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§ 0 - Introduction

Recently, Hoffman and Osserman [4] have studied degenerate minimal surfaces in $\mathbb{R}^4$ in great details. In this paper we continue the study from a different point of view. We first analyse the geometry of the complex quadric Q_2 in CP^3 by looking at its intersections with hyperplanes in CP^3, as studied in [4], but we emphasize on their intrinsic aspects and their relations with the euclidean geometry in $\mathbb{R}^4$ since each point in Q_2 represents an oriented plane in $\mathbb{R}^4$. Most important of all, we will show that there are natural ways to assign normal directions to each intersection so that when we study degenerate minimal surfaces in $\mathbb{R}^4$ there are natural normal vector fields to facilitate understanding the second fundamental form. And we will apply the study, in particular, to the stability of such minimal surfaces.

§ 1 - Preliminary

The quadric in CP^3 is defined to be

(1.1) $$Q_2 = \{Z\varepsilon\ CP^3 / Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 = 0\}$$

Each point of Q_2 can be viewed naturally as an oriented plane in $\mathbb{R}^4$, generated by its real and imaginary parts:

(1.2) $$X = (x_1, x_2, x_3, x_4), \quad Y = (y_1, y_2, y_3, y_4)$$

where $x_j + i\, y_j = z_j, \quad 1 \le j \le 4$

To each orientable minimal surface in $\mathbb{R}^4$ given by the immersion

(1.3) $$x : M^2 \longrightarrow \mathbb{R}^4,$$

the generalized Gauss map

(1.4) $$[\phi] : M^2 \longrightarrow Q_2$$

is definid by

(1.5) $$\phi = \frac{\partial x}{\partial \xi} - \frac{\partial x}{\partial \eta},$$

where $\zeta = \xi + i\eta$ is the conformal structure on M defined by isothermal parameters (ξ,η);
x is said to be degenerate if its Gauss image lies in some hyperplane in CP^3, and, is said to be h-degenerate if h is the largest integer such that the Gaussian image lies in a projective subspace of codimension h. Therefore there are only three kinds of degeneracy : 1-degenerate, 2-degenerate and 3-degenerate. It's known [4] that 3-degenerate minimal surfaces in $\mathbb{R}^4$ are just planes. We will, therefore, only concentrate on 1-degenerate and 2-degenerate minimal surfaces in $\mathbb{R}^4$.

Finally, for a normal variation

(1.6) $$F : M \times (-\varepsilon,\varepsilon) \longrightarrow \mathbb{R}^4$$

with compact support, the formula for the second variation is given [6] by

(1.7) $$I(v,v) = \int_M < -\Delta v - \tilde{A}(v),\ v > dM$$

where V is the variational vector field. If $V = hn$ for some unitary normal vector field n and some smooth function h, then (1.7) can be rewritten as

$$(1.8) \qquad I(V,V) = \int_M (-h\Delta h - h^2\langle A^n, A^n\rangle + h^2\langle \nabla n, \nabla n\rangle)\, dM$$

A domain D of M is said to be stable if for all non-trivial normal variations with compact support in D, $I(V,V)$ is always positive.

§ 2 - The geometry of the quadric Q_2

1) It's known [4] that the intersection of Q_2 with a hyperplane H in CP^3 is congruent in Q_2 to the one by assuming H be determined by

$$(2.1) \qquad cZ_3 - Z_4 = 0$$

with

$$(2.2) \qquad c = it\ , \quad 0 \le t \le 1$$

For $t = 1$, $S = Q_2 \cap H$ is the union of two projective lines:

$$(2.3) \qquad L_1 : Z_1 - iZ_2 = 0, \quad iZ_3 - Z_4 = 0$$

$$L_2 : Z_1 + iZ_2 = 0, \quad iZ_3 - Z_4 = 0$$

with only one common point: $[0,0,1,i]$, and S has constant curvature $K = 2$.

