Regular Articles

Quasi-invariant measures for generalized approximately proper equivalence relations

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A R T I C L E   I N F O

A B S T R A C T

We introduce a generalization of the notion of approximately proper equivalence relations studied by Renault and with it, we build an étale groupoid. Choosing a suitable set of continuous functions to play the role of a potential, we construct a cocycle in that groupoid and discuss the corresponding Radon-Nikodym problem.

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1. Introduction

In [11], Renault introduced the notion of an approximately proper equivalence relation on a compact topological space X, consisting of an increasing sequence $\{R_n\}_{n\in\mathbb{N}}$ of equivalence relations on X, each of which is proper in the sense that the corresponding quotient map is a local homeomorphism. When equipped with the inductive limit topology, the union $R = \bigcup_n R_n$ becomes an étale groupoid, and if one is moreover given a suitable sequence of continuous real-valued functions on X, a cocycle[1] may be defined on R.

As its title suggests, the main goal of [11] is to study the corresponding Radon-Nikodym problem, i.e., to find the probability measures on X which are quasi-invariant with Radon-Nikodym derivative equal to the aforementioned cocycle.

Among other things, the relevance of solving the Radon-Nikodym problem lies in the fact that the solutions lead to KMS states on the groupoid C*-algebra and hence have a profound relevance to Statistical

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 [1] In this paper the term *cocycle* will always be taken to mean a *one*-cocycle.

Mechanics. For compact subshifts of $\mathbb{Z}^d$, these solutions for the Gibbs relation are precisely the Gibbs measures, and they are KMS states for the respective C*-algebra [3].

As mentioned in [11, Section 7], approximately proper equivalence relations arise naturally in the study of local homeomorphisms from a compact topological space to itself. Precisely speaking, given a compact topological space X, and a local homeomorphism $\sigma : X \to X$, one lets,

$$R_n = \big\{(x,y) \in X \times X : \sigma^n(x) = \sigma^n(y)\big\},$$

for each $n \geq 0$, and it is not hard to see that each R_n is a proper equivalence relation so that $\{R_n\}_{n \in \mathbb{N}}$ is an approximately proper equivalence relation in the sense of [11].

Prominent examples of local homeomorphisms on compact topological spaces are given by Markov shifts. On the other hand, in a recent paper [2], we have focused on a generalization of Markov shifts introduced by M. Laca and the second named author in [6], which in turn have been shown by Renault [10] to consist essentially of a generalized *shift space*, with the notable difference that the *shift map* is no longer defined on the whole space, but only on a proper open subset.

The precise setup of [10] is that of a locally compact space X, an open subset $U \subseteq X$, and a local homeomorphism

$$\sigma : U \to X.$$

However, if one starts from this data, it is not possible to build an approximately proper equivalence relation by the procedure indicated above, not least because σ^n fails to be defined on the whole space X. If one wants to make sense of the relation

$$x \sim y \iff \sigma^n(x) = \sigma^n(y),$$

one must restrict attention to elements x and y for which $\sigma^n(x)$ and $\sigma^n(y)$ make sense, namely elements of the domain of σ^n, which we shall henceforth denote by U_n. We thus define

$$R_n = \big\{(x,y) \in U_n \times U_n : \sigma^n(x) = \sigma^n(y)\big\},$$

which is clearly a proper equivalence relation on U_n. If $n \leq m$, observe that

$$\sigma^n(x) = \sigma^n(y) \implies \sigma^m(x) = \sigma^m(y),$$

as long as all of the above terms are defined, i.e., as long as x and y lie in the smaller set U_m. This may be more succinctly expressed by saying that

$$R_n \cap (U_m \times U_m) \subseteq R_m. \tag{1.1}$$

If one misreads the above inclusion, ignoring the intersection with $U_m \times U_m$, one will be left with the impression that the R_n are *increasing*, just as in [11], although this is evidently not true given that the sets where these relations are defined in fact *decrease*.

Another distinctive feature of the R_n is the fact that, still under the hypothesis that $n \leq m$, one has that U_m is invariant under R_n, meaning that

$$\big((x,y) \in R_n\big) \wedge (y \in U_m) \implies x \in U_m,$$

which may be expressed by saying that

$$R_n \cap (U_n \times U_m) \subseteq U_m \times U_m. \tag{1.2}$$

The reader may easily verify that, together, (1.1) and (1.2) are equivalent to

$$R_n \cap (U_n \times U_m) \subseteq R_m,$$

which might not have an immediately intuitive interpretation, but due to its sheer simplicity, is adopted in this work as the main axiom in our generalization of Renault's notion of approximately proper equivalence relations, given in full detail in (5.1), below, and referred to as a GAP, for short.

The main aim of the present work is to conduct a study of GAPs along the lines of Renault's study of approximately proper equivalence relations. We thus show that the union $R = \bigcup_n R_n$ is an equivalence relation, hence a principal groupoid, which becomes étale when given the inductive limit topology. A suitable notion of potential is introduced, leading up to a cocycle relative to which the Radon-Nikodym problem may be investigated.

Since each R_n is assumed to be proper, one has that R is the (not necessarily increasing) union of proper equivalence relations, so it is not surprising that the study of proper relations is as important here as it is in [11]. Should our U_n be compact we would be able to borrow the results of the first few sections of [11], but the example of infinite state Markov shifts, our main motivation, requires an understanding of the Radon-Nikodym problem for proper relations on non-compact spaces. The lack of compactness indeed brings several complications, most of them stemming from the fact that equivalence classes no longer need to be finite. For example, the normalization achieved in [11, Proposition 3.1.iii] by means of replacing a potential ρ' by

$$\rho(x) := \frac{\rho'(x)}{\sum_{y \sim x} \rho'(y)}$$

needs to be dealt with in a more careful way if equivalence classes are allowed to be infinite.

Once the proper case is taken care of, we apply our results for GAPs, showing, among other things, that quasi-invariant measures may be characterized, much in the same way as DLR measures, as those which are fixed by a family of conditional expectations. See section (6), and in particular Corollary (6.8), for full details.

The existence part of the Radon-Nikodym problem, which follows easily from compactness when that property is present, e.g. as in [7, 8.2], turns out to be a delicate question here. In fact, existence may already fail in the proper case, but it is nevertheless easy to determine precisely when this happens. The crucial point is to analyze the partition function

$$\zeta(x) = \sum_{y \sim x} \rho(y),$$

defined in (4.2), which may well return infinite values, should equivalence classes be infinite. The set of points x for which $\zeta(x) = \infty$, which we denote by Z_ρ, is a forbidden zone for finite quasi-invariant measures in the sense that any such measure assigns zero mass to Z_ρ. Thus, if ζ is identically infinite, a situation very easy to arrange, there are no nontrivial solutions for the Radon-Nikodym problem. Excluding this extreme situation, i.e. when ζ is finite on at least one point, one may easily show the existence of quasi-invariant measures. See section (4) for more details in the proper case.

Unfortunately, we have no definitive answer for the existence question in the most general situation of GAPs treated here, which is perhaps to be expected given that similar results rely heavily on compactness. However we can offer several partial existence results which the reader will find in section (7).

2. Proper equivalence relations

As mentioned above, we start by analyzing proper equivalence relations, avoiding the compactness assumption.

2.1. Standing hypothesis

Throughout these notes we will assume that X is a locally compact, second countable, metrizable space.

We will denote the σ-algebra of Borel measurable subset of X by $\mathcal{B}(X)$, and the set of all Borel measurable functions

$$f : X \to [0, +\infty]$$

by $\mathcal{M}^+\big(X, \mathcal{B}(X)\big)$.

2.2. Definition. An equivalence relation $R \subseteq X \times X$ is said to be *proper*, provided the quotient space X/R is Hausdorff, and the quotient map

$$\pi : X \to X/R$$

is a local homeomorphism.[2]

▶ From now on we shall fix a proper equivalence relation R on X.

Given any x in X, we will denote its equivalence class by $R(x)$, in symbols

$$R(x) = \{y \in X : (x, y) \in R\}.$$

2.3. Definition. For each f in $\mathcal{M}^+\big(X, \mathcal{B}(X)\big)$, we will let

$$E(f)\big|_x = \sum_{y \in R(x)} f(y).$$

Observe that the above sum could very well diverge, in which case we of course set $E(f)\big|_x$ to be ∞. Therefore, like f, one has that $E(f)$ is a function taking values in $[0, \infty]$.

We will soon prove that $E(f)$ is $\mathcal{B}(X)$-measurable, but so far we will see it simply as an element of $\mathcal{M}^+\big(X, \mathcal{P}(X)\big)$, where $\mathcal{P}(X)$ is the σ-algebra of *all* subsets of X (with respect to which any function is measurable). In other words, E may be seen as a map

$$E : \mathcal{M}^+\big(X, \mathcal{B}(X)\big) \to \mathcal{M}^+\big(X, \mathcal{P}(X)\big).$$

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[2] A map $\varphi : X \to Y$, between topological spaces X and Y, is said to be a *local homeomorphism* provided for every x in X, there are open subsets $A \subseteq X$, and $B \subseteq Y$, such that $x \in A$ and φ is a homeomorphism from A onto B.

2.4. Proposition. *E is σ-additive, in the sense that it is positively homogeneous and*

$$E\Big(\sum_{n=1}^{\infty} f_n\Big) = \sum_{n=1}^{\infty} E(f_n),$$

for any sequence $\{f_n\}_n$ in $\mathcal{M}^+(X, \mathcal{B}(X))$.

Proof. It is evident that E is positively homogeneous. Given any sequence $\{f_n\}_n$ in $\mathcal{M}^+(X, \mathcal{B}(X))$, for every x in X, we have

$$E\big(\sum_{n=1}^{\infty} f_n\big)\big|_x = \sum_{y \in R(x)} \sum_{n=1}^{\infty} f_n(y) = \sum_{n=1}^{\infty} \sum_{y \in R(x)} f_n(y) = \sum_{n=1}^{\infty} E(f_n)\big|_x. \quad \square$$

2.5. Proposition. *If f is in $\mathcal{M}^+(X, \mathcal{B}(X))$, then so is $E(f)$.*

Proof. Let $\{U_n\}_{n \in \mathbb{N}}$ be a countable open cover of X, such that the quotient map

$$\pi : X \to X/R$$

is a homeomorphism when restricted to each U_n. Also let $\{\psi_n\}_{n \in \mathbb{N}}$ be a partition of unit subordinate to this cover.

Given f in $\mathcal{M}^+(X, \mathcal{B}(X))$, put $f_n = f\psi_n$, so that $f = \sum_n f_n$, pointwise. Using (2.4), we then have that

$$E(f) = E\Big(\sum_{n=1}^{\infty} f_n\Big) = \sum_{n=1}^{\infty} E(f_n),$$

so it suffices to prove that each $E(f_n)$ is Borel-measurable. Write

$$\tau : \pi(U_n) \to U_n$$

for the inverse of the restriction of π to U_n, and let $V_n = \pi^{-1}\big(\pi(U_n)\big)$. We then claim that

$$E(f_n)\big|_x = \begin{cases} f_n\big(\tau(\pi(x))\big), & \text{if } x \in V_n, \\ 0, & \text{otherwise.} \end{cases}$$

Indeed, when x is not in V_n, then $\pi(x)$ is not in $\pi(U_n)$. So, while f_n vanishes outside U_n, there is no y in U_n such that $(x, y) \in R$. The sum defining $E(f_n)\big|_x$ therefore has no nonzero terms, and hence $E(f_n)\big|_x = 0$.

On the other hand, if x is in V_n, then $\pi(x) \in \pi(U_n)$, so $\pi(x) = \pi(y)$, for a unique y in U_n, namely $y = \tau\big(\pi(x)\big)$, whence $f(y)$ is the only possibly nonzero term in the aforementioned sum. Therefore

$$E(f_n)\big|_x = f(y) = f_n\big(\tau(\pi(x))\big),$$

as claimed. Since the correspondence $x \mapsto f_n\big(\tau(\pi(x))\big)$ is easily seen to be Borel-measurable on V_n, we have that $E(f_n)$ is Borel-measurable on X. $\quad \square$

Notice that, in view of the above result, E may be viewed as a map from $\mathcal{M}^+(X, \mathcal{B}(X))$ to itself. We therefore no longer need to consider the σ-algebra $\mathcal{P}(X)$, and we shall henceforth use the simplified notation

$$\mathcal{M}^+(X) := \mathcal{M}^+(X, \mathcal{B}(X)).$$

A few other useful properties of E are as follows:

2.6. Proposition. *Given $f, g \in \mathcal{M}^+(X)$, one has that:*

(i) *$E(f)$ is R-invariant, meaning that if $(x, y) \in R$, then $E(f)\big|_x = E(f)\big|_y$,*
(ii) *if g is R-invariant, then $E(gf) = gE(f)$,*
(iii) *if $f \leq g$, then $E(f) \leq E(g)$,*
(iv) *if f vanishes outside a subset $A \subseteq X$, then $E(f)$ vanishes outside*

$$Orb(A) := \{y \in X : \exists x \in A, \ (x, y) \in R\}.$$

Proof. We prove only (iv). Given $x \in X$, suppose that

$$0 \neq E(f)\big|_x = \sum_{y \in R(x)} f(y).$$

Then it is easy to see that there exists at least one y such that $(x, y) \in R$, and $f(y) \neq 0$. Consequently $y \in A$, whence $x \in \text{Orb}(A)$. This proves that

$$E(f)\big|_x \neq 0 \Rightarrow x \in \text{Orb}(A),$$

from where the conclusion follows immediately. $\square$

When multiplying extended real numbers, as in the multiplication "gf" above, we adopt the convention according to which $0 \times \infty = \infty \times 0 = 0$. A trivial, but highly relevant fact to be noted regarding this convention is that the multiplication of positive extended real numbers is both associative and infinitely distributive, i.e.,

$$c \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} ca_n,$$

for every c and every sequence $\{a_n\}_n$ in $[0, \infty]$. Incidentally, the above choice for the value of $0 \times \infty$ is necessary for the validity of the distributive property, since

$$0 \times \infty = 0 \times \sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} 0 \times \frac{1}{n} = 0.$$

Given ρ, f in $\mathcal{M}^+(X)$, we have that $E(\rho f) \in \mathcal{M}^+(X)$. Fixing ρ we may then define the map

$$E_\rho : f \in \mathcal{M}^+(X) \mapsto E(\rho f) \in \mathcal{M}^+(X),$$

which is clearly also σ-additive. Therefore, for every measure ν on $\mathcal{B}(X)$, we may consider the measure $E_\rho^*(\nu)$ given by (A.3). Some elementary observations regarding $E_\rho^*(\nu)$ are in order:

2.7. Proposition. *Given a function ρ in $\mathcal{M}^+(X)$ as well as a measure ν on $\mathcal{B}(X)$, one has that*

(i) *$E_\rho^*(\nu)$ is a finite measure if and only if $E(\rho)$ is ν-integrable,*
(ii) *if A is any R-invariant[3] Borel-measurable subset of X with $\nu(A) = 0$, then $E_\rho^*(\nu)(A) = 0$ as well.*

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[3] A subset $A \subseteq X$ is said to be *R-invariant* if, whenever $x \in A$, and $(x, y) \in R$, one has that $y \in A$.

Proof. The first point follows immediately from

$$E_\rho^*(\nu)(X) = \int_X 1 \, dE_\rho^*(\nu) = \int_X E_\rho(1) \, d\nu = \int_X E(\rho) \, d\nu.$$

Regarding (ii), and denoting the characteristic function of A by 1_A, it is obvious that $\rho 1_A$ vanishes outside A. Therefore $E(\rho 1_A)$ vanishes outside $\mathrm{Orb}(A)$ by (2.6.iv). However, since A is invariant, we have that $\mathrm{Orb}(A) = A$, so in fact $E(\rho 1_A)$ vanishes outside A. Therefore

$$E_\rho^*(\nu)(A) = \int_X 1_A \, dE_\rho^*(\nu) = \int_X E(\rho 1_A) \, d\nu = \int_{X \setminus A} E(\rho 1_A) \, d\nu = 0. \quad \square$$

3. The operator E on $C_c(X)$

In this section we continue assuming that X satisfies (2.1) and that $R \subseteq X \times X$ is a proper equivalence relation on X. Whenever we speak of R as a topological space, we will be referring to the topology induced on R by the product topology of $X \times X$.

We will often view R as a *groupoid* under the multiplication operation according to which the product $(x, y) \cdot (z, w)$ is defined if and only if $y = z$, in which case it is set to be (x, w). The unit space of such a groupoid is, therefore, the diagonal $\{(x, x) : x \in X\}$, which we identify with X in the obvious way. The *range* and *source* maps are then given respectively by

$$r(x, y) = x, \quad \text{and} \quad s(x, y) = y, \quad \forall (x, y) \in R.$$

It is well known that R is then a Hausdorff étale groupoid.

