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ON THE RICCI CONDITION AND
MINIMAL SURFACES WITH CONSTANTLY CURVED GAUSS MAP

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§1 - INTRODUCTION

A riemannian metric ds^2 on a surface is said to satisfy the Ricci condition if its Gaussian curvature K satisfies $K < 0$, and if the new metric $d\hat{s}^2 = \sqrt{-K}ds^2$ is flat, i.e., its Gaussian curvature $\hat{K} \equiv 0$. It's known that every metric on a minimal surface in $\mathbb{R}^3$ satisfies this condition away from the points where $K = 0$. As a matter of fact, Ricci [1, p.124] showed that every metric satisfying this condition can be locally realized on a minimal surface in $\mathbb{R}^3$. Lawson [7] has discovered that a minimal surface S in $\mathbb{R}^n$ whose induced metric satisfies the Ricci condition off the set of isolated points where $K = 0$ if and only if the Gauss image of S has constant curvature 1, and, that the constancy of the curvature of the Gauss image will impose a necessary and sufficient condition on the induced metric. We shall use that condition to generalize the Ricci condition. And we will show that under this new definition the classical Ricci-Curbastro theorem can be extended. On the other hand, the generalized Gauss map of a minimal surface in $\mathbb{R}^n$ represents a holomorphic curve in the complex projective space $P^{n-1}(\mathbb{C})$. With a normalized Fubini-Study metric on $P^{n-1}(\mathbb{C})$, holomorphic curves with constante curvature in $P^{n-1}(\mathbb{C})$ have been classified by Calabi

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[2]. This fact, together a general theory of Calabi [3] about minimal surfaces isometric to regular holomorphic curves in $\mathbb{C}^m$, can therefore be used to study minimal surfaces in euclidean spaces whose Gauss images possess constant curvatures. In this paper, we will also discuss some properties of these minimal surfaces, including some results of Lawson [7] and Hoffmann-Osserman [5].

§2 - PRELIMINAIRES

We first recall some elementary properties of regular orientable minimal surfaces in $\mathbb{R}^n$. Detailed discussion can be found in [4].

Let S be a minimal surface given by the immersion $x: M^2 \longrightarrow \mathbb{R}^n$. The isothermal parameters (ξ,η) give a complex structure $\zeta = \xi + i\eta$ on M. Then the induced metric is given by

(1) $$ds^2 = \lambda^2 |d\zeta|^2$$

where

(2) $$\lambda^2 = \frac{1}{2} \sum_{k=1}^{n} |\phi_k|^2$$

with

(3) $$\phi_k(\zeta) = \frac{\partial x_k}{\partial \xi} - i \frac{\partial x_k}{\partial \eta}, \qquad k=1,2,\ldots,n.$$

The Gaussian curvature of the minimal surface, calculated by the formula

(4) $$K = -\frac{\Delta \log \lambda}{\lambda^2}$$

can be expressed by

$$K = -\frac{4|\phi\wedge\phi'|^2}{|\phi|^6} \tag{5}$$

where

$$\phi = (\phi_1,\ldots,\phi_k), \quad |\phi|^2 = \sum_{k=1}^{n} |\phi_k|^2 \tag{6}$$

and

$$|\phi\wedge\phi'|^2 = \sum_{1\le j<k\le n} |\phi_j\phi_k' - \phi_j'\phi_k|^2 . \tag{7}$$

The generalized Gauss map is defined to be the complex analytic map

$$g(\zeta) = [\phi_1(\zeta),\ldots,\phi_n(\zeta)] \in Q_{n-2} \subset P^{n-1}(\mathbb{C}) \tag{8}$$

where

$$Q_{n-2} = \{[z_1,\ldots,z_n] \in P^{n-1}(\mathbb{C}) \,\Big|\, \sum_{k=1}^{n} z_k^2 = 0\} \tag{9}$$

is the hyperquadric in the complex projective space $P^{n-1}(\mathbb{C})$. The Gauss map defined in (8) is conjugate to the usual one.

With respect to the normalized Fubini-Study metric

$$d\bar{s}^2 = 2\frac{|Z\wedge dZ|^2}{|Z|^4} \tag{10}$$

on $P^{n-1}(\mathbb{C})$, the metric induced on the Gauss image $\hat{S}$ of S is given by

$$d\hat{s}^2 = -Kds^2 = -K\lambda^2|d\zeta|^2 \tag{11}$$

From (4), the Gaussian curvature $\hat{K}$ of $\hat{S}$ can be expressed by

(12) $$\hat{K} = \frac{\Delta \log\sqrt{-K\lambda}}{K\lambda^2}$$

We say that S is a minimal surface with constantly curved Gauss map if $\hat{K}$ is constant, which is an intrinsic condition: it depends only on the metric ds^2.