For $0 \le t \le 1$, S is isometric to the quadric $\tilde{Q}_1$:

$$(2.4) \qquad \tilde{Z}_1^2 + \tilde{Z}_2^2 + K\tilde{Z}_3^2 = 0$$

in CP^3, where

$$(2.5) \qquad k = \frac{1-t^2}{1+t^2}$$

Now set

$$(2.6) \qquad Z_1 = \tilde{Z}_1 , \quad Z_2 = \tilde{Z}_2, \quad Z_3 = \sqrt{k}\ \tilde{Z}_3$$

which identified $\tilde{Q}_1$ with $Q_1 = \{Z \in CP^2/Z_1^2 + Z_2^2 + Z_3^2 = 0\}$. Through the identification

$$(2.7) \qquad \zeta \in C-\{0\} \longleftrightarrow [\tfrac{1}{2}(\zeta - \tfrac{1}{\zeta}), \tfrac{-i}{2}(\zeta + \tfrac{1}{\zeta}), 1]$$

$$\zeta = 0 \longleftrightarrow [1,i,0]$$

$$\zeta = \infty \longleftrightarrow [1,-i,0]$$

betwen $\hat{C}$ and Q_1, we see that the Fubini-Study metric on CP^2

$$(2.8) \qquad d\tilde{s}^2 = 2 \frac{|\tilde{Z} \wedge d\tilde{Z}|^2}{|\tilde{Z}|^4}$$

induces a metric in C given by

$$(2.9) \qquad ds^2 = \frac{4}{k} \frac{|\zeta|^4 + 2k|\zeta|^2 + 1}{(|\zeta|^4 + \frac{2}{k}|\zeta|^2 + 1)^2} |d\zeta|^2 \equiv \lambda^2 |d\zeta|^2$$

Using the formula for the Gaussian curvature

$$(2.10) \qquad K = - \frac{2\, \partial\bar{\partial} \log \lambda^2}{\lambda^2}$$

where $\partial = \frac{\partial}{\partial \zeta}$, $\bar{\partial} = \frac{\partial}{\partial \bar{\zeta}}$, we get

$$(2.11) \qquad K = 2 - k^2 \frac{(|\zeta|^4 + \frac{2}{k}|\zeta|^2 + 1)^3}{(|\zeta|^4 + 2k|\zeta|^2 + 1)^3}$$

From (2.11) and the fact that

$$(2.12)\qquad |\tilde{z}_1|^2 + |\tilde{z}_2|^2 + |\tilde{z}_3|^2 = \frac{1}{2|\zeta|^2}\left(|\zeta|^4 + \frac{2}{k}|\zeta|^2 + 1\right)$$

$$|\tilde{z}_1|^2 + |\tilde{z}_2|^2 + K^2|\tilde{z}_3|^2 = \frac{1}{2|\zeta|^2}\left(|\zeta|^4 + 2k|\zeta|^2 + 1\right)$$

for $\zeta \neq 0$, we therefore obtain the Ness' formula [5, p60] for the Gaussian curvature on $\tilde{Q}_1$:

$$(2.13)\qquad K = 2 - k^2 \frac{(|\tilde{z}_1|^2 + |\tilde{z}_2|^2 + |\tilde{z}_3|^2)^3}{(|\tilde{z}_1|^2 + |\tilde{z}_2|^2 + k^2|\tilde{z}_3|^2)^3}$$

To determine the extremum of K, note that K depends only on $r = |\zeta|^2$. Studing the function

$$(2.14)\qquad f(r) = \frac{r^2 + \frac{2}{k}r + 1}{r^2 + 2kr + 1}, \quad r \geq 0,$$

we find that K achieves its maximum at $|\zeta| = 1$ and its minimum at $\zeta = 0$ or $\zeta = \infty$. And we obtain the result of Hoffman-Osserman [4]

$$(2.15)\qquad \max K = 2 - k^2$$

$$\min K = 2 - \frac{1}{k}$$

Furthermore, from (2.9) we see that

$$ds^2\Big|_{|\zeta|=r} = ds^2\Big|_{|\zeta|=\frac{1}{r}}$$ and therefore, $K(\zeta) = K(\frac{1}{\zeta})$.

