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Abstract

We show that the family of recognizable $\mathcal{Z}$-subsets of A^* is closed under (integer) division by a positive integer. The technique that we use to prove this result is constructive and, by generalizing this construction, we obtain a characterization of recognizable $\mathcal{Z}$-subsets of A^+ as a sum of finitely many simple $\mathcal{Z}$-subsets of A^+. We also show that the family of recognizable $\mathcal{Z}$-subsets of A^* is not closed under division by a negative integer, or under taking the remainder of the division by a non-zero integer.

1 Introduction

In the seventies, S. Eilenberg [2] studied the recognizable subsets with multiplicities in an arbitrary semiring K, paying special attention to the cases of

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the Boolean semiring and the semiring of natural numbers. A more algebraic treatment of recognizable K-subsets is given by Berstel and Reutenauer [1].

In [7] and [8], we studied some properties of $\mathcal{M}$-subsets of A^*, where $\mathcal{M}$ is the tropical semiring. For background and the most important results about $\mathcal{M}$-subsets see Simon [12, 13, 14, 15, 16], Hashiguchi [3, 4, 5, 6], Leung [11] and Krob [9, 10].

This paper is concerned with the corresponding theory for the semiring $\mathcal{Z}$, which is just an extension of $\mathcal{M}$ to the set of all integers; that is, $\mathcal{Z}$ consists of the integer numbers extended with ∞ and equipped with the minimum and addition operations.

Here, by considering a certain construction of $\mathcal{Z}$-automata, we prove two results concerning to the recognizable $\mathcal{Z}$-subsets of A^*. The first of these (see Theorem 2) asserts that, if X is a recognizable $\mathcal{Z}$-subset of A^* and d is a positive integer, the $\mathcal{Z}$-subset $Y = X$ div d is recognizable, where wY is the division quotient of wX by d, for all w in A^*. The second result (see Theorem 15) gives a characterization of recognizable $\mathcal{Z}$-subsets through simple $\mathcal{Z}$-subsets. More precisely, we show that every recognizable $\mathcal{Z}$-subset of A^+ is the sum of a finite number of simple $\mathcal{Z}$-subsets of A^+.

We also show (see Lemma 8) that if d is a negative integer, X div d is not always a recognizable $\mathcal{Z}$-subset, and (see Lemma 10) if d is a non-zero integer, $Y = X$ mod d is not always a recognizable $\mathcal{Z}$-subset, where wY is the division remainder of wX by d, for all w in A^*.

Eilenberg [2] showed that if X is a recognizable $\mathbf{N}$-subset of A^*, where $\mathbf{N}$ is the semiring of the natural numbers, and d is a positive integer, the $\mathbf{N}$-subsets $Y_1 = X$ div d and $Y_2 = X$ mod d are recognizable. Moreover, in [2] it is proved that $X = dY_1 + Y_2$, and hence X can be 'recovered' from Y_1 and Y_2. However, we prove that for the recognizable $\mathcal{Z}$-subsets, such an 'inversion operation' to division does not exist.

2 The semiring $\mathcal{Z}$, $\mathcal{Z}$-subsets and $\mathcal{Z}$-A-automata

The *semiring* $\mathcal{Z}$ has as support $\mathbf{Z} \cup \infty$ and as operations the minimum and the addition. The minimum plays the rôle of semiring addition and the addition plays the rôle of semiring multiplication. Note that $\mathcal{Z}$ is a commutative semiring and the identities with respect to minimum and addition are ∞ and 0, respectively.

The subsemiring $\mathcal{Z}^-$ of $\mathcal{Z}$ consists of the nonpositive integers and ∞. It is isomorphic to $\mathcal{M}^d$, the dual of $\mathcal{M}$, whose support is $\mathbb{N} \cup -\infty$ and whose operations are the maximum and the addition.

Let A be a finite alphabet. A *$\mathcal{Z}$-subset* X of A^* is a function $X : A^* \to \mathcal{Z}$. For each w in A^*, wX is called the *multiplicity* with which w *belongs* to X. If $1X = \infty$ then we also say that X is a $\mathcal{Z}$-subset of A^+.

The following operations are defined over $\mathcal{Z}$-subsets of A^*, where $\{ X_i \mid i \in I\}$ is a family of $\mathcal{Z}$-subsets of A^* indexed by a set I, X and Y are $\mathcal{Z}$-subsets of A^*, and $m \in \mathcal{Z}$. For (a) and (b) we assume that I is finite, and for (e) and (f) we assume that $1X = \infty$.

(a) $\forall w \in A^*, \quad w(\min_{i\in I} X_i) = \min_{i\in I}(wX_i)$ (minimum)

(b) $\forall w \in A^*, \quad w(\sum_{i\in I} X_i) = \sum_{i\in I}(wX_i)$ (addition)

(c) $\forall w \in A^*, \quad w(XY) = \min_{xy=w}(xX + yY)$ (concatenation)

(d) $\forall w \in A^*, \quad w(m + X) = m + wX$

(e) $\forall w \in A^*, \quad wX^+ = w(\min_{n\geq 1} X^n) = \min_{n\geq 1}(wX^n)$

(f) $X^* = \min(\mathbf{1}, X^+)$, where the $\mathcal{Z}$-subset $\mathbf{1}$ is defined by $\forall w \in A^*. \quad w\mathbf{1} = 0$ if $w = 1$ and $w\mathbf{1} = \infty$, otherwise.

Recall that, for any semiring K, one naturally has the operations of addition, intersection, and multiplication of K-subsets. In the case in which $K = \mathcal{Z}$, these operations are, respectively, the ones given in (a), (b) and (c) above.

Observe that if $I = \emptyset$, $\min_{i\in I}(m_i) = \infty$ and $\sum_{i\in I} m_i = 0$.

The family $\mathcal{Z}\langle\langle A\rangle\rangle$ of all $\mathcal{Z}$-subsets of A^* with the minimum (a) and concatenation (c) operations constitutes a semiring, whose identities are, respectively, the $\mathcal{Z}$-subset $\emptyset$ (where, for all $w \in A^*, w\emptyset = \infty$) and the $\mathcal{Z}$-subset $\mathbf{1}$.