§3 - RICCI CONDITIONS AND MINIMAL SURFACES WITH CONSTANTLY CURVED GAUSS MAP

From (8) we see that the Gauss image of a minimal surface represents a holomorphic curve in $P^{n-1}(\mathbb{C})$. Therefore the study of regular holomorphic curves in $P^{n-1}(\mathbb{C})$ with constant curvature will be of central importance in our discussion. Calabi [2] classified these curves by proving

THEOREM 3.1 - i) The curve c_n: $P^1(\mathbb{C}) \longrightarrow P^n(\mathbb{C})$ given in homogeneous coordinates by

(13) $$[z_0, z_1] \longmapsto [z_0^n, \sqrt{n} z_0^{n-1} z_1, \ldots, \sqrt{\binom{n}{j}}\, z_0^{n-j} z_1^j, \ldots, z_1^n]$$

has constant curvature 2/n.

ii) Any holomorphic curve ψ: $D \longrightarrow P^n(\mathbb{C})$ which has constant curvature and does not lie in any linear subspace of $P^n(\mathbb{C})$ must be unitarily equivalent to the curve $\tilde{c}_n$: $D \longrightarrow P^n(\mathbb{C})$ given by

(14) $$z \longmapsto [1, \sqrt{n} z, \ldots, \sqrt{\binom{n}{j}}\, z^j, \ldots, z^n]$$

iii) If ψ: $D \longrightarrow P^n(\mathbb{C})$ has constant curvature $\hat{K}$, then $\hat{K} = \frac{2}{k}$ for some k in the range $1 \le k \le n$. Any two such curves are unitarily equivalent. Furthermore $\hat{K} = 2/k$ if and only if ψ takes values in some k-dimensional linear subspace of $P^n(\mathbb{C})$ and is unitarily equivalent to

(15) $$z \longmapsto [1, \sqrt{k}z, \ldots, \sqrt{\binom{k}{j}}\, z^j, \ldots, z^k, 0, \ldots, 0]$$

REMARK - Therefore we can see that for a minimal surface with constantly curved Gauss map, the Gaussian curvature $\hat{K}$ of the Gauss image has to be of the value $2/k$, $k=1,2,\ldots$. This property is equivalent to a condition on the metric ds^2 of the minimal surface, which generalizes the classical condition of Ricci. In fact, Lawson [7] observed:

PROPOSITION 3.2 - Let S be a minimal surface in $\mathbb{R}^n$. Then the Gauss image has constant curvature $\hat{K} = 2/k$ $(k\geq 1)$ if and only if the new metric $d\tilde{s}^2 = (-K)^{k/k+2} ds^2$ in flat off the set where the Gaussian curvature K of S vanishes.

PROOF - From (2), (5), (12) and the formula

(16) $$\Delta \log|\phi|^2 = 4\frac{\partial}{\partial\zeta}\frac{\partial}{\partial\bar{\zeta}}\log|\phi|^2 = 4\frac{|\phi\wedge\phi'|^2}{|\phi|^4},$$

we see that

(17) $$\hat{K} = 2/k \iff \Delta\log|\phi\wedge\phi'|^2 = 8\left(1-\frac{1}{k}\right)\frac{|\phi\wedge\phi'|^2}{|\phi|^4}$$

And from (1), (2), (4), (16), we have

(18) $$d\tilde{s}^2 \text{ is flat} \iff \Delta\log|\phi\wedge\phi'|^2 = \frac{8k-8}{k}\,\frac{|\phi\wedge\phi'|^2}{|\phi|^4}$$

Q.E.D.

Therefore we define that a C^{ω} riemannian metric ds^2 defined on a Riemann surface is said to satisfy the Ricci condition of type k $(k \geq 1)$ if $K < 0$ and the new metric $d\tilde{s}^2 = (-K)^{k/k+2} ds^2$ is flat.

First let's consider the question: whethere there exists minimal surface whose Gauss image has constante curvature $2/k$ for each $k \geq 1$. Such examples can be easily constructed by considering the minimal surface

$$(19) \quad x^k(z) = \left(z, \frac{\sqrt{k}}{2} z^2, \ldots, \sqrt{\binom{k}{j}}\, z^{j+1}/j+1, \ldots, z^{k+1}/k+1\right) \in \mathbb{C}^{k+1} = \mathbb{R}^{2k+2}$$

whose Gauss map is

$$(20) \quad g^k(z) = [1, \sqrt{k}z, \ldots, \sqrt{\binom{k}{j}}\, z^j, \ldots, z^k, -i, -i\sqrt{k}z, \ldots, -i\sqrt{\binom{k}{j}}\, z^j, \ldots, -iz^k]$$

which is unitarily equivalent to the curve (15) under the unitary transformation

$$\frac{\sqrt{2}}{2} \begin{bmatrix} I_{k+1} & I_{k+1} \\ -iI_{k+1} & iI_{k+1} \end{bmatrix}$$