3.1. Proposition. *Given any continuous, complex-valued function f on R, suppose that f vanishes outside a given subset $L \subseteq R$, such that $s(L)$ is relatively compact. Then:*

(i) *The expression*

$$g(y) = \sum_{\gamma : r(\gamma) = y} f(\gamma)$$

gives a well-defined and continuous function on X.

(ii) *If $r(L)$ is also relatively compact, then g has compact support.*

Proof. We first claim that R is closed in $X \times X$. In order to see this let

$$\pi : X \to X/R$$

denote the quotient map and observe that

$$R = \{(x, y) \in X \times X : \pi(x) = \pi(y)\} = \{(x, y) \in X \times X : \big(\pi(x), \pi(y)\big) \in \Delta\},$$

where Δ is the diagonal in $X/R \times X/R$. Since X/R is Hausdorff, we have that Δ is closed, and hence R is closed in $X \times X$.

Given y_0 in X, let K be a compact neighborhood of y_0, and observe that

$$r^{-1}(K) \cap L \subseteq K \times s(L),$$

so $r^{-1}(K) \cap L$ is relatively compact in $X \times X$, and hence also in the closed subspace R. For each

$$\gamma \in \overline{r^{-1}(K) \cap L},$$

namely the closure of $r^{-1}(K) \cap L$ within R, let $U_\gamma \subseteq R$ and $V_\gamma \subseteq X$ be open sets such that $\gamma \in U_\gamma$, $r(\gamma) \in V_\gamma$, and such that the restriction of the range map r to U_γ gives a homeomorphism onto V_γ.

We shall also insist that, whenever $r(\gamma) \neq y_0$, the open neighborhood V_γ of $r(\gamma)$ is chosen such that $y_0 \notin \overline{V}_\gamma$. We therefore get an open cover $\{U_\gamma\}_\gamma$ of the compact set $\overline{r^{-1}(K) \cap L}$, from which one may extract a finite subcover, say $\{U_\gamma\}_{\gamma \in F}$, where F is some finite set of γ's.

Splitting F according to whether or not $r(\gamma) = y_0$, we define

$$F_1 = \{\alpha \in F : r(\alpha) = y_0\}, \quad \text{and} \quad F_2 = \{\beta \in F : r(\beta) \neq y_0\}.$$

We then put

$$V = \text{int}(K) \cap \bigcap_{\alpha \in F_1} V_\alpha \cap \bigcap_{\beta \in F_2} X \setminus \overline{V}_\beta,$$

and we claim that $y_0 \in V$. To see this, notice that y_0 lies in $\text{int}(K)$ because K is a neighborhood of y_0. Moreover, for every α in F_1, we have that

$$y_0 = r(\alpha) \in V_\alpha,$$

and finally, for every β in F_2, we have explicitly chosen V_β so that $y_0 \notin \overline{V}_\beta$.

We next claim that, for every η in R,

$$r(\eta) \in V \ \wedge \ f(\eta) \neq 0 \Rightarrow \eta \in U_\alpha, \text{ for some } \alpha \in F_1. \tag{3.1.1}$$

Indeed, given η satisfying the above antecedent, we clearly have that

$$\eta \in r^{-1}(K) \cap L,$$

so there exists some γ in F, such that $\eta \in U_\gamma$, and we would now like to decide whether γ lies in F_1 or in F_2. The key observation here is that

$$r(\eta) \in V \cap r(U_\gamma) = V \cap V_\gamma,$$

so V_γ has a nonempty intersection with V, and this can only happen when $\gamma \in F_1$, thus completing the proof of (3.1.1).

Next, use the fact that R is Hausdorff to produce a collection of pairwise disjoint open sets $\{W_\alpha\}_{\alpha \in F_1}$ such that each $\alpha \in W_\alpha$ and finally put

$$\Omega = V \cap \bigcap_{\alpha \in F_1} r(W_\alpha \cap U_\alpha).$$

Noticing that $W_\alpha \cap U_\alpha$ is open in U_α, we see that $r(W_\alpha \cap U_\alpha)$ is open in $r(U_\alpha) = V_\alpha$, so Ω is an open subset of X. Also, since $\alpha \in W_\alpha \cap U_\alpha$, we have that $y_0 = r(\alpha) \in r(W_\alpha \cap U_\alpha)$, so $y_0 \in \Omega$.

For each α in F_1, denote by t_α the inverse of the homeomorphism

$$r{\restriction}_{U_\sigma} : U_\alpha \to V_\alpha,$$

and, regarding the function g referred to in the statement, we claim that for every y in Ω, one has that

$$g(y) = \sum_{\alpha \in F_1} f(t_\alpha(y)). \tag{3.1.2}$$

To prove this claim it is enough to show that

$$\{\gamma \in R : r(\gamma) = y, \ f(\gamma) \neq 0\} = \{t_\alpha(y) : \alpha \in F_1, \ f(t_\alpha(y)) \neq 0\}, \tag{3.1.3}$$

and that the $t_\alpha(y)$ in the description of the set in the right hand side above are pairwise distinct.

With respect to this last statement, notice that for each α in F_1,

$$t_\alpha(y) \in t_\alpha\big(r(W_\alpha \cap U_\alpha)\big) = W_\alpha \cap U_\alpha \subseteq W_\alpha,$$

so the $t_\alpha(y)$ lie in pairwise disjoint sets and hence are necessarily pairwise distinct.

We next observe that the inclusion "$\supseteq$" in (3.1.3) is evident, so we focus on the reverse inclusion "$\subseteq$". For this, pick γ in R such that $r(\gamma) = y$, and $f(\gamma) \neq 0$, and notice that by (3.1.1) it follows that $\gamma \in U_\alpha$ for some $\alpha \in F_1$. Therefore $\gamma = t_\alpha(y)$, so we see that γ lies in the set in the right hand side of (3.1.3).

This proves (3.1.3) and hence also (3.1.2), from where it is clear that the sum defining g has finitely many nonzero terms, so that g is well defined, and moreover that g is continuous.

In order to prove (ii), it is enough to observe that if $g(y) \neq 0$, there must be at least one γ with $r(\gamma) = y$, and $f(\gamma) \neq 0$, whence $\gamma \in L$, and then $y \in r(L)$. Viewing through the contrapositive

$$y \notin r(L) \quad \Rightarrow \quad f(\gamma) = 0, \ \text{for all } \gamma \text{ such that } r(\gamma) = y,$$

which in turn implies that $g(y) = 0$. Thus g vanishes outside the relatively compact set $r(L)$, and hence it is compactly supported. $\square$

3.2. Corollary. *Given $f \in C_c(R)$, the correspondence*

$$x \mapsto \sum_{y \in R(x)} f(x, y)$$

defines a compactly supported, continuous function on X.

Proof. Follows immediately from (3.1.ii) upon choosing L to be the support of f. $\square$

Recall that the operator E defined in (2.3) is only defined for non-negative functions. This is due to the fact that the summation involved in its definition is not supposed to converge but, as long as the summands are non-negative, we may always assign a sensible value to the sum, that value being ∞ in the divergent case. In the case of compactly supported functions, the situation is however much better behaved:

3.3. Proposition. *Given f in $C_c(X)$, and for every x in X, the sum*

$$\sum_{y \in R(x)} f(y),$$

has at most finitely many nonzero terms. Moreover, defining

$$E(f)\big|_x = \sum_{y \in R(x)} f(y), \quad \forall\, x \in X,$$

one has that $E(f)$ is a continuous function on X.

Proof. Since R is a proper equivalence relation, we have that $R(x)$ is a closed, discrete set for every x in X. Therefore, if K is the compact support of f, one has that $R(x) \cap K$ is finite from where the first assertion follows immediately.

Addressing the last assertion, consider the continuous function

$$g : (x, y) \in R \mapsto f(y) \in \mathbb{C}.$$

Denoting the support of f by L, notice that g vanishes outside the set

$$K := (X \times L) \cap R.$$

Since $s(K) \subseteq L$, we see that $s(K)$ is relatively compact. We may therefore employ (3.1.i) to conclude that the function g defined there is continuous, namely

$$g(x) = \sum_{\gamma : r(\gamma) = x} g(\gamma) = \sum_{y \in R(x)} g(x, y) = \sum_{y \in R(x)} f(y) = E(f)\big|_x,$$

concluding the proof. □

In view of the above result, we get a map

$$E : C_c(X) \to C(X).$$

On the other hand, recall that in (2.3) we defined an operator

$$E : \mathcal{M}^+(X) \to \mathcal{M}^+(X),$$

using the exact same formula as in (3.3). Clearly the two operators referred to above coincide on the intersection of their domains, so there is no ambiguity in using the same notation "E" for these maps.

Some of the main properties of E on $C_c(X)$ reflect those listed in (2.6):

3.4. Proposition. *For every f and g in $C_c(X)$, one has that:*

  (i) *$E(f)$ is R-invariant,*
 (ii) *if g is R-invariant, then $E(gf) = gE(f)$,*
(iii) *$E(f)$ is continuous,*
 (iv) *$E(f)$ is bounded.*

Proof. Leaving the easy proofs of (i) and (ii) to the reader, we notice that (iii) was already proved in (3.3).

Regarding (iv), let K be the compact support of f, and let M be the supremum of $|E(f)|$ on K, which is finite by (iii). We will then prove that $|E(f)|$ is bounded by M on all of X. In order to prove that

$$\big|E(f)\big|_x\big| \leq M, \tag{3.4.1}$$

for any given x in X, we may evidently assume that $E(f)\big|_x \neq 0$. In this case

$$0 \neq E(f)\big|_x = \sum_{y \in R(x)} f(y),$$

so there exists at least one y in K such that $(x, y) \in R$. Therefore

$$\big|E(f)\big|_x \overset{\text{(i)}}{=} \big|E(f)\big|_y \big| \leq M,$$

proving (3.4.1). □

Complementing (2.7), we may now describe a few other relevant properties of $E_\rho^*(\nu)$ under the extra hypothesis that ρ is finitely valued and continuous.

3.5. Proposition. *Let ν be a measure on $\mathcal{B}(X)$, and let $\rho : X \to \mathbb{R}$, be a non-negative, continuous function. Setting $\mu = E_\rho^*(\nu)$, one has that:*

(i) *if $\nu(X) < \infty$, then μ is a Borel measure (i.e. finite on compact sets),*

(ii) *if μ' is any measure on $\mathcal{B}(X)$ such that*

$$\int_X f \, d\mu' = \int_X E(\rho f) \, d\nu < \infty, \quad \forall f \in C_c^+(X),$$

then $\mu = \mu'$, and in particular the above identity holds for every f in $\mathcal{M}^+(X)$.

Proof. In order to verify (i), and using (B.4), it is enough to prove that every f in $C_c^+(X)$ is μ-integrable. Given such an f, notice that the continuity of ρ implies that ρf lies in $C_c(X)$, whence $E(\rho f)$ is bounded by (3.4.iv). Therefore

$$\int_X f \, d\mu = \int_X E(\rho f) \, d\nu \leq \nu(X) \|E(\rho f)\|_\infty < \infty.$$

Addressing (ii), observe that the hypothesis says that every f in $C_c^+(X)$ is μ'-integrable, so another application of (B.4) tells us that μ' is a Borel measure. By hypothesis, we then have that

$$\int_X f \, d\mu' = \int_X f \, d\mu, \quad \forall f \in C_c^+(X),$$

from where we deduce that μ is also a Borel measure. Since any f in $C_c(X)$ may be written as the linear combination of functions in $C_c^+(X)$, we deduce that the identity displayed above holds for every f in $C_c(X)$, so $\mu = \mu'$, by (B.3) and the uniqueness part of the Riesz-Markov Theorem. □

4. Proper equivalence relations and quasi-invariant measures

▶ As before, throughout this section we fix a space X satisfying (2.1), as well as a proper equivalence relation R on X. We will moreover fix a continuous function

$$\rho : X \to \mathbb{R},$$

which will henceforth be supposed *strictly positive*, i.e.,

$$\rho(x) > 0, \quad \forall x \in X,$$

and which will be referred to as the *potential* for R.

The relevance of ρ is that it leads to a multiplicative cocycle on R via the formula

$$D(x,y) := \rho(x)/\rho(y), \quad \forall\, (x,y) \in R, \tag{4.1}$$

and the goal of this section is to study *quasi-invariant* measures relative to this cocycle. See (4.9) below for the precise definition.

In some applications of our theory, the role of ρ is played by the function $\rho(x) = e^{\beta h(x)}$, where $\beta > 0$ and h is a continuous, real-valued function on X. The assumption that ρ is strictly positive then holds automatically. Another reason why we need to assume that ρ is never zero is that, otherwise, the above definition of D would run into trouble.

4.2. Definition. For each x in X, define

$$\zeta(x) := E(\rho)\big|_x = \sum_{y \in R(x)} \rho(y).$$

We will refer to ζ as the *partition function* for the potential ρ.

4.3. Proposition. ζ *is bounded below by* ρ, *and consequently*

$$0 < \zeta(x) \le \infty, \quad \forall\, x \in X.$$

Proof. Obvious. $\quad\square$

Since ζ is defined to be $E(\rho)$, we have by (2.5) that ζ lies in $\mathcal{M}^+(X)$. We may in fact prove that ζ satisfies a stronger regularity property:

4.4. Proposition. ζ *is lower semi-continuous.*

Proof. Let $\{\varphi_n\}_n$ be as in (B.5). Then

$$\zeta = E(\rho) = E\Big(\lim_{n\to\infty} \rho\varphi_n\Big) \overset{\text{(A.3.i)}}{=} \lim_{n\to\infty} E(\rho\varphi_n),$$

whence ζ is the limit of an increasing sequence of continuous functions by (3.3), from where the conclusion follows. $\quad\square$

4.5. Corollary. *The set*

$$Z_\rho = \{x \in X : \zeta(x) = \infty\}$$

is a G_δ, *hence a Borel set.*

Proof. Noting that

$$Z_\rho = \bigcap_{k \in \mathbb{N}} \{x \in X : \zeta(x) > k\},$$

the result is an immediate consequence of (4.4). $\quad\square$

In what follows we will make frequent references to the function ζ^{-1}, so it is worth discussing it briefly now. Recall from (4.3) that $\zeta(x) > 0$, for all x in X, so we will never run into the trouble of considering the inverse of zero. On the other hand, when $\zeta(x) = \infty$, we evidently put $\zeta^{-1}(x) = 0$.

In view of our convention that $\infty \times 0 = 0$, observe that

$$\zeta(x)\zeta(x)^{-1} = \begin{cases} 0, & \text{if } x \in Z_\rho, \\ 1, & \text{otherwise,} \end{cases} \tag{4.6}$$

so we have that

$$\zeta\zeta^{-1} = 1_{X \setminus Z_\rho}. \tag{4.7}$$

In particular, we note the following partial-isometric-like property of ζ, to be used shortly:

$$\zeta^{-1}\zeta\zeta^{-1} = \zeta^{-1}. \tag{4.8}$$

All things considered, we will see that the somewhat unusual fact that $\zeta\zeta^{-1}$ vanishes on Z_ρ will not be so crucial. For example, we will soon encounter expressions such as

$$\int_X \zeta\zeta^{-1} \, d\mu,$$

but often the measure μ will also vanishes on Z_ρ, so the funny behavior of $\zeta\zeta^{-1}$ on Z_ρ becomes totally irrelevant.

We next recall the definition of a quasi-invariant measure in the special case of étale groupoids.

4.9. Definition *([9, I.3.15]).* Let $\mathcal{G}$ be an étale groupoid and let $D : \mathcal{G} \to \mathbb{R}_+^*$ be a multiplicative cocycle. A measure μ on $\mathcal{G}^{(0)}$ is said to be *quasi-invariant* relative to D when

$$\int_{\mathcal{G}^{(0)}} \sum_{\gamma \in r^{-1}(x)} f(\gamma) \, d\mu(x) = \int_{\mathcal{G}^{(0)}} \sum_{\gamma \in s^{-1}(x)} f(\gamma) D(\gamma) \, d\mu(x),$$

for every f in $C_c(\mathcal{G})$.

For the case of our groupoid R, the above quasi-invariance condition becomes

$$\int_X \sum_{y \in R(x)} f(x,y) \, d\mu(x) = \int_X \sum_{x \in R(y)} f(x,y) D(x,y) \, d\mu(y), \tag{4.10}$$

for every f in $C_c(R)$.

The following result lists several equivalent conditions for a measure to solve the Radon-Nikodym problem.