A *$\mathcal{Z}$-A-automaton* $\mathcal{A} = (Q, I, T)$ is an automaton over A, with a finite set Q of states, two $\mathcal{Z}$-subsets I and T of Q and a $\mathcal{Z}$-subset $E_{\mathcal{A}}$ of $Q \times A \times Q$.

If $pI \neq \infty$ (resp. $pT \neq \infty$), we say that p is an *initial state* (resp. *final state*) of $\mathcal{A}$.

If (p, a, q) is an *edge* in $\mathcal{A}$, we say that its *label* is a and that its *multiplicity* is $(p, a, q)E_{\mathcal{A}}$. If $(p, a, q)E_{\mathcal{A}} \neq \infty$, the edge (p, a, q) is said to be a *useful edge* of $\mathcal{A}$.

If P is a *path* of length n in $\mathcal{A}$, with *origin* p_0 and *terminus* p_n, that is

$$P = (p_0, a_1, p_1)(p_1, a_2, p_2)\ldots(p_{n-1}, a_n, p_n) \ .$$

then its *label* is $|P| = a_1a_2\ldots a_n$ and its *multiplicity* $\|P\|$ is the sum of the multiplicities of its edges, that is

$$\|P\| = \sum_{i=1}^{n}(p_{i-1}, a_i, p_i)E_{\mathcal{A}} \ .$$

For convenience, if P is the path above, we also write

$$P = (p_0, a_1a_2\ldots a_n, p_n) \quad \text{and} \quad P : p_0 \xrightarrow{a_1a_2\ldots a_n} p_n \ .$$

Concatenations, factorizations and factors of paths are defined as usual.

A *path* P is *useful* if $\|P\| \neq \infty$. A useful path, whose origin i and terminus t satisfy $iI \neq \infty$ and $tT \neq \infty$, is called *successful*.

The *behavior* of $\mathcal{A}$ is the $\mathcal{Z}$-subset $\|\mathcal{A}\|$ of A^* that associates a multiplicity to each word as follows. Let w be in A^* and let C be the set of successful paths P in $\mathcal{A}$ with label $|P| = w$. Then,

$$w\|\mathcal{A}\| = \min_{P\in C}(iI + \|P\| + tT) \ ,$$

where i and t are the origin and the terminus of the path P, respectively.

A successful path P in $\mathcal{A}$, with label w, origin i and terminus t, is called *victorious*, if $iI + \|P\| + tT = w\|\mathcal{A}\|$.

The structure $\mathcal{C} = (Q, E_{\mathcal{C}})$ over A, consisting of a finite set of states Q, a set of edges $Q \times A \times Q$ and a $\mathcal{Z}$-subset $E_{\mathcal{C}}$ of $Q \times A \times Q$, is called a *$\mathcal{Z}$-A-semiautomaton*. From $\mathcal{C}$ we can construct a $\mathcal{Z}$-A-automaton $\mathcal{A}$ by introducing two $\mathcal{Z}$-subsets I and T of Q. In this case, $\mathcal{A}$ can also be denoted by $\mathcal{A} = (\mathcal{C}, I, T)$.

We say that a $\mathcal{Z}$-A-automaton $\mathcal{A} = (Q, I, T)$ is *normalized* if $\mathcal{A}$ has a unique initial state i and a unique final state t, with $t \neq i$ and $iI = tT = 0$.

and, moreover, there are neither edges with terminus i nor edges with origin t.

We say that a $\mathcal{Z}$-A-*automaton* $\mathcal{A} = (Q, I, T)$ is *simple* if

$$(Q \times A \times Q)E_{\mathcal{A}} \subseteq \{0, 1, -1, \infty\}, \quad QI \subseteq \{0, \infty\} \quad \text{and} \quad QT \subseteq \{0, \infty\} .$$

It is important to observe that in a normalized or simple $\mathcal{Z}$-A-automaton $\mathcal{A}$, every victorious path P with label w satisfies $\|P\| = w\|\mathcal{A}\|$ (because $QI, QT \subseteq \{0, \infty\}$) and every successful path P' with label w is such that $w\|\mathcal{A}\| \leq \|P'\|$ (because $\|P\| \leq \|P'\|$). These properties will be frequently used in the proofs.

A $\mathcal{Z}$-*subset* of A^* is *recognizable* if it is the behavior of some $\mathcal{Z}$-A-automaton. It is well known that every recognizable $\mathcal{Z}$-subset of A^+ is the behavior of a normalized $\mathcal{Z}$-A-automaton. The family of all recognizable $\mathcal{Z}$-subsets of A^* is denoted by $\mathcal{Z}\operatorname{Rec} A^*$.

A class of recognizable $\mathcal{Z}$-subsets of A^* that has received some attention is that of simple $\mathcal{Z}$-subsets of A^*, denoted by $\mathcal{Z}\operatorname{SRec} A^*$. A $\mathcal{Z}$-*subset* of A^* is *simple* if it is the behavior of some simple $\mathcal{Z}$-A-automaton. We showed [7, 8] that the family of simple $\mathcal{M}$-subsets of A^* is a proper subfamily of all recognizable $\mathcal{M}$-subsets of A^*. This result can be easily extended to the family of recognizable $\mathcal{Z}$-subsets of A^*; that is, $\mathcal{Z}\operatorname{SRec} A^* \subsetneq \mathcal{Z}\operatorname{Rec} A^*$.

Let us denote by $\mathbf{A}^+$ the $\mathcal{Z}$-subset of A^* such that

$$\forall w \in A^*, \quad w\mathbf{A}^+ = \begin{cases} \infty & \text{if } w = 1 \\ 0 & \text{otherwise} . \end{cases}$$

Then, one can easily verify the following result.

Proposition 1 *For every recognizable $\mathcal{Z}$-subset X of A^* there exists a normalized $\mathcal{Z}$-A-automaton $\mathcal{A}$ such that $\|\mathcal{A}\| = X + \mathbf{A}^+$.*

■

3 Closure of $\mathcal{Z}\operatorname{Rec} A^*$ under the division by an integer

We studied the closure properties of the family of recognizable $\mathcal{M}$-subsets of A^* and of two of its subfamilies under several operations. These results can be found in our doctoral thesis [7] and in [8]. Here, we investigate the closure properties of families of $\mathcal{Z}$-subsets under taking the quotient and the remainder of the division by an integer different from zero.

