i_3\alpha^{-1}(m)s_3(\rho(g))^{-1}s_3(\rho(g))s_3(\rho(gh)) \\
&= \left(i_3\alpha^{-1}(n)\right)\left(i_3((f_3(\rho(g^{-1})))(\alpha^{-1}(m)))\right)s_3(\rho(gh)).
\end{aligned}$$

Hence, because i_3 is injective, to prove that $\Psi(ab) = \Psi(a)\Psi(b)$ it is sufficient to show that

$$f_1(g^{-1})(m) = \alpha(f_3(\rho(g^{-1})))(\alpha^{-1}(m)).$$

But, by Lemma 2.7,

$$\alpha f_3(\rho(g^{-1}))\alpha^{-1} = f_2(r_2(\rho(g^{-1})))$$

and thus by the assumption

$$\alpha f_3(\rho(g^{-1}))\alpha^{-1} = f_2(r_1(g^{-1})).$$

So, indeed by Lemma 2.5,

$$\alpha f_3(\rho(g^{-1}))\alpha^{-1} = f_1(g^{-1}.$$

Using the previous Lemmas it is easy to check that Ψ is indeed an isomorphism.□

Proposition 2.9 *Let $N \xrightarrow{i_1} E_1 \to G$ be a split exact sequence of a finite group N and a torsion free abelian group G and let s_1 be a transversal homomorphism. Let $A \lhd G$ and $E_2 = E_1/B$, supposing of course that $B = s_1(A) \lhd E_1$. Suppose that there exist a sequence $M \xrightarrow{i_2} E_3 \xrightarrow{\pi_2} E_2$ with i_2 a monomorphism, π_2 an epimorphism and $\pi_2 \circ i_2 : M \to \pi_1 \cdot \imath_1(N)$ is an isomorphism where $\pi_1 : E_1 \to E_2$ is the canonical projection. If E_1 and E_3 have the same Hirsch number and $E_3/i_2(M)$ is abelian, then E_1 is isomorphic to a subgroup of E_3.*

Proof. The only thing to check is that there exists a monomorphism $\rho : G \longrightarrow G_3$ satisfying the conditions of Lemma 2.8. Since G is free abelian the epimorphisms $r_2 : G_3 \to G_1$ and $r_1 : G \to G_1$ define an epimorphism $\rho : G \to G_3$ such that $r_2 \circ \rho = r_1$. Because of the construction of G_3 and because of the assumptions, the abelian groups G and G_3 have the same Hirsch number (namely the same Hirsch number as that of both E_1 and E_3). It follows that the kernel of ρ has Hirsch number zero. So it is a periodic subgroup of G and therefore is trivial. Hence ρ is a monomorphism. □

3 Isomorphisms of Direct Products

In this section we prove the isomorphism problem for some classes of polycyclic-by-finite groups and give necessary and sufficient conditions for this problem to hold for a certain class. This extends the result of [9].

We shall say that a group G satisfies the *normalizer condition* if $\mathcal{N}_{\mathcal{U}_1(\mathbf{Z}G)}(G) = \mathcal{Z}(\mathcal{U}_1(\mathbf{Z}G))G$. We need the following lemma.

Lemma 3.1 *Let G be the direct product of groups N_1 and N_2. Then G satisfies the normalizing condition if and only if N_1 and N_2 satisfy the normalizing condition.*

Proof. Denote by $\psi_i : G \longrightarrow N_i$ the natural projection of G onto N_i. Its extension to the group rings will also be denoted by ψ_i. Observe that $N_i = Ker(\psi_j)$ if $i \neq j$.

Suppose that G satisfies the normalizing condition and let $u \in \mathbf{Z}N_1$ be an augmented unit in the normalizer of N_1. Then certainly u also normalizes G and hence $u = wg$ with $g \in G$ and $w \in \mathcal{Z}(\mathbf{Z}G)$. It follows that $u = \psi_1(u) = \psi_1(w).\psi_1(g)$ and thus N_1 satisfies the normalizer condition.

Conversely suppose that N_1 and N_2 satisfy the normalizer condition and let $u \in \mathcal{U}_1(\mathbf{Z}G)$ be in the normalizer of G. Denote by $u_i = \psi_i(u)$. Then u_i normalizes N_i and thus $u_i = w_i n_i$ with w_i a central unit in $\mathbf{Z}N_i$ and $n_i \in N_i$. Since G is the direct product of the N_i it follows that w_i is central in G. Define $w = uu_1^{-1}u_2^{-1}$. We claim that w is a central unit. Suppose this for the moment; then $u = wu_1u_2 = (ww_1w_2)(n_1n_2)$. Since ww_1w_2 is a central unit, the Lemma follows.