4.11. Theorem. *Let X be a topological space satisfying (2.1). Also let R be a proper equivalence relation on X, seen as an étale groupoid. Given a continuous, strictly positive function $\rho : X \to \mathbb{R}$, consider the cocycle defined on R by $D(x,y) = \rho(x)/\rho(y)$. Then, for every finite measure μ on X, the following are equivalent:*

(i) *μ is D-quasi-invariant,*

(ii) $\displaystyle\int_X f E(\rho g) \, d\mu = \int_X E(\rho f) g \, d\mu, \quad \forall f, g \in C_c(X),$

(iii) $\displaystyle\int_X f\zeta\,d\mu = \int_X E(\rho f)\,d\mu, \forall \quad f \in C_c^+(X),$

(iv) $\displaystyle\int_X f\,d\mu = \int_X E(f\rho\zeta^{-1})\,d\mu, \quad \forall\, f \in C_c^+(X),$

(v) *there exists a positive measure ν on X, with respect to which ζ is integrable, and*

$$\int_X f\,d\mu = \int_X E(\rho f)\,d\nu, \quad \forall\, f \in C_c(X).$$

In addition, if any of the above equivalent conditions hold, then $\mu(Z_\rho) = 0$.

Proof. (i) $\Rightarrow$ (ii). Pick f and g in $C_c(X)$, and consider the function F on R given by the formula

$$F(x,y) = f(x)g(y)\rho(y).$$

Plugging F in (4.10) we have

$$\int_X \sum_{y \in R(x)} f(x)g(y)\rho(y)\,d\mu(x) = \int_X \sum_{x \in R(y)} f(x)g(y)\rho(x)\,d\mu(y),$$

which translated precisely into (ii).

(ii) $\Rightarrow$ (iii). Given f in $C_c^+(X)$, let $\{\varphi_n\}_n$ be as in (B.5). Then

$$\int_X f\zeta\,d\mu = \int_X f E(\rho)\,d\mu = \int_X f E(\lim_{n\to\infty} \rho\varphi_n)\,d\mu \overset{\text{(A.3.i)}}{=}$$

$$= \lim_{n\to\infty} \int_X f E(\rho\varphi_n)\,d\mu \overset{\text{(ii)}}{=} \lim_{n\to\infty} \int_X E(\rho f)\varphi_n\,d\mu = \int_X E(\rho f)\,d\mu.$$

(iii) $\Rightarrow$ (iv). We will first prove that $\mu(Z_\rho) = 0$. In order to do it suppose by way of contradiction that $\mu(Z_\rho) > 0$. Since μ is finite, it is regular by (B.3), so there exists a compact set $K \subseteq Z_\rho$, such that $\mu(K) > 0$. Using Uryhson, take f in $C_c^+(X)$, such that $f|_K = 1$, so that

$$\infty = \infty \times \mu(K) = \int_K \zeta\,d\mu = \int_K f\zeta\,d\mu \le$$

$$\le \int_X f\zeta\,d\mu \overset{\text{(iii)}}{=} \int_X E(\rho f)\,d\mu \le \|E(\rho f)\|_\infty\,\mu(X) \overset{\text{(3.4.iv)}}{<} \infty.$$

Arriving at a contradiction we conclude that $\mu(Z_\rho) = 0$, as desired.

We next claim that (iii) indeed holds for all f in $\mathcal{M}^+(X)$, namely that

$$\int_X f\zeta\,d\mu = \int_X E(\rho f)\,d\mu, \quad \forall\, f \in \mathcal{M}^+(X). \tag{4.11.1}$$

Letting $\zeta\mu$ and μ play the roles of μ' and ν, respectively, in (3.5.ii), we only need to prove that every f in $C_c^+(X)$ is integrable with respect to $\zeta\mu$, but this follows from

$$\int_X f\zeta\,d\mu \overset{\text{(iii)}}{=} \int_X E_\rho(f)\,d\mu \le \mu(X)\|E_\rho(f)\|_\infty \overset{\text{(3.4.iv)}}{<} \infty.$$

Therefore (4.11.1) is verified so, for any f in $C_c^+(X)$, we may plug in $f\zeta^{-1}$ there, obtaining

$$\int_X E(\rho f \zeta^{-1}) \, d\mu = \int_X f\zeta^{-1}\zeta \, d\mu = \int_X f \, d\mu,$$

where the last equality is justified by (4.6), which says that $\zeta\zeta^{-1} = 1$ on $X \setminus Z_\rho$, and by the fact that μ vanishes on Z_ρ. This proves (iv).

(iv) $\Rightarrow$ (v). Defining $\nu := \zeta^{-1}\mu$, and given f in $C_c^+(X)$, we have that

$$\int_X f \, d\mu = \int_X E(\rho f \zeta^{-1}) \, d\mu \overset{(2.6)}{=} \int_X E(\rho f)\zeta^{-1} \, d\mu = \int_X E(\rho f) \, d\nu.$$

Since $C_c(X)$ is linearly spanned by $C_c^+(X)$, the last assertion in (v) follows. Furthermore, employing (3.5.ii) once more, one has that

$$\int_X f \, d\mu = \int_X E(\rho f) \, d\nu, \quad \forall f \in \mathcal{M}^+(X),$$

so we are allowed to plug $f = 1$ above, whence

$$\int_X \zeta \, d\nu = \int_X E(\rho) \, d\nu = \int_X 1 \, d\mu = \mu(X) < \infty,$$

hence proving the remaining first assertion of (v).

(v) $\Rightarrow$ (i). In order to prove that μ is D-quasi-invariant, we need to check (4.10) for every f in $C_c(R)$. As a notational aid, let us temporarily write

$$A(x) = \sum_{y \in R(x)} f(x,y), \quad \text{and} \quad B(y) = \sum_{x \in R(y)} f(x,y)D(x,y),$$

so that our goal is to prove that A and B have the same integral relative to μ. En passant, notice that A and B lie in $C_c(X)$ by (3.2).

Starting from the left-hand-side of (4.10), observe that

$$\int_X \sum_{y \in R(x)} f(x,y) \, d\mu(x) = \int_X A \, d\mu \overset{(v)}{=} \int_X E(\rho A) \, d\nu =$$

$$= \int_X \sum_{z \in R(x)} \rho(z) \sum_{y \in R(z)} f(z,y) \, d\nu(x) = \int_X \sum_{z \in R(x)} \sum_{y \in R(z)} \rho(z) f(z,y) \, d\nu(x). \qquad (4.11.2)$$

On the other hand, starting from the right-hand-side of (4.10), we have

$$\int_X \sum_{z \in R(y)} f(z,y)D(z,y) \, d\mu(y) = \int_X B \, d\mu \overset{(v)}{=} \int_X E(\rho B) \, d\nu =$$

$$= \int_X \sum_{y \in R(x)} \rho(y) \sum_{z \in R(y)} f(z,y)D(z,y) \, d\nu(x) = \int_X \sum_{y \in R(x)} \sum_{z \in R(y)} \rho(z) f(z,y) \, d\nu(x). \qquad (4.11.3)$$

Notice that the difference between (4.11.2) and (4.11.3) is simply that, in the former, the sum ranges over all pairs (z, y) such that

$$x \underset{R}{\sim} z \underset{R}{\sim} y,$$

while, in the latter, the pairs (y, z) considered are those for which

$$x \underset{R}{\sim} y \underset{R}{\sim} z.$$

Being an equivalence relation, R is transitive, whence in both cases above the sum ranges over all y and all z in the equivalence class of x, and therefore we see that (4.11.2) and (4.11.3) coincide. This proves (4.10) and hence that μ is D-quasi-invariant. $\square$

The characterization given by (4.11.v) may be used to produce D-quasi-invariantmeasures, as we now show:

4.12. Corollary. *Given a measure ν on X such that ζ is ν-integrable, there exists a unique finite, D-quasi-invariantmeasure μ on X such that*

$$\int_X f \, d\mu = \int_X E(\rho f) \, d\nu, \quad \forall f \in C_c(X).$$

Proof. Given ν, let $\mu = E_\rho^*(\nu)$, so

$$\mu(X) = \int_X 1 \, d\mu = \int_X E(\rho) \, d\nu = \int_X \zeta \, d\nu < \infty,$$

so μ is indeed a finite measure and it is D-quasi-invariantbecause it satisfies (4.11.v). The uniqueness of μ now follows from the uniqueness part of the Riesz-Markov Theorem. $\square$

The next result settles the question regarding the existence of nontrivial D-quasi-invariant measures.

4.13. Corollary. *The following are equivalent:*

 (i) *there exists at least one D-quasi-invariant probability measure on X,*
 (ii) *ζ is not identically infinite.*

Proof. Recall that Z_ρ is the set of points where ζ is infinite, so (ii) is equivalent to saying that $Z_\rho \neq X$, or equivalently that $X \backslash Z_\rho$ is nonempty.

Assuming (i), let μ be a D-quasi-invariant probability measure on X. By the last sentence in (4.11) we have that $\mu(Z_\rho) = 0$, and hence that $\mu(X \backslash Z_\rho) = 1$, so $X \backslash Z_\rho \neq \emptyset$, proving (ii). Conversely, if $X \backslash Z_\rho$ is nonempty, it is easy to exhibit a measure ν on X satisfying

$$\int_X \zeta \, d\nu = 1.$$

Take, for example, any point $y_0 \in X \backslash Z_\rho$ and, observing that $0 < \zeta(y_0) < \infty$ by (4.3), it is enough to choose

$$\nu = \zeta(y_0)^{-1} \delta_{y_0},$$

where δ_{y_0} is the Dirac measure on y_0. Given any such ν, the measure μ built in (4.12) in terms of ν is a D-quasi-invariant probability measure, proving (i). $\quad\square$

The remainder of this section will be devoted to a closer look at the fourth condition of (4.11).

4.14. Proposition. *Consider the operator* $P_\rho : \mathcal{M}^+(X) \to \mathcal{M}^+(X)$*, given by*

$$P_\rho(f) = E(f\rho\zeta^{-1}), \quad \forall\, f \in \mathcal{M}^+(X).$$

Then

(i) $P_\rho(1) = 1_{X\setminus Z_\rho}$,
(ii) $P_\rho^2 = P_\rho$,
(iii) *the range of* P_ρ *coincides with the set* $\mathcal{M}_{R,\rho}^+(X)$*, consisting of all* R-*invariant functions* f *in* $\mathcal{M}^+(X)$ *which vanish on* Z_ρ.

Proof. We should first observe that, since ρ and ζ^{-1} lie in $\mathcal{M}^+(X)$, the range of P_ρ is indeed a subset of $\mathcal{M}^+(X)$ by (2.5).

In order to prove the first assertion, we compute

$$P_\rho(1) = E(\rho\zeta^{-1}) \stackrel{(2.6.\mathrm{ii})}{=} E(\rho)\zeta^{-1} = \zeta\zeta^{-1} \stackrel{(4.6)}{=} 1_{X\setminus Z_\rho}.$$

We next claim that:

(a) the range of P_ρ is *contained* in $\mathcal{M}_{R,\rho}^+(X)$, and
(b) $P_\rho(f) = f$, for every f in $\mathcal{M}_{R,\rho}^+(X)$.

In order to verify (a), pick any f in $\mathcal{M}^+(X)$. Since

$$P_\rho(f) = E(f\rho)\zeta^{-1}$$

by (2.6.ii), and since ζ^{-1} vanishes on Z_ρ, then $P_\rho(f)$ also vanishes on Z_ρ. The fact that $P_\rho(f)$ lies in $\mathcal{M}_{R,\rho}^+(X)$ then follows immediately from (2.6.i).

To prove (b), let $f \in \mathcal{M}_{R,\rho}^+(X)$. Then

$$P_\rho(f) = E(f\rho\zeta^{-1}) \stackrel{(2.6.\mathrm{ii})}{=} f E(\rho)\zeta^{-1} = f\zeta\zeta^{-1} = f,$$

where the last step relies on the fact that f vanishes on Z_ρ. This said, (ii) and (iii) follow trivially from (a) and (b). $\quad\square$

Among the characterizations of D-quasi-invariant measures given by (4.11), a particularly useful one is (4.11.iv), given the nice properties of the operator P_ρ described in (4.14). For that reason, and also for future reference, we restate part of the conclusions of (4.11) in a way as to emphasize the importance of P_ρ.

4.15. Corollary. *Under the conditions of (4.11) one has that* μ *is* D-*quasi-invariant if and only if* $P_\rho^*(\mu) = \mu$.

Some further important facts involving P_ρ^* are as follows.

4.16. Proposition. *Let* ν *be any finite measure on* X*. Then*

(i) *if $A \subseteq X$ is an invariant Borel subset, then $P_\rho^*(\nu)(A) = \nu(A\backslash Z_\rho)$,*
(ii) *if ν vanishes on an invariant Borel set $A \subseteq X$, then the same is true for $P_\rho^*(\nu)$,*
(iii) *$P_\rho^*(\nu)$ is finite,*
(iv) *$P_\rho^*(\nu)$ is nonzero if and only if $\nu(X\backslash Z_\rho)$ is nonzero,*
(v) *if ν is a probability measure vanishing on Z_ρ, then so is $P_\rho^*(\nu)$,*
(vi) *$P_\rho^*\big(P_\rho^*(\nu)\big) = P_\rho^*(\nu)$,*
(vii) *$P_\rho^*(\nu)$ is D-quasi-invariant.*

Proof. Given A as in (i), we have

$$P_\rho^*(\nu)(A) = \int_X 1_A \, dP_\rho^*(\nu) = \int_X E(1_A \rho \zeta^{-1}) \, d\nu \stackrel{(2.6.\text{ii})}{=} \int_X 1_A E(\rho)\zeta^{-1} \, d\nu =$$

$$= \int_X 1_A \zeta\zeta^{-1} \, d\nu \stackrel{(4.6)}{=} \int_X 1_A 1_{X\backslash Z_\rho} \, d\nu = \nu(A\backslash Z_\rho).$$

Points (ii–v) then follow immediately from (i). Regarding (vi), it is an easy consequence of (4.14.ii). Finally, let us prove (vii). For this, set $\mu = P_\rho^*(\nu)$, so we see from (vi) that $P_\rho^*(\mu) = \mu$, and the conclusion follows from (4.15). □

5. Generalized approximately proper equivalence relations

As before, throughout this section, we assume that X is a locally compact, second countable, metrizable space.

5.1. Definition. By a *generalized approximately proper equivalence relation* on X, a GAP for short, we shall mean a pair

$$\mathcal{R} = \Big(\{U_n\}_{n\in\mathbb{N}}, \{R_n\}_{n\in\mathbb{N}}\Big),$$

where each U_n is an open subset of X, and each R_n is a proper equivalence relation on U_n, such that

(i) $X = U_0 \supseteq U_1 \supseteq U_2 \supseteq \cdots$
(ii) R_0 is the identity relation on U_0, that is, R_0 is the diagonal in $U_0 \times U_0$,
(iii) if $n \leq m$, then $R_n \cap (U_n \times U_m) \subseteq R_m$.

5.2. Remark. Two immediate consequences of the definition are as follows: if $\mathcal{R} = \big(\{U_n\}_{n\in\mathbb{N}}, \{R_n\}_{n\in\mathbb{N}}\big)$ is a GAP on X then, whenever $n \leq m$, one has that

(i) the restriction of R_n to U_m, namely $R_n \cap (U_m \times U_m)$, is contained in R_m, because

$$R_n \cap (U_m \times U_m) \subseteq R_n \cap (U_n \times U_m) \stackrel{(5.1.\text{iii})}{\subseteq} R_m.$$

(ii) If $n \leq m$, then U_m is *invariant* under R_n in the sense that if a point x in U_n is equivalent under R_n to a point y in U_m, then x lies in U_m. In fact,

$$(x,y) \in R_n \cap (U_n \times U_m) \stackrel{(5.1.\text{iii})}{\subseteq} R_m \subseteq U_m \times U_m,$$

so $x \in U_m$.

It is not hard to see that also (5.2.i–ii) imply (5.1.iii), so the reader might think of the latter as subsuming the former, which some may consider a more natural set of conditions.

The main motivation and the main source of examples for GAPs is described in detail in Section (8), below.

▶ From now on we fix a GAP $\mathcal{R} = \big(\{U_n\}_{n\in\mathbb{N}},\ \{R_n\}_{n\in\mathbb{N}}\big)$ on X.

5.3. Proposition. *Setting*

$$R = \bigcup_{n\in\mathbb{N}} R_n,$$

one has that R is an equivalence relation on X.