The quotient (div) and the remainder (mod) of the integer division over the natural numbers can be extended to the semiring $\mathcal{Z}$ by putting

$$\forall d \neq 0, \quad \infty \text{ div } d = \infty, \quad \infty \bmod d = \infty \quad \text{and}$$

$$\forall m \in \mathbf{Z}, \quad m \text{ div } d = k \quad \text{and} \quad m \bmod d = r \ .$$

where k and r are the unique integers such that $kd + r = m$ and $0 \leq r < |d|$.

Observe that in this definition the remainder is always non-negative and the following properties are satisfied:

$$m \text{ div } d = -(m \text{ div } -d) \quad \text{and} \quad m \bmod d = m \bmod -d \ .$$

We can extend the operations div and mod to the $\mathcal{Z}$-subsets of A^* as follows. Let X be a $\mathcal{Z}$-subset of A^* and let $d \neq 0$. The $\mathcal{Z}$-subsets X div d and X mod d of A^* are defined by:

$$\forall w \in A^*, \quad w(X \text{ div } d) = wX \text{ div } d \quad \text{and} \quad w(X \bmod d) = wX \bmod d \ .$$

Theorem 2 *Let d be a positive integer. If X is a recognizable $\mathcal{Z}$-subset of A^+ then X div d is a recognizable $\mathcal{Z}$-subset of A^+.*

In the proof of Theorem 2 we will construct a $\mathcal{Z}$-A-automaton $\mathcal{B} = (Q, I, T)$ from a normalized $\mathcal{Z}$-A-automaton $\mathcal{A} = (Q_\mathcal{A}, I_\mathcal{A}, T_\mathcal{A})$ such that $\|\mathcal{B}\| = \|\mathcal{A}\|$ div d. The idea is to construct $\mathcal{B}$ from d 'copies' of $\mathcal{A}$.

Let us first construct a $\mathcal{Z}$-A-semiautomaton $\mathcal{C}$, depending on $\mathcal{A}$, which will also be used in the next section. For convenience, for an integer $d \geq 1$, put $[1, d] = \{1, \ldots, d\}$. Let $\mathcal{C} = (Q, E_\mathcal{C})$, where $Q = Q_\mathcal{A} \times [1, d]$ and the useful edges of $\mathcal{C}$ with their respective multiplicities are defined as follows.

Let $\alpha' = (p, a, q)$ be a useful edge of $\mathcal{A}$. Let us consider

$$k = \alpha' E_\mathcal{A} \text{ div } d \quad \text{and} \quad r = \alpha' E_\mathcal{A} \bmod d \ .$$

Then $\alpha' E_\mathcal{A} = kd + r$.

For each $i \in [1, d]$, $\alpha = ((p, i), a, (q, j))$ is a useful edge of $\mathcal{C}$, satisfying

- if $i > r$, $j = i - r$ and $\alpha E_C = k$; thus,

$$\alpha' E_{\mathcal{A}} = kd + r = d(\alpha E_C) + i - j :$$

- if $i \leq r$, $j = i - r + d$ and $\alpha E_C = k + 1$; thus,

$$\alpha' E_{\mathcal{A}} = kd + r = kd + d + r - d = d(k+1) + r - d = d(\alpha E_C) + i - j .$$

In both cases, $j \in [1, d]$ and $\alpha' E_{\mathcal{A}} = d(\alpha E_C) + i - j$. Note that this condition uniquely defines both j and αE_C, for every i and $\alpha' E_{\mathcal{A}}$.

Let us see what we can say about the edge multiplicities of C when d is the maximum of the absolute values of the multiplicities of the useful edges of $\mathcal{A}$. This is, in fact, the situation we will have in the next section. If each useful edge α' of $\mathcal{A}$ is such that $0 \leq |\alpha' E_{\mathcal{A}}| \leq d$, it results that $k = \alpha' E_{\mathcal{A}}$ div d is 0 or 1 or -1. Also, $k = 1$ if and only if $\alpha' E_{\mathcal{A}} = d$ and, in this case, $r = \alpha' E_{\mathcal{A}} \bmod d = 0$. Then,

$$\forall i \in [1, d], \qquad \alpha E_C = \begin{cases} k \text{ (which can be 0 or 1 or } -1) & \text{if } i > r \\ k \text{ (which can be 0 or } -1) + 1 & \text{if } i \leq r \end{cases} .$$

Thus the edge multiplicities of C are in $\{0, 1, -1, \infty\}$.

In the sequel, we study some properties relating paths in $\mathcal{A}$ with the corresponding paths in C and vice versa.

Let $P_{\mathcal{A}}$ and P_C be the sets of useful paths in $\mathcal{A}$ and in C, respectively. Let us define a function $\Psi : P_C \to P_{\mathcal{A}}$ as follows. If

$$P = ((p_0, i_0), a_1, (p_1, i_1))((p_1, i_1), a_2, (p_2, i_2)) \ldots ((p_{n-1}, i_{n-1}), a_n, (p_n, i_n))$$

is a useful path in C, then

$$P\Psi = (p_0, a_1, p_1)(p_1, a_2, p_2) \ldots (p_{n-1}, a_n, p_n) .$$

It is easy to see that $P\Psi$ is a useful path in $\mathcal{A}$ and we say that $P\Psi$ is the *projection* of P in $\mathcal{A}$. On the other hand, one can see that for each useful path P' in $\mathcal{A}$ and for each $i \in [1, d]$, there exists a unique useful path P in C, with origin in $Q_{\mathcal{A}} \times \{i\}$, whose projection in $\mathcal{A}$ is P'. Such a path P will be called the *i-lifting* of P' in C. The following lemma relates the multiplicities of a useful path in C and of its projection.

Lemma 3 *Let P be a useful path in $\mathcal{C}$ from (p,i) to (q,j), i and $j \in [1,d]$. Then its projection P' in $\mathcal{A}$ satisfies*

$$\|P'\| = d\|P\| + i - j \ .$$

Proof. Let P be a useful path in $\mathcal{C}$ from (p,i) to (q,j). Let w be the label of P, $w = w_1 \dots w_t$, with $w_l \in A$, $(1 \le l \le t)$. The proof is by induction on the length t of the path P.

