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by

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A Discrete Variational Problem Related to Ising Droplets at Low Temperatures

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Abstract

We consider a variational problem on the d-dimensional lattice Z^d which has applications in the study of the metastable behavior of the stochastic Ising model [Ligg, NS1, NS2, S]. The problem, an isoperimetric one, is to find what is the smallest area a finite subset of Z^d can have for each fixed volume. If ϕ is one of these subsets we define its volume as the number of points in it and its area as the number of pairs of points in Z^d which are neighbors and such that only one of them belongs to ϕ.

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In this note we consider the following variational problem on the lattice: to find, within the class of all subsets of Z^d with fixed volume v, what is the smallest possible area. For each $\phi \subset Z^d$, finite, we define its volume as the cardinality of this set, denoted by $|\phi|$, and its area, denoted by $A^d(\phi)$, as the number of edges with only one endpoint in ϕ

$$A^d(\phi) = |\{\{x,y\} \in Z^d \times Z^d : d(x,y) = 1, x \in \phi, y \notin \phi\}| \qquad (1)$$

where $d(x,y) \equiv \sum_{i=1}^{d} |x_i - y_i|$ is the lattice distance. If ϕ is a subset of a i-dimensional subspace S of Z^d it is also convenient to define the area of ϕ inside this subspace by considering only pairs in it

$$A^i(\phi) = |\{\{x,y\} \in S \times S : d(x,y) = 1, x \in \phi, y \notin \phi\}|. \tag{2}$$

To our knowledge this discrete isoperimetric problem was not solved before though similar questions appear to be natural in the context of combinatorics [B1, B2]. The motivation for this problem arises from the analysis of the metastable behavior of the finite volume d-dimensional stochastic Ising model for very low temperatures [Ligg, NS1, NS2, S]. In this statistical mechanics model to each subset ϕ, which is called a *configuration*, of $\{1,2,\ldots,N\}^d \subset Z^d$, for some fixed and large N, we associate a real number, *the energy of configuration* ϕ, as follows

$$H(\phi) = A(\phi) - h|\phi|$$

where $h \geq 0$. The system moves on the set of configurations as a continuous time stochastic process such that, after a long time, the proportion of time the system spent in a given configuration φ is proportional to $\exp(-H(\phi)/T)$, the Gibbs weight, where $T > 0$ is the temperature. For low temperatures the system spends most of the time on the configuration $\phi_0 = \{1,2,\ldots,N\}^d$ which is the global minimum of H. In the so called Glauber Dynamics the system can only change, in one step, between configurations that differ by a single point. From a given configuration the system choses what will be the next one in such a way that transitions that decrease the energy are favored. For low temperatures if the initial configuration is a local minimum of energy the system may be "trapped" in its vicinity for a long time before reaching ϕ_0 which is a behavior associated with metastability. The variational problem posed here provides some of the information about the energy profile on the space of configurations necessary for the analysis of this kind of problem.

For each volume v, a configuration that solves the variational problem corresponds, in a sense to be made precise shortly, to the best approximation of a d-dimensional cube with the given volume. To clarify this statement we need some definitions.

If $\phi \subset Z^d$, let $\mathcal{E}(\phi)$ be the class of all configurations that can be obtained from ϕ by lattice translations, lattice rotations and lattice reflections which are therefore equivalent with respect to the variational problem. Write $\partial\phi$ for the *external boundary* of $\phi \subset Z^d$

$$\partial\phi \equiv \{y \notin \phi : \text{ there exists } x \in \phi \text{ with } d(x,y) = 1\}$$

Let Σ^d be the class of configurations defined as follows: $\emptyset \in \Sigma^d$ and if $\phi \neq \emptyset$

i)$x = (x_1, \ldots, x_d) \in \phi \Rightarrow x_i \geq 1, 1 \leq i \leq d$

ii)$\{x_i = 1\} \cap \phi \neq \emptyset, 1 \leq i \leq d$

iii)$l_i(\phi) \geq l_j(\phi)$ if $i \leq j$ where, for $\rho \subset Z^d$, finite,

$$l_i(\rho) = \max\{j : \rho \cap \{x_i = j\} \neq \emptyset\} - \max\{k : \rho \cap \{x_i = k\} = \emptyset\} \qquad (3)$$

is the lenght of ρ along direction i. To simplify the notation we write $\{x_i = k\}$ instead of $\{x = (x_1, \ldots, x_d) \in Z^d : x_i = k\}$.

A set $\phi \subset Z^d$ is called a *j-dimensional block* if it is a paralellepiped with d-j sides with lenght 1 and the remaining j sides either all equal or assuming two successive positive integers. That is, a j-dimensional block is a set

$$\phi \in \mathcal{E}(\{x \in Z^d : 1 \leq x_i \leq L_i\})$$

with $L_i = 1$ if $i > j$ and $L_i \in \{L, L-1\}$, $1 \leq i \leq j$, for some positive integer L.