Calculating the area for $D = \{\zeta \in \hat{C} / |\zeta| \leq 1\}$ we find

$$(2.16)\qquad A(D) = 2\pi$$

Hence S can be viewed, intrinsically, as in Figure 1

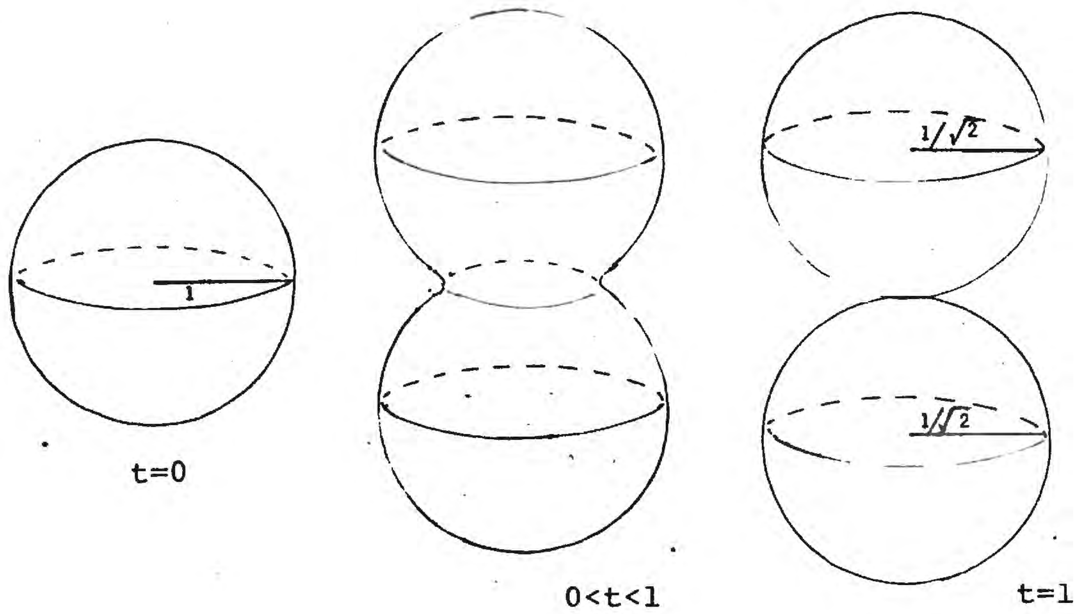

FIGURE 1

2) Since each point in Q_2 represents an oriented plane in $\mathbb{R}^4$ and isometries on Q_2 are induced by isometries on $\mathbb{R}^4$, $O(4)$, we now study S in terms of the 4-dimensional euclidean geometry.

For t = 1, S decomposes in two complex projective lines. We take one of them, say,

(2.17) $\quad L: \; Z_2 = i\, Z_1, \quad Z_4 = i\, Z_3$

write

(2.18) $\quad Z_1 = \alpha + i\,\beta, \quad Z_3 = \gamma + i\,\delta$

Then the homogeneous vector

(2.19) $\quad [Z_1, Z_2, Z_3, Z_4] = X + i\,Y$

satisfies

(2.20) $X = (\alpha, -\beta, \gamma, -\delta)$, $Y = (\beta, \alpha, \delta, \gamma)$

Set

(2.21) $V = (\gamma, \delta, -\alpha, -\beta)$, $W = (\delta, -\gamma, -\beta, \alpha)$

natural orthogonal complements to X, Y in $\mathbb{R}^4$. Then

(2.22) $V + i\,W = [Z_3, -iZ_3, -Z_1, iZ_1]$

represents the oriented plane normal to (2.19) and describes a complex projective line in Q_2:

(2.23) $\tilde{L}$: $\tilde{Z}_2 = -i\,\tilde{Z}_1$, $\tilde{Z}_4 = -i\,\tilde{Z}_3$

We can conclude now that

<u>Proposition 2.1</u> - For each complex projective line in Q_2, there is a natural correspondence to another complex projective line in Q_2 such that any two corresponding planes are mutually orthogonal in $\mathbb{R}^4$.

For $0 \le t < 1$, we will show that there exist two natural orthogonal normal fields defined on S. We start with some algebraic considerations.

Let

(2.24) $Z = (Z_1, Z_2, Z_3, Z_4) \in C^4-\{0\}$

Write

(2.25) $Z = X + i\,Y$

with

(2.26) $X = (x_1, x_2, x_3, x_4), \quad Y = (y_1, y_2, y_3, y_4)$

where $Z_j = x_j + i\, y_j, \quad 1 \le j \le 4$

Define

(2.27)
$$N_1 = \mathrm{Im}\,(Z_2Z_3,\ Z_3Z_1,\ Z_1Z_2, 0)$$
$$N_2 = \mathrm{Im}\,(Z_2Z_4,\ Z_4Z_1,\ 0,\ Z_1Z_2)$$

It's trivial to see that

<u>Lemma 2.2</u> - $\mathrm{Im}\,(Z_1Z_2) = 0 \iff \det\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = 0$

Let π be the canonical projection from $\mathbb{R}^4$ to the x_1x_2 - plane. Then we have

<u>Lemma 2.3</u> - $\mathrm{Im}\,(Z_1Z_2) = 0 \longleftrightarrow \pi(X)$ e $\pi(Y)$ are linearly dependent.