If $t = 1$, then $P = ((p,i), w, (q,j))$ is a useful edge of $\mathcal{C}$. Let $P' = (p, w, q)$ be the projection of P in $\mathcal{A}$. By the construction of $\mathcal{C}$, we can verify that

$$\|P'\| = d\|P\| + i - j \ ,$$

as required. Hence, let us suppose that $t > 1$ and that the lemma is valid for useful paths in $\mathcal{C}$ of length less than t. Then, the path $P = ((p,i), w, (q,j))$ can be decomposed in the path $P_1 = ((p,i), w_1 \dots w_{t-1}, (s,l))$ and in the edge $\alpha = ((s,l), w_t, (q,j))$, for some $s \in Q$ and $l \in [1,d]$, such that $P = P_1\alpha$.

By the induction hypothesis applied to the path $P_1 = ((p,i), w_1 \dots w_{t-1}, (s,l))$, its projection ${P_1}' = (p, w_1 \dots w_{t-1}, s)$ in $\mathcal{A}$ satisfies

$$\|{P_1}'\| \ = \ d\|P_1\| + i - l \ .$$

Let $\alpha' = (s, w_t, q)$ be the projection of $\alpha = ((s,l), w_t, (q,j))$ in $\mathcal{A}$. By the construction of $\mathcal{C}$, we have that

$$\|\alpha'\| \ = \ d\|\alpha\| + l - j \ .$$

Thus, the path $P' = {P_1}'\alpha' = (p, w_1 \dots w_{t-1}, s)(s, w_t, q)$ from p to q is the projection of P in $\mathcal{A}$ and

$$\|P'\| = \|{P_1}'\| + \|\alpha'\| = d\|P_1\| + i - l + d\|\alpha\| + l - j = d(\|P_1\| + \|\alpha\|) + i - j \ .$$

Therefore, $\|P'\| = d\|P\| + i - j$.

∎

The crucial property of the construction of $\mathcal{C}$ is stated in Lemma 3 above; it says that for every useful path P in $\mathcal{C}$ and its projection P' in $\mathcal{A}$, the difference $\|P'\| - d\|P\|$ only depends on the origin and the terminus of the path P.

Corollary 4 *Let P be a useful path in $\mathcal{C}$ from (p,i) to (q,j), i and $j \in [1,d]$. Let P' be the projection of P in $\mathcal{A}$. Then*

$$\|P\| = \begin{cases} \|P'\| \text{ div } d & \text{if } i - j \geq 0 \\ 1 + \|P'\| \text{ div } d & \text{if } i - j < 0 \ . \end{cases}$$

■

Proof of Theorem 2. For $d = 1$ we have nothing to prove.

Let $d \geq 2$. Let X be a recognizable $\mathcal{Z}$-subset of A^+ and let $\mathcal{A} = (Q_\mathcal{A}, I_\mathcal{A}, T_\mathcal{A})$ be a normalized $\mathcal{Z}$-A-automaton such that $\|\mathcal{A}\| = X$.

Let us construct a $\mathcal{Z}$-A-automaton $\mathcal{B} = (\mathcal{C}, I, T)$ from the $\mathcal{Z}$-A-semiautomaton $\mathcal{C} = (Q, E_\mathcal{C})$, whose construction and properties we just described. For this, let us define the $\mathcal{Z}$-subsets I and T of Q:

$$(q,d)I = qI_\mathcal{A} \ (\forall q \in Q_\mathcal{A}) \quad \text{and} \quad (q,j)I = \infty \quad (\forall q \in Q_\mathcal{A}, \quad \forall j \in [1, d-1]) \ ;$$

$$(q,j)T = qT_\mathcal{A} \quad (\forall q \in Q_\mathcal{A}, \quad \forall j \in [1,d]) \ .$$

We wish to prove that $\|\mathcal{B}\| = \|\mathcal{A}\|$ div d. Let $w \in A^+$ be such that $w\|\mathcal{A}\| \neq \infty$ and let P' be a victorious path in $\mathcal{A}$, with label w. By Corollary 4, the d-lifting P of P' in $\mathcal{B}$ satisfies

$$\|P\| = \|P'\| \text{ div } d \ ;$$

hence,

$$w\|\mathcal{B}\| \leq \|P\| = w\|\mathcal{A}\| \text{ div } d \ . \tag{1}$$

Let now P_1 be a victorious path in $\mathcal{B}$, with label w. Let P_1' be the projection of P_1 in $\mathcal{A}$. Then, remembering that the origin of P_1 lies in $Q_\mathcal{A} \times \{d\}$, and using Corollary 4, we have that

$$w\|\mathcal{B}\| = \|P_1\| = \|P_1'\| \text{ div } d \geq \|P'\| \text{ div } d = w\|\mathcal{A}\| \text{ div } d \ . \tag{2}$$

Thus, from (1) and (2), we have that $w\|\mathcal{B}\| = w\|\mathcal{A}\|$ div d.

Moreover, we observe that $1\|\mathcal{B}\| = \infty$ and if $w\|\mathcal{A}\| = \infty$ then Corollary 4 implies that $w\|\mathcal{B}\| = \infty$. Thus, $\|\mathcal{B}\| = \|\mathcal{A}\|$ div $d = X$ div d. Therefore, X div d is a recognizable $\mathcal{Z}$-subset of A^+.

■

In the proof of Theorem 2, if $\mathcal{A}$ is an $\mathcal{M}$-A-automaton (resp. $\mathcal{Z}^-$-A-automaton), $\mathcal{B}$ will be an $\mathcal{M}$-A-automaton (resp. $\mathcal{Z}^-$-A-automaton). Thus, Theorem 2 is also valid to the recognizable $\mathcal{M}$-subsets (resp. $\mathcal{Z}^-$-subsets).