Call $\phi \cap \{x_i = k\}$ a *slice* of ϕ along direction i at position k. Call it an *external slice* on the positive (negative) direction i if it is non empty but $\phi \cap \{x_i = k+1\} = \emptyset$ ($\phi \cap \{x_i = k-1\} = \emptyset$). Clearly any slice of a block is itself a block.

Remark 1: The external boundary of a block ϕ, $\partial\phi$, is the union of $2d$ disjoint blocks $\partial\phi = \cup_{k=1}^{2d} b_k$. Each b_k may be obtained by tranlation of one lattice unit of each external slice towards the outside of ϕ. Moreover these blocks are not connected to each other as, if x and y are points in different blocks, then $d(x,y) > 1$ (the notion of connectivity is the usual one in percolation theory: a set $S \subset Z^d$ is connected if for any pair of its points, say x and y, there exists a sequence $\{z_i\}_{i=1}^{N}$, for some N, in S with $x_1 = x$, $x_N = y$ and $d(x_i, x_{i+1}) = 1$ for $1 \leq i < N$).

Remark 2: If ϕ and ψ are different blocks with $\phi \subset \psi$ then at least one of the blocks in $\partial\phi$ (as in Remark 1 above) is contained in ψ.

If $A_i = \{a :$ there exists an i-dimensional block with volume $a\}$, for $1 \leq i \leq d$, and for any positive integer v let

$$\underline{v}^i = \max\{a \in A_i : a \leq v\}$$

and

$$\overline{v}^i = \min\{a \in A : a \geq v\}$$

.

Define $\mathbf{b}_i(v)$ ($\mathbf{B}_i(v)$) as the i-dimensional block in Σ^d with volume $\underline{v}^i$ ($\overline{v}^i$). $\mathbf{b}_i(v)$ ($\mathbf{B}_i(v)$) is the largest (smallest) i-dimensional block with volume not larger (smaller) than v. We now give an explicit construction of $\mathbf{b}_i(v)$.

If $v > 1$, let $L_i(v)$ and $M_i(v)$ be given by

$$L_i(v) = \min\{l \in N : l^i \geq v\} \text{ and } \tag{4}$$

$$M_i(v) = \max\{m \in \{0, 1, \ldots, i\} : L_i^m(v)(L_i(v) - 1)^{i-m} \leq v\}. \tag{5}$$

Set $\mathbf{b}_i(0) = \emptyset$, $\mathbf{b}_i(1) = \{x_j = 1, 1 \leq j \leq d\}$ and, if $v \geq 2$

$$\mathbf{b}_i(v) = \{1 \leq x_j \leq \ell_j; \text{ for } 1 \leq j \leq d\} \tag{6}$$

with $\ell_j = L_i(v)$ for $1 \leq j \leq M_i(v)$, $\ell_j = L_i(v) - 1$ for $M_i(v) + 1 \leq j \leq i$ and $\ell_j = 1$ for $i < j \leq d$.

Remark 3: Clearly if b_1 and b_2 are i-dimensional blocks with volumes a_1 and a_2 with $a_1 \leq a_2$ then one can find $b \in \mathcal{E}(b_2)$ such that $b_1 \subset b$.

For any positive integer v, let $\{v_i\}_{1 \leq i \leq d}$, be defined by

$$v_d = |\mathbf{b}_d(v)| \text{ and } v_i = |\mathbf{b}_i(v - \sum_{j>i} v_j)| \tag{7}$$

so that $v = \sum_{i=1}^d v_i$.

We can now define a class of configurations $\mathcal{B}^d(v)$ which corresponds to *the best approximation of a d-dimensional cube* with volume v and, as we will show, realizes the smallest area for a given volume v. A configuration ξ belongs to $\mathcal{B}^d(v)$ if $|\xi| = v$ and it has a decomposition in blocks

$$\xi = \cup_{i=1}^d \xi_i \tag{8}$$

satisfying

Condition a: $\xi_i \cap \xi_j = \emptyset$ if $i \neq j$

Condition b: $\xi_i \in \mathcal{E}(b_i(v_i)), v_i$ as before

Condition c: For all pairs $1 \leq i < j \leq d$ we have $\xi_i \subset \partial\xi_j$.

Therefore elements of $\mathcal{B}^d(v)$ may be constructed as follows: start with a block $\xi_d \in \mathcal{E}(\mathbf{b}_d(v_d))$; let $\partial\xi_d = \cup_{j=1}^{2d} b_j^d$, with $\{b_j^d\}_{j=1}^{2d}$ being the collection of disjoint blocks, each one attached to one of the external slices of ξ_d; chose one of them, call it b^d, and take $\xi_{d-1} \subset b^d$; this inclusion must be strict (otherwise one could add a slice to ξ_d, obtaining a larger block with volume still smaller than v); therefore at least one of the disjoint blocks in $\partial\xi_{d-1} = \cup_{j=1}^{2d} b_j^{d-1}$, say b^{d-1}, is also strictly included in b^d (follows from remark 2 above); thus, if $d > 2$ we may take ξ_{d-2} in b^{d-1}; this can go on until ξ_1 is chosen and clearly a), b) and c) will be satisfied.