We now prove the claim: Let $n \in N_i$ and let $g = w^{-1}nw$. Since w normalizes G we have that $g \in G$. As $g - n = w^{-1}nw - n = [w^{-1}n, w]$ it follows from [18, 41.1] that g and n are conjugate in G. But N_i is normal in G and thus $g \in N_i$. Hence $w^{-1}N_iw = N_i$. Using this we have that $w^{-1}n_iw = \psi_i(w^{-1}n_iw) = n_i$ for all $n_i \in N_i$ and hence the claim is proved. $\square$

Theorem 3.2 *Let $G = N \times A$, the direct product of a finite group N and a non periodic finitely generated abelian group A. Suppose the isomorphism problem holds for N. Then, the normalizer condition holds for N if and only if the isomorphism problem holds for G.*

Proof. We first prove that we may suppose that A is torsion free. To do this we show that we can reduce to this case. So suppose that the Theorem holds when A is torsion free. Write $A = B \times A_1$ with $B = T(A)$ and A_1 is torsion free. So $G = N_1 \times A_1$ where $N_1 = N \times B$. Using the normal subgroup correspondence one readily shows that the isomorphism problem still holds for N_1. If N satisfies the normalizer condition then, by Lemma 3.1, N_1 also satisfies the normalizer condition and since A_1 is torsion free it follows that the isomorphism problem holds for G. Conversely suppose that the isomorphism problem holds for G. Then, since A_1 is torsion free, we have that N_1 satisfies the normalizer condition. Lemma 3.1 now gives us that N satisfies the normalizer condition. So for the rest of the proof we shall assume that A is torsion free.

Assume $\mathcal{N}_{\mathcal{U}_1(\mathbf{Z}N)}(N) = \mathcal{Z}(\mathcal{U}_1(\mathbf{Z}N))N$. Let $H \subseteq \mathcal{U}_1(\mathbf{Z}G)$ be a group basis. We shall see at the end of the proof that H also satisfies the condition on the normalizer. Assume this for the moment. By [17, Theorem III.4.17], H has a finite normal subgroup M with $|M| = |N|$ and $\Delta(G, N) = \Delta(H, M)$. Since $\mathbb{Z}A \cong \mathbb{Z}G/\Delta(G, N)$ is a domain, it follows, as before, that $M = T(H)$. Write $A = \langle x_1, x_2, \cdots, x_n\rangle$, with n the rank of A, and choose $y_1, y_2, \cdots, y_n \in H$ such that $\mathbb{Z}G/\Delta(G, N) = \mathbb{Z}\langle x_1, \cdots, x_n\rangle = \mathbb{Z}\langle\overline{y_1}, \cdots, \overline{y_n}\rangle$. Using Corollary 2.3, for every $1 \leq i \leq n$, we can find $v_i \in \mathcal{Z}(\mathcal{U}_1(\mathbb{Z}(M)))$ and $h_i \in \mathcal{Z}(C_H(M))$ such that $x_i = v_i h_i$. Since $\langle x_1, \cdots, x_n\rangle = \langle\overline{y_1}, \cdots, \overline{y_n}\rangle$ (as these are the trivial augmented units in $\mathbb{Z}A = \mathbb{Z}(H/T(H))$), we may suppose that each $y_i = h_i$. Thus $\langle y_1, \cdots, y_n\rangle$ is a finitely generated abelian group. Write $\langle y_1, \cdots, y_n\rangle = T(\langle y_1, \cdots, y_n\rangle) \times B$, where B is a free abelian group of rank n. As $H/M = \overline{B}$ and $B \subseteq C_H(M)$, we get $H = M \times B$. So to prove that $H \cong G$ it is now sufficient to show that $N \cong M$. To see this, let $\psi : \mathbb{Z}G \rightarrow \mathbb{Z}N$ be the natural epimorphism. By [7, Corollary 2.3], this map is injective on torsion subgroups. Since $|N| = |M|$, it follows therefore (see [18, Lemma 37.4]) that $\mathbb{Z}N = \mathbb{Z}\psi(M)$. Thus, since by assumption N satisfies the isomorphism problem, $N \cong \psi(M) \cong M$ and $G \cong H$. Let Ψ be any isomorphism between M and N and extend it to an isomorphism of the group rings. It is now clear that N satisfies the condition on the normalizer if and only if M satisfies this condition.

To proof the converse we just have to exhibit a counterexample in case G does not satisfy the condition on the normalizer. This will be done in Theorem 3.6. □

Corollary 3.3 *Let $G = N \times A$, the direct product of a finite group N and a non trivial finitely generated free abelian group A. Then, the isomorphism problem holds for G if and only if both the normalizer condition and the isomorphism problem hold for N.*

Proof. This follows immediately from Theorem 3.2 and the fact that if $G = N \times A$ satisfies the isomorphism problem, and because A is free abelian, then N also satisfies the isomorphism problem. This can be shown easily by extending isomorphisms defined on $\mathbb{Z}N$ to $\mathbb{Z}G$ and by using that N is the torsion subgroup of G. □

As mentioned in [10], the normalizer condition for a finite group N is equivalent to $Aut_{\mathbb{Z}}(G) = Inn(G)$, where $Aut_{\mathbb{Z}}(G)$ is the group of automorphisms of G which are induced by conjugation by units of $\mathbb{Z}N$ that normalize N. It is a well known conjecture that the normalizer condition holds for any finite group but this was disproved by Hertweck.