Proof. The only slightly nontrivial point regards the transitivity of R. In order to prove it, suppose that (x,y) and (y,z) lie in R. We may then pick n and m such that $(x,y) \in R_n$ and $(y,z) \in R_m$, and we may assume without loss of generality that $n \leq m$. In that case we have that

$$(x,y) \in R_n \cap (U_n \times U_m) \overset{(5.1.iii)}{\subseteq} R_m.$$

Since R_m is transitive we have that

$$(x,z) \in R_m \subseteq R. \quad \square$$

5.4. Lemma. *Equipping each R_n with the topology induced from the product topology on $X \times X$, one has that $R_n \cap R_m$ is open in R_n, for all n and m in $\mathbb{N}$.*

Proof. Given (x,y) in $R_n \cap R_m$, by the definition of the product topology on R_n we must prove the existence of open subsets $V, W \subseteq X$, such that

$$(x,y) \in R_n \cap (V \times W) \subseteq R_n \cap R_m.$$

Assuming that $n \leq m$, choose $V = U_n$, and $W = U_m$, and observe that the above inclusion is then immediately verified thanks to (5.1.iii). On the other hand, proving the result under the opposite assumption, i.e. that $n \geq m$, is equivalent to maintaining the assumption that $n \leq m$ (with which the reader must be used to by now) and proving instead that

$$(x,y) \in R_m \cap (V \times W) \subseteq R_n \cap R_m. \tag{5.4.1}$$

For each $k \in \mathbb{N}$, denote by π_k the quotient map

$$\pi_k : U_k \to U_k/R_k,$$

and for each $k \in \{n,m\}$, let us choose an open set $W_k \subseteq U_k$, such that $y \in W_k$, and such that π_k restricts to a homeomorphism from W_k to the open set $\pi_k(W_k)$. Replacing both W_n and W_m by

$$W := W_n \cap W_m,$$

we may assume that $W_n = W_m$.

Notice that $\pi_n(x) = \pi_n(y) \in \pi_n(W)$, so we have that $x \in \pi_n^{-1}\big(\pi_n(W)\big)$, and upon setting

$$V := \pi_n^{-1}\big(\pi_n(W)\big) \cap U_m,$$

we see that V is an open subset of X, and we moreover claim that (5.4.1) holds. The first part, namely that $(x, y) \in R_m \cap (V \times W)$, is evident and, in order to prove that

$$R_m \cap (V \times W) \subseteq R_n \cap R_m, \tag{5.4.2}$$

let us pick (z, w) in the set appearing in the left-hand side above. It follows that $z \in V$, whence $\pi_n(z) \in \pi_n(W)$, so there exists some w' in W such that $\pi_n(z) = \pi_n(w')$. Another way to express this is by saying that $(z, w') \in R_n$, but since (z, w') also lies in $U_m \times U_m$, we deduce that

$$(z, w') \in R_n \cap (U_m \times U_m) \subseteq R_m.$$

Recall that $(z, w) \in R_m$, as well, so transitivity yields $(w, w') \in R_m$. Observing that both w and w' lie in W, and using that π_m is injective on W, we see that $w = w'$, whence

$$(z, w) = (z, w') \in R_n.$$

This finishes the verification of (5.4.2), and hence also of (5.4.1), concluding the proof. $\quad\square$

Recall that the inductive limit topology on the union of an increasing sequence of topological spaces

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$$

is the topology according to which a subset $U \subseteq \bigcup_n X_n$ is open if and only if $U \cap X_n$ is open in X_n, for every n. In our situation, where $R = \bigcup_n R_n$, the R_n do not form an increasing sequence of subsets but one may nevertheless equip X with the topology defined as above, that is, in which a subset $U \subseteq R$ is open if and only if $U \cap R_n$ is open in R_n, for every n. Even though this might constitute a slight abuse of the language, we shall refer to that topology as the *inductive limit topology* on R.

5.5. Lemma. *Equipping R with the inductive limit topology we have that each R_n is open in R.*

Proof. Follows immediately from (5.4). $\quad\square$

The two previous Lemmas form the key to showing the following result, whose otherwise easy proof we leave for the reader.

5.6. Proposition. *Given a generalized approximately proper equivalence relation on X, one has that R is an étale groupoid when equipped with the inductive limit topology.*

6. Quasi-invariant measures and GAPs

As before, throughout this section we assume that X is a locally compact, second countable, metrizable space. We will also assume that we are given a GAP

$$\mathcal{R} = \Big(\{U_n\}_{n \in \mathbb{N}}, \{R_n\}_{n \in \mathbb{N}}\Big)$$

on X.

6.1. Definition. By a *potential* for $\mathcal{R}$ we shall mean a collection $\{k_n\}_{n \geq 1}$, of continuous functions

$$k_n : U_n \to \mathbb{R},$$

such that for every $n \geq 1$, one has that

$$(x, y) \in R_{n-1} \cap (U_n \times U_n) \Rightarrow k_n(x) = k_n(y). \tag{6.1.1}$$

It is perhaps worth pointing out that a potential involves no k_0. On the other hand, the lowest case of (6.1.1) is tautological, that is, when $n = 1$, we have that R_{n-1}, also known as R_0, is the identity relation, and it is no surprise that $x = y$ implies that $k_n(x) = k_n(y)$.

Regarding (6.1.1), notice that

$$R_{n-1} \cap (U_n \times U_n) = R_{n-1} \cap R_n,$$

because

$$R_{n-1} \cap R_n \subseteq R_{n-1} \cap (U_n \times U_n) \subseteq R_{n-1} \cap (U_{n-1} \times U_n) \overset{(5.1.\mathrm{iii})}{\subseteq} R_{n-1} \cap R_n.$$

An equivalent way to state (6.1.1) is therefore to require that $k_n(x) = k_n(y)$, for all (x, y) in $R_{n-1} \cap R_n$.

▶   From now on we assume that we are given a potential $\{k_n\}_{n \geq 1}$ for $\mathcal{R}$.

Our next goal is to use a potential to produce a cocycle on the groupoid $R = \bigcup_{n \in \mathbb{N}} R_n$. As a first step, we introduce the following notation:

6.2. Definition. For all $n \geq 0$, let $h_n : U_n \to \mathbb{R}$, be defined recursively by $h_0 = 0$, and

$$h_n = h_{n-1}\lceil_{U_n} + k_n, \quad \forall n \geq 1.$$

In addition, for all $n \geq 0$, we define

$$c_n : (x, y) \in R_n \mapsto h_n(x) - h_n(y) \in \mathbb{R}.$$

Of course one may alternatively define h_n by

$$h_n = \sum_{i=1}^{n} k_i \lceil_{U_n},$$

observing that, when $n = 0$, the usual convention about sums without any summands gives $h_0 = 0$, as expected.

6.3. Proposition. *For every $n \geq 1$, and all (x, y) in $R_{n-1} \cap R_n$, one has that*

$$c_{n-1}(x, y) = c_n(x, y).$$

Proof. The difference between $c_n(x, y)$ and $c_{n-1}(x, y)$ is precisely $k_n(x) - k_n(y)$, but since (x, y) lies in $R_{n-1} \cap R_n$, condition (6.1.1) applies.   □

6.4. Proposition. *There exists a (necessarily unique) continuous cocycle c on R, such that $c = c_n$ on each R_n.*

Proof. We first claim that, whenever $0 \leq n \leq m$, one has that

$$R_n \cap R_m = R_n \cap R_{n+1} \cap \ldots \cap R_{m-1} \cap R_m. \tag{6.4.1}$$

In order to see this, it is clearly enough to show that $R_n \cap R_m \subseteq R_k$, for every k with $n \leq k \leq m$, and in turn this follows from

$$R_n \cap R_m \subseteq R_n \cap (U_m \times U_m) \subseteq R_n \cap (U_n \times U_k) \overset{(5.1.\text{iii})}{\subseteq} R_k.$$

Given any γ in R, choose n such that $\gamma \in R_n$, and put

$$c(\gamma) = c_n(\gamma).$$

To see that this is well defined, suppose that $\gamma \in R_m$, for some other m, and let us prove that $c_n(\gamma) = c_m(\gamma)$. Assuming without loss of generality that $n \leq m$, it follows from (6.4.1) that $\gamma \in R_n \cap R_{n+1} \cap \ldots \cap R_{m-1} \cap R_m$, so we may apply (6.3) to show that

$$c_n(\gamma) = c_{n+1}(\gamma) = \cdots = c_{m-1}(\gamma) = c_m(\gamma).$$

This shows that c is well defined and we leave it as an easy exercise to show that c is a continuous cocycle on R. $\square$

We shall next present two general results about quasi-invariant measures on étale groupoids, to be used later.

6.5. Proposition. *Let $\mathcal{G}$ be an étale groupoid and suppose that we are given a collection $\{\mathcal{G}_i\}_{i \in I}$ of open subgroupoids $\mathcal{G}_i \subseteq \mathcal{G}$, such that $\mathcal{G} = \bigcup_{i \in I} \mathcal{G}_i$. Suppose moreover that $D : \mathcal{G} \to \mathbb{R}_+^*$ is a continuous multiplicative cocycle and that μ is a finite measure on $\mathcal{G}^{(0)}$. Then μ is D-quasi-invariant if and only if the restriction (see (A.4) for a discussion regarding the concept of restricting a measure to a subset) of μ to $\mathcal{G}_i^{(0)}$ is quasi-invariant relative to the restriction of D to $\mathcal{G}_i$, for every i.*

Proof. We prove only the "if" part, leaving the "only if" part to the reader. We must therefore check (4.9) for every f in $C_c(\mathcal{G})$. By [5, 3.10] (which holds even if $\mathcal{G}$ is non-Hausdorff), we have that f may be written as a finite linear combination of functions f_j, each of which lies in $C_c(U_j)$, for some open bisection U_j. Therefore, since both sides of (4.9) are clearly linear with respect to f, it suffices to prove (4.9) under the assumption that $f \in C_c(U)$, for some open bisection U.

Letting K be the compact support of f, recall that the hypotheses imply that $\{U \cap \mathcal{G}_i\}_i$ is an open cover for K. Choosing a finite subcover $\{U \cap \mathcal{G}_{i_k}\}_{k=1}^n$ and a partition of unit $\{\varphi_k\}_{k=1}^n$ subordinate to it [13, 21.1.5], we may write

$$f = \sum_{k=1}^n f \varphi_k,$$

observing that $f \varphi_k$ lies in $C_c(\mathcal{G}_{i_k})$.

The upshot of this argument is that we may further reduce (4.9) by assuming that f is supported on a single $\mathcal{G}_i$. Under this assumption, observe that the integrands in both sides of (4.9) vanish whenever x is not in $\mathcal{G}_i^{(0)}$, so it suffices to verify a variant of (4.9), namely where both occurrences of $\mathcal{G}^{(0)}$ are replaced by $\mathcal{G}_i^{(0)}$. The resulting expression is then seen to hold because the restriction of μ to $\mathcal{G}_i^{(0)}$ is D-quasi-invariant by hypothesis. $\square$

6.6. Lemma. *Let $\mathcal{G}$ be an étale groupoid with a continuous multiplicative cocycle $D : \mathcal{G} \to \mathbb{R}_+^*$, and let μ be a finite, D-quasi-invariant measure on $\mathcal{G}^{(0)}$.*

(i) *If φ is a bounded, invariant,[4] Borel-measurable function on $\mathcal{G}^{(0)}$, then $\varphi\mu$ is also D-quasi-invariant.*

(ii) *If E is an invariant,[5] Borel subset of $\mathcal{G}^{(0)}$, then $\mu_E := 1_E\,\mu$ is also D-quasi-invariant.*

Proof. In order to prove (i) we pick any f in $C_c(\mathcal{G})$ and we set out to verify (4.9) relative to the measure $\varphi\mu$. Starting from the left-hand side, we have

$$
\int_{\mathcal{G}^{(0)}} \sum_{\gamma \in r^{-1}(x)} f(\gamma)\,d\varphi\mu(x) = \int_{\mathcal{G}^{(0)}} \sum_{\gamma \in r^{-1}(x)} f(\gamma)\varphi(x)\,d\mu(x) =
$$

$$
= \int_{\mathcal{G}^{(0)}} \sum_{\gamma \in r^{-1}(x)} f(\gamma)\varphi\big(r(\gamma)\big)\,d\mu(x) \overset{(4.9)}{=} \int_{\mathcal{G}^{(0)}} \sum_{\gamma \in s^{-1}(x)} f(\gamma)\varphi\big(s(\gamma)\big)D(\gamma)\,d\mu(x) =
$$

$$
= \int_{\mathcal{G}^{(0)}} \sum_{\gamma \in s^{-1}(x)} f(\gamma)D(\gamma)\varphi(x)\,d\mu(x) = \int_{\mathcal{G}^{(0)}} \sum_{\gamma \in s^{-1}(x)} f(\gamma)D(\gamma)\,d\varphi\mu(x),
$$

proving (i). Point (ii) now follows from (i) upon taking φ to be the characteristic function of E. $\square$

Returning to the GAP we have fixed at the beginning of this section, and assuming we are given a potential $\{k_n\}_{n\geq 1}$, leading up to the cocycle c of (6.4), consider the multiplicative cocycle

$$
D : \gamma \in R \mapsto e^{c(\gamma)} \in \mathbb{R}_+^*,
$$

as well as the multiplicative cocycles

$$
D_n : \gamma \in R_n \mapsto e^{c_n(\gamma)} \in \mathbb{R}_+^*.
$$

We then have the following immediate consequence of (6.5):

6.7. Corollary. *Let μ be a finite measure on X. Then μ is D-quasi-invariant if and only if $\mu\!\restriction_{U_n}$ is D_n-quasi-invariant for every n in $\mathbb{N}$.*

Once the question of the quasi-invariance of a measure μ on X is reduced to the quasi-invariance of measures on proper equivalence relations, namely the R_n's in the above Corollary, the results of Section (4) apply. Our goal in what follows is to patch the conclusions of these results for the various R_n in a meaningful way from the point of view of R. For each $n \in \mathbb{N}$, we shall let

$$
\rho_n : x \in U_n \mapsto e^{h_n(x)} \in \mathbb{R}_+^*,
$$

and we will henceforth let ζ_n be the partition function given by (4.2) in terms of ρ_n.

Alongside ρ_n, D_n and ζ_n, all of the other ingredients introduced in Section (4) will also be relevant here, such as Z_{ρ_n} and P_{ρ_n}, as well as the operator E_n on $\mathcal{M}^+(U_n)$ given by (2.3) relative to the equivalence relation R_n.

As a first use of these notations, we have the following immediate consequence of (6.7) and (4.15).

---

[4] A function φ defined on $\mathcal{G}^{(0)}$ is said to be invariant when $\varphi(r(\gamma)) = \varphi(s(\gamma))$, for every γ in $\mathcal{G}$.

[5] A subset $E \subseteq \mathcal{G}^{(0)}$ is said to be invariant when $r(\gamma) \in E \Leftrightarrow s(\gamma) \in E$, for every γ in $\mathcal{G}$.

6.8. Corollary. *Let μ be a finite measure on X. Then μ is D-quasi-invariant if and only if*

$$P^*_{\rho_n}(\mu\upharpoonright_{U_n}) = \mu\upharpoonright_{U_n}, \quad \forall n \in \mathbb{N}.$$

Part of the difficulty in simultaneously dealing with so many maps and sets is the fact that they each refer to a different equivalence relation. Attempting to bring everything to a common environment we introduce the following:

6.9. Definition. Let $n \in \mathbb{N}$ be given.

(i) For any f in $\mathcal{M}^+(U_n)$, we will denote by $\iota_n(f)$ the extension of f to the whole of X obtained by setting it to be zero outside U_n. When no confusion is likely to arise, we shall denote that extension simply by f, by abuse of language.

(ii) We will write Z_n and Y_n for Z_{ρ_n} and $U_n \backslash Z_{\rho_n}$, respectively, and we will view both Z_n and Y_n as subsets of X (which of course they are).

(iii) We will denote by F_n the map from $\mathcal{M}^+(X)$ to itself given, for every f in $\mathcal{M}^+(X)$, and for every x in X, by

$$F_n(f)\big|_x = \begin{cases} \displaystyle\sum_{y \in R_n(x)} f(y), & \text{if } x \in U_n, \\ 0, & \text{otherwise.} \end{cases}$$

(iv) We will denote by Q_n the map from $\mathcal{M}^+(X)$ to itself given, for every f in $\mathcal{M}^+(X)$, and for every x in X, by

$$Q_n(f)\big|_x = \begin{cases} \displaystyle\sum_{y \in R_n(x)} f(y)\rho_n(y)\zeta_n(y)^{-1}, & \text{if } x \in U_n, \\ 0, & \text{otherwise.} \end{cases}$$

(v) We will say that a given f in $\mathcal{M}^+(X)$ is R_n-*invariant* if $f(x) = f(y)$, whenever $(x, y) \in R_n$.

6.10. Remarks.

(a) Since $h_0 = 0$, we have that $\rho_0 = 1$, and clearly also $\zeta_0 = 1$. Therefore $Z_0 = \emptyset$.

(b) Notice that $F_n(f)$ and $Q_n(f)$ may be alternatively defined as

$$F_n(f) = \iota_n\big(E_n(f\upharpoonright_{U_n})\big), \quad \text{and} \quad Q_n(f) = \iota_n\big(P_{\rho_n}\big(f\upharpoonright_{U_n}\big)\big).$$

For that reason F_n and Q_n should be seen as natural extensions of E_n and P_{ρ_n} to $\mathcal{M}^+(X)$, respectively. Notice also that

$$Q_n(f) = F_n\big(f\iota_n(\rho_n\zeta_n^{-1})\big), \quad \forall f \in \mathcal{M}^+(X).$$

(c) Observe that the invariance of a function under an equivalence relation is a concept usually considered when the relation is defined on the *whole domain* of said function. However, the fact that R_n is an equivalence relation on U_n, rather than on X, does not prevent us from introducing the invariance notion expressed in (6.9.v). An example of a function obeying this property is given by $\iota_n(f)$, where f is any function in $\mathcal{M}^+(U_n)$ which is constant on each R_n-equivalence class.