Write $\mathcal{B}^d = \cup_{v=0}^{\infty}\mathcal{B}^d(v)$.

It is clear that, with this definition, if one adds a slice to a block the result may not be a block. It will be convenient in what follows to distinguish a subset $\widetilde{\mathcal{B}^d}$ of $\mathcal{B}^d$ of what we call *canonical elements* or *canonical configurations* of those on which each *protuberance* $\cup_{i\leq k}\xi_i$ attached to the block ξ_{k+1} (and ξ itself) is "almost a block", that is, is contained in a block equivalent to $B_k(|\cup_{i\leq k}\xi_i|)$ (the smallest k-dimensional block with volume not smaller than $|\cup_{i\leq k}\xi_i| = \sum_{i\leq k} v_i$).

Condition d: $\{x \in Z^d : 1 \leq x_j \leq l_j(\cup_{i\leq k}\xi_i)\} \in \mathcal{E}(\mathbf{B}_k(|\cup_{i\leq k}\xi_i|))$ for $1 \leq k \leq d$.

Areas of elements of $\mathcal{B}^d$ are simple to compute. Take $\xi \in \mathcal{B}^d(v)$ with $\xi_i = \cup_{i=1}^{d}\xi_i$, $\xi_i \in \mathcal{E}(b_i(v_i))$ as in (8).

Then

$$A^d(\xi) = \sum_{i=1}^{d} A^i(\xi_i) \tag{9}$$

where $A^i(\xi_i)$ is the area of the block ξ_i within the corresponding i-dimensional subspace of Z^d.

Write a_v^d for the smallest possible area in the class of configurations in Z^d with volume v

$$a_v^d = \min\{A^d(\phi); \phi \subset Z^d, |\phi| = v\}. \tag{10}$$

Our main result is as follows

Theorem: *In any dimension d*

$$a_v^d = A^d(\xi) \textit{ for all } \xi \in \mathcal{B}^d(v)$$

This result identifies a class of configurations with volume v that we can use to compute a_v^d. It is not true, however, that all subsets of Z^d with volume v and area a_v^d belong to $\mathcal{B}^d(v)$. A simple example in $d = 2$ of a set with smallest area not in $\mathcal{B}^2$ is obtained from a square of lenght larger or equal 3 by removing two points at the corners on the same diagonal.

This theorem provides the answer for the variational problem. For a numerical example, it follows from it that the smallest possible area of a subset of Z^5 with volume 15141 is 22410.

Rather than working with *area of a configuration* it is simpler and sufficient for our purposes to introduce the auxiliary notion of *projected area of a configuration* ϕ, denoted by $PA^d(\phi)$. We define it as twice the number of lines in Z^d which intersect the given configuration. With this definition $PA^d(\phi)$ is equal to $A^d(\phi)$ for a class of configurations which includes $\mathcal{B}^d(v)$. More precisely if l is a line in Z^d, that is

$$l = \{x \in Z^d : x = x_0 + ke_i; k \in Z\}$$

where e_i is the unit vector along direction i, $1 \leq i \leq d$. $x_0 \in Z^d$ and if $\mathcal{L}$ is the set of all lines of Z^d we define

$$PA^d(\phi) = 2|\{l \in \mathcal{L} : l \cap \phi \neq \emptyset\}| \tag{11}$$

If ϕ belongs to an j-dimensional subspace S of Z^d we also define its projected area within this subspace by considering only lines in S

$$PA^j(\phi) = 2|\{l \in \mathcal{L} : l \subset S; l \cap \phi \neq \emptyset| \tag{12}$$

It is simple to verify the following

Lemma 1:

a) $PA^d(\xi) \leq A^d(\xi)$, *for all* $\xi \in Z^d$
b) $PA^d(\xi) = A^d(\xi)$ *if* $\xi \in \mathcal{B}^d$
c) $PA^d(\xi) = PA^d(\phi)$ *if both* ξ *and* ϕ *belong to* $\mathcal{B}^d$

The Theorem is a consequence of the following
Lemma 2: *For* $d \geq 1$ *let*

$$p_v^d = \min\{PA^d(\xi) : \xi \subset Z^d, |\xi| = v\}. \tag{13}$$

then $p_v^d = PA^d(\xi)$ *for all* $\xi \in \mathcal{B}^d(v)$.

That Lemma 2 implies the theorem is a consequence of Lemma 1: if we assume Lemma 2 and $\xi \in \mathcal{B}^d(v)$ then $p_v^d = PA^d(\xi) \leq PA^d(\phi)$ for any $\phi \in Z^d$, $|\phi| = v$; by parts a) and b) of Lemma 1 $PA^d(\phi) \leq A^d(\phi)$ and $PA^d(\xi) = A^d(\xi)$ so that $A^d(\xi) \leq A^d(\phi)$ for any $\phi \subset Z^d$, $|\phi| = v$ and the theorem is true.

We prove Lemma 2 using induction in the dimension d. Let P(d) be the property that Lemma 2 is true for dimension d, that is

For lattice dimension d we have $p_v^d = PA^d(\xi)$ *for all* $\xi \in \mathcal{B}^d(v)$.