Since

(2.28)
$$N_1 = (x_2y_3 - x_3y_2,\ x_3y_1 - x_1y_3,\ x_1y_2 - x_2y_1,\ 0)$$
$$N_2 = (x_2y_4 - x_4y_2,\ x_4y_1 - x_1y_4,\ 0,\ x_1y_2 - x_2y_1)$$

it can be proved that if $x_1y_2 - x_2y_1 = 0$ then N_1 and N_2 are linearly dependent. Together with Lemmas 2.2, 2.3 we obtain

<u>Lemma 2.4</u> - N_1 and N_2 are linearly dependent if and only if $\pi(X)$ and $\pi(Y)$ are linearly dependent.

Furthermore, by straightforward calculation, we have

<u>Lemma 2.5</u> - $\langle N_j, X\rangle = 0 = \langle N_j, Y\rangle$ for $j = 1,2$

Now in case

$$(2.29) \qquad z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0, \qquad z_4 = it\, z_3$$

for $0 \leq t < 1$, we have

$$(2.30) \qquad z_1^2 + z_2^2 + (\sqrt{1-t^2}\, z_3)^2 = 0$$

and

$$(2.31) \qquad (x_1, x_2 \sqrt{1-t^2}\, x_3), \qquad (y_1, y_2, \sqrt{1-t^2}\, y_3)$$

are orthogonal and possess the same positive norm.

Since $\mathrm{Im}(\bar{z}_2 \sqrt{1-t^2}\, z_3, \sqrt{1-t^2}\, \bar{z}_3 z_2, \bar{z}_1 z_2) \neq 0$,

we have

Lemma 2.6 - N_1 will never vanish if (2.29) holds.

And, in this case, with

$$(2.32) \qquad x_4 = -t\, y_3, \qquad y_4 = t x_3$$

and

$$(2.33) \qquad x_1^2 + x_2^2 + x_3^2 + t^2 y_3^2 = y_1^2 + y_2^2 + y_3^2 + t^2 x_3^2$$

$$x_1 y_1 + x_2 y_2 + x_3 y_3 - t^2 x_3 y_3 = 0$$

we have

Lemma 2.7 - $\langle N_1, N_2 \rangle = 0$ if (2.29) holds.

Combining these results and the fact that the directions of N_1 and N_2 are independent of the choice of the homogeneous coordinates, we conclude that

Proposition 2.8 - For $0 \le t < 1$, there are two natural unitary normal vector fields, n_1, n_2, defined on S, such that

a) $n_1 \perp n_2$

b) $n_1 \mathbin{/\!/} N_1$ and

c) $n_2 \mathbin{/\!/} N_2$ when $\operatorname{Im}(\bar{Z}_1 Z_2) \neq 0$

In order to get a more precise and useful description of the normal fields we set

$$(2.34)\qquad w = \frac{Z_3 - i\, Z_4}{Z_1 - i\, Z_2} = (1+t)\,\frac{Z_3}{Z_1 - i\, Z_2}$$

$$\tilde{w} = \frac{Z_3 + i\, Z_4}{Z_1 - i\, Z_2} = (1-t)\,\frac{Z_3}{Z_1 - i\, Z_2}$$

Then $w = \frac{1+t}{1-t}\,\tilde{w}$ and it can be easily seen that

$$(2.35)\qquad (Z_1,Z_2,Z_3,Z_4) = (Z_1 - iZ_2)\left(\frac{1}{2}\left(1 - \frac{1-t}{1+t}\, w^2\right),\ \frac{i}{2}\left(1 + \frac{1-t}{1+t}\, w^2\right),\ \frac{1}{1+t}\, w,\ \frac{it}{1+t}\, w\right)$$

and, at $Z_1 - i\, Z_2 \neq 0$, n_1 and n_2 are parallel to

$$(2.36)\qquad \tilde{n}_1 = (2 \operatorname{Re} w,\ 2 \operatorname{Im} w,\ (1-t)\,|w|^2 - (1+t),\ 0)$$

$$\tilde{n}_2 = (-2t \operatorname{Im} w,\ 2t \operatorname{Re} w,\ 0,\ (1-t)\,|w|^2 + (1+t))$$

respectively, which satisfy obviously $\langle \tilde{n}_1, \tilde{n}_2 \rangle = 0$ and

$$(2.37)\qquad |\tilde{n}_1| = |\tilde{n}_2| = (1-t)^2\,|w|^4 + 2(1+t^2)\,|w|^2 + (1+t)^2$$

Remark - The directions of $\tilde{n}_1$ and $\tilde{n}_2$ extend naturally over $Z_1 - i\, Z_2 = 0$ to the directions of $(0,0,1,0)$ and $(0,0,0,1)$, respectively, since w extends naturally to ∞ at $Z_1=iZ_2$.