Some elementary properties of these extended notions are in order.

6.11. Proposition. *Given $n \in \mathbb{N}$, one has for all $f, g \in \mathcal{M}^+(X)$, that*

(i) $F_n\big(\iota_n(\rho_n)\big) = \iota_n(\zeta_n),$

(ii) $Q_n(1) = 1_{Y_n},$

(iii) $F_n(f) = F_n(1_{U_n} f),$

(iv) $Q_n(f) = Q_n(1_{U_n} f) = Q_n(1_{Y_n} f),$

(v) $Q_n(f) = 1_{Y_n} Q_n(f),$

(vi) $F_n(f)$ and $Q_n(f)$ are R_n-invariant and vanish off U_n,

(vii) *if f is R_n-invariant and vanishes off U_n, then f is R_k-invariant for every $k \leq n$,*

(viii) *if g is R_n-invariant, then $F_n(gf) = gF_n(f)$, and $Q_n(gf) = gQ_n(f)$,*

(ix) *if A is an R_n-invariant subset of U_n, then, as a subset of X, A is R_k-invariant for every $k \leq n$,*

(x) F_n and Q_n are σ-additive.

Proof. Left for the reader. $\square$

Recalling that

$$Z_n = Z_{\rho_n} = \{x \in U_n : \zeta_n(x) = \infty\},$$

and that

$$Y_n = U_n \backslash Z_{\rho_n} = \{x \in U_n : \zeta_n(x) < \infty\},$$

we will now study certain relations among these sets, and we begin with the following auxiliary result.

6.12. Lemma. *For every $0 \leq n \leq m$, and for every x in U_m, there exists a subset $\Lambda \subseteq R_m(x)$, such that $x \in \Lambda$, and*

$$R_m(x) = \bigsqcup_{\lambda \in \Lambda} R_n(\lambda),$$

the square cup denoting disjoint union.

Proof. We first claim that if C_n and C_m are equivalence classes for R_n and R_m, respectively, then

$$C_n \cap C_m \neq \emptyset \Rightarrow C_n \subseteq C_m.$$

To see this, choose $z \in C_n \cap C_m$, and let $y \in C_n$. Then

$$(y, z) \in R_n \cap (U_n \times U_m) \overset{(5.1.\text{iii})}{\subseteq} R_m,$$

so $y \in C_m$. This said, we see that the R_m-equivalence class of x splits as the union of R_n-equivalence classes, whence the conclusion. $\square$

The promised relations among the Z_n and the Y_n are in order.

6.13. Proposition.

(a) *For every $m \geq 1$, and every x in U_m, one has that* $\quad e^{k_m(x)}\zeta_{m-1}(x) \leq \zeta_m(x)$.
(b) *If $0 \leq n \leq m$, then $Z_n \cap U_m \subseteq Z_m$,*
(c) *If $0 \leq n \leq m$, then $Y_m \subseteq Y_n$.*

Proof. In order to prove (a) write

$$R_m(x) = \bigsqcup_{\lambda \in \Lambda} R_{m-1}(\lambda),$$

where $x \in \Lambda \subseteq R_m(x)$, by (6.12). So

$$\zeta_m(x) = \sum_{y \in R_m(x)} e^{h_m(y)} = \sum_{\lambda \in \Lambda} \sum_{y \in R_{m-1}(\lambda)} e^{h_m(y)} = \sum_{\lambda \in \Lambda} \sum_{y \in R_{m-1}(\lambda)} e^{h_{m-1}(y)}e^{k_m(y)} = \cdots$$

For every $\lambda \in \Lambda$, and $y \in R_{m-1}(\lambda)$, notice that

$$(y, \lambda) \in R_{m-1} \cap (U_m \times U_m) \overset{(6.1.1)}{\Rightarrow} k_m(y) = k_m(\lambda),$$

so the above equals

$$\cdots = \sum_{\lambda \in \Lambda} e^{k_m(\lambda)} \sum_{y \in R_{m-1}(\lambda)} e^{h_{m-1}(y)} = \sum_{\lambda \in \Lambda} e^{k_m(\lambda)}\zeta_{m-1}(\lambda) \geq e^{k_m(x)}\zeta_{m-1}(x).$$

This proves (a). In order to prove (b), observe that under the hypothesis of (a) we have that

$$\zeta_{m-1}(x) = \infty \Rightarrow \zeta_m(x) = \infty,$$

from where we trivially deduce that $Z_{m-1} \cap U_m \subseteq Z_m$. Assuming now that $0 \leq n \leq m$, we will prove (b) by induction on $m - n$.

In order to do this, notice that the case "$m - n = 0$" is immediate, while the case "$m - n = 1$" has just been proved. When $m - n > 1$, we then have that

$$Z_n \cap U_m \subseteq Z_n \cap U_{m-1} \cap U_m \subseteq Z_{m-1} \cap U_m \subseteq Z_m,$$

taking care of (b). With respect to (c), we have

$$Y_m = U_m \backslash Z_m = U_m \cap Z_m^c \overset{(a)}{\subseteq} U_m \cap (Z_n \cap U_m)^c \subseteq U_m \cap (Z_n^c \cup U_m^c) =$$
$$= (U_m \cap Z_n^c) \cup (U_m \cap U_m^c) = U_m \cap Z_n^c \subseteq U_n \cap Z_n^c = U_n \backslash Z_n = Y_n. \quad \square$$

There are many situations in the present context in which not necessarily increasing sequences satisfy some *increasing-like* property as we look inside the appropriate U_m. For example, when $n \leq m$, there is no comparison between R_n and R_m, as sets, but when we restrict R_n to U_m, that is, when we consider $R_n \cap (U_m \times U_m)$, we get a subset of R_m. Similarly, there is no comparison between Z_n and Z_m, but as seen above, $Z_n \cap U_m \subseteq Z_m$.

We next present some crucial properties of the F_n and the Q_n.

6.14. Proposition. *If $0 \leq n \leq m$, and if $f, g \in \mathcal{M}^+(X)$, then*

(i) $F_m\big(f F_n(g)\big) = F_m\big(F_n(f)g\big),$

(ii) $Q_m\big(Q_n(f)\big) = Q_m(f) = Q_n\big(Q_m(f)\big).$

Proof. Addressing (i), since both sides vanish on $X \backslash U_m$, by definition, it is enough to prove that they agree on U_m. Given x in U_m, we have

$$F_m\big(f F_n(g)\big)\big|_x = \sum_{y \in R_m(x)} f(y) \sum_{z \in R_n(y)} g(z) = \cdots \tag{6.14.1}$$

Employing (6.12) we write

$$R_m(x) = \bigsqcup_{\lambda \in \Lambda} R_n(\lambda),$$

where $\Lambda \subseteq R_m(x)$, so (6.14.1) equals

$$\cdots = \sum_{\lambda \in \Lambda} \sum_{y \in R_n(\lambda)} f(y) \sum_{z \in R_n(y)} g(z) = \sum_{\lambda \in \Lambda} \sum_{y \in R_n(\lambda)} \sum_{z \in R_n(\lambda)} f(y)g(z) =$$

$$= \sum_{\lambda \in \Lambda} \sum_{z \in R_n(\lambda)} g(z) \sum_{y \in R_n(z)} f(y) = \sum_{z \in R_m(x)} g(z) \sum_{y \in R_n(z)} f(y) = F_m\big(g F_n(f)\big)\big|_x.$$

This proves (i). In order to prove (ii), recall that $h_m = \sum_{i=1}^{m} k_i \restriction_{U_m}$. So, defining

$$\ell = \sum_{i=n+1}^{m} k_i \restriction_{U_m},$$

we then have that $h_m = h_n + \ell$. We next claim that

$$(x,y) \in R_n \cap (U_m \times U_m) \Rightarrow \ell(x) = \ell(y).$$

This is because, for every $i = n+1, \ldots, m$, we have that

$$(x,y) \in R_n \cap (U_m \times U_m) \subseteq R_n \cap (U_n \times U_{i-1}) \subseteq R_{i-1},$$

so (x,y) also lies in $R_{i-1} \cap (U_i \times U_i)$, and this implies that $k_i(x) = k_i(y)$, according to (6.1.1). In other words, ℓ, or rather $\iota_m(\ell)$, is R_n-invariant.

To prove (ii) we start with its left-hand-side, taking full advantage of the abuse of language announced in (6.9.i):

$$Q_m\big(Q_n(f)\big) \overset{(6.11.\text{viii})}{=} F_m\big(F_n(f\rho_n\zeta_n^{-1})\rho_m\big)\zeta_m^{-1} \overset{(i)}{=} F_m\big(f\rho_n\zeta_n^{-1}F_n(e^\ell \rho_n)\big)\zeta_m^{-1} \overset{(6.11.\text{viii})}{=}$$

$$= F_m\big(f\rho_n\zeta_n^{-1}e^\ell F_n(\rho_n)\big)\zeta_m^{-1} = F_m\big(f\rho_m\zeta_m^{-1}\zeta_n\big)\zeta_m^{-1} \overset{(4.7)}{=} F_m(f 1_{Y_n}\rho_m)\zeta_m^{-1} =$$

$$= Q_m(f 1_{Y_n}) \overset{(6.11.\text{iv})}{=} Q_m(f 1_{Y_m} 1_{Y_n}) \overset{(6.13)}{=} Q_m(f 1_{Y_m}) = Q_m(f).$$

With respect to the second equality in (ii), we have

$$Q_n\big(Q_m(f)\big) = Q_n\big(1 Q_m(f)\big) \overset{(6.11.\text{vi,vii\&viii})}{=} Q_n(1)Q_m(f) \overset{(6.11.\text{ii})}{=} 1_{Y_n} Q_m(f) \overset{(6.11.\text{v})}{=}$$

$$= 1_{Y_n} 1_{Y_m} Q_m(f) \overset{(6.13)}{=} 1_{Y_m} Q_m(f) = Q_m(f). \quad \square$$

We next present some useful properties of the dual operators Q_n^*.

6.15. Proposition. *Given n in $\mathbb{N}$, and given any finite measure μ on X, one has that*

(i) $Q_n^*(g\mu) = gQ_n^*(\mu)$, *for every R_n-invariant function g in $\mathcal{M}^+(X)$,*

(ii) $Q_n^*(\mu) = Q_n^*(1_{Y_n}\mu) = 1_{Y_n}Q_n^*(\mu)$,

(iii) $Q_n^*(\mu)\restriction_{U_n} = P_{\rho_n}^*(\mu\restriction_{U_n})$,

(iv) *if A is an R_n-invariant, Borel subset of X, then $Q_n^*(\mu)(A) = \mu(A \cap Y_n)$,*

(v) *if m is another integer with $n \leq m$, then $Q_m^*(Q_n^*(\mu)) = Q_m^*(\mu) = Q_n^*(Q_m^*(\mu))$.*

Proof. The first point follows easily from (6.11.viii).

(ii): Pick any f in $\mathcal{M}^+(X)$. Then

$$\int_X f \, dQ_n^*(\mu) = \int_X Q_n(f) \, d\mu \stackrel{(6.11.v)}{=} \int_X Q_n(f)1_{Y_n} \, d\mu =$$

$$= \int_X Q_n(f) \, d1_{Y_n}\mu = \int_X f \, dQ_n^*(1_{Y_n}\mu),$$

proving the first identity in (ii). As for the second one, we have

$$\int_X f \, dQ_n^*(\mu) = \int_X Q_n(f) \, d\mu \stackrel{(6.11.iv)}{=} \int_X Q_n(1_{Y_n}f) \, d\mu = \int_X f1_{Y_n} \, dQ_n^*(\mu),$$

taking care of (ii).

(iii): Given $f \in \mathcal{M}^+(U_n)$, we have

$$\int_{U_n} f \, dP_{\rho_n}^*(\mu\restriction_{U_n}) = \int_{U_n} P_{\rho_n}(f) \, d\mu\restriction_{U_n} = \int_X \iota_n\big(P_{\rho_n}(f)\big) \, d\mu =$$

$$= \int_X \iota_n\big(P_{\rho_n}(\iota_n(f)\restriction_{U_n})\big) \, d\mu \stackrel{(6.10.b)}{=} \int_X Q_n\big(\iota_n(f)\big) \, d\mu =$$

$$= \int_X \iota_n(f) \, dQ_n^*(\mu) = \int_{U_n} f \, dQ_n^*(\mu)\restriction_{U_n},$$

(iv): We have

$$Q_n^*(\mu)(A) = \int_X 1_A \, dQ_n^*(\mu) = \int_X Q_n(1_A) \, d\mu \stackrel{(6.11.viii)}{=}$$

$$\stackrel{(6.11.viii)}{=} \int_X 1_A Q_n(1) \, d\mu \stackrel{(6.11.ii)}{=} \int_X 1_A 1_{Y_n} \, d\mu = \mu(A \cap Y_n).$$

(v): This is a direct consequence of (6.14.ii). □

We may now state an important quasi-invariance condition for measures on X.

6.16. Theorem. *Let μ be a finite measure on X. Then μ is D-quasi-invariantif and only if, for every $n \in \mathbb{N}$, one has that*

$$Q_n^*(\mu) = 1_{U_n}\mu.$$

Proof. We have already seen in (6.7) that μ is D-quasi-invariant if and only each if $\mu\!\restriction_{U_n}$ is D_n-quasi-invariant. By (4.15) this is in turn equivalent to saying that $P_{\rho_n}^*(\mu\!\restriction_{U_n}) = \mu\!\restriction_{U_n}$, but in view of (6.15.iii), this is now the same as

$$Q_n^*(\mu)\!\restriction_{U_n} = \mu\!\restriction_{U_n}.$$

Since we know that $Q_n^*(\mu)$ vanishes on $X\backslash U_n$ by (6.15.iii), the proof is concluded. $\square$

7. Existence of quasi-invariant measures

Having characterized quasi-invariantmeasures in a concise way in (6.16), we now discuss their existence. This is a multi-faceted question manifesting itself in different ways on different parts of X. It is therefore convenient to break X down into simpler pieces, so we shall henceforth consider the following subsets

$$V_n := U_n\backslash U_{n+1}, \quad \forall\, n \in \mathbb{N},$$
$$V_\infty := \bigcap_{n\in\mathbb{N}} U_n, \quad Z := \bigcup_{n\in\mathbb{N}} Z_n,$$
$$W_n := V_n\backslash Z, \quad \forall\, n \in \mathbb{N}\cup\{\infty\},$$

which we represent in the following Diagram (7.1). Please note that each Z_n should be thought of as the largest shaded rectangle possessing the indicated lower-left-hand corner.
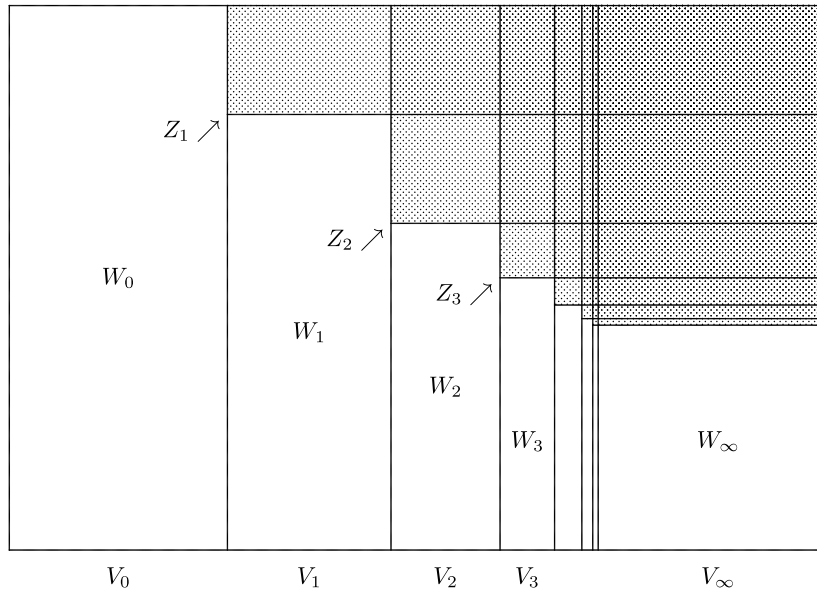
Diagram (7.1). The sets V_n, V_∞, W_n, W_∞ and Z_n inside of X.