To study the general situation, from now on, we only consider simply connected holomorphic curves, by passing, if necessary, to the universal covering spaces. Let ψ be any regular holomorphic curve in $\mathbb{C}^m = \mathbb{R}^{2m}$ whose Gauss image has constant curvature $2/k$. It's clear that its Gauss map $[\psi' - i\psi']$ is isometric to $[\psi']$. Therefore, form Thm. 3.1, $[\psi']$ is unitarily equivalent to $[1, \sqrt{k}z, \ldots, \sqrt{\binom{k}{j}} z^j, \ldots, z^k]$. Thus we may assume that ψ is of the form

$$(22) \quad \psi(z) = \int f(z) \left(1, \sqrt{k}z, \ldots, \sqrt{\binom{k}{j}}\, z^j, \ldots, z^k\right) dz$$

where f(z) is holomorphic function which never vanishes. And it's clear that ψ lies fully in $\mathbb{C}^{k+1}$.

Next we turn our attention to see whether any metric satisfying the Ricci condition of type k can be realized on some minimal surfaces in euclidean spaces. This can be easily answered by using a theorem of Calabi [3].

THEOREM 3.3 - Let $ds^2 = F|d\zeta|^2$ be a real analytic metric on a Riemann surface M. Then ds^2 is induced by a linearly full holomorphic immersion $\phi: M \longrightarrow \mathbb{C}^m$ if and only if the functions

$$F_k = \det\left[(\partial^p \bar{\partial}^q F)_{p,q=0}^{k-1}\right], \qquad k=1,2,3,\ldots$$

satisfy

$$\begin{cases} F_k \geq 0 \text{ and not } \equiv 0 \text{ for } k \leq m \\ F_k \equiv 0 \text{ for } k > m \end{cases}$$

And further, the functions F_k can be computed by a recursive formula:

$$F_{k+1} = \frac{F_k^2}{F_{k-1}} \partial\bar{\partial} \log F_k$$

where $F_0 = 1$ and $F_1 = F$.

Now using the definition of the Ricci condition of type k and the recursive formula in Thm. 3.3, a straightforward computation shows that $F_1, \ldots, F_{k+1} \geq 0$ but not $\equiv 0$, and $F_{k+2} \equiv 0$. Therefore we have:

PROPOSITION 3.4 - Any metric ds^2 satisfying the Ricci condition of type k can be realized on a holomorphic curve lying fully in $\mathbb{C}^{k+1}$.

And we can further show a theorem which generalizes the classical Ricci-Curbastro theorem:

THEOREM 3.5 - For k even, any metric satisfying the Ricci condition of type k can be realized on a minimal surface in $\mathbb{R}^{k+1}$.

Before proving Thm. 3.5, we need another theorem of Calabi [3] to determine the dimension of the smallest affine subspace which contains a given minimal surface isometric to a regular holomorphic curve:

THEOREM 3.6 - Let $\psi(\zeta)$ be a simply connected regular holomorphic **curve lying fully in $\mathbb{C}^m$. The space $V(\psi)$ of non-congruent** minimal **immersions in euclidian spaces which are isometric to ψ** is naturally described as the set of all complex symmetric m×m matrices P such that

(i) $I_m - P\bar{P} \geq 0$, and

(ii) $\psi' \cdot P\psi'^t \equiv 0$.

Furthermore, let n be the dimension of the smallest affine subspace containing S, where S is a minimal surface corresponding to P. Then

$$n - m = \text{rank}(I_m - P\bar{P})$$

In particular, $m \leq n \leq 2m$, and $n = m$ iff $I_m = P\bar{P}$.

PROOF OF THM. 3.5 - From Prop. 3.2 & Prop. 3.4, we see that ds^2 can be realized on a holomorphic curve ψ in $\mathbb{C}^{k+1} = \mathbb{R}^{2k+2}$, and whose Gauss image has constant curvature $2/k$. And from the discussions immediately after Prop. 3.2, we may also assume that ψ is of the form (22). Now applying Thm. 3.6 and let P be the $(k+1)\times(k+1)$ symmetric matrix given by

$$P = \begin{bmatrix} 0 & & & & 1 \\ & \ddots & & -1 & \\ & & (-1)^{k/2} & & \\ & -1 & & \ddots & \\ 1 & & & & 0 \end{bmatrix}$$

We know then that there exists a minimal surface isometric to ψ and lying fully in $\mathbb{R}^{k+1}$, since

$$P\bar{P} = I_{k+1} \text{ and } \psi'\cdot P\cdot\psi'^t = z^k\left(\sum_{j=0}^{k}(-1)^j\binom{k}{j}\right) \equiv 0$$

Q.E.D.