§ 3 - 1-degenerate minimal surfaces in $\mathbb{R}^4$

Let $x: M^2 \longrightarrow \mathbb{R}^4$ be an 1-degenerate minimal surface. Without loss of generality (see [4]), we may assume its generalized Gauss map satisfies:

$$(3.1) \qquad \phi_4 = it\,\phi_3$$

for some $0 \leq t < 1$

Set

$$(3.2) \qquad g = \frac{\phi_3 - i\phi_4}{\phi_1 - i\phi_2} = (1+t)\,\frac{\phi_3}{\phi_1 - i\phi_2}$$

which is a meromorphic function on M. Comparing (2.35), ϕ can be written as

$$(3.3) \qquad \phi = \frac{\phi_1 - i\phi_2}{2}\left(1 - \frac{1-t}{1+t}\,g^2, i\left(1 + \frac{1-t}{1+t}\,g^2\right); \frac{2}{1+t}\,g, \frac{2it}{1+t}\,g\right)$$

Set

$$(3.4) \qquad f(\zeta)\,d\zeta = (\phi_1 - i\phi_2)\,d\zeta$$

which is a global holomorphic differential on M. Then the induced metric

$$(3.5) \qquad ds^2 = \lambda^2\,|d\zeta|^2$$

is given by

$$(3.6) \qquad \lambda^2 = \frac{1}{2}\,|\phi|^2 \equiv \frac{1}{2}\sum_{j=1}^{4}|\phi_j|^2 = \frac{|f|^2}{4}\left\{1 + \frac{2(1+t^2)}{(1+t)^2}\,|g|^2 + \left(\frac{1-t}{1+t}\right)^2|g|^4\right\}$$

And the Gaussian curvature K given by the formula (see [3])

(3.7) $$K = -4 \frac{|\phi\wedge\phi'|^2}{|\phi|^6}$$

can be computed to be

(3.8) $$K = -\frac{16|g'|^2}{|f|^2}(1+t)^4 \frac{(1+t)^2 + 2(1-t)^2|g|^2 + \frac{(1-t)^2(1+t^2)}{(1+t)^2}|g|^4}{\{(1+t)^2 + 2(1+t^2)|g|^2 + (1-t)^2|g|^4\}^3}$$

From (2.36), (2.37) and (3.6), setting $g=u+iv$, we get that

(3.9) $$N_1 = (-2u, -2v, -(1-t)|g|^2 + (1+t), 0)$$

$$N_2 = (-2tv, 2tu, 0, (1-t)|g|^2 + (1+t))$$

are two normal vector fields defined off the isolated set: $\phi_1 - i\phi_2 = 0$, with

(3.10) $$N_1 \perp N_2$$

$$|N_1| = |N_2| = (1-t)^2|g|^4 + 2(1+t^2)|g|^2 + (1+t)^2 = \frac{4\lambda^2(1+t)^2}{|f|^2}$$

Furthermore,

(3.11) $$n_1 = \frac{N_1}{|N_1|}, \quad n_2 = \frac{N_2}{|N_2|}$$

are two mutually orthogonal unitary normal vector fields which extend over all M.

To study the induced second fundamental form we set

(3.12) $$e_1 = \frac{1}{\lambda}\frac{\partial u}{\partial \xi}, \quad e_2 = \frac{1}{\lambda}\frac{\partial u}{\partial \eta}$$

which form a local tangent frame for x. Note that

$$\left(\frac{\partial N_1}{\partial \xi}\right)^t = \left\langle\frac{\partial N_1}{\partial \xi}, e_1\right\rangle e_2 + \left\langle\frac{\partial N_1}{\partial \xi}, e_2\right\rangle e_2 =$$

$$= \frac{1}{\lambda}\left\{\left(1+\frac{1-t}{1+t}|g|^2\right)\left(u_\xi \operatorname{Re}f+u_\xi \operatorname{Im}f\right)e_1+\left(1+\frac{1-t}{1+t}|g|^2\right)\left(-u_\xi \operatorname{Im}f-v_\xi \operatorname{Re}f\, e_2\right)\right\}$$

