A CONJECTURE OF H.J. ZASSENHAUS ABOUT UNITS OF FINITE ORDER IN INTEGRAL GROUP RINGS

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§1. Introduction: The isomorphism problem

Let R be a commutative ring with identity and G a finite group. We shall denote by RG the group ring of G over R and by $\varepsilon: RG \longrightarrow R$ the augmentation mapping i.e. the ring homomorphism given by $\varepsilon(\sum_g a(g).g) = \sum_g a(g)$. The kernel of this map, called the augmentation ideal of RG will be denoted by $\Delta(G)$.

More generally, if A is a normal subgroup of G we shall denote by $\Delta(G:A)$ the kernel of the homomorphism $\omega: RG \longrightarrow RG/A$ induced by the natural projection $G \longrightarrow G/A$ i.e. the mapping given by

$$\omega(\sum_g a(g)g) = \sum_g a(g)\bar{g} ,$$

where $\bar{g} = gA$ is the left coset of g modulo A.

A natural question, when working with group rings, is to decide whether the structure of the group ring RG determines the structure of R or, more precisely, if given two groups G,H such that $RG \cong RH$, this will imply that $G \cong H$.

Stated in such a loose way, the question has a negative answer. Actually, a classical paper by E.C. Dade [3] gives an example of two non-isomorphic groups G,H such that $FG \cong FH$ for all fields F.

The natural context to consider this question is when working with integral group rings. In fact, if G and H are groups, the assumption that $\mathbb{Z}G \cong \mathbb{Z}H$ is the strongest possible since it can be easily shown that it also implies $RG \cong RH$ for every ring with identity R.

An isomorphism $\phi: \mathbb{Z}H \longrightarrow \mathbb{Z}G$ is said to be normalized if it preserves augmentations, i.e., if $\varepsilon(\phi(\alpha)) = \varepsilon(\alpha)$ for every $\alpha \in \mathbb{Z}H$. It is easy to see that if $\mathbb{Z}H \cong \mathbb{Z}G$ then there exists a normalized isomorphism $\phi: \mathbb{Z}H \longrightarrow \mathbb{Z}G$.

An invertible element $\mu \in \mathbb{Z}G$ such that $\varepsilon(\mu) = 1$ is called a normalized unit of $\mathbb{Z}G$; the set of all normalized units in a subset $X \subset \mathbb{Z}G$ will be denoted by $V(X)$.

Notice that if we set $n = |H|$ and $\phi: \mathbb{Z}H \longrightarrow \mathbb{Z}G$ is a normalized isomorphism, since $h^n = 1$ we have that $\phi(h)^n = 1$ and also $\varepsilon(\phi(h)) = \varepsilon(h) = 1$ i.e. $\phi(H) \subset V(\mathbb{Z}G)$.

The first important result regarding the isomorphism problem was the following.

<u>Theorem</u> (Higman [4]) - Let G be a finite abelian group and let H be another group such that $\mathbb{Z}G \cong \mathbb{Z}H$. Then $G \cong H$.

We recall that a group G is called <u>metabelian</u> if it contains a normal subgroup A such that both A and G/A are abelian.

The next result regarding this problem might be considered an extension of the one above.

Theorem (Whitcomb [10]) - Let G be a finite metabelian group and let H be another group such that $\mathbb{Z}G \cong \mathbb{Z}H$. Then $G \cong H$.

The main step in its proof consists in showing that if G is metabelian and $\mu \in V(\mathbb{Z}G)$ in of finite order then there exists a unique element $g \in G$ such that

$$\mu \equiv g (\text{mod } \Delta(A).\Delta G)$$

where we regard $\Delta(A)$ as imbedded in RG.

§2. The conjecture

As we noticed above, progress on the isomorphism problem depends on information about units of finite order. Hence, one would like to obtain as much information as possible on these elements.