In dimension d=1 this property is trivial and $\mathcal{B}^1(v)$ corresponds to the set of all intervals of lenght v in Z.

We now prove that P(d-1) implies P(d) for $d \geq 2$. To do this we start with an arbitrary initial configuration in Z^d and use P(d-1) to modify it step by step into a configuration in $\mathcal{B}^d$ with the same volume but smaller or equal projected area. From this the validity of P(d) follows. We use sucessive Greek letters to denote the configurations in each step of this process.

Let $\alpha \subset \Sigma^d$, $|\alpha| = v$. As mentioned before this is the most general case as any configuration in Z^d has a equivalent one in Σ^d. Write $a_i = |\alpha \cap \{x_1 = i\}|$ for the volume of α in the i-th slice across direction 1.

If $A = \max\{a_i; 1 \leq i \leq l_1(\alpha)\}$ with $l_i(\alpha)$ given by (3) and $\mathcal{L}^1 \subset \mathcal{L}$ is the set of lines in Z^d which are perpendicular to direction 1, then

$$PA^d(\alpha) = 2|\{l \in \mathcal{L} \setminus \mathcal{L}^1 : l \cap \alpha \neq \emptyset\}| + \sum_{i=1}^{l_1(\alpha)} PA^{d-1}(\alpha \cap \{x_1 = i\}) \tag{14}$$

$$\geq 2A + \sum_{i=1}^{l_1(\alpha)} PA^{d-1}(\alpha \cap \{x_1 = i\})$$

since the number of lines along direction 1 intersecting α can not be smaller than A and if $l \in \mathcal{L}^1$ then $l \subset \{x_1 = i\}$ for some i.

By (13) we have

$$PA^d(\alpha) \geq 2A + \sum_{i=1}^{l_1(\alpha)} p_{a_i}^{d-1} \tag{15}$$

Now we verify that if P(d-1) is true then the right hand side of (15) actually corresponds to the projected area of a subset of Z^d.

Lemma 3: *If $\widetilde{\mathcal{B}^d}(v)$ is the set of canonical elements of $\mathcal{B}^d(v)$, as defined before, then*

a) $\widetilde{\mathcal{B}^d}(v) \neq \emptyset$ for all $d \geq 1$ and $v \geq 1$.

b) If $\xi \in \widetilde{\mathcal{B}^d}(v-1)$ for some $d \geq 1$ and $v > 1$ then there exists $\eta \in \widetilde{\mathcal{B}^d}(v)$ with $\xi \subset \eta$.

Proof(of Lemma 3): First note that any set with a single point of Z^d is in $\widetilde{\mathcal{B}^d}(1)$ for $d \geq 1$. Therefore the lemma is proven if we verify b).

Let $\{v_i\}_{i=1}^d$ and $\{(v-1)_i\}_{i=1}^d$ be the decompositions of v and $v-1$, respectively, as in (7) and let $I = \max\{i : (v-1)_i \neq v_i\} \geq 1$. If $I > 1$ and $L \equiv L_I(\sum_{i=1}^I v_i)$ and $M \equiv M_I(\sum_{i=1}^I v_i)$, $L_I(\sum_{i=1}^I v_i)$ and $M_I(\sum_{i=1}^I v_i)$ as defined in (4) we have $v_i = 0$ for $i < I$ as $v_I = L^M(L-1)^{I-M}$, $\sum_{i=1}^I v_i = \sum_{i=1}^I (v-1)_i + 1$ and $\sum_{i=1}^I (v-1)_i < L^M(L-1)^{I-M}$ (the first equality is the definition of v_I, the second holds because $v_i = (v-1)_i$ for $i > I$ and the third follows from the definition of I). Therefore $v_I = \sum_{i \leq I}(v-1)_i + 1$.

For an arbitrary $\xi = \cup_{i=1}^d \xi_i \in \widetilde{\mathcal{B}^d}(v-1)$ for $v > 1$ with $\{\xi_i\}_{i=1}^d$ as in (8) we are going to construct $\eta \in \widetilde{\mathcal{B}^d}(v)$ with $\xi \subset \eta$.

If $\cup_{i=1}^I \xi_i = \emptyset$ take $\eta_i = \xi_i$ for $i > I$ and $\eta_I = \{x\}$ with $x \in \cap_{i>I} \partial \xi_i$. The existence of such a point follows from remark 2 and the fact that the inclusion in condition c must be strict.

If $|\cup_{i=1}^I \xi_i| = c > 0$ take $\eta_i = \xi_i$ for $i > I$ (if $I < d$) and $\eta_I \in \mathcal{E}(\mathbf{b}_I(c+1)$ such that $\eta_I \supset \cup_{i=1}^I \xi_i$. To see that this is possible first note that there exists a block b such that $\cup_{i=1}^I \xi_i \subset b \in \mathcal{E}(\mathbf{B}_I(c))$ by condition d in the definition of $\widetilde{\mathcal{B}}^d$. Now $|b| \geq c$ (by definition of $\mathbf{B}_I(c)$) but since $v_I = c+1 \in A_I$ (there exists an I-dimensional block with volume c) we also must have $|b| \leq c+1$. Therefore, by remark 3, there exists a block with volume $c+1$ that contains b and hence $\cup_{i=1}^I \xi_i$.