The proof of Theorem 3.2 gives quite a bit information even without the normalizer condition assumption. Note that in this more general context we only get that the elements $v_i \in \mathcal{N}_{\mathcal{U}_1(\mathbf{Z}M)}(M)$. The precise information obtained this way is written in the following result. This answers a question raised in [9] and extends the main result of that paper.

Theorem 3.4 *Let $G = N \times A$ where N is a finite group and $A = \langle x_1 \rangle \times \cdots\cdots \times \langle x_n \rangle$ is a finitely generated free abelian group. Let H be a group. Then, $\mathbf{Z}G \cong \mathbf{Z}H$ if and only if the following holds:*

1. *$M = T(H)$ is a subgroup and $\mathbf{Z}N \cong \mathbf{Z}M$.*
2. *there exist $y_1, \cdots, y_n \in H$ so that $H/M = \langle y_1 M \rangle \times \cdots \times \langle y_n M \rangle$ is free abelian of rank n.*
3. *For each i there exists $v_i \in \mathcal{N}_{\mathcal{U}_1(\mathbf{Z}M)}(M)$ such that $v_i y_i$ is a central element, i.e., the action of y_i on M is given by conjugation by v_i. Moreover $\langle v_1 y_1, \cdots, v_n y_n \rangle \cong A$.*

Proof. Suppose that $\mathbf{Z}G \cong \mathbf{Z}H$. The proof of Theorem 3.2 gives us the proof in one direction.

So suppose now that we have a group H with all these conditions. Then,

$$\mathbf{Z}G = \mathbf{Z}(N \times A) = (\mathbf{Z}N)[x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}],$$

the Laurent polynomial ring over the ring $\mathbf{Z}N$ and in n commuting variables.. Let $B = \langle v_1 y_1, \cdots, v_n y_n \rangle$, a free abelian subgroup of $\mathcal{U}(\mathbf{Z}(H))$. Then

$$\mathbf{Z}H = \mathbf{Z}\langle M, y_1, \cdots, y_n \rangle$$

and it is readily verified that, as a ring, it is generated by the group ring $\mathbf{Z}M$ and the group B. Since B is central in $\mathbf{Z}H$ it also follows easily that

$$\mathbf{Z}H = (\mathbf{Z}M)B.$$

As, by assumption, $B \cong A$ and $\mathbf{Z}M \cong \mathbf{Z}N$, one gets that $\mathbf{Z}G \cong \mathbf{Z}H$. $\square$

Now we show how to construct counterexamples to the isomorphism problem for infinite groups.

Let $G = N \rtimes_\sigma \langle z \rangle$ with N a finite group and $\langle z \rangle$ the infinite cyclic group. Take $u \in \mathcal{N}_{\mathcal{U}_1(\mathbf{Z}N)}(N)$. Denote by τ the automorphism of N induced by conjugation by u, and let $v = zu$. If $n \in N$, then

$$v^{-1} n v = u^{-1}(z^{-1} n z) u = u^{-1} \sigma(n) u = \tau \circ \sigma(n).$$

Hence $H = N \rtimes_{\tau \circ \sigma} \langle v \rangle \subseteq \mathcal{U}_1(\mathbf{Z}G)$.

Lemma 3.5 *H is a group basis for $\mathbf{Z}G$.*

Proof. Each element of H can be written uniquely as nv^k for some $n \in N$ and $k \in \mathbf{Z}$. Suppose that $\sum_k \sum_n c_{n,k} n v^k = 0$, where each $c_{n,k} \in \mathbf{Z}$. We want to prove that all all integers $c_{n,k}$ are zero. Now $v = u^z z$. Since G is a group basis and $\sum_n c_{n,k} n (u^z z)^k \in \mathbf{Z}[N] z^k$ for all k, it follows that the latter sum is zero. As $(u^z z)^k$ is invertible it follows that $\sum_n c_{n,k} n = 0$ and thus all the $c_{n,k}$ are zero, as required.

Note that $z = vu^{-1} = v(\sum_{n \in N} u_n^{-1} n) = \sum_{n \in N} u_n^{-1}(vnv^{-1})v$. Hence $z \in \mathbf{Z}H$ and thus $\mathbf{Z}H = \mathbf{Z}G$.□

It is well known (see [13, 11.1.1]) that $G = N \rtimes_\sigma \langle z \rangle \cong H = N \rtimes_{\tau\circ\sigma} \langle v \rangle$ if and only if $\tau = (\tau \circ \sigma)\sigma^{-1}$ is inner. The latter means that $n_0^{-1} n n_0 = \tau(n) = u^{-1} n u$ for some $n_0 \in N$; or equivalently $n_0^{-1} u \in \mathcal{Z}(\mathcal{U}_1(\mathbf{Z}N))$. Of course this implies $u \in \mathcal{Z}(\mathcal{U}_1(\mathbf{Z}N))N$.

So we have shown the following.