Corollary 5 *Let d be a positive integer. If X is a recognizable $\mathcal{M}$-subset (resp. $\mathcal{Z}^-$-subset) of A^+ then X div d is a recognizable $\mathcal{M}$-subset (resp. $\mathcal{Z}^-$-subset) of A^+.*

■

Theorem 2 can be easily extended for the family of all recognizable $\mathcal{Z}$-subsets as can Corollary 5 for the family of all recognizable $\mathcal{M}$-subsets and $\mathcal{Z}^-$-subsets of A^*.

Corollary 6 *Let d be a positive integer. $\mathcal{Z}\,\mathrm{Rec}\,A^*$, $\mathcal{M}\,\mathrm{Rec}\,A^*$ and $\mathcal{Z}^-\,\mathrm{Rec}\,A^*$ are closed under* div d.

Proof. Let $X \in \mathcal{Z}\,\mathrm{Rec}\,A^*$. By Theorem 2, $(X + \mathbf{A}^+)$ div d is a recognizable $\mathcal{Z}$-subset of A^+. Thus, X div $d = \min((X + \mathbf{A}^+)$ div $d, (1X$ div $d) + \mathbf{1})$ is a recognizable $\mathcal{Z}$-subset of A^*.

The proof for X in $\mathcal{M}\,\mathrm{Rec}\,A^*$ or $\mathcal{Z}^-\,\mathrm{Rec}\,A^*$ is similar.

■

It follows from Theorem 2 and Corollary 6 that the family of simple $\mathcal{Z}$-subsets of A^* is also closed under integer division by a positive integer.

Corollary 7 *Let d be a positive integer. $\mathcal{Z}\,\mathrm{SRec}\,A^*$ is closed under* div d.

■

In the sequel, we verify that if d is a negative integer, $\mathcal{Z}\,\mathrm{Rec}\,A^*$ is not closed under div d.

Lemma 8 *Let d be a negative integer. $\mathcal{Z}\,\mathrm{Rec}\,A^*$ is not closed under* div d.

Proof. Let $A = \{a, b\}$ and let X be the $\mathcal{Z}$-subset of A^* defined by $1X = \infty$ and $\forall w \in A^+, wX = \min\{-|w|_a, -|w|_b\}$. It is clear that $X \in \mathcal{Z}\,\mathrm{Rec}\,A^*$.

Let us consider the $\mathcal{Z}$-subset $Y = X$ div -1. By the definition of Y, we have $1Y = \infty$ and $\forall w \in A^+$, $wY = w(X \text{ div } -1) = wX \text{ div } -1 = -wX = -\min\{-|w|_a, -|w|_b\} = \max\{|w|_a, |w|_b\}$.

Suppose that Y is a recognizable $\mathcal{Z}$-subset of A^*. Then, there is a normalized $\mathcal{Z}$-A-automaton $\mathcal{A}$ such that $\|\mathcal{A}\| = Y$.

Consider the word $w = a^n b^n$, where n is the number of the states of $\mathcal{A}$. Then $wY = n$.

Let P be a victorious path in $\mathcal{A}$, spelling w. Then, $\|P\| = w\|\mathcal{A}\| = wY = n$ and exist naturals r, s and t, with $s > 0$ and $r + s + t = n$ such that the path P can be factorized as

$$P : q_0 \xrightarrow{a^n} q_1 \xrightarrow{b^r} q_2 \xrightarrow{b^s} q_2 \xrightarrow{b^t} q_3 \ .$$

Consider the factor $P_1 = (q_2, b^s, q_2)$ of P. If $\|P_1\| \leq 0$, exists a successful path P' in $\mathcal{A}$,

$$P' : q_0 \xrightarrow{a^n} q_1 \xrightarrow{b^r} q_2 \xrightarrow{b^s} q_2 \xrightarrow{b^s} q_2 \xrightarrow{b^t} q_3 \ ,$$

spelling the word $w' = a^n b^{n+s}$ such that

$$\|P'\| \leq \|P\| = n \ .$$

Then $w'\|\mathcal{A}\| \leq \|P'\| \leq n$. This is a contradiction because

$$w'Y = \max\{n, n+s\} \geq n+1 \ .$$

Thus, $\|P_1\| > 0$. And, in this case, exists a successful path P'' in $\mathcal{A}$,

$$P'' : q_0 \xrightarrow{a^n} q_1 \xrightarrow{b^r} q_2 \xrightarrow{b^t} q_3 \ ,$$

spelling the word $w'' = a^n b^{n-s}$ such that

$$\|P''\| < \|P\| = n \ .$$

Then $w''\|\mathcal{A}\| \leq \|P''\| < n$. This is a contradiction because

$$w''Y = \max\{n, n-s\} = n \ .$$

Hence, Y is not a recognizable $\mathcal{Z}$-subset of A^*. Therefore, $\mathcal{Z}$ Rec A^* is not closed under div d, when d is a negative integer. ∎

A consequence of the proof of the previous lemma is the statement in the next lemma (see [8]) which was also showed by Krob [10] in another context.

Lemma 9 *There is a $\mathcal{Z}$-subset X of A^* such that X is recognizable but $-X$ is not.*

■

We saw that $\mathcal{Z}\,\mathrm{Rec}\,A^*$ is closed under the div d operation when d is a positive integer. It turns out that, however, this is not true for the mod operation.

Lemma 10 *Let d be an integer, $d \neq 0$. $\mathcal{Z}\,\mathrm{Rec}\,A^*$ is not closed under* mod *d.*

Proof. Let $A = \{a, b, c\}$ and let X be the $\mathcal{Z}$-subset of A^* defined by

$$\forall w \in A^*, \quad wX = \min\{|w|_c, -2|w|_a, -2|w|_b - 1\} \ .$$

It is clear that $X \in \mathcal{Z}\,\mathrm{Rec}\,A^*$.

Let us consider the $\mathcal{Z}$-subset $Y = X \bmod 2 = X \bmod -2$. Then,

$$\forall w \in A^*, \quad wY = \begin{cases} 0 & \text{if } (w \in c^* \text{ and } |w|_c \text{ is even) or } |w|_a > |w|_b \\ 1 & \text{if } (w \in c^* \text{ and } |w|_c \text{ is odd) or } |w|_a \leq |w|_b \end{cases} .$$

Suppose that Y is a recognizable $\mathcal{Z}$-subset of A^*. In this case, there is a $\mathcal{Z}$-A-automaton $\mathcal{A} = (Q, I, T)$ such that $\|\mathcal{A}\| = Y$.