Following the same spirit of the proof above, we can show that

PROPOSITION 3.7 - For $k = 1$ or 3, any metric ds^2 satisfying the Ricci condition of type k can only be realized on some minimal surface in $\mathbb{R}^n$ with $n > k+1$.

PROOF

It's a simple computation to show that there exists no $(k+1)\times(k+1)$ symmetric complex matrix P such that

$$\tilde{c}_k\cdot P\cdot\tilde{c}_k^t \equiv 0 \text{ and } P\bar{P} = I_{k+1} \text{ for } k=1 \text{ or } 3.$$

Q.E.D.

REMARK

1. In fact, for k=1, the only symmetric complex matrix which satisfies

$$\tilde{c}_1\cdot P\cdot\tilde{c}_1^t \equiv 0$$

is the zero matrix.

2. It's interesting to know whether Prop. 3.7 continues to hold for general odd integer k.

Finally, we would like to treat some simplest cases of minimal surfaces with constantly curved Gauss map.

Case 1. $k=1$, i.e., $\hat{K}=2$

From the proof of Thm. 3.5, we observe that any minimal surface S with $\hat{K}=2$ is isometric to a linearly full holomorphic curve $\psi: M \longrightarrow \mathbb{C}^2$ whose tangent line is of the form

$$\psi'(\zeta) = f(\zeta)(1,\zeta)$$

for some non-vanishing holomorphic function f. From Thm. 3.6 and remark 1 of Prop. 3.7, we see that S is, in fact, congruent to ψ. Thus we have:

THEOREM 3.8 - Any minimal surface S in $\mathbb{R}^n$ whose Gauss image has constant curvature 2 must lie in some 4-dim affine subspace of $\mathbb{R}^n$, and is congruent to some holomorphic curve in $\mathbb{C}^2 = \mathbb{R}^4$ of the form

(23) $$\psi(\zeta) = \int (f(\zeta), \zeta f(\zeta))d\zeta$$

where $f(\zeta)$ is a non-vanishing holomorphic function.

Case 2. $k=2$, i.e., $\hat{K}=1$

Lawson [7] has given a complete description of this case:

THEOREM 3.9 - Let $x: M \longrightarrow \mathbb{R}^n$ be a simply connected minimal surface whose Gauss image has constant curvature 1. Then there is an isometric minimal immersion $x_0: M \longrightarrow \mathbb{R}^3$ and a number $\beta \in [0,2\pi]$ such that

(24) $$x = x_\beta \equiv \cos\beta\, x_0 \oplus \sin\beta\, x_0^*: M \longrightarrow \mathbb{R}^6$$

where x_0^* is the immersion conjugate to x_0. Furthermore, up to congruence, every minimal immersion, which is isometric to x, is associate to one of the surfaces x_β. In particular,

(i) if $n \leq 5$, then $x(M) \subseteq \mathbb{R}^3$

(ii) any two isometric minimal immersions of M into $\mathbb{R}^3$ are associate.

Case 3. $k>2$

This general case, to our best knowledge, is far from being known. We only know that from Thm 3.5 that for k even there exist such surfaces S in $\mathbb{R}^{k+1}$, and hence all the surfaces

$$S_\beta = \cos\beta \; S \oplus \sin\beta \; S^* \tag{25}$$

belong to the same class, where S* is the conjugate surface of S. But, in general, they don't exhaust the whole class. If we take

$$P = \begin{bmatrix} 0 & & & & -1 \\ & & & 1/8 & \\ & & 1/6 & & \\ & 1/8 & & & \\ -1 & & & & 0 \end{bmatrix}$$

And apply Thm. 3.6 for $\psi(\zeta) = (\zeta, \zeta^2, \frac{\sqrt{6}}{3}\zeta^3, \frac{1}{2}\zeta^4, \frac{1}{5}\zeta^5)$, then we get a minimal surface S lying fully in $\mathbb{R}^8$ whose Gaussian image has constant curvature 2/4. And S, in no way, can be represented in the form (25).

For k odd, we can see from Thm. 3.6 & Prop. 3.7 that, in no way, a minimal surface whose Gaussian image possesses constant curvature 2/3 can be represented by the formula (25). A general description of these surfaces would be very interesing.

REMARK. The case of $k = 1$ or 3 was first observed by Hoffman-Osserman [5], our presentation here is slightly different.

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