Theorem 3.6 *Let N be a finite group. If $N_{\mathcal{U}_1(\mathbf{Z}N)}(N) \neq \mathcal{Z}(\mathcal{U}_1(\mathbf{Z}N))N$, then the isomorphism problem does not hold for $G = N \rtimes_\sigma \langle z \rangle$ with z of infinite order. Hence, it also does not hold for $G = N \times A$ where A is a non-periodic finitely generated abelian group.*

Note that the example of Hertweck in [4] shows that finite non nilpotent groups N with the properties listed in the theorem exist. Hence the previous Theorem provides a counterexample to the isomorphism problem for infinite groups. Note that Zimmerman and Roggenkamp, using the main result of Mazur, in [14] gave a counterexample to the isomorphism problem for group rings of groups $N \rtimes \mathbf{Z}$ over a ring R which is the ring of integers of a global algebraic number field. The ring $R \neq \mathbf{Z}$.

Although the isomorphism problem fails in general for infinite groups, the next proposition indicates that maybe the right property to prove is that group bases are subisomorphic.

Proposition 3.7 *Let $G = N \rtimes A$ be the semidirect of a finite group N with a finitely generated abelian group A. Suppose that G is nilpotent and that $\mathcal{Z}(\mathcal{U}_1(\mathbf{Z}G)) = \mathcal{Z}(G)$. Let H be a group. If $\mathbf{Z}G \cong \mathbf{Z}H$, then G and H are subisomorphic, i.e., G is isomorphic to a subgroup of H and vice versa.*

Proof. Clearly without loss of generality we may assume that H is a group basis of $\mathbf{Z}G$ and that A is free abelian of finite rank. Obviously, G satisfies the assumptions of Theorem 2.1. Hence Corollary 2.4 implies that $\mathcal{Z}(G) = \mathcal{Z}(H)$. Let $C = C_A(N)$ and note that $C \subseteq \mathcal{Z}(G) = \mathcal{Z}(H)$ has finite index in G. Let $\psi : \mathbf{Z}G \to \mathbf{Z}(G/C)$ be the natural epimorphism, and write $\overline{G} = \psi(G)$ and $\overline{H} = \psi(H)$. As before, $\Delta(G,C) = \Delta(H,C)$ and thus $\mathbf{Z}\overline{H} = \mathbf{Z}\overline{G} \cong \mathbf{Z}G/\Delta(G,C)$ and both $\overline{G}$ and $\overline{H}$ are finite nilpotent groups. Since, the isomorphism problem holds for finite nilpotent groups we get that $\overline{H} \cong \overline{G} \cong \overline{N} \rtimes (A/C) \cong N \rtimes (A/C)$. and $\Delta(\overline{G},\overline{N}) = \Delta(\overline{H},\overline{M})$. Hence, because of Roggenkamp and Scott's result (or more generally Weiss' result, see for example [18, Theorem 40.4]), there exists $u \in \mathcal{U}(\mathbf{Q}\overline{G})$ such that $u^{-1}\overline{G}u = \overline{H}$. Since $\overline{N}$ is normal in $\overline{G}$, it follows that $\Delta(\overline{G},\overline{N})$ and therefore

$$\Delta(\overline{H},\overline{M}) = \Delta(\overline{G},\overline{N}) = u^{-1}\Delta(\overline{G},\overline{N})u = \Delta(\overline{H},u^{-1}\overline{N}u).$$

Consequently, $\overline{M} = u^{-1}\overline{N}u$. Thus, $\overline{H} = \overline{M} \rtimes \overline{B}$ with $\overline{B} = u^{-1}(A/C)u$. Let $B = \psi^{-1}(\overline{B}) \cap H$. Since $(ker\psi) \cap H = C$ is torsion free and $\overline{B} \cap \overline{M} = \{1\}$ we must have $B \cap M = \{1\}$. Therefore B is embedded in H/M. Since $\mathbf{Z}H/\Delta(H,M) = \mathbf{Z}G/\Delta(G,N) \cong \mathbf{Z}A$ we also get that B is abelian. So it follows that $H = M \rtimes B$ and $H/C \cong G/C$. So, in particular, H and G have the same Hirsch number (namely that of C). The Theorem now follows as an application of Proposition 2.9. □

Note that in the proof of the Proposition it is actually shown that shown that $H \cong N \rtimes A_1 \subseteq G$, for some subgroup A_1 of A.

Corollary 3.8 *Let $G = N \rtimes_\sigma \langle z \rangle$ be a nilpotent group which is a semidirect product of a finite group N with an infinite cyclic group. If $\mathcal{U}_1(\mathbf{Z}G)$ has trivial centre and $\sigma^2 = 1$, then the isomorphism problem holds for G.*

Proof. Note that the assumption $\sigma^2 = 1$ is equivalent with z^2 is trivial. Let H be a group basis for $\mathbf{Z}G$. By Proposition 3.7 (and the previous remark), $H \cong N \rtimes \langle z^k \rangle$, for some positive integer k. We deal with two cases. First if k is even, then $H \rtimes N \times \langle z \rangle$, a direct product. Since N is finite nilpotent we know that the isomorphism problem holds for N as well as the normalizer condition (see [18, Corollary 9.2]). Hence, by Corollary 3.3, the isomorphism problem holds for H and thus $G \cong H$. Second, assume k is odd. Since $\sigma = \sigma^k$ one gets immediately that $H \cong N \rtimes \langle z^k \rangle \cong N \rtimes \langle z \rangle = G$. □.