$\square_{Lemma 3}$

Let $A_1 \geq A_2 \geq \ldots \geq A_{l_1(\alpha)}$ be an ordering of the numbers $a_1, a_2, \ldots, a_{l_1(\alpha)}$. For each A_i, $1 \leq i \leq l_1(\alpha)$, find a $\beta_i \subset \{x_1 = i\}$ in $\widetilde{\mathcal{B}^{d-1}}(A_i)$ so that if $l \in \mathcal{L}_1$ and $l \cap \beta_i \neq \emptyset$ then $l \cap \beta_j \neq \emptyset$ for all $1 \leq j \leq i$ so that each β_i is smaller than β_j if $1 \leq i < j \leq l_1(\alpha)$. This can be done by Lemma 3. In this case we say that $\{\beta_i\}_{i=1}^{l_1(\beta)}$ is a non increasing sequence of elements in $\widetilde{\mathcal{B}^{d-1}}$.

Then $\beta = \cup_{i=1}^{l_1(\alpha)} \beta_i$ satisfyies

$$PA^d(\alpha) \geq 2A + \sum_{i=1}^{l_1(\alpha)} p_{a_i}^{d-1} = PA^d(\beta) \tag{16}$$

Configuration β is the union of $l_1(\beta) \leq l_1(\alpha)$ (d-1)-dimensional configurations which may be different. In the next step we see that the projected area is smaller if all the (d-1)-dimensional blocks in each β_i are equal for $i < l_1(\beta)$.

The idea is that if two slices are smaller than the first and larger one, β_1, the best is to enlarge one as much as possible at the expense of the other with the restriction that both should remain smaller than β_1. This restriction assures that the number of lines along direction 1 intersecting the configuration remains equal to A. The following results are used to make these remarks precise.

Lemma 4: *Let $P(n)$ be true and ξ_1 and ξ_2 be two arbitrary configurations in Z^n, $d \geq 1$. Then*

$$PA^n(\xi_1) + PA^n(\xi_2) \geq PA^n(\eta_1) + PA^n(\eta_2) \tag{17}$$

for any η_1, η_2 such that $\eta_1 \in \mathcal{B}^n(|\xi_1 \cap \xi_2|)$ and $\eta_2 \in \mathcal{B}^n(|\xi_1 \cup \xi_2|)$.

Proof:(of Lemma 4) If P(n) is true it is enough to prove (17) with $\eta_1 = \xi_1 \cap \xi_2$ and $\eta_2 = \xi_1 \cup \xi_2$.

Let l be a line in Z^n. First consider the case on which $l \cap \xi_1 \cap \xi_2 = \emptyset$. If $l \cap (\xi_1 \cup \xi_2)$ is also empty this line does not contribute to any of the projected areas in the inequality. If $l \cap (\xi_1 \cup \xi_2)$ is not empty this line contributes to at least one of the projected areas in the left hand side while it does the same only for the second term in the right hand side.

If $l \cap \xi_1 \cap \xi_2$ is not empty then l intersects ξ_1, ξ_2 and $\xi_1 \cup \xi_2$ therefore contributing to all terms in the inequality (17).

$\square_{Lemma4}$

For $\rho \subset Z^d$, finite, define

$$b(\rho) = \{1 \leq x_i \leq l_i(\rho); 1 \leq i \leq d\} \tag{18}$$

Corollary 1: *Let $P(n)$ be true. ξ and ϕ be elements of $\widetilde{\mathcal{B}}^n$ such that*

$$\emptyset \neq \xi \subset \phi$$

and b be a n-dimensional block satisfying $b(\phi) \subset b$, $b(\phi) \neq b$.

Then there exist σ and τ in $\widetilde{\mathcal{B}^n}$ satisfying

$$|\sigma| < |\xi|,$$

$$\tau \subset b,$$

$$|\xi| + |\phi| = |\sigma| + |\tau| \text{ and}$$

$$PA^n(\xi) + PA^n(\phi) \geq PA^n(\sigma) + PA^n(\tau)$$

Proof:(of Corollary 1) As $b(\phi)$ is smaller than b let

$$J = \min\{i; l_i(\phi) < l_i(b)\} \in \{1, \ldots, n\}.$$

Define $\hat{\phi} = \{x \in Z^n; z - x + e_J \in \phi\}$, where $z = (l_1(\phi)+1, \ldots, l_n(\phi)+1)$ and e_J is the unit vector on direction J. This is the configuration obtained from ϕ by inversion on all lattice directions inside $b(\phi)$ followed by a translation of one unit along direction J.