(3.13)

$$\left(\frac{\partial N_2}{\partial \xi}\right)^t = \frac{1}{\lambda}\left\{-t\left(1-\frac{1-t}{1+t}|g|^2\right)\left(v_\xi \operatorname{Re}f+u_\xi \operatorname{Im}f\right)e_1+t\left(1-\frac{1-t}{1+t}|g|^2\right)\left(v_\xi \operatorname{Im}f-u_\xi \operatorname{Re}f\right)e_2\right\}$$

$$\left(\frac{\partial N_1}{\partial \eta}\right)^t = \frac{1}{\lambda}\left\{-\left(1+\frac{1-t}{1+t}|g|^2\right)\left(v_\xi \operatorname{Re}f+u_\xi \operatorname{Im}f\right)e_1+\left(1+\frac{1-t}{1+t}|g|^2\right)\left(v_\xi \operatorname{Im}f-u_\xi \operatorname{Re}f\right)e_2\right\}$$

$$\left(\frac{\partial N_2}{\partial \eta}\right)^t = \frac{1}{\lambda}\left\{t\left(1-\frac{1-t}{1+t}|g|^2\right)\left(-u_\xi \operatorname{Re}f+v_\xi \operatorname{Im}f\right)e_1+t\left(1-\frac{1-t}{1+t}|g|^2\right)\left(u_\xi \operatorname{Im}f+v_\xi \operatorname{Re}f\right)e_2\right\}$$

Therefore we obtain

$$\langle A^{n_1}, A^{n_1}\rangle = \langle A^{n_1}e_1, A^{n_1}e_1\rangle + \langle A^{n_1}e_2, A^{n_1}e_2\rangle =$$

$$= \frac{1}{\lambda^2|N_1|^2}\left\{\left\langle\left(\frac{\partial N_1}{\partial \xi}\right)^t, \left(\frac{\partial N_1}{\partial \xi}\right)^t\right\rangle + \left\langle\left(\frac{\partial N_1}{\partial \eta}\right)^t, \left(\frac{\partial N_1}{\partial \eta}\right)^t\right\rangle\right\} =$$

(3.14)
$$= \frac{|f|^4\,|g'|^2}{2\,\lambda^6\,(1+t)^2}\left(1+\frac{1-t}{1+t}|g|^2\right)^2$$

$$\langle A^{n_2}, A^{n_2}\rangle = \frac{t^2|f|^4\,|g'|^2}{2\,\lambda^6\,(1+t)^2}\left(1-\frac{1-t}{1+t}|g|^2\right)^2$$

$$\langle A^{n_1}, A^{n_2}\rangle = \langle A^{n_1}e_1, A^{n_2}e_1\rangle + \langle A^{n_1}e_2, A^{n_2}e_2\rangle = 0$$

$$\langle A,A\rangle = \langle A^{n_1}, A^{n_1}\rangle + \langle A^{n_2}, A^{n_2}\rangle = \frac{|f|^4\,|g'|^2}{2\lambda^6(1+t)^2}\left\{\left(1+\frac{1-t}{1+t}|g|^2\right)^2 + t^2\left(1-\frac{1-t}{1+t}|g|^2\right)^2\right\}$$

From (3.14) we see immediately $\langle A^{n_1}, A^{n_1}\rangle - \langle A^{n_2}, A^{n_2}\rangle \geq 0$ and for any unitary normal vector $n = n_1 \cos\alpha + n_2 \operatorname{sen}\alpha$, $\langle A^n, A^n\rangle = \cos^2\alpha \,\langle A^{n_1}, A^{n_1}\rangle + \operatorname{sen}^2\alpha\, \langle A^{n_2}, A^{n_2}\rangle$

Therefore we have

Proposition 3.1 - The two natural unitary normal vector fields n_1, n_2 defined in (3.11) satisfy

(3.15) $\langle A^{n_2}, A^{n_2}\rangle \leq \langle A^n, A^n\rangle \leq \langle A^{n_1}, A^{n_1}\rangle$

for any unitary vector field n normal to x.

And since $\langle A, A\rangle = -\frac{1}{2}K$ we have

Proposition 3.2 - The two orthogonal unitary normal vector fields

(3.16) $V = \frac{\sqrt{2}}{2}(n_1 + n_2)$, $W = \frac{\sqrt{2}}{2}(n_1 - n_2)$

satisfy

(3.17) $\langle A^V, A^V\rangle = \langle A^W, A^W\rangle = -K$

Next, considering the normal connection ∇ and the fact $\langle \nabla_{e_j} n_1, n_2\rangle = -\langle \nabla_{e_j} n_2, n_1\rangle$ for $j = 1,2$, we can easily prove that

Proposition 3.3 - For any unitary normal vector field n, $\langle \nabla n, \nabla n\rangle$ is a well-defined function, independent of the direction of n.