4 Automorphisms of Direct Products

In this section we show how the results of the previous sections give information about the automorphisms of certain integral group rings. The main ingredient used is the construction of group bases different from the one started with (see Lemma 3.5). For convenience we recall the Automorphism Conjecture [18]:

Let G be a group and Φ an automorphism of $\mathbf{Z}G$; then there exists an automorphism ϕ of G such that $\Phi \circ \phi^{-1}$ is induced by conjugation with a unit of $\mathbf{Q}G$.

The following observation is clear.

Lemma 4.1 *Let G be any group and suppose that the Automorphism Conjecture holds for G. If $H \subseteq \mathcal{U}_1(\mathbf{Z}G)$ is a group basis which is isomorphic to G then $\mathcal{Z}(G) = \mathcal{Z}(H)$.*

The next result gives a criterium for a subgroup to be linearly independent over $\mathbf{Z}$.

Lemma 4.2 *Let G be a residually finite group and $N \subseteq \mathcal{U}_1(\mathbf{Z}G)$ a finite subgroup. Let $A \subseteq G$ be a subgroup and let $H = \langle N, A\rangle$. If $S = G \bigcap H$ has finite index in G then H is linearly independent over $\mathbf{Z}$.*

Proof. Suppose that $\sum c_h h = 0$ with the $0 \neq c_h \in \mathbf{Z}$. Let X be the union of the supports (with respect to G) of the $h \in H$ appearing in this sum. Since G is residually finite there exists a normal subgroup M of finite index in G such that the canonical map $\Psi : \mathbf{Z}G \longrightarrow \mathbf{Z}(G/M)$ is injective on X. On the other hand, because S and M are of finite index in G, the subgroup $M \cap H$ is also of finite index in G. Hence $\Psi(H)$ is finite. Therefore the elements of $\Psi(H)$ are linearly independent (see [18, Lemma 37.1]). Hence all the $c_h = 0$, a contradiction. □

Theorem 4.3 *Let $G = N \times A$ be the direct product of a finite group N and a non-trivial finitely generated free abelian group A. The Automorphism Conjecture holds for G if and only if the following properties hold:*

1. *$\mathcal{Z}(\mathcal{U}_1(\mathbf{Z}G)) = \mathcal{Z}(G)$ (or equivalently, $\mathcal{Z}(\mathcal{U}_1(\mathbf{Z}T(G)))$ is trivial).*

2. *Subgroups of $\mathcal{U}_1(\mathbf{Z}G)$ which are isomorphic to N are rationally conjugate to N.*

Proof. Suppose the automorphism conjecture holds for G. Because of Corollary 2.2, to prove that property 1. holds, it is sufficient to show that all central units of $\mathbf{Z}N$ are trivial. So suppose $u \in \mathcal{U}_1(\mathbf{Z}N)$ is central. We have to prove that u is trivial, that is $u \in N$. Suppose the contrary. Since central torsion units are trivial ([17, Corollary I.1.7]), the unit u is not periodic. Write $A = A_1 \times \langle z \rangle$, with A_1 a subgroup of A and $\langle z \rangle$ infinite cyclic. Let $B = A_1 \times \langle uz \rangle$. By Lemma 3.5, $N \times B$ is also a group basis for $\mathbf{Z}G$ and clearly $N \times A \cong N \times B$. Since, by assumption, the Automorphism conjecture holds for G, Lemma 4.1 implies that $\mathcal{Z}(G) = \mathcal{Z}(N \times B)$, in particular $uz \in \mathcal{Z}(G)$. Hence $u \in (\mathcal{Z}(G) \cap \mathbf{Z}N) = \mathcal{Z}(N)$, a contradiction.

To prove the second property, let $M \subseteq \mathcal{U}_1(\mathbf{Z}G)$ and suppose M is isomorphic to N. Let $H = M \times A$. Note that G is polycyclic-by-finite and thus residually finite (see [16, Theorem 4.3]). By Lemma 4.2 the elements of H are linearly independent over $\mathbf{Z}$ and hence, by Lemma 4 in [9], the group H is a group basis for $\mathbf{Z}G$. Since $G \cong H$, it therefore follows that there exists an automorphism Φ of $\mathbf{Z}G$ which map G onto H. Because the automorphism conjecture holds for G and both N and M are characteristic subgroups (they are the respective torsion subgroups) it follows that they are rationally conjugate.