Apply Lemma 4 with $\xi_1 = \hat{\phi}$ and $\xi_2 = \xi$ and take $\eta_1 = \sigma$ and $\eta_2 = \tau$. Then $|\sigma| = |\hat{\phi} \cap \xi| < |\xi|$ because the point with all coordinates equal one is in ξ (as $\emptyset \neq \xi \in \Sigma^n$) but does not belong to $\hat{\phi}$ (as it would imply that the point with coordinates $x_i = l_i(\phi)$, $i \neq J$, and $x_J = l_J(\phi) + 1$ belongs to ϕ). It is also clear that $\hat{\phi} \cup \xi$ can not have more than one additional slice on direction J. As $\tau \in \widetilde{\mathcal{B}^n}(|\hat{\phi} \cup \xi|)$ the same is true for it, and $\tau \subset b$.

$\square_{Corollary 1}$

Corollary 2: *Let $P(n)$ be true, ξ and ϕ be elements of $\widetilde{\mathcal{B}^n}$ such that*

$$\emptyset \neq \xi \subset \phi$$

and b be a n-dimensional block satisfying $\phi \neq b(\phi) = b$.

Then there exist σ and τ in $\widetilde{\mathcal{B}^n}$ satisfying

$$|\sigma| < |\xi|,$$

$$\tau \subset b,$$

$$|\xi| + |\phi| = |\sigma| + |\tau| \text{ and}$$

$$PA^n(\xi) + PA^n(\phi) \geq PA^n(\sigma) + PA^n(\tau)$$

Proof:(of Corollary 2) It is very similar to the proof of Corollary 1. Define $\check{\phi} = \{x \in Z^n; z - x \in \phi\}$, where $z = (l_1(\phi)+1, \ldots, l_n(\phi)+1)$, for the configuration obtained from ϕ by inversion on all lattice directions.

Apply Lemma 4 with $\xi_1 = \check{\phi}$ and $\xi_2 = \xi$ and take $\eta_1 = \sigma$ and $\eta_2 = \tau$. Again $|\sigma| = |\check{\phi} \cap \xi| < |\xi|$ because the point with all coordinates equal one is in ξ (as $\emptyset \neq \xi \in \Sigma^n$) but does not belong to $\check{\phi}$ (as it would imply that the point with coordinates $x_i = l_i(\phi)$, $1 \leq i \leq d$ belongs to η contradicting the hypothesis that $\phi \neq b(\phi)$). It is also clear that $\check{\phi} \cup \xi$ can not be larger than b and the same is also true for $\tau \in \widetilde{\mathcal{B}^n}(|\check{\phi} \cup \xi|)$.

$\square_{Corollary 2}$

We now apply these results to $\beta = \cup_{i=1}^{l_1(\beta)} \beta_i$, as in (16), and write $\beta_i = \cup_{j=1}^{d-1} \beta_{i,j}$, $\{\beta_{i,j}\}_{j=1}^{d-1}$ for their decomposition in blocks.

Let us say that β_j is *large* if $\beta_{j,d-1}$ is equal to $\beta_{1,d-1}$ translated to $\{x_1 = j\}$ along direction 1 and that it is *small* otherwise.

A β_j small can be of two types

1) $b(\beta_j)$ is smaller than $\beta_{1,d-1}$, that is, $b(\beta_j) \subset \rho, b(\beta_j) \neq \rho$, for ρ equal the translation of $\beta_{1,d-1}$ to the subspace $\{x_1 = j\}$.

2) $b(\beta_j)$ is equal to the traslation of $\beta_{1,d-1}$ to $\{x_1 = j\}$.

Suppose β is such that there is a β_j, $1 \leq j < l_1(\beta)$ which is small of type 1. In this case apply Corollary 1 with $n = d-1$, ξ equal $\beta_{l_1(\beta)}$ translated along direction 1 to $\{x_1 = 1\}$, ϕ equal β_j translated to $\{x_1 = 1\}$ and b equal $\beta_{1,d-1}$. Let γ be obtained by replacing β_j with τ and $\beta_{l_1(\beta)}$ by σ. More precisely

$$\gamma = \cup_{i=1}^{l_1(\beta)} \gamma_i$$

$\gamma_i \in \widetilde{\mathcal{B}^{d-1}}$ with

$$\gamma_i = \beta_i \text{ if } i \notin \{j, l_1(\beta)\},$$

$$\gamma_j = \{x : x - (j-1)e_1 \in \tau\} \text{ is the translation of } \tau \text{ to } \{x_1 = j\} \text{ and}$$

$$\gamma_{l_1(\beta)} = \{x : x - (l_1(\beta) - 1)e_1 \in \sigma\}$$

is the translation of σ along the positive direction 1 to get a configuration in $\{x_1 = l_1(\beta)\}$.

The number of lines of Z^d along direction 1 which intersects γ is still equal to A and Lemma 4 implies that the modification on slices j and $l_1(\beta)$ does not increase the projected area. Therefore

$$PA^d(\alpha) \geq PA^d(\beta) \geq PA^d(\gamma) \tag{19}$$

The procedure used to go from β to γ can be repeated as long as there is a β_j small of type 1 for some $1 \leq j < l_1(\beta)$. As it always remove points from the last slice, $\beta_{l_1(\beta)}$, it is possible that eventually this last slice is emptied. In this case, the argument proceeds for the new configuration which is shorter along direction 1 and with removals now occuring at the current last slice.