There is one obvious way of constructing such units. If $\alpha \in U(RG)$ and g is any element in G, then $\gamma = \alpha^{-1} g\alpha$ is a unit of finite order. I. Hughes and K.R. Pearson [5] and C. Polcino Milies [6] have shown that not all units of finite order can be obtained in this way, in the cases where $G = S_3$ and $G = D_4$ respectively. However, it is easily seen that we can obtain them all if we do allow α to take values also in $\mathbb{Q}G$. Hence, we have the following:

Conjecture (Zassenhaus) - Let μ be a normalized unit of finite order in $\mathbb{Z}G$. Then, there exists an element $\alpha \in U(\mathbb{Q}G)$ such that $\alpha^{-1}\mu\alpha \in G$.

This conjecture was verifyied for the first time by A. K. Bhandari and I. S. Luthar [1] for split metacyclic groups $G = \langle a \rangle \rtimes \langle x \rangle$ where $o(a) = p$, $o(x) = q$ with p,q prime integers such that q divides p-1.

Next, C. Polcino Milies and S.K. Sehgal [7] extended this result for split metacyclic groups of the form above where $\langle a \rangle$ is a cyclic p-group and $o(x)$ does not divide the order of a. Here, it was also assumed that the action of x on a is faithful. Also, a more general result was obtained:

<u>Theorem [7]</u> - Let $G = A \rtimes X$ where A and X are <u>abelian</u>. Let $\mu \in V(\mathbb{Z}G)$ be a unit of finite order such that $(o(\mu), |A|) = 1$. Then, there exists $\gamma \in V(\mathbb{Q}G)$ and an element $x \in X$ such that $\mu = \gamma^{-1}x\gamma$.

The next step in this sequence of results was given by J. Ritter and S.K. Sehgal [9] who removed the restriction about the action of x on a being faithful. In that same paper it was shown that the Conjecture of Zassenhaus holds for a wider class of groups.

<u>Theorem [9]</u> - Let G be a nilpotent class 2 group. If $\mu \in V(\mathbb{Z}G)$ is a torsion unit, then there exists $\gamma \in V(\mathbb{Q}G)$ such that $\mu = \gamma^{-1}g\gamma$ for some $g \in G$. Moreover, there exists an element $x \in G$ such that $\mu \equiv g^x \pmod{\Delta(G')\Delta(G)}$.

Finally, the result about split metacyclic groups was brought to a more definite form:

<u>Theorem [8]</u> - Let G be the split metacyclic group $G = \langle a \rangle \rtimes \langle x \rangle$ where $o(a) = n$, $o(x) = t$ and $(u,t) = 1$. Then, for any unit of finite order $\mu \in V(\mathbb{Z}G)$ there exist $\gamma \in V(\mathbb{Q}G)$ and $g \in G$ such that $\mu = \gamma^{-1}x\gamma$.

We see that all results that have been obtained so far, regarding the conjecture, are positive. However, they are still restricted to special types of groups. Considering the state of the theory, it should be expected to push these results further and prove it, at least, for split metabelian groups.

Final comments

It was shown in [2] that there exist ideals I_k of the integral group ring such that there exist unique group elements g_k with $\mu \equiv g_k (\mathrm{mod}\ I_k)$; moreover, μ and g_k have the same order.

One might be tempted to believe that one of these elements might be the one to solve the conjecture. However, J. Ritter and S.K. Sehgal gave in [9] the following example:

$$\text{Set} \quad G = D_{10} = \langle a, x \mid a^5 = x^2 = 1 \ , \ a^x = a^{-1} \rangle$$

In this case, the variety of ideals I_k reduces to the ideals $\Delta(A)\ \Delta(G)$ and $\Delta(G)\ \Delta(A)$, where $A = \langle a \rangle$.

The unit $\mu = -a^2+a^3+a^4+(-1+a^3)x$

is such that

$$\mu \equiv a^3 \quad \mathrm{mod}\ \Delta A\ \Delta G$$

$$\mu \equiv a^2 \quad \mathrm{mod}\ \Delta G\ \Delta A$$

but μ is conjugate to a and a^4 in $\mathbb{Q}G$.

Finally, let as mention that one might state the stronger :

<u>Conjecture</u> - Every finite subgroup of units in $V(\mathbb{Z}G)$ is conjugate, in $\mathbb{Q}G$, to a subgroup of G.

No results are known, so far, regarding this problem.

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