We now prove the converse. Let Φ be an automorphism of $\mathbf{Z}G$ and $M = \Phi(N)$. By condition 1. it follows that $\Phi(G) = M \times A$. By condition 2. there exists a rational unit $u \in \mathbf{Q}G$ such that $u^{-1}Mu = N$. Thus $u^{-1}\Phi(G)u = G$ and from this the result follows. □

Since the torsion units are trivial in the integral group ring $\mathbf{Z}N$ of a finite abelian group N, it is clear that the automorphism conjecture holds for $\mathbf{Z}N$. Also, for a finitely generated free abelian group A, the units of $\mathbf{Z}(N \times A)$ are trivial if and only if the units if $\mathbf{Z}N$ are trivial (this is well known and also follows from Corollary 2.2). Hence as an immediate consequence of the previous theorem we obtain a characterization of when the automorphism conjecture holds for finitely generated abelian groups.

Corollary 4.4 *Let G be a finitely generated abelian group. The Automorphism Conjecture holds for G if and only if one of the following properties holds:*

1. *G is finite.*

2. *$\mathbf{Z}T(G)$ has only trivial units (that is, the exponent of $T(G)$ divides 4 or 6).*

In case G is infinite the Automorphism Conjecture holds if and only if $\mathbb{Z}G$ has only trivial units.

Hence there are many commutative infinite counterexamples to the automorphism conjecture. Note that there already exist noncommutative counterexamples due to Roggenkamp and Scott (see also Klinger [8]) and Sehgal and Zalesskii ([18, Theorem 45.13]). The (finite) counterexample of Roggenkamp and Scott can easily be used to show that Condition 2. of Theorem 4.3 does not hold in general. In [1] and [7] one can find results related to this question. Finally, Ritter and Sehgal in [12] determined when all central units of the integral group ring of a finite group are trivial. It follows that Condition 1. can be described completely internally within the group.

5 Isomorphisms For Nilpotent Groups

In this section we are concerned with Sehgal's Question 46 in [18], that is, we consider the isomorphism problem for integral group rings of nilpotent groups G. We recall some well known results. Röhl in [15] showed that any group basis for $\mathbb{Z}G$ is also nilpotent. Ritter and Sehgal in [11] showed that if G and H are finitely generated nilpotent class two groups with isomorphic integral group rings then $G \cong H$.

In this section we first show that the property "nilpotence class two" is determined by the integral group ring. Note that we do not assume that the involved groups are finitely generated.

For this we need to introduce some notation. For a group G we denote by $\gamma_n(G)$ the n-term in the lower central series of G. If G is nilpotent, then by $\gamma(G)$ we denote its nilpotency class.

Recall that the n-th dimension subgroup of a group G is defined to be the subgroup $D_n(G) = G \bigcap (1 + \Delta^n(G))$.

Theorem 5.1 *Let G be a nilpotent class two group and let H be a group basis of $\mathbb{Z}G$. Then H is a nilpotent group of nilpotency class two.*

In particular, the isomorphism problem holds for any finitely generated nilpotent class two group.

Proof. Because $H \subseteq \mathcal{U}_1(\mathbb{Z}G)$ we have that $\Delta(G, T(G)) = \Delta(H, T(H))$. Since $G/T(G)$ and $H/T(H)$ are ordered groups, the units of $\mathbb{Z}(G/T(G)) \cong$

$\mathbb{Z}G/\Delta(G,T(G))$ and $\mathbb{Z}(H/T(H) \cong \mathbb{Z}H/\Delta(H,T(H))$ are trivial (see [18, Lemma 45.3]), and thus it follows that $G/T(G) \cong G/T(H)$.

Let $L = \gamma_n(H)$ be the last non-trivial term of the lower central series of H. If L is not periodic, then $L \not\subseteq T(H)$ and thus H and $H/T(H)$ have the same nilpotency class. Hence the result follows in this case.

So we may suppose for the rest of the proof that L is torsion. Since L is torsion and central, we have that $L \subseteq G$ (see [17, Corollary I.1.7]). To prove the result, it is now sufficient to show that $n \leq 2$. Suppose the contrary, that is $n > 2$. Hence $L \subseteq \gamma_3(G)$. Since $\gamma_3(H)$ is equal to the third dimension subgroup of H(see [3]) and because $\Delta(G) = \Delta(H)$, we have that $L \subseteq G \bigcap (1 + \Delta^3(G)) = \gamma_3(G)$, where the last equality follows again by [3, Corollary IV.1.10]. Since G has nilpotency class two it follows that L is trivial, a contradiction. Hence $n = 2$. □

It follows from the Theorem that if G is a polycyclic-by-finite group with $\gamma_3(G)$ a torsion group then $G/\gamma_3(G)$ is determined by $\mathbb{Z}G$ and thus also the Hirsch length of G is an invariant of the integral group ring.

Next we show that any nilpotency class is determined by the integral group ring but provided our group is nilpotent and finitely generated. For this we need some more notation.