The all important sets U_n's may then be described as

$$U_n = \bigsqcup_{n\le m\le\infty} V_m, \quad \forall\, n \in \mathbb{N},$$

the square cup denoting disjoint union. In particular

$$X = U_0 = \bigsqcup_{n \in \mathbb{N} \cup \{\infty\}} V_n,$$

whence also

$$X = Z \sqcup \bigsqcup_{n \in \mathbb{N} \cup \{\infty\}} W_n. \tag{7.2}$$

7.3. Proposition. *The following sets are R-invariant:*

(i) U_n, *for all* $n \in \mathbb{N}$,
(ii) V_n *and* W_n, *for all* $n \in \mathbb{N} \cup \{\infty\}$,
(iii) Z.

Proof. Let us prove first that U_m is R-invariant, for every m. For this suppose that $(x, y) \in R$, and $y \in U_m$, so we may pick some n such that $(x, y) \in R_n$. Assuming initially that $n \leq m$, we have that

$$(x, y) \in R_n \cap (U_n \times U_m) \overset{(5.1.\text{iii})}{\subseteq} R_m \subseteq U_m \times U_m,$$

proving that $x \in U_m$, as desired. Assuming now that $m \leq n$, the conclusion comes even easier because

$$(x, y) \in R_n \subseteq U_n \times U_n \subseteq U_m \times U_m,$$

so again we have that $x \in U_m$.

Let us next prove that Z is R-invariant. So we pick $(x, y) \in R$, with $y \in Z$, whence there are n and m such that $(x, y) \in R_n$, and $y \in Z_m$. Assuming initially that $n \leq m$, we have that

$$(x, y) \in R_n \cap (U_n \times U_m) \overset{(5.1.\text{iii})}{\subseteq} R_m.$$

Since ζ_m is known to be R_m-invariant on U_m, it follows that $\zeta_m(x) = \zeta_m(y) = \infty$, whence $x \in Z_m$, as required.

Assuming now that $m \leq n$, notice that

$$y \in Z_m \cap U_n \overset{(6.13.\text{b})}{\subseteq} Z_n,$$

so the R_n-invariance of ζ_n implies that

$$\infty = \zeta_n(y) = \zeta_n(x),$$

and we conclude that $x \in Z_n \subseteq Z$.

Since the R-invariant sets clearly form a complete Boolean algebra, the remaining statements follow. $\square$

The following result further streamlines the various quasi-invariance conditions and it will be instrumental in the study of the existence question.

7.4. Theorem. *Let μ be a finite measure on X, and for every $k \in \mathbb{N} \cup \{\infty\}$, set $\mu_k = 1_{W_k}\mu$. Then μ is D-quasi-invariant if and only if all of the following conditions hold:*

(i) $\mu(Z) = 0$,

(ii) $Q_k^*(\mu_k) = \mu_k$, for every $k \geq 1$,

(iii) $Q_i^*(\mu_\infty) = \mu_\infty$, for every $i \geq 1$.

Proof. Assuming that μ is D-quasi-invariant , we have by (6.7) that $\mu{\restriction}_{U_k}$ is D_k-quasi-invariant, for every $k \in \mathbb{N}$, whence we deduce from (4.11) that

$$0 = \mu{\restriction}_{U_k}(Z_k) = \mu(Z_k),$$

from where (i) follows.

Given $k \geq 1$, recall that W_k is R-invariant, so μ_k is D-quasi-invariant by (6.6). In then follows from (6.16), that[6]

$$Q_k^*(\mu_k) = 1_{U_k}\mu_k = 1_{U_k}1_{W_k}\mu = 1_{W_k}\mu = \mu_k,$$

proving (ii).

By the reasoning in the first sentence of the paragraph above, we also have that μ_∞ is D-quasi-invariant. So, again by (6.16), we have for all $k \in \mathbb{N}$, that

$$Q_k^*(\mu_\infty) = 1_{U_k}\mu_\infty = 1_{U_k}1_{W_\infty}\mu = 1_{W_\infty}\mu = \mu_\infty,$$

whence (iii).

Conversely, assuming that μ satisfies (i–iii), we will initially prove that μ_k is D-quasi-invariant for all $k \in \mathbb{N} \cup \{\infty\}$, via the characterization provided by (6.16). We must therefore prove that

$$Q_n^*(\mu_k) = 1_{U_n}\mu_k, \quad \forall n \in \mathbb{N}. \tag{7.4.1}$$

When $k = \infty$, this is provided for by (iii), and the fact that $W_\infty \subseteq U_n$, so it remains to prove (7.4.1) for $k \in \mathbb{N}$. Assuming first that $k < n$, we have that

$$Q_n^*(\mu_k) \overset{(6.15.\text{ii})}{=} Q_n^*(1_{Y_n}\mu_k) = Q_n^*(1_{Y_n}1_{W_k}\mu_k) = 0,$$

because $Y_n \cap W_k \subseteq U_n \cap W_k = \emptyset$, and likewise $1_{U_n}\mu_k = 0$. Assuming now that $n \leq k$, we have

$$Q_n^*(\mu_k) \overset{(\text{ii})}{=} Q_n^*\big(Q_k^*(\mu_k)\big) \overset{(6.15.\text{i})}{=} Q_k^*(\mu_k) \overset{(\text{ii})}{=} \mu_k = 1_{U_k}\mu_k.$$

Note that the above use of (ii) is not quite correct because it has only been assumed for $k \geq 1$. However, when $k = 0$, that condition holds trivially because Q_0^* is the identity transformation. This concludes the verification of (7.4.1), hence showing that μ_k is D-quasi-invariant.

Employing (7.2) we have that

$$\mu = 1_Z\mu + \sum_{k \in \mathbb{N} \cup \{\infty\}} 1_{W_k}\mu \overset{(\text{i})}{=} \sum_{k \in \mathbb{N} \cup \{\infty\}} \mu_k,$$

which is seen to be a D-quasi-invariant measure since each factor has this property, which in turn is clearly preserved under sums. $\quad\square$

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[6] Of course $Q_n^*(\mu_k) = 1_{U_n}\mu_k$, for all n, but so far we only need the case $n = k$.

Before we discuss the result above, let us introduce the notion of a measure living on a measurable set, which will be used from now on.

7.5. Definition. Given a measure space $(X, \mathcal{F}, \mu)$ and a set $M \in \mathcal{F}$. We say that μ *lives in* M when $\mu(X \backslash M) = 0$.

We explain the importance of using Definition (7.5) instead of the notion of measure *supported on a measurable set*: the concrete example that motivates this paper is the development of the thermodynamic formalism on *generalized countable Markov shifts* $X_A = \Sigma_A \cup Y_A$, started in [2], these spaces are the compactification of the countable Markov shift space Σ_A such that both, the space Σ_A, and its complement Y_A are dense in the whole space X_A. To characterize conformal measures that give mass for different parts of the space, we needed to abandon measures supported on sets, since these are closed sets. In section (10), we present examples of DLR measures living in different parts of the space as well, and some of these parts are dense.

Now, back to Theorem (7.4), observe that any measure μ on X which assigns zero mass to Z satisfies

$$\mu = \sum_{k \in \mathbb{N} \cup \{\infty\}} \mu_k, \tag{7.6}$$

where each measure μ_k lives in W_k (meaning that $\mu_k(X \backslash W_k) = 0$), namely $\mu_k = 1_{W_k} \mu$. Conversely, if we are given a collection $\{\mu_k\}_{k \in \mathbb{N} \cup \{\infty\}}$ of measures on X, such that μ_k lives in W_k, then (7.6) may be used to define a measure μ on X which assigns measure zero to Z. In other words, there is a one-to-one correspondence between the μ's and the collections $\{\mu_k\}_{k \in \mathbb{N} \cup \{\infty\}}$. This said, observe that the conditions characterizing a D-quasi-invariant measure in (7.4) consist of independent conditions on each "coordinate" μ_k.

In particular, if we fix any k in $\mathbb{N} \cup \{\infty\}$, and if we pick any measure μ_k living on W_k, and satisfying the corresponding condition, namely

(a) condition (7.4.ii), in case $k \geq 1$,
(b) condition (7.4.iii), in case $k = \infty$, or
(c) no condition at all, when $k = 0$,

then μ_k, itself, is a D-quasi-invariant measure. En passant, we note that any measure living in W_0 is automatically D-quasi-invariant. In any case, the existence question for quasi-invariantmeasures should be split into separate questions regarding the existence of quasi-invariantmeasures living in each W_k. In case k is finite, this question has a very simple answer:

7.7. Proposition. *Given an integer n, with $0 \leq n < \infty$, there exists a D-quasi-invariant probability measure living in W_n if and only if W_n is nonempty.*

Proof. Ignoring the blatantly obvious "only if" part, we deal only with the "if" part. Assuming that W_n is nonempty, choose any probability measure ν living in W_n, e.g. a Dirac measure based on any point chosen in W_n. Setting $\mu = Q_n^*(\nu)$, notice that

$$\mu(X) = \int_X 1 \, dQ_n^*(\nu) = \int_X Q_n(1) \, d\nu \overset{(6.11.\text{ii})}{=} \int_X 1_{Y_n} \, d\nu = \nu(Y_n) = 1,$$

where the last equality is a consequence of the fact that

$$W_n = V_n \backslash Z \subseteq U_n \backslash Z_n = Y_n.$$

This shows that μ is a probability measure.

Since W_n is R_n-invariant by (7.3), and since ν lives in W_n, then μ also lives in W_n by (6.15.iv). For that reason we have that $\mu_k := 1_{W_k}\mu = \delta_{nk}\mu$, for every k in $\mathbb{N}$. So, in order to prove that μ is D-quasi-invariant, we need only check (7.4.ii) for $k = n$, given that all of the other conditions hold trivially. To do this we compute

$$Q_n^*(\mu) = Q_n^*\big(Q_n^*(\nu)\big) \overset{(6.15.i)}{=} Q_n^*(\nu) = \mu,$$

proving that μ is D-quasi-invariant. $\square$

The existence question for quasi-invariantmeasures living in W_∞ is much more subtle, not least because (7.4.iii) involves infinitely many conditions. The following result might have excessively rigid hypotheses, but it is the best existence result we can offer in this generality:

7.8. Theorem. *Suppose that W_∞ contains a compact, R-invariant, nonempty subset K. Suppose also that $\zeta_n^{-1}{\restriction}_K$ is continuous[7] for every $n \in \mathbb{N}$. Then there exists a D-quasi-invariant probability measure living in K.*

Proof. We begin by observing that $W_\infty \subseteq Y_n$, for every $n \in \mathbb{N}$, because

$$W_\infty = V_\infty \backslash Z \subseteq U_n \backslash Z_n = Y_n.$$

Denote by $P(X, K)$ the set of all probability measures on X living in K. This is clearly a nonempty set given that it contains any Dirac measure δ_{x_0} with $x_0 \in K$. We claim that, for every n in $\mathbb{N}$, one has that

$$Q_n^*\big(P(X, K)\big) \subseteq P(X, K).$$

In fact, if ν is in $P(X, K)$, then

$$Q_n^*(\nu)(X) \overset{(6.15.iv)}{=} \nu(Y_n) = \nu(Y_n \cap K) = \nu(K) = 1,$$

so we see that $Q_n^*(\nu)$ is a probability measure. Moreover

$$Q_n^*(\nu)(X \backslash K) \overset{(6.15.iv)}{=} \nu\big((X \backslash K) \cap Y_n\big) \leq \nu(X \backslash K) = 0,$$

so $Q_n^*(\nu)$ lives in K. Identifying $P(X, K)$ with the set $P(K)$ of all probability measures on K via the correspondence

$$\mu \in P(X, K) \mapsto \mu{\restriction}_K \in P(K),$$

we claim that, for every g in $C(K)$, the function

$$\mu \in P(X, K) \mapsto \int_K g \, dQ_n^*(\mu) \in \mathbb{C}$$

is continuous on $P(K)$ relative to the topology induced by the weak* topology of the dual of $C(K)$.

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[7] Please note that by saying that $\zeta_n^{-1}{\restriction}_K$ is continuous we mean that it belongs to $C(K)$. This should not be confused with the much more stringent requirement that ζ_n^{-1} be continuous at all points of K.

To prove it we first use Tietze's extension Theorem to produce a continuous function f, defined on the whole of X, and whose restriction to X coincides with g. We further use Uhrysohn's Lemma to obtain a continuous function φ on X such that $\varphi\!\restriction_K = 1$, and whose support is compact and contained in U_n. Replacing f by φf we may then assume that the support of our originally chosen f is compact and contained in U_n. We then have

$$\int\limits_K g\, dQ_n^*(\mu) = \int\limits_X f\, dQ_n^*(\mu) = \int\limits_X Q_n(f)\, d\mu = \int\limits_X i_n\big(P_{\rho_n}(f\!\restriction_{U_n})\big)\, d\mu =$$

$$= \int\limits_{U_n} P_{\rho_n}(f\!\restriction_{U_n})\, d\mu\!\restriction_{U_n} = \int\limits_{U_n} E_n(f\!\restriction_{U_n}\rho_n\zeta_n^{-1})\, d\mu\!\restriction_{U_n} \overset{(3.4.\text{ii})}{=}$$

$$\overset{(3.4.\text{ii})}{=} \int\limits_{U_n} E_n(f\!\restriction_{U_n}\rho_n)\zeta_n^{-1}\, d\mu\!\restriction_{U_n} = \int\limits_K E_n(f\!\restriction_{U_n}\rho_n)\zeta_n^{-1}\, d\mu.$$

Recalling that $E_n(f\!\restriction_{U_n}\rho_n)$ is continuous by (3.3), and that ζ_n^{-1} is continuous on K by hypothesis, the claim follows.

We will next prove that the fixed points of Q_n^* in $P(X, K)$ form a closed subset. To see this let μ be any measure in $P(X, K)$. By the uniqueness part of the Riesz-Markov, to say that $Q_n^*(\mu) = \mu$ is to say that

$$\int\limits_K g\, d\mu = \int\limits_K g\, dQ_n^*(\mu), \quad \forall\, g \in C(K). \tag{7.8.1}$$

Viewed as functions of the variable μ, both sides of the above expression are now known to be weak*-continuous, so the set of solutions to this system on equations, as g ranges in $C(K)$, form a closed set, hence proving that the set of fixed points of Q_n^* in $P(X, K)$ is indeed weak*-closed.

Choosing any ν in $P(X, K)$, we define

$$\mu_n = Q_n^*(\nu),$$

for every n in $\mathbb{N}$. Using Alaoglu's Theorem we may then find a limit point, say

$$\mu_\infty \in P(X, K),$$

for the sequence $\{\mu_n\}_n$. Given two integers n and m, with $0 \leq n \leq m$, observe that

$$Q_n^*(\mu_m) = Q_n^*\big(Q_m^*(\nu)\big) \overset{(6.15.\text{v})}{=} Q_m^*(\nu) = \mu_m,$$

so we see that all but finitely many μ_m's are fixed points for Q_n^*, hence the same holds for μ_∞, thanks to the closedness of the set fixed points just proved.

Observing that μ_∞ lives in K, and that $K \subseteq U_n$, for every n, we then have that

$$Q_n^*(\mu) = \mu = 1_{U_n}\mu,$$

so μ_∞ is D-quasi-invariant by (6.16). $\quad\square$

8. Renault-Deaconu groupoids

In this section we will describe an example of GAP coming from generalized Renault-Deaconu groupoids [10]. This is in fact the main motivation for introducing and studying GAPs.

We will henceforth fix a space X satisfying (2.1) and we will suppose we are given an open subset $U \subseteq X$, and a map

$$\sigma : U \to X,$$

which we will assume to be a local homeomorphism.

Let $U_0 = X$, and for each $n \geq 1$, put

$$U_n = \{x \in U : \sigma(x) \in U_{n-1}\}.$$

It is then easy to see that U_n is effectively the domain of σ^n, and that the U_n form a collection of open subsets of X satisfying (5.1.i).

The generalized Renault-Deaconu groupoid $\mathcal{G}_\sigma$ associated to σ was defined by Renault in [10], and it consists of all triples (x, n, y) in $X \times \mathbb{Z} \times X$, such that there exist $k, l \in \mathbb{N}$, satisfying $n = k - l$, $x \in U_k$, $y \in U_l$, and $\sigma^k(x) = \sigma^l(y)$.

The multiplication of two elements (x, n, y) and (z, m, w) in $\mathcal{G}_\sigma$ is defined only when $y = z$, in which case the product is set to be $(x, n + m, w)$. The topology of $\mathcal{G}_\sigma$ is generated by the collection of subsets

$$W_{k,l,A,B} = \{(x, k - l, y) : x \in A, \ y \in B, \ \sigma^k(x) = \sigma^l(y)\},$$

for all $k, l \in \mathbb{N}$, and all open subsets $A \subseteq U_k$, and $B \subseteq U_l$. With this structure $\mathcal{G}_\sigma$ becomes an étale groupoid and we refer the reader to [10] for more on $\mathcal{G}_\sigma$.