Let $n = |Q|$ and let us consider the word $w = a^{n+1}b^n$. Then, there is a victorious path P in $\mathcal{A}$, with $|P| = w$ and $\|P\| = w\|\mathcal{A}\| = wY = 0$. Moreover, there are naturals r, s and t, with $s > 0$ and $r + s + t = n + 1$ such that the path P can be decomposed as

$$P : i \xrightarrow{a^r} p \xrightarrow{a^s} p \xrightarrow{a^t} q \xrightarrow{b^n} f \ .$$

Consider the factor (p, a^s, p) of P. If $\|(p, a^s, p)\| \geq 0$, the path

$$P_1 : i \xrightarrow{a^r} p \xrightarrow{a^t} q \xrightarrow{b^n} f$$

spells the word $w' = a^{n+1-s}b^n$ and we have that

$$w'\|\mathcal{A}\| \leq \|P_1\| \leq \|P\| = 0 \ .$$

But $w'Y = 1$, contradicting that $Y = \|\mathcal{A}\|$.

Hence, $\|(p, a^s, p)\| < 0$ and in this case, the path

$$P_2 : i \xrightarrow{a^r} p \xrightarrow{a^s} p \xrightarrow{a^s} p \xrightarrow{a^t} q \xrightarrow{b^n} f$$

spells the word $w'' = a^{n+1+s}b^n$ and we have that

$$w''\|\mathcal{A}\| \leq \|P_2\| < \|P\| = 0 \ .$$

But $w''Y = 0$, contradicting that $Y = \|\mathcal{A}\|$.

Therefore, $Y = X \bmod 2 = X \bmod -2$ is not a recognizable $\mathcal{Z}$-subset of A^*.

■

The quotient (div) and the remainder (mod) were defined in such a way that the remainder is always non-negative. However, there are cases in which one defines the integer division (div′ and mod′) so that the remainder has the same sign as the dividend. That is,

$$\forall d \neq 0, \quad \infty \text{ div}' d = \infty, \quad \infty \text{ mod}' d = \infty \quad \text{and}$$

$$\forall m \in \mathbf{Z}, \quad m \text{ div}' d = k \quad \text{and} \quad m \text{ mod}' d = r \ ,$$

where k and r are the unique integers such that $kd + r = m$, $0 \leq |r| < |d|$ and $rm \geq 0$.

The following properties are also satisfied:

$$m \text{ div}' d = -(m \text{ div}' - d) \quad \text{and} \quad m \text{ mod}' d = m \text{ mod}' - d \ .$$

As before, we can extend the operation div′ and mod′ to the $\mathcal{Z}$-subsets of A^*. Let X be a recognizable $\mathcal{Z}$-subset of A^* and let $d \neq 0$. The $\mathcal{Z}$-subsets X div′ d and X mod′ d of A^* are defined by

$$\forall w \in A^*, \quad w(X \text{ div}' d) = wX \text{ div}' d \quad \text{and} \quad w(X \text{ mod}' d) = wX \text{ mod}' d \ .$$

Let us see how the two operations div′ and mod′ relate to the standard div and mod.

Let $d \neq 0$ and $m \in \mathbf{Z}$ be given, and let

$$m \text{ div } d = k_1, \quad m \text{ mod } d = r_1,$$

$$m \operatorname{div}' d = k_2, \qquad m \operatorname{mod}' d = r_2 .$$

Then

$$k_2 = \begin{cases} k_1 & \text{if } m \geq 0 \text{ or } r_1 = 0 \\ k_1 + 1 & \text{if } m < 0 \text{ and } r_1 > 0 \text{ and } d > 0 \\ k_1 - 1 & \text{if } m < 0 \text{ and } r_1 > 0 \text{ and } d < 0 \end{cases}$$

$$\text{and} \quad r_2 = \begin{cases} r_1 & \text{if } m \geq 0 \text{ or } r_1 = 0 \\ r_1 - |d| & \text{if } m < 0 \text{ and } r_1 > 0 . \end{cases}$$

Let us show that there is a recognizable $\mathcal{Z}$-subset X of A^* such that $X \operatorname{mod}' d$ and $X \operatorname{div}' d$ are not recognizable $\mathcal{Z}$-subsets of A^*.

Theorem 11 *Let d be a positive integer. $\mathcal{Z}\operatorname{Rec} A^*$ is not closed under* $\operatorname{div}' d$.

Proof. Let $A = \{a, b\}$ and let X be the $\mathcal{Z}$-subset of A^* defined by

$$\forall w \in A^*, \quad wX = 2(|w|_a - |w|_b) + 1 .$$

It is clear that X is a recognizable $\mathcal{Z}$-subset and we observe that $\forall w \in A^*$, $wX \bmod 2 = 1$.

Consider the $\mathcal{Z}$-subsets of A^*, $F = X \operatorname{div} 2$ and $G = X \operatorname{div}' 2$. Then, from the observations in the definition of div', we have that

$$\forall w \in A^*, \quad wG = \begin{cases} wF & \text{if } wF \geq 0 \\ wF + 1 & \text{if } wF < 0 . \end{cases}$$

But, we can observe that, $\forall w \in A^*$, $wF = |w|_a - |w|_b$. Therefore, G can be described by

$$\forall w \in A^*, \quad wG = \begin{cases} |w|_a - |w|_b & \text{if } |w|_a \geq |w|_b \\ |w|_a - |w|_b + 1 & \text{if } |w|_a < |w|_b . \end{cases}$$

We will show that G is not a recognizable $\mathcal{Z}$-subset of A^*.

Let us suppose that G is a recognizable $\mathcal{Z}$-subset of A^*. In this case, there is a $\mathcal{Z}$-A-automaton $\mathcal{A} = (Q, I, T)$ such that $\|\mathcal{A}\| = G$.

Let $n = |Q|$ and let us consider the word $w = a^n b^n$. Then, there is a victorious path P in $\mathcal{A}$, with $|P| = w$ and $\|P\| = w\|\mathcal{A}\| = wG = 0$.