Let $\mathcal{A}$ be the set of finitely generated nilpotent groups whose integral group ring contains a group basis H with $\gamma(G) \neq \gamma(H)$. Since torsion free finitely generated nilpotent groups are ordered it follows by [18, Lemma 45.3] that the isomorphism problem is true for torsion free nilpotent groups and thus elements of $\mathcal{A}$ are not torsion free. Denote by $h(G)$ the Hirsch number of G. Let $\delta(G) = h(G) + \gamma(G) + |T(G)|$ and let $\mathcal{G}$ denote the set of those groups G in $\mathcal{A}$ which have smallest possible $\delta(G)$. Since $G/T(G)$ is an ordered group it follows by [18, 45.3] that $h(G) = h(H)$. On the other hand [7, Corollary 3.5] tells us that $T(G)$ and $T(H)$ are isomorphic and in particular they have the same order. Hence it follows that $\delta(G) - \delta(H) = \gamma(G) - \gamma(H)$ for any group basis H of $\mathbb{Z}G$.

Lemma 5.2 *Let $G \in \mathcal{G}$, then $T(G)$ has prime power order and if H is a group basis of $\mathbb{Z}G$ then $\gamma(G) \leq \gamma(H) \leq \gamma(G) + 1$. Moreover if $\gamma(H) = \gamma(G) + 1$ then the last non-trivial term of the lower central series of G is non-torsion, that of H is torsion of prime order and is the only normal subgroup of prime order; in particular $\mathcal{Z}(G)$ has cyclic torsion.*

Proof. Set $n = \gamma(G)$ and $m = \gamma(H)$. Since $G \in \mathcal{G}$ it is clear that $n \leq m$. So to prove the result we may assume throughout the proof that $n < m$.

Suppose that $A = \gamma_n(G)$ and $B = \gamma_m(H)$ are both torsion. Note that in this case A and B being central and torsion are simultaneously in both G and H (see [18]).

Now $B \not\subseteq A$, then $n - 1 = \gamma(G/A) = \gamma(H/A) = m$, a contradiction. So, $B \subset A$ and thus $n \geq \gamma(G/B) = \gamma(H/B) = m - 1$. Since also $n < m$ we get that $n = m - 1$. However, $n - 1 = \gamma(G/A) = \gamma(H/A)$ and thus $\gamma_n(H) \subseteq A$ is central. So $m = \gamma(H) = n$, again a contradiction.

If both A and B were non-torsion we could use [18, Lemma 45.3] to get that $n = \gamma(G/T(G)) = \gamma(H/T(H)) = m$.

We now prove that $A = \gamma_n(G)$ is non-torsion. Suppose the contrary, then by the previous B is non-torsion. Since $A \subseteq G \bigcap H$ we have that $\Delta(G, A) = \Delta(H, A)$ and thus $\mathbf{Z}(G/A) \cong \mathbf{Z}(H/A)$. Since $G \in \mathcal{G}$ we therefore get $m = \gamma(H/A) = \gamma(G/A) = n - 1$, a contradiction. This proves that A is indeed non-torsion.

Now, since G contains torsion, the group $T(\mathcal{Z}(G))$ is not trivial. Let C be a non-trivial finite central subgroup of G. Then, as before, $C \subseteq (G \cap H)$ and $\Delta(G, C) = \Delta(H, C)$. So $\mathbf{Z}(G/C) \cong \mathbf{Z}(H/C)$ and $|T(G/C)| < |T(G)|$. Hence, as $G \in \mathcal{G}$, we get $\gamma(G/C) = \gamma(H/C)$. Since A is not torsion, $\gamma(G/C) = n$. If $B \not\subseteq C$, then $\gamma(H/C) = m$, and thus $n = m$, a contradiction. Therefore $B \subseteq C$. Since C is arbitrary, it follows that $T(\mathcal{Z}(G))$ is cyclic of prime power order and B is cyclic of prime order. Hence $T(G)$ also has prime power order. Also, $n = \gamma(G/B) = \gamma(H/B) = m - 1$, and thus $m = n + 1$. The result now follows. □

Proposition 5.3 *Let $G \in \mathcal{G}$ and choose a group basis H of $\mathbf{Z}G$ with $n = \gamma(G) \neq \gamma(H) = m$. Then $\gamma_m(H) \subseteq D_{m+1}(G)$ and thus G is a counterexample to the dimension subgroup problem. Moreover there exists a finite p-group which is a homomorphic image of G and also a counterexample to the dimension subgroup problem.*

Proof. Lemma 5.2 tells us that $\gamma_m(H)$ has order p, a prime, and that $\gamma_n(G)$ is non-torsion. We also know that in this case $m = n + 1$. Lemma III.6.8 of [17] states that $\Delta^{[k]}(G) + \Delta^{k+1}(G) = \Delta(G, \gamma_k(G)) + \Delta^{k+1}(G)$ for all $k \geq 1$. Since $\mathbf{Z}G = \mathbf{Z}H$ the left hand side of the previous equation does not depend on G and thus $\Delta(G, \gamma_k(G)) + \Delta^{k+1}(G) = \Delta(H, \gamma_k(H)) + \Delta^{k+1}(H)$ for all $k \geq 1$. Since $\gamma_m(G) = \{1\}$ we therefore get with $k = m$ that $\Delta^{m+1}(G) = \Delta(H, \gamma_m(H)) + \Delta^{m+1}(G)$ and thus $\Delta(H, \gamma_m(H)) \subseteq \Delta^{m+1}(G)$. Hence, $\gamma_m(H) \subset (1 + \Delta^{m+1}(G))$. But, since $\gamma_m(H)$ is torsion and central we also have $\gamma_m(H) \subseteq G$. So $\gamma_m(H) \subseteq D_{m+1}(G)$. It follows that $\gamma_m(G) \neq$