Let us next consider, for each $n \in \mathbb{N}$, the subset R_n of $U_n \times U_n$ defined by

$$R_n = \{(x, y) \in U_n \times U_n : \sigma^n(x) = \sigma^n(y)\}.$$

Recalling that σ is a local homeomorphism, it is clear that σ^n is a local homeomorphism from U_n to X, so R_n is easily seen to be a proper equivalence relation on U_n.

8.1. Proposition. *One has that*

$$\mathcal{R} = \Big(\{U_n\}_{n \in \mathbb{N}}, \{R_n\}_{n \in \mathbb{N}}\Big),$$

is a GAP on X.

Proof. All we need at this point is to verify (5.1.iii). So, suppose that $n \leq m$, and let $(x, y) \in R_n \cap (U_n \times U_m)$. The first conclusion to be drawn is that both x and y lie in U_n, and that $\sigma^n(x) = \sigma^n(y)$. Besides, y lies in U_m, so y is also in the domain of σ^m. This implies in particular that $\sigma^n(y)$ lies in the domain of σ^{m-n}, so the same holds for $\sigma^n(x)$. Consequently

$$\sigma^m(y) = \sigma^{m-n}\big(\sigma^n(y)\big) = \sigma^{m-n}\big(\sigma^n(x)\big) = \sigma^m(x),$$

thus showing that $(x, y) \in R_m$.   □

The Renault-Deaconu groupoid admits a continuous cocycle

$$\delta : (x, n, y) \in \mathcal{G}_\sigma \mapsto n \in \mathbb{Z},$$

whose kernel is therefore an open subgroupoid, which is clearly isomorphic and homeomorphic to the GAP groupoid $R = \bigcup_n R_n$, via the homeomorphism sending each (x, y) in R to $(x, 0, y)$.

In order to speak of quasi-invariantmeasures on $\mathcal{G}_\sigma$, we need to introduce a real-valued cocycle. Recall from [2] that if $h : U \to \mathbb{R}$ is a continuous function, often thought of as a *potential* function, one may define

$$b(x, n - m, y) = \sum_{i=0}^{n-1} h\big(\sigma^i(x)\big) - \sum_{i=0}^{m-1} h\big(\sigma^i(y)\big), \tag{8.2}$$

whenever $\sigma^n(x) = \sigma^m(y)$, obtaining in this way a well defined continuous cocycle on $\mathcal{G}_\sigma$, taking values in the additive group of real numbers.

The restriction of c to the subgroupoid R is then evidently a continuous cocycle on R, but we would instead like to introduce it from a different point of view, in line with (6.4). For each $n \geq 1$, let us define

$$k_n(x) = h\big(\sigma^{n-1}(x)\big), \quad \forall\, x \in U_n.$$

Notice that, for x in U_n, the largest integer i for which the expression $h\big(\sigma^i(x)\big)$ is guaranteed to be well defined is $i = n - 1$. This is because $U_n = \mathrm{dom}(\sigma^n)$, so $\sigma^n(x)$ is well defined, which in turn implies that $\sigma^{n-1}(x)$ is in U, and hence it does makes sense to apply h to $\sigma^{n-1}(x)$. However, there is no reason for $\sigma^n(x)$ to lie in U, so $h(\sigma^n(x))$ may not be well defined.

8.3. Proposition. *The collection $\{k_n\}_{n \geq 1}$ defined above is a potential for $\mathcal{R}$, in the sense of (6.1).*

Proof. To verify (6.1.1), let $n \geq 1$ and choose $(x, y) \in R_{n-1} \cap (U_n \times U_n)$. Then $\sigma^{n-1}(x) = \sigma^{n-1}(y)$, whence $k_n(x) = k_n(y)$, as needed. □

The associated cocycle is then given on any $(x, y) \in R_n$, by

$$c(x, y) = c_n(x, y) = h_n(x) - h_n(y) = \sum_{i=1}^{n} k_n(x) - k_n(y) =$$

$$= \sum_{i=1}^{n} h\big(\sigma^{i-1}(x)\big) - h\big(\sigma^{i-1}(y)\big) = \sum_{i=0}^{n-1} h\big(\sigma^i(x)\big) - h\big(\sigma^i(y)\big), \tag{8.4}$$

which happens to be the restriction to R of the cocycle b defined in (8.2).

Observe that both the unit space of $\mathcal{G}_\sigma$ and that of R may be naturally identified with X. So a given finite measure μ on X may be tested for quasi-invariance either relatively to the cocycle e^b on $\mathcal{G}_\sigma$, or to the cocycle e^c on R.

8.5. Definition. Let μ be a finite measure on X. We will say that μ is

 (i) a *conformal* measure when it is quasi-invariantrelatively to the cocycle e^b on $\mathcal{G}_\sigma$,
(ii) a *DLR* measure when it is quasi-invariantrelatively to the cocycle e^c on R.

Since R is a subgroupoid of $\mathcal{G}_\sigma$, and since e^c is the restriction of e^b to R, it is immediate that:

8.6. Proposition. *Every conformal measure on X is a DLR measure.*

9. Eigenmeasures

As in the previous section we let X be a space satisfying (2.1), $U \subseteq X$ be an open set, and $\sigma : U \to X$ be a local homeomorphism. Our goal here is to show that every eigenmeasure for Ruelle's operator is a DLR measure.

For each f in $\mathcal{M}^+(U)$, and for every x in X, define

$$L(f)\big|_x = \sum_{t \in \sigma^{-1}(x)} f(t),$$

so that L becomes a map

$$L : \mathcal{M}^+(U) \to \mathcal{M}^+(X).$$

Regarding the expression defining $L(f)$ above, observe that if x is not in $\sigma(U)$, then $\sigma^{-1}(x)$ is the empty set, whence there are no summands at all, hence the sum turns out to be zero. In other words, $L(f)$ vanishes outside $\sigma(U)$.

We also consider the operator

$$\alpha : \mathcal{M}^+(X) \to \mathcal{M}^+(U),$$

given by $\alpha(f) = f \circ \sigma$. We finally define

$$E : \mathcal{M}^+(U) \to \mathcal{M}^+(U),$$

by

$$E(f)\big|_x = \sum_{\sigma(t)=\sigma(x)} f(t), \quad \forall\, x \in U.$$

For f in $\mathcal{M}^+(U)$ and any x in U, observe that,

$$\alpha\big(L(f)\big)\big|_x = L(f)\big|_{\sigma(x)} = \sum_{t \in \sigma^{-1}(\sigma(x))} f(t) = \sum_{\sigma(t)=\sigma(x)} f(t) = E(f)\big|_x,$$

so we see that

$$\alpha \circ L = E. \tag{9.1}$$

Another useful property is

$$L(\alpha(g)f) = gL(f), \quad \forall\, g \in \mathcal{M}^+(X), \quad \forall\, f \in \mathcal{M}^+(U), \tag{9.2}$$

which the reader will have no difficulty in checking.

We shall also fix a continuous function

$$\rho : U \to \mathbb{R},$$

satisfying $\rho(x) > 0$, for all x in U. Here ρ is supposed to play the role of e^h, where h is the function used for creating the cocycle b in (8.2). The operator

$$L_\rho : \mathcal{M}^+(U) \to \mathcal{M}^+(X),$$

defined by the formula $L_\rho(f) = L(\rho f)$, is then the analogue of Ruelle's operator in the present situation.

9.3. Lemma. *Suppose that*

(i) *μ is a measure on X,*
(ii) *λ is a nonzero scalar,*

such that

$$\int_X L(\rho f)\, d\mu = \lambda \int_U f\, d\mu, \quad \forall f \in \mathcal{M}^+(U). \tag{9.3.1}$$

Then

$$\int_U E(\rho) f\, d\mu = \int_U E(\rho f)\, d\mu, \quad \forall f \in \mathcal{M}^+(U). \tag{9.3.2}$$

Proof. Given g in $\mathcal{M}^+(X)$, plug $f = \alpha(g)$ in (9.3.1), to get

$$\int_U \alpha(g)\, d\mu = \lambda^{-1} \int_X L\big(\alpha(g)\rho\big)\, d\mu \overset{(9.2)}{=} \lambda^{-1} \int_X g L(\rho)\, d\mu. \tag{9.3.3}$$

Working from the right-hand-side of (9.3.2), we have

$$\int_U E(\rho f)\, d\mu \overset{(9.1)}{=} \int_U \alpha\big(L(\rho f)\big)\, d\mu \overset{(9.3.3)}{=} \lambda^{-1} \int_X L(\rho f) L(\rho)\, d\mu \overset{(9.2)}{=}$$

$$\overset{(9.2)}{=} \lambda^{-1} \int_X L\big(\rho f \alpha(L(\rho))\big)\, d\mu \overset{(9.1)}{=} \lambda^{-1} \int_X L\big(\rho f E(\rho)\big)\, d\mu \overset{(9.3.1)}{=} \int_U f E(\rho)\, d\mu. \quad \square$$

For each $n \geq 1$, let U_n be the domain of σ^n, so that the map

$$\sigma^n : U_n \to X,$$

is a local homeomorphism so it could be used in place of σ in order to define all of the above ingredients. To be precise these are:

- $L_n : \mathcal{M}^+(U_n) \to \mathcal{M}^+(X)$, given by $\quad L_n(f)\big|_x = \displaystyle\sum_{t \in \sigma^{-n}(x)} f(t),$
- $\alpha_n : \mathcal{M}^+(X) \to \mathcal{M}^+(U_n)$, given by $\quad \alpha_n(f) = f \circ \sigma^n,$
- $E_n : \mathcal{M}^+(U_n) \to \mathcal{M}^+(U_n)$, given by $\quad E_n(f)\big|_x = \displaystyle\sum_{\sigma^n(t) = \sigma^n(x)} f(t).$

We shall also let

$$\rho_n = \rho\, \alpha\big(\rho\big) \alpha_2(\rho) \cdots \alpha_{n-1}(\rho).$$

Regarding the product defining ρ_n above, observe that each $\alpha_i(\rho)$ is a member of $\mathcal{M}^+(U_{i+1})$, so they all may be restricted to U_n before the multiplication is performed, resulting of course in a member of $\mathcal{M}^+(U_n)$.

9.4. Proposition. *For every $n, m \geq 0$, one has that*

(i) $L_n\big(\mathcal{M}^+(U_{n+m})\big) \subseteq \mathcal{M}^+(U_m)$, *and*

(ii) $L_m \circ L_n = L_{n+m}$.

Proof. Given f in $\mathcal{M}^+(U_{n+m})$, and x in $X \setminus U_m$, we must prove that $L_n(f)\big|_x = 0$. Arguing by contradiction, suppose this is not so. Therefore there exists at least one $t \in \sigma^{-n}(x)$, such that $f(t) \neq 0$, so it follows that

$$x = \sigma^n(t) \in \sigma^n(U_{n+m}) \subseteq U_m,$$

a contradiction, proving (i). In order to prove (ii), pick f in $\mathcal{M}^+(U_{n+m})$, and x in X. We then have

$$L_m\big(L_n(f)\big)\big|_x = \sum_{t \in \sigma^{-m}(x)} L_n(f)\big|_t = \sum_{t \in \sigma^{-m}(x)} \sum_{s \in \sigma^{-n}(t)} f(s) = \sum_{t \in \sigma^{-n-m}(x)} f(s) = L_{n+m}(f)\big|_x. \quad \square$$

9.5. Lemma. *Let μ be a measure on X satisfying (9.3.1). Then*

$$\int_X L_n(\rho_n f)\, d\mu = \lambda^n \int_{U_n} f\, d\mu, \quad \forall f \in \mathcal{M}^+(U_n).$$

Proof. For f in $\mathcal{M}^+(U_n)$, we have by induction that

$$\int_X L_n(\rho_n f)\, d\mu = \int_X L_{n-1}\big(L(\rho\alpha(\rho_{n-1})f)\big)\, d\mu = \int_X L_{n-1}\big(\rho_{n-1}L(\rho f)\big)\, d\mu =$$

$$= \lambda^{n-1} \int_{U_{n-1}} L(\rho f)\, d\mu = \lambda^{n-1} \int_U L(\rho f)\, d\mu \overset{(9.3.1)}{=} \lambda^n \int_U f\, d\mu = \lambda^n \int_{U_n} f\, d\mu. \quad \square$$

9.6. Corollary. *Let μ be a measure on X satisfying (9.3.1). Then for any n in $\mathbb{N}$, one has that*

$$\int_{U_n} E_n(\rho_n) f\, d\mu = \int_{U_n} E_n(\rho_n f)\, d\mu, \quad \forall f \in \mathcal{M}^+(U_n).$$

Proof. Follows immediately by applying (9.3) to σ^n and ρ_n. $\quad \square$

Given a continuous function $h : U \to \mathbb{R}$, let $\rho = e^h$, and recall from (8.5) that a finite measure μ on X is said to be a DLR measure for h if it is quasi-invariantrelative to the cocycle e^c on R, where c is given in terms of h by (8.4). The cocycle e^c is in fact a common extension of the cocycles e^{c_n} defined on each R_n by

$$e^{c_n(x,y)} = \exp\Big(\sum_{i=0}^{n-1} h\big(\sigma^i(x)\big) - h\big(\sigma^i(y)\big)\Big) = \frac{\rho(x)\rho(\sigma(x))\cdots\rho(\sigma^{n-1}(x))}{\rho(y)\rho(\sigma(y))\cdots\rho(\sigma^{n-1}(y))} = \frac{\rho_n(x)}{\rho_n(y)}.$$

We then see that, if μ is a finite measure satisfying the conclusions of (9.6), then $\mu{\restriction}_{U_n}$ satisfies (4.11.iii) relative to ρ_n, so it is quasi-invariantfor e^{c_n} by (4.11). Employing (6.7) it then follows that μ is e^c-quasi-invariant, hence a DLR measure. Summarizing we obtain the following:

9.7. Theorem. *Let X be a locally compact metric space, U be an open subset of X, and $\sigma : U \to X$ be a local homeomorphism. Choosing any continuous potential $h : U \to \mathbb{R}$, let $\rho = e^h$. Then any finite measure on X which is an eigenvalue for the corresponding Ruelle operator, meaning that it satisfies (9.3.1) with a nonzero eigenvalue λ, is necessarily a DLR measure.*

Proof. Follows immediately by applying (9.6) to σ^n and ρ_n. $\quad\square$

It should be noted that Corollary (9.6) may be seen as a generalization of Theorem (9.7) to measures which are not necessarily finite, as long as we redefine the notion of DLR measures as those which satisfy the conclusions of (9.6). However, since our theory of DLR measures was developed only for finite measures, the various equivalent conditions for a measure to be DLR have not been proved here for infinite measures.

10. Generalized countable Markov shift spaces and KMS states

This section is dedicated to showing how an explicit application of the theory of GAPs and the DLR measures developed in this work. In [2], we started the construction of the thermodynamic formalism for generalized Markov shift spaces. These are compactifications of the usual Markov shift spaces, realized by adding some finite words, that sometimes can have multiplicity. For simplicity, the major example in this section is the renewal shift, but we emphasize that our results for DLR measures can be used in full generality on the transitive shift spaces. In particular, the generalized renewal shift does not present a multiplicity of extra words, that is, there are not two different points related to the same word, and then we can represent them by their respective stems only, for details see [2] and [8].

Let A be the $\mathbb{N}_* \times \mathbb{N}_*$ transition matrix, where $\mathbb{N}_* = \mathbb{N} \setminus \{0\}$ given by

$$A_{i,j} = \begin{cases} 1, & \text{if } i = 1 \text{ or } i = j + 1, \\ 0, & \text{otherwise.} \end{cases}$$

The generalized renewal shift space is the compact Polish space $X_A = \Sigma_A \cup Y_A$, where

$$\Sigma_A = \{x \in \mathbb{N}_*^{\mathbb{N}} : A_{x_i, x_{i+1}} = 1 \text{ for every } i \in \mathbb{N}\}$$

is the usual renewal shift, and

$$Y_A = \{x = x_0 x_1 \dots x_{n-1} : n \in \mathbb{N}_*, x_{n-1} = 1, A_{x_i, x_{i+1}} = 1 \text{ for every } 0 \le i \le n-2\} \cup \{\xi^0\}$$

is the set of finite words, where ξ^0 is the *empty word*. The *length* of an element $x \in X_A$, denoted by $|x| \in \mathbb{N} \cup \{\infty\}$, denotes the number of letters of x, and $|\xi^0| = 0$ by default. Both Σ_A and Y_A are dense in X_A. The shift map is a local homomorphism $\sigma : U \to X_A$, where $U = X_A \setminus \{\xi^0\}$ is an open set. The GAP on X_A is given as follows:

(i) $U_0 = X_A$, $U_n = \{x \in X_A : n \le |x|\} = \sigma^{-n}(\xi^0)$.
(ii) $R_0 = \text{diag } X_A \times X_A$, $R_n = \{(x, y) \in X_A \times X_A : n \le |x| = |y|, \sigma^n(x) = \sigma^n((y)\}$.
  In addition, the sets introduced at the beginning of section (7) are given by
(iii) $V_n = \{x \in X_A : |x| = n\}$, $V_\infty = \Sigma_A$, $Z_n = Z = \emptyset$, $W_n = V_n$.