PROOF - It's sufficient to consider the function

$$F(\alpha) = \langle \nabla(\ell \cos\alpha + n \,\text{sen}\,\alpha), \nabla(\ell \cos\alpha + m \,\text{sen}\,\alpha)\rangle$$

where $\{\ell, m\}$ is any normal frame, and show that
$F'(\alpha) = 0$ q.e.d.

Now we treat the problem of stability. First we find that the stability of a domain depends only on the normal direction n_1.

Theorem 3.4 - Let D be a domain of an 1-degenerate minimal surface in $\mathbb{R}^4$. Then D is stable if and only if $I(V,V) > 0$, $\forall$ $V = hn_1$ where h is a non-zero differentiable function with compact support in D.

PROOF - The necessity of the hypothesis is obvious by definition. On the other hand, if D is unstable then it's known [7] that there exist a compact sub-domain D' of D and a Jacobi field $\tilde{V}$ defined on D' which vanishes on $\partial D'$. Since the zeros of $\tilde{V}$ are analytic curves or isolated points, if we set $h = |\tilde{V}|$, $n = \frac{\tilde{V}}{|\tilde{V}|}$ then from Prop. 3.1 and 3.3, we have

$$0 = I(\tilde{V},\tilde{V}) = \int_M - h\Delta h - h^2 \langle A^n, A^n\rangle + h^2 \langle \Delta n, \Delta n\rangle$$

$$\geq \int_M - h\Delta h - h^2 \langle A^{n_1}, A^{n_1}\rangle + h^2 \langle \Delta n_1, \Delta n_1\rangle = I(V,V)$$

with $V = hn_1$. This leads to a contradiction.

q.e.d.

On the other hand, if we look at the problem of stability intrinsically, following the arguments of Barbosa - do Carmo [1,2], we get

Theorem 3.5 - Let D be a simply connected domain of an 1-degenerate minimal surface in $\mathbb{R}^4$. If the area of the generalized Gauss map on $\bar{D}$ is less than $\frac{4\pi}{3-k^2}$, where k is given by (2.5), then D is stable.

PROOF - Since the generalized Gauss map is a branching covering on its image and it's known [3] that $d\hat{s}^2 = -k\, ds^2$ in the induced Fubini - Study metric on the image whose Gaussian curvature

$$\hat{K} \leq 2-k^2,$$

by (2.15). From (3.4) and (3.10) in [2], we compute that if the Gauss image has area less than $\frac{4\pi}{3-k^2}$, then the first eigenvalue λ_1 with respect to $d\hat{s}^2$ on D is greater than 2. The rest of the proof then follows the same argument in [1].

q.e.d.

Remarks

1 - For t = 0, we have k = 1. This gives the same result in [1], which is sharp. It would be interesting to know whether our result is sharp for arbitrary general t. If it were true, then the result obtained by Barbosa - do Carmo [2] for minimal surfaces in $\mathbb{R}^4$ would therefore be sharp also.

2 - From our discussions in this section, we see clearly that 1-degenerate minimal surfaces in $\mathbb{R}^4$ possess many properties similar to those in $\mathbb{R}^3$, as observed first by Hoffman-Osserman [4].

§ 4 - 2-degenerate minimal surfaces in $\mathbb{R}^4$

It's known [4] that any 2-degenerate minimal surface in $\mathbb{R}^4$ is a regular complex analytic curve lying in $C^2 = \mathbb{R}^4$, with respect to some orthogonal complex structure on $\mathbb{R}^4$. Now let

(4.1) $\psi = (f, g) : M^2 \longrightarrow C^2 = \mathbb{R}^4$

be a regular holomorphic curve, where M is a Riemann surface and $C^2 = \mathbb{R}^2 \oplus i\,\mathbb{R}^2$ is the canonical identification to $\mathbb{R}^4$. Then the real coordinates of ψ are given by

(4.2) $x = \mathrm{Re}\,(f, -if, g, -ig)$

And with respect to a local complex parameter $\zeta = u + iv$, the generalized Gauss map is given by

(4.3) $\phi(\zeta) = \left(f'(\zeta), -if'(\zeta), g'(\zeta), -ig'(\zeta)\right)$

Writing

(4.4) $f'(\zeta) = \alpha+i\beta, \qquad g'(\zeta) = \gamma+i\delta$

in real and imaginary parts, then $\phi = X + iY$ with

(4.5) $X = (\alpha, \beta, \gamma, \delta), \qquad Y = (\beta, -\alpha, \delta, -\gamma)$

which satisfy

(4.6) $\langle X,Y\rangle = 0, \qquad |X|^2 = |Y|^2 = |f'|^2 + |g'|^2 = \lambda^2$