Moreover, there are naturals r, s and t, with $s > 0$ and $r + s + t = n$ such that the path P can be decomposed as

$$P : i \xrightarrow{a^r} p \xrightarrow{a^s} p \xrightarrow{a^t} q \xrightarrow{b^n} f \ .$$

First, suppose that the multiplicity of the factor (p, a^s, p) of P is zero or negative. In this case, the path

$$P_1 : i \xrightarrow{a^r} p \xrightarrow{a^s} p \xrightarrow{a^s} p \xrightarrow{a^t} q \xrightarrow{b^n} f$$

spells the word $w' = a^{n+s}b^n$ and $\|P_1\| \leq \|P\|$. Then, we have that

$$w'\|\mathcal{A}\| \leq \|P_1\| \leq \|P\| = 0 \ .$$

But

$$w'G = |w'|_a - |w'|_b = s > 0 \ ,$$

contradicting that $G = \|\mathcal{A}\|$.

Now, suppose that $0 < \|(p, a^s, p)\| < s$. In this case, for the path P_1 and the word w' described above, we have that

$$w'\|\mathcal{A}\| \leq \|P_1\| < \|P\| + s = s \ .$$

But $w'G = s$, contradicting that $G = \|\mathcal{A}\|$.

If $\|(p, a^s, p)\| \geq s$, the path

$$P_2 : i \xrightarrow{a^r} p \xrightarrow{a^t} q \xrightarrow{b^n} f$$

spells the word $w'' = a^{n-s}b^n$ and $\|P_2\| = \|P\| - \|(p, a^s, p)\| \leq -s$. Then, we have that

$$w''\|\mathcal{A}\| \leq \|P_2\| \leq -s \ .$$

But

$$w''G = |w''|_a - |w''|_b + 1 = -s + 1 \ ,$$

contradicting that $G = \|\mathcal{A}\|$.

Therefore, there can not exist a $\mathcal{Z}$-A-automaton whose behavior is G. Thus, G is not a recognizable $\mathcal{Z}$-subset of A^*.

■

A consequence of the proof of Theorem 11 is given in the sequence.

Corollary 12 *There is a recognizable $\mathcal{Z}$-subset X of A^* such that the $\mathcal{Z}$-subset Y defined by*

$$\forall w \in A^*, \quad wY = \begin{cases} wX & \text{if } wX \geq 0 \\ wX + 1 & \text{if } wX < 0 \end{cases}$$

is not recognizable.

■

Lemma 13 *Let d be a negative integer. $\mathcal{Z}\,\mathrm{Rec}\,A^*$ is not closed under* $\mathrm{div}'\, d$.

Proof. For all $m \in \mathcal{Z}$, m div $-1 = m$ div$'$ -1, because in this case the remainder is zero. Thus, from the proof of Lemma 8, we can conclude that $\mathcal{Z}\,\mathrm{Rec}\,A^*$ is not closed under div$'$ d, when d is a negative integer. ■

Lemma 14 *Let d be an integer, $d \neq 0$. $\mathcal{Z}\,\mathrm{Rec}\,A^*$ is not closed under* $\mathrm{mod}'\, d$.

Proof. Let us consider the $\mathcal{Z}$-subset X in the proof of Lemma 10:

$$\forall w \in A^*, \quad wX = \min\{|w|_c, -2|w|_a, -2|w|_b - 1\}$$

and take $Y = X \bmod' 2 = X \bmod' -2$. Then,

$$\forall w \in A^*, \quad wY = \begin{cases} 0 & \text{if } (w \in c^* \text{ and } |w|_c \text{ is even}) \text{ or } |w|_a > |w|_b \\ 1 & \text{if } w \in c^* \text{ and } |w|_c \text{ is odd} \\ -1 & \text{if } |w|_a \leq |w|_b \end{cases}.$$

Suppose that Y is a recognizable $\mathcal{Z}$-subset of A^*. In this case, there is a $\mathcal{Z}$-A-automaton $\mathcal{A} = (Q, I, T)$ such that $\|\mathcal{A}\| = Y$.

Let $n = |Q|$ and let us consider the word $w = a^n b^n$. Then, there is a victorious path P in $\mathcal{A}$, with $|P| = w$ and $\|P\| = w\|\mathcal{A}\| = wY = -1$. Moreover, there are naturals r, s and t, with $s > 0$ and $r + s + t = n$ such that the path P can be decomposed as

$$P : i \xrightarrow{a^r} p \xrightarrow{a^s} p \xrightarrow{a^t} q \xrightarrow{b^n} f .$$

Consider the factor (p, a^s, p) of P. If $\|(p, a^s, p)\| \leq 0$, the path

$$P_1 : i \xrightarrow{a^r} p \xrightarrow{a^t} p \xrightarrow{a^s} p \xrightarrow{a^t} q \xrightarrow{b^n} f$$

spells the word $w' = a^{n+s}b^n$ and we have that

$$w'\|\mathcal{A}\| \leq \|P_1\| \leq \|P\| = -1 \ .$$

But $w'Y = 0$, contradicting that $Y = \|\mathcal{A}\|$.

Hence, $\|(p, a^s, p)\| > 0$ and in this case, the path

$$P_2 : i \xrightarrow{a^r} p \xrightarrow{a^t} q \xrightarrow{b^n} f$$

spells the word $w'' = a^{n-s}b^n$ and we have that

$$w''\|\mathcal{A}\| \leq \|P_2\| < \|P\| = -1 \ .$$

But $w''Y = -1$, contradicting that $Y = \|\mathcal{A}\|$.

Therefore, $Y = X \bmod' 2 = X \bmod' -2$ is not a recognizable $\mathcal{Z}$-subset of A^*.

■

The closure properties of $\mathcal{Z}\,\mathrm{Rec}\,A^*$ that we have seen in this section are summarized in Table 1.