Let $\delta = \cup_{i=1}^{l_1(\delta)} \delta_i$, $l_1(\delta) \leq l_1(\beta)$, be the final configuration on which this procedure can no longer be applied because no δ_i is small of type 1 for $1 \leq i < l_1(\delta)$.

Suppose that there exists a δ_i which is small of type 2. Apply Corollary 2 again with $n = d-1$, ξ equal $\beta_{l_1(\beta)}$ translated along direction 1 to $\{x_1 = 1\}$, ϕ equal δ_i translated to $\{x_1 = 1\}$ and b equal $\beta_{1,d-1}$. Let ϵ be the configuration obtained by replacing δ_i with τ and $\beta_{l_1(\beta)}$ by σ given by the Corollary. As before we have

$$PA^d(\gamma) \geq PA^d(\delta) \geq PA^d(\epsilon) \tag{20}$$

Apply this procedure as many times as possible. Eventually it can no longer be applied because all slices are large (except, possibly, the last one). Call $\zeta = \cup_{j=1}^{l_1(\zeta)} \zeta_j$ this final subset of Z^d. Let K be the direction along which the protuberances $\cup_{i \leq d-2} \zeta_{j,i}$ ($\zeta_j = \cup_{i \leq d-1} \zeta_{j,i}$) are attached to $\zeta_{j,d-1}$ for $1 \leq j \leq l_1(\zeta) - 1$

$$K = \min\{k : l_k(\zeta_{j,d-1}) = l_d(\zeta_{j,d-1}), 1 < k \leq d\}$$

for $1 \leq j \leq l_1(\zeta) - 1$. It does not depend on j.

Therefore ζ may not be a d-dimensional parallelepiped with sides $l_i(\zeta)$, $1 \leq i \leq d$ only because two ((d-1)-dimensional) of its "faces" may be "eroded", that is, are partially full, namely, the ones at $\{x_1 = l_1(\zeta)\}$ and at $\{x_K = l_K(\zeta)\}$. More precisely

$$\zeta \supset \{1 \leq x_i \leq a_i\} \tag{21}$$

with $a_1 = l_1(\zeta) - 1, a_K = l_K(\zeta) - 1$ and $a_i = l_i(\zeta)$ for i different from 1 and K.

As each ζ_j is in $\widetilde{\mathcal{B}^{d-1}}$, its lenghts on $\{x_1 = j\}$ can assume at most two consecutive positive numbers.

Suppose that $b(\zeta)$ is not a block ($b(\zeta)$ as in (18)). That may be so because $l_1(\zeta)$ is either too short or too long.

Consider first the case on which $b(\zeta)$ is not a block and $l_1(\zeta) \leq l_K(\zeta) - 1$. Then

$$a \equiv |\zeta \cap \{x_K = l_K(\zeta)\}| \leq l_1(\zeta)\Pi_{i\neq 1,K} l_i(\zeta) \leq$$
$$(l_K(\zeta) - 1)\Pi_{i\neq 1,K} l_i(\zeta) \leq |\zeta \cap \{x_1 = 1\}| \equiv b$$

.

In this case define a new configuration η obtained from ζ by excluding the points in $\zeta \cap \{x_K = l_K(\zeta)\}$ and adding them on the (d-1)-dimensional subspace $\{x_1 = l_1(\zeta) + 1\}$ such that

$$\eta_{l_1(\eta)} \equiv \eta \cap \{x_1 = l_1(\zeta) + 1\} \in \mathcal{B}^{d-1}(|\zeta \cap \{x_K = l_K(\zeta)\}|)$$

and every line along direction 1 intersecting $\eta \cap \{x_1 = l_1(\eta) + 1\}$ also intersects $\zeta \cap \{x_1 = 1\}$.

This can be done as a consequence of the inequality between a and b above and Lemma 3. Note that

$$\eta_j = \zeta_j \setminus \{x_K = l_K(\zeta)\}$$

is a (d-1)-dimensional block by (21). Therefore η is also a non increasing sequence of elements in $\widetilde{\mathcal{B}^d} \cap \Sigma^d$ with $l_1(\eta) = l_1(\zeta) + 1$ and

$$PA^d(\epsilon) \geq PA^d(\zeta) \geq PA^d(\eta)$$

By repeating this procedure if necessary we eventualy reach a configuration with the appropriate lenght on direction 1.

Consider now the case $l_1(\zeta)$ is too big or, more precisely, that $b(\zeta)$ is not a block and $l_1(\zeta) \geq l_2(\zeta) + 1$. In this case we remove points on $\zeta \cap \{x_1 = l_1(\zeta)\}$ and add them to the subspace $\{x_K = l_K(\zeta) + 1\}$.