It's easy to see that

(4.7) $N_1 = (-\delta, -\gamma, \beta, \alpha), \qquad N_2 = (\gamma, -\delta, -\alpha, \beta)$

are normal to x and are mutually orthogonal. Then

$$(4.8)\qquad e_1 = \frac{1}{\lambda} X, \qquad e_2 = \frac{1}{\lambda} Y$$

$$n_1 = \frac{1}{\lambda} N_1 \qquad n_2 = \frac{1}{\lambda} N_2$$

form a local adapted frame for x . To calculate the second fundamental form, using the Cauchy-Riemann equations

$$(4.9)\qquad \alpha_u = \beta_v , \qquad \alpha_v = -\beta_u$$

$$\gamma_u = \delta_v \qquad \gamma_v = -\delta_u$$

we get

$$A^{n_1} e_1 = -\frac{1}{\lambda^3}\left(-\alpha\delta_u - \beta\gamma_u + \gamma\beta_u + \delta\alpha_u\right) e_1 - \frac{1}{\lambda^3}\left(\beta\delta_u - \alpha\gamma_u - \delta\beta_u + \alpha_u\right) e_2$$

$$A^{n_1} e_2 = -\frac{1}{\lambda^3}\left(-\alpha\gamma_u + \beta\delta_u + \gamma\alpha_u - \delta\beta_u\right) e_1 - \frac{1}{\lambda^3}\left(\beta\gamma_u + \alpha\delta_u - \delta\alpha_u - \gamma\beta_u\right) e_2$$

$$(4.10)$$

$$A^{n_2} e_1 = -\frac{1}{\lambda^3}\left(\alpha\gamma_u - \beta\delta_u - \gamma\alpha_u + \delta\beta_u\right) e_1 - \frac{1}{\lambda^3}\left(-\beta\gamma_u - \alpha\delta_u + \delta\alpha_u + \gamma\beta_u\right) e_2$$

$$A^{n_2} e_2 = -\frac{1}{\lambda^3}\left(-\alpha\delta_u - \beta\gamma_u + \gamma\beta_u + \alpha_u\right) e_1 - \frac{1}{\lambda^3}\left(\beta\delta_u - \alpha\gamma_u - \delta\beta_u + \gamma\alpha_u\right) e_2,$$

and

$$\langle A^{n_1}, A^{n_1}\rangle = \langle A^{n_2}, A^{n_2}\rangle = \frac{2}{\lambda^6}\Big\{ |f'|^2 |g''|^2 + |g'|^2 |f''|^2 - \bar{f}' f'' g' \bar{g}'' - f' \bar{f}'' \bar{g}' g$$

$$(4.11)$$

$$= \frac{2}{\lambda^6} \left| (f', g') \wedge (f'', g'') \right|^2$$

$$\langle A^{n_1}, A^{n_2}\rangle = 0$$

Therefore, we have the Gaussian curvature

$$(4.12)\qquad K = -\langle A^n, A^n\rangle$$

for any unitary normal vector n. Summing up, we have

Theorem 4.1 - Let $x : M^2 \longrightarrow \mathbb{R}^4$ be a 2-degenerate minimal surface in $\mathbb{R}^4$. Then, with respect to any orthonormal normal vectors, v,w, the second fundamental form satisfies

$$(4.13) \qquad \langle A^v, A^v\rangle = \langle A^w, A^w\rangle = -K$$

$$\langle A^v, A^w\rangle = 0$$

where K is the Gaussian curvature of the surface.

Remarks

1 - From (4.7) we see that $\psi = (f,g)$ is orthogonal and isometric to $\tilde{\psi} = (\bar{g}, -\bar{f})$ as 2-degenerate minimal surfaces in $\mathbb{R}^4$. We thus conclude that $|K| = |K^{\perp}|$, where $K^{\perp}$ is the normal curvature of x. In fact, this is a characteristic property for holomorphic curves in C^2.

2 - The property (4.13) is, in fact, also characteristic. We will discuss them elsewhere.

3 - If M is simply connected then, using a fixed uniform parameter, we can construct global unitary normal vector fields.

4 - By Wirtinger's inequality [8], we know that any 2-degenerate minimal surface in $\mathbb{R}^4$ is stable. From our discussions in (1.8) Prop. 3.3 and (4.12), we see that for any unitary normal vector field n and any normal variational vector field V,

$$I(V,V) = I(hu, hu)$$

with $h = |V|$. This probably explain, from the variational viewpoint, why holomorphic curves in $C^2 = \mathbb{R}^4$ are always sta ble.

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