$D_m(G)$ and thus G is a counterexample to the dimension subgroup problem. Since G is residually finite if follows that G has a finite homomorphic image which is a counterexample to the dimension subgroup problem and, G being nilpotent, an easy argument shows that the latter can be assumed to be of prime power order. □

The Proposition 5.3 gives a link between the dimension subgroup problem and the problem under consideration in this section, that is, the nilpotency class is determined by the integral group ring. In the following theorem we show that if the dimension subgroup problem holds for a finitely generated nilpotent group G, then we get a positive solution to our problem. Recall that the dimension subgroup problem does not hold in general; even for nilpotent groups of nilpotency class three there are counterexamples. The smallest such counterexample is due to Rips (see [3]). This example has the same properties as those listed in Proposition 5.3. However, if G has nilpotency class three and $T(G)$ has no 2-torsion, then we can give an elementary proof for a positive solution to our problem.

For this recall that Losey (see for example [3, Corollary iV.1.11]) proved that $D_4^2(G) \subseteq \gamma_4(G)$, for any group G. Hence, if G is a nilpotent group of class three and $T(G)$ does not contain 2-torsion, then $D_4(G)$ is trivial. So, if H is as in Proposition 5.3 with $\gamma(H) \neq 3$, then, by Lemma 2.2, $4 = \gamma(H)$. Hence by Proposition 5.3 $\{1\} \neq \gamma_4(G) \subseteq D_5(G) = \{1\}$, a contradiction. So we have shown that indeed the nilpotency class of G is determined by its integral group ring.

Theorem 5.4 *Let G be a finitely generated nilpotent group and suppose that one of the following properties holds.*

1. *G satisfies the dimension subgroup conjecture.*
2. *Central units of $\mathbb{Z}G$ are trivial and $\gamma(G) \leq 3$.*
3. *$\mathbb{Z}T(G)$ has only trivial units that are central in $\mathbb{Z}G$ and $\gamma(G) \leq 3$.*

Then group bases of $\mathbb{Z}G$ all have the same nilpotency class.

Proof The first part follows at once from Proposition 5.3.

Because of the results in Section 5, to prove the second statement we only have to deal with $\gamma(G) = 3$ and a group a group basis H for $\mathbb{Z}G$ with $\gamma(H) \geq 3$. We first prove that if $\gamma_3(G)$ is torsion then $\gamma(H) = 3$ and $\gamma_3(G) = \gamma_3(H)$, without any assumption on the center of the group ring.

In fact since $A = \gamma_3(G)$ is torsion and central it is also contained in H. Thus from the previous section we get that $2 = \gamma(G/A) = \gamma(H/A)$. But this means that $\gamma_k(H) \subseteq A$ for all $k \geq 3$ and thus $\gamma(H) \leq 3$. So indeed $\gamma(H) = 3$ and $\gamma_3(H) \subseteq \gamma_3(G)$. Since $\gamma(H/\gamma_3(H)) = 2$ and $\mathbf{Z}(H/\gamma_3(H)) \cong \mathbf{Z}(G/\gamma_3(H))$, it follows that $\gamma(G/\gamma_3(H)) = 2$ and thus $\gamma_3(G) \subseteq \gamma_3(H)$. Hence $\gamma_3(G) = \gamma_3(G)$.

If on the other hand $\gamma_3(G)$ is non-torsion we write it as the product of a finite group and a non-trivial free abelian group B and let $A = B^2$. Since A is central and because of the assumption on the centre, we have by Corollary 2.4 that $A \subseteq G \bigcap H$. Hence $\Delta(G, A) = \Delta(H, A)$ and $\overline{H}$, the image of H in $\mathbf{Z}(G/A)$, is a group basis of $\mathbf{Z}(G/A)$. Note that $\gamma_3(G/A)$ is a torsion group and thus by what we proved above $3 = \gamma(G/A) = \gamma(H/A)$ and $\gamma_3(G/A) = \gamma_3(H/A)$. ¿From this it follows that $\gamma_3(H) \subseteq \gamma_3(G)$. So $\gamma_3(H)$ is central and thus $\gamma(H) = 3$.

The third statement follows as a consequence of the second statement and Corollary 2.2. □

Corollary 5.5 *Let G be a finitely generated nilpotent group¿ If G is without 2-torsion, then all group bases of $\mathbf{Z}G$ have the same nilpotency class.*

Proof. In a recent paper Gupta has shown that $D_k(G)/\gamma_k(G)$ is a 2-group, for any $k \geq 1$. Hence the result follows from Theorem 5.4. □

Finally we remark that all our results are stated for integral group rings but it is clear that they hold for more general coefficient rings as in [9].

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