Given a potential $h : U \to \mathbb{R}$, the sets Z_n (and consequently Z) above are empty because $R_n(x)$ is finite for every $x \in U_n$ and $n \in \mathbb{N}$. The next lemma is a property of X_A that will be used in item (b) of (10.2).

10.1. Lemma. *For every $n > 0$,*

$$\sum_{y \in W_n} y_0 < 2^n.$$

Proof. We prove by induction. The case $n = 1$ is straightforward, and Proposition 4.73 of [8] gives $|W_n| = 2^{n-1}$ for every $n > 0$. We assume that the result holds for n, and in addition observe that $y \in W_{n+1}$ if and only if $y = (y_0' + 1)y'$ or $y = 1y'$ for some $y' \in W_n$. Then,

$$\sum_{y \in W_{n+1}} y_0 = \sum_{y \in W_n} 1 + \sum_{y \in W_n} (1 + y_0) = 2^n + \sum_{y \in W_n} y_0 < 2^{n+1}. \quad \square$$

Again, just for matters of simplicity, we fix $h \equiv 1$ and present several different examples of DLR measures.

10.2. Examples.

(a) *Eigenmeasure living in Y_A.* Although the Ruelle's operator in [2] is defined on compactly supported functions on U, the formulations of eigenmeasure in there and here are equivalent via equivalence to conformality. The latter notion only involves the measure-theoretic aspect of the theory. By Theorem 90 of [2], a Borel probability μ is an eigenmeasure for the Ruelle's operator L_{e^h} and eigenvalue $\lambda > 0$ if and only if μ is $e^{\log \lambda - F}$-conformal (sense of Denker-Urbański). Moreover, identity (6.50) from the same work above states that such a probability exists and lives on Y_A if and only if the series

$$\sum_{n \in \mathbb{N}} \sum_{x \in Y_A, |x| = n} e^{-n \log \lambda + h_n(x)} = \sum_{n \in \mathbb{N}} e^{-n(\log \lambda - \log 2 - 1)}$$

converges, i.e., $\lambda > 2e$. By Theorem (9.7), the eigenmeasure is a DLR measure.

(b) *DLR measure living on a proper subset of Y_A that is not an eigenmeasure.* By the proof of Proposition (7.7), for fixed $n > 0$, take $\nu_n = \delta_{n,n-1,\dots,1}$, then the measure $\mu_n = Q_n^*(\nu_n)$ is a DLR measure. Note that, given $\lambda > 0$,

$$\lambda \int f d\mu_n = \lambda \int Q_n(f) d\nu_n = \lambda Q_n(f)|_{n,n-1,\dots,1},$$

$$\int L_{e^h}(f) d\mu_n = \int Q_n(L_{e^h}(f)) d\nu_n = Q_n(L_{e^h}(f))|_{n,n-1,\dots,1}.$$

Then, a necessary condition for μ be an eigenmeasure for the eigenvalue λ is the following: for every $f > 0$ (and hence $L_{e^h}(f) > 0$), we must have

$$\frac{\lambda Q_n(f)|_{n,n-1,\dots,1}}{Q_n(L_{e^h}(f))|_{n,n-1,\dots,1}} = 1.$$

By exchanging h by 1, one gets

$$Q_n(f)|_{n,n-1,\dots,1} = 2^{-(n-1)} \sum_{y \in W_n} f(y),$$

$$Q_n(L_e(f))|_{n,n-1,\dots,1} = 2^{-(n-1)} e \sum_{y \in W_n} [f(1,y) + f(y_0 + 1, y)].$$

By setting $f \equiv 1$, we have

$$1 = \frac{\lambda Q_n(f)|_{n,n-1,\dots,1}}{Q_n(L_{e^h}(f))|_{n,n-1,\dots,1}} = \frac{\lambda}{2e},$$

and then, $\frac{\lambda}{2e} = 1$. On the other hand, if we set $f(x) = x_0$, we obtain

$$1 = \frac{\lambda Q_n(f)|_{n,n-1,\dots,1}}{Q_n(L_{e^h}(f))|_{n,n-1,\dots,1}} = \frac{\lambda \sum_{y \in W_n} y_0}{e \sum_{y \in W_n}(y_0 + 2)} = \frac{2}{\frac{2^n}{\sum_{y \in W_n} y_0} + 1}, \tag{10.3}$$

where we used that $|W_n| = 2^{n-1}$ (see Proposition 4.73 of [8]). By Lemma (10.1), the right-handed side of (10.3) satisfies

$$1 = \frac{2}{\frac{2^n}{\sum_{y \in W_n} y_0} + 1} < 1,$$

a contradiction, and then μ_n is not an eigenmeasure, for any $\lambda > 0$.

(c) *DLR measure living on the whole Y_A that is not an eigenmeasure.* For each $n > 0$, let μ_n as in (b). The Borel probability measure

$$\mu = \sum_{n=1}^{\infty} 2^{-n} \mu_n$$

is a DLR measure and it is not an eigenmeasure for every $\lambda > 0$.

As presented above, we have examples of DLR measures that are not eigenmeasures for any eigenvalue. This phenomenon also occurs for the standard countable Marov shifts: in [1, Example 30] is presented a case for the *reverse renewal shift*, whose transition matrix is the transpose of A above. For every Walters' potential with finite first variation and finite Gurevich pressure, the probability measure $\mu = \delta_{\overline{x}}$, $\overline{x} = (1, 2, 3, 4, \dots)$, is a DLR measure which is not an eigenmeasure for every eigenvalue. In this case, the countable shift space Σ_{A^T} is locally compact, and then its respective generalized shift space X_{A^T} coincides with Σ_{A^T} because of [2, Proposition 32], and then the DLR measure lives in Σ_{A^T}.

Finally, we can consider continuous cocycles, for instance, those which arise from a continuous potential $h : U \to \mathbb{R}$ as in section (8), here denoted by $c : R \to \mathbb{R}$, one gets a KMS state on the groupoid C*-algebra $C^*(R)$ from each DLR measure, via the cocycle dynamics for the 1-parameter group of automorphisms $\{\tau_t\}_{t \in \mathbb{R}}$ where

$$\tau_t(f)(g) = e^{itc(g)} f(g), \quad \forall f \in C_c(R).$$

In this case, the groupoid $c^{-1}(0)$ is principal, and by [12, Theorem 3.3.12], the KMS states for the C*-dynamical system above are in the form $\varphi = \varphi_\mu$, where μ is a DLR measure, and it is given by

$$\varphi(f) = \int_{X_A} h(x, x) d\mu(x), \quad \forall h \in C_c(R),$$

where we identified the set of units of R with X_A, via the homeomorphism $x \mapsto (x, x)$.

Acknowledgments

Appendix A. Elementary remarks about measure theory

In sections 8 and 9 of this work, we make some elementary remarks, mainly to fix our notation. By a measurable space we shall mean a pair $(X, \mathcal{B})$, consisting of a nonempty set X, and a σ-algebra $\mathcal{B}$ of subsets of X.

A.1. Definition. Given a measurable space $(X, \mathcal{B})$, we shall denote by $\mathcal{M}^+(X, \mathcal{B})$ the set of all $\mathcal{B}$-measurable functions

$$f : X \to [0, +\infty].$$

A.2. Definition. Given measurable spaces $(X, \mathcal{B})$ and $(Y, \mathcal{C})$, a positively homogeneous map

$$T : \mathcal{M}^+(X, \mathcal{B}) \to \mathcal{M}^+(Y, \mathcal{C})$$

is said to be *σ-additive* if for any sequence $\{f_n\}_n$ in $\mathcal{M}^+(X, \mathcal{B})$, we have that

$$T\Big(\sum_{n=1}^{\infty} f_n\Big) = \sum_{n=1}^{\infty} T(f_n),$$

all sums being interpreted as pointwise sums.

A.3. Proposition. *Let $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ be measurable spaces, and let*

$$T : \mathcal{M}^+(X, \mathcal{B}) \to \mathcal{M}^+(Y, \mathcal{C})$$

be a σ-additive map. Then:

(i) *If $\{f_n\}_n$ is a non-decreasing sequence of functions in $\mathcal{M}^+(X, \mathcal{B})$, then*

$$T\big(\lim_{n \to \infty} f_n\big) = \lim_{n \to \infty} T(f_n),$$

   all limits being interpreted as pointwise limits.
(ii) *For every measure[8] ν on $(Y, \mathcal{C})$, there exists a measure $T^*(\nu)$ on $(X, \mathcal{B})$, such that*

$$\int_X f \, dT^*(\nu) = \int_Y T(f) \, d\nu, \quad \forall f \in \mathcal{M}^+(X, \mathcal{B}).$$

Proof. Regarding (i), define $g_1 = f_1$, and for each $n \geq 2$, define

$$g_n = f_n(x) - f_{n-1}(x), \quad \forall x \in X.$$

Observe that, in case $f_{n-1}(x) = \infty$, then necessarily also $f_n(x) = \infty$, in which case we adopt the convention according to which $g_n(x) = 0$. The functions g_n so defined are then $\mathcal{B}$-measurable and non-negative, and we have that $f_{n-1} + g_n = f_n$. Consequently $f_n = \sum_{i=1}^{n} g_i$, so

$$T\big(\lim_{n \to \infty} f_n\big) = T\Big(\sum_{i=1}^{\infty} g_i\Big) = \sum_{i=1}^{\infty} T(g_i) = \lim_{n \to \infty} \sum_{i=1}^{n} T(g_i) = \lim_{n \to \infty} T(f_n),$$

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[8] All measures in this work are assumed to be σ-additive and positive.

proving (i). In order to prove (ii), for every f in $\mathcal{M}^+(X, \mathcal{B})$, define

$$I(f) = \int_X T(f)\, d\nu.$$

We then claim that, given any sequence $\{f_n\}_n$ in $\mathcal{M}^+(X, \mathcal{B})$, one has that

$$I\left(\sum_{n=1}^{\infty} f_n\right) = \sum_{n=1}^{\infty} I(f_n).$$

This is a consequence of the σ-additivity of T and the monotone convergence Theorem, as follows:

$$I\left(\sum_{n=1}^{\infty} f_n\right) = \int_X T\left(\sum_{n=1}^{\infty} f_n\right) d\nu = \int_X \sum_{n=1}^{\infty} T(f_n)\, d\nu = \sum_{n=1}^{\infty} \int_X T(f_n)\, d\nu = \sum_{n=1}^{\infty} I(f_n),$$

thus proving the claim. Defining $\mu(E) = I(1_E)$, for every E in $\mathcal{B}$, it then follows that μ is a σ-additive measure on X, and clearly

$$I(f) = \int_X f\, d\mu, \qquad (A.3.1)$$

for every simple function f in $\mathcal{M}^+(X, \mathcal{B})$.

We then claim that $(A.3.1)$ holds for every f in $\mathcal{M}^+(X, \mathcal{B})$. To see this, recall that every such f may be written as the pointwise limit of a non-decreasing sequence $\{s_n\}_n$ of simple functions in $\mathcal{M}^+(X, \mathcal{B})$ [13, Section 18.1]. So

$$I(f) = \int_X T\left(\lim_{n\to\infty} s_n\right) d\nu \overset{(i)}{=} \int_X \lim_{n\to\infty} T(s_n)\, d\nu \overset{(*)}{=} \lim_{n\to\infty} \int_X T(s_n)\, d\nu =$$

$$= \lim_{n\to\infty} I(s_n) = \lim_{n\to\infty} \int_X s_n\, d\mu = \int_X f\, d\mu.$$

In time, we observe that $(*)$, above, is justified by the monotone convergence Theorem and the easily proven fact that $\{T(s_n)\}_n$ is a non-decreasing sequence.

This proves our claim and we then have for every f in $\mathcal{M}^+(X, \mathcal{B})$ that

$$\int_X f\, d\mu = I(f) = \int_X T(f)\, d\nu,$$

so it is enough to put $T^*(\nu) := \mu$. $\square$

Before closing this section let us comment on two related notions which will be used often.

A.4. Remark. In this work we shall consider two similar, but inequivalent, ways of restricting a measure on a space X to a Borel subset $Y \subseteq X$. The first one, officially called the *restriction of μ to Y*, consists of the measure denoted $\mu{\upharpoonright}_Y$, defined on the σ-algebra $\mathcal{B}(Y)$ of all Borel subsets of Y, by

$$\mu{\upharpoonright}_Y(A) = \mu(A), \quad \forall A \in \mathcal{B}(Y).$$

The second one, which we will denote by $1_Y\mu$, is nothing but the well-known measure obtained by multiplying the measurable function 1_Y by the measure μ. Recall that the domain of $1_Y\mu$ is still $\mathcal{B}(X)$, as opposed to $\mathcal{B}(Y)$, and

$$1_Y\mu(A) = \int_A 1_Y\,d\mu = \mu(A \cap Y),$$

for all $A \in \mathcal{B}(X)$.

Appendix B. Elementary remarks about measure theory and topology

B.1. Proposition. *Every space X satisfying (2.1) is σ-compact.*

Proof. For every x in X, let U_x be a relatively compact, open neighborhood of x. By reducing U_x a bit we may assume that it belongs to some previously chosen countable base B of open sets for X. It then follows that

$$\{\overline{U}_x : x \in X\}$$

is a family of compact sets covering X. This family is countable (even though it might be indexed on an uncountable set) because the U_x's belong to the countable base B. $\square$

The reason for restricting ourselves to (2.1) is to simplify some aspects of measure theory. In this short section, we will explain exactly what we mean by this.

Recall that the *Borel σ-algebra*, denoted $\mathcal{B}(X)$, is the σ-algebra of subsets of X generated by the closed subsets. On the other hand, the *Baire σ-algebra* [13, 21.6], denoted $\mathcal{B}a(X)$, is the smallest σ-algebra of subsets of X for which the functions in $C_c(X)$ are measurable.

If one is interested in the measurability properties of none other than continuous, compactly supported functions, the Baire σ-algebra is the most appropriate one to be considered. The Baire σ-algebra is known to be generated by the compact G_δ subsets of X [13, Theorem 21.21].

The first advantage of working with (2.1) comes from the fact that X is metrizable and we have the following:

B.2. Lemma *(See [4, Corollary 6.3.5]). Suppose that X is as in (2.1). Then the Baire and Borel σ-algebras on X coincide.*

A measure μ defined on $\mathcal{B}(X)$ is called a *Borel measure* when it assigns finite measure to every compact set [13, Section 21.3]. If μ is instead defined on $\mathcal{B}a(X)$, it is called a *Baire measure* provided it is finite on compact (Baire-measurable) sets [13, Section 21.6].

One of the main applications of Borel measures in Analysis is the Riesz-Markov Theorem [13, Section 21.6] which states that each positive linear functional on $C_c(X)$ is given by the integration against a unique *regular* Borel measure (a regular Borel measure is also called a *Radon* measure).

Another reason to work under (2.1) is that, in this case, every Baire measure is regular [13, Theorem 21.27]. Since we now know that the Baire and Borel σ-algebras coincide, we deduce that:

B.3. Lemma *(See also [13, Theorem 21.20]). Under (2.1), every Borel measure on $\mathcal{B}(X)$ is regular.*

B.4. Proposition. *Let μ be any measure on $\mathcal{B}(X)$. Then μ is a Borel measure if and only if*

$$\int_X f \, d\mu < \infty,$$

for all f in $C_c^+(X)$.

Proof. We verify only the "if" part. For this, let K be any compact subset of X. Using local compactness one may find a relatively compact, open set U such that $K \subseteq U$. By Uryshon's Theorem let $f : X \to [0, 1]$ be a continuous function vanishing off U, and such that $f = 1$ on K. If follows that $f \in C_c^+(X)$, whence

$$\mu(K) = \int_X 1_K \, d\mu \le \int_X f \, d\mu < \infty.$$

Being finite on compact sets, μ is a Borel measure. $\quad\square$

B.5. Proposition. *There exists a sequence $\{\varphi_n\}_n$ of continuous, compactly supported functions*

$$\varphi_n : X \to [0, 1],$$

such that $\varphi_n \le \varphi_{n+1}$, for every n, and $\lim_{n\to\infty} \varphi_n = 1$, pointwise.

Proof. Let $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of compact subsets of X such that

$$K_n \subseteq \operatorname{int}(K_{n+1}), \quad \text{and} \quad X = \bigcup_n K_n.$$

Using Uryhson, for each $n \in \mathbb{N}$, let

$$\varphi_n : X \to [0, 1]$$

be a continuous function with $\varphi_n = 1$ on K_n and $\varphi_n = 0$ on $X \setminus \operatorname{int}(K_{n+1})$. It is then easy to see that $\varphi_n \le \varphi_{n+1}$, and that $\{\varphi_n\}_n$ converges pointwise to 1 on X. $\quad\square$

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