As all lines along direction 1 which intersect $\zeta_{l_1(\zeta)}$ also intersect ζ_1 we have

$$PA^d(\zeta) = PA^d(\zeta \setminus \zeta_{l_1(\zeta)}) + PA^{d-1}(\zeta_{l_1(\zeta)}). \tag{22}$$

By (21) we also have

$$(\zeta \setminus \zeta_{l_1(\zeta)}) \cap \{x_K = 1\} \supset \{1 \le x_i \le b_i\} \tag{23}$$

with $b_1 = l_1(\zeta) - 1$, $b_K = 1$ and $b_i = l_i(\zeta)$ for $i \notin \{1, K\}$. Moreover, as we assume $l_1(\zeta) \ge l_2(\zeta) + 1$,

$$\{1 \le x_i \le b_i\} \supset \{1 \le x_i \le c_i\} \tag{24}$$

with $c_1 = l_2(\zeta)$, $c_K = 1$ and $c_i = l_i(\zeta)$. The right hand side of (24) is a (d-1)-dimensional block (therefore an element of $\widetilde{\mathcal{B}^{d-1}}$) with volume that is larger or equal $|\zeta_{l_1(\zeta)}|$. Thus, by Lemma 3, we can find $\omega \in \mathcal{E}(\psi)$ such that a) $\omega \in \{x_K = l_K(\zeta) + 1\}$, b) $\psi \in \widetilde{\mathcal{B}^{d-1}}(|\zeta_{l_1(\zeta)}|)$ and c) all lines along direction K intersecting ω also intersects $\zeta \setminus \zeta_{l_1(\zeta)}$. If we define $\theta = (\zeta \setminus \zeta_{l_1(\zeta)}) \cup \omega$ we have

$$PA^d(\zeta) = PA^d(\zeta \setminus \zeta_{l_1(\zeta)}) + PA^{d-1}(\omega) = PA^d(\theta) \tag{25}$$

The first equality in (25) is true by (22), a) and b) of the definition of ω and part b) of Lemma 1. The second equality holds by c) (as in (14) above, with $\mathcal{L}^1$ replaced by $\mathcal{L}^K$, the set of lines of Z^d perpendicular to direction K). Now organize the slices of θ as done in transforming α to ζ and repeat the previous argument until the appropriate lenght along direction 1 is obtained.

Therefore if $b(\zeta)$ in (21) is not a block it can be modified as described above to a configuration, say ι, so that $\{1 \le x_i \le l_i(\iota), 1 \le i \le d\}$, is a d-dimensional block. By construction we have that ι is almost a block in the following sence

$$\{1 \le x_i \le e_i; 1 \le i \le d\} \subset \iota \subset \{1 \le x_i \le E_i; 1 \le i \le d\} \tag{26}$$

where $e_i = E_i = l_i(\iota)$ for $i \notin \{1, Q\}$, $e_i + 1 = E_i = l_i(\iota)$ for $i \in \{1, Q\}$ with $1 \le Q \le d$. Q is just the direction such that $\iota_j \setminus \{x_Q = l_Q(\iota)\}$ is a (d-1)-dimensional block with $\iota_j \equiv \iota \cap \{x_1 = j\}$ is chosen in $\widetilde{\mathcal{B}^{d-1}} \cap \Sigma^{d-1}$.

Suppose that the element of $\widetilde{\mathcal{B}^{d-1}}$ chosen on the subspace $\{x_1 = l_1(\iota)\}$, $u_{l_1(\zeta)}$, has lenght along direction Q that is smaller than $l_Q(\iota)$. In this case $u_{l_1(\iota)}$ and $\iota \cap \{x_Q = l_Q(\iota)\}$ are disjoint.

Therefore

$$PA^d(\iota) = PA^d(\chi) + PA^{d-1}(\iota_{l_1(\iota)}) + PA^{d-1}(\iota \cap \{x_Q = l_Q(\iota)\}) \qquad (27)$$

where $\chi = \{1 \leq x_i \leq e_i; 1 \leq i \leq d\}$ as in (ref1) is the block defined in (26).

To verify (27) note that the lines that could contribute to both (d-1)-dimensional projected areas in the right hand side while contributing only once to $PA^d(\iota)$ would have to be subset of $\{x_1 = l_1(\iota, x_Q = l_Q(\iota)\}$ which is disjoint from ι.

We then apply Corollaries 1 and 2 to $\iota_{l_1(\iota)}$ and $\iota \cap \{x_Q = l_Q(\iota)\}$, increasing one and decreasing the other until the largest one, say the one on $\{x_1 = l_1(\iota)\}$, which is in $\widetilde{\mathcal{B}^{d-1}}$, has its (d-1)-dimensional block equal to the face of χ. The resulting configuration is in $\mathcal{B}^d$ and Lemma 2 is proven in this case.

The last possibility to be considered is that $\iota_{l_1(\iota)}$ and $\iota \cap \{x_Q = l_Q(\iota)\}$ are not disjoint. In this case $\iota_{l_1(\iota)}$ must have its (d-1)-dimensional block equal to a face of χ and ι is already in $\mathcal{B}^d$. This finishes the proof of Lemma 2 and therefore of the Theorem.

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