Table 1: Closure properties of $\mathcal{Z}\,\mathrm{Rec}\,A^*$ under quotient and remainder by a non-zero integer

Operator	$d > 0$	$d < 0$
div	yes	no
mod	no	no
div′	no	no
mod′	no	no

4 A characterization of recognizable $\mathcal{Z}$-subsets of A^+

Eilenberg [2] showed that, for any semiring K, the family of recognizable K-subsets is closed under intersection. But we showed [7] that the family of simple $\mathcal{M}$-subsets, $\mathcal{M}\,\mathrm{SRec}\,A^*$, is not closed under addition. (Recall that the addition of $\mathcal{M}$-subsets plays the rôle of intersection of K-subsets for a general semiring K.) This fact led us to investigate the following question:

> Is every recognizable $\mathcal{M}$-subset of A^+ the sum of a finite number of simple $\mathcal{M}$-subsets of A^+?

For instance, one can verify that the recognizable $\mathcal{M}$-subset X defined by

$$\forall w \in \{a,b\}^+, \quad wX = 2|w|_a + 3|w|_b$$

is not a simple $\mathcal{M}$-subset, but it can be described as the sum of five simple $\mathcal{M}$-subsets X_1, X_2, X_3, X_4 and X_5 defined by

$$\forall w \in \{a,b\}^+, \quad wX_1 = wX_2 = |w|_a \quad \text{and} \quad wX_3 = wX_4 = wX_5 = |w|_b \ .$$

In fact, X may also be written as the sum of three simple $\mathcal{M}$-subsets Y_1, Y_2 and Y_3 defined by

$$\forall w \in \{a,b\}^+, \quad wY_1 = wY_2 = |w| \quad \text{and} \quad wY_3 = |w|_b \ .$$

We obtained an affirmative answer for this question (see [7] and [8]). The next theorem generalizes this result to the semiring $\mathcal{Z}$.

Theorem 15 *A $\mathcal{Z}$-subset of A^+ is recognizable if and only if it is the sum of a finite number of simple $\mathcal{Z}$-subsets of A^+.*

Proof. Let X be a recognizable $\mathcal{Z}$-subset of A^+. Let $\mathcal{A} = (Q_{\mathcal{A}}, I_{\mathcal{A}}, T_{\mathcal{A}})$ be a normalized $\mathcal{Z}$-A-automaton such that $\|\mathcal{A}\| = X$ and let d be the maximum of the absolute values of the multiplicities of the useful edges of $\mathcal{A}$.

Let us construct d $\mathcal{Z}$-A-automata $\mathcal{A}_i$ $(1 \le i \le d)$ such that $\sum_{i=1}^{d} \|\mathcal{A}_i\| = \|\mathcal{A}\|$.

For each $i \in [1, d]$, the $\mathcal{Z}$-A-automaton $\mathcal{A}_i = (\mathcal{C}, I_i, T)$ is constructed from the $\mathcal{Z}$-A-semiautomaton $\mathcal{C} = (Q, Ec)$ which was introduced in the previous section. We define the $\mathcal{Z}$-subsets I_i and T of Q:

$$(\forall q \in Q_{\mathcal{A}}, \quad \forall j \in [1, d]) \qquad (q, j)I_i = \delta(i, j) + qI_{\mathcal{A}} ,$$

$$\text{where} \quad \delta(i, j) = \begin{cases} 0 & \text{if } i = j \\ \infty & \text{otherwise} \end{cases} ;$$

$$(\forall q \in Q_{\mathcal{A}}, \quad \forall j \in [1, d]) \qquad (q, j)T = qT_{\mathcal{A}} .$$

Note that $QI_i, QT \subseteq \{0, \infty\}$ and as the edge multiplicities of $\mathcal{C}$ are in $\{0, 1, -1, \infty\}$ (as we saw in the previous section). $\mathcal{A}_i$ is a simple $\mathcal{Z}$-A-automaton. We can also observe that the $\mathcal{Z}$-A-automata $\mathcal{A}_i$ $(1 \leq i \leq d)$ differ from each other only in the initial states.

Before we continue the proof of this theorem, we study, through the next lemmas, the properties which relate the paths in each $\mathcal{A}_i$ $(1 \leq i \leq d)$ with their projections in $\mathcal{A}$. We also study the relations existing between the paths in $\mathcal{A}_i$ and in $\mathcal{A}_j$, for $i \neq j$.

Let $i \in [1, d]$. Let P be a victorious path in $\mathcal{A}_i$ with terminus in $Q_{\mathcal{A}} \times \{j\}$, for some $j \in [1, d]$. We say that P is a *tallest victorious path* in $\mathcal{A}_i$, if there are no victorious paths in $\mathcal{A}_i$ with the same label of P and with terminus in $Q_{\mathcal{A}} \times \{k\}$, for $k \in [1, d]$, $k > j$.

Lemma 16 *Let P be a tallest victorious path in $\mathcal{A}_i$, $i \in [1, d]$. Then its projection P' is a victorious path in $\mathcal{A}$.*

Proof. Let P be a tallest victorious path in $\mathcal{A}_i$. Then, P has its origin in $Q_{\mathcal{A}} \times \{i\}$. Let us suppose that the terminus of P lies in $Q_{\mathcal{A}} \times \{j\}$, for some $j \in [1, d]$. If the projection P' of P is not a victorious path in $\mathcal{A}$, there is a victorious path P_1' in $\mathcal{A}$ such that $|P_1'| = |P'|$ and $\|P_1'\| < \|P'\|$.

Let P_1 be the i-lifting of P_1' in $\mathcal{A}_i$ and we suppose that P_1 terminates in $Q_{\mathcal{A}} \times \{k\}$, for some $k \in [1, d]$. Then, using Lemma 3,

$$d\|P_1\| = \|P_1'\| - i + k < \|P'\| - i + k = d\|P\| + i - j - i + k = d\|P\| + k - j .$$

Therefore,

$$d(\|P_1\| - \|P\|) < k - j .$$

Moreover, P_1 is a successful path in $\mathcal{A}_i$. In fact, its origin (p,i) and its terminus (q,k) satisfy $(p,i)I_i = pI_{\mathcal{A}} \neq \infty$ and $(q,k)T = qT_{\mathcal{A}} \neq \infty$, since that its projection P_1' is a victorious path in $\mathcal{A}$. But, as P is a victorious path in $\mathcal{A}_i$, $\|P_1\| \geq \|P\|$.

If $\|P_1\| = \|P\|$ then P_1 is also a victorious path in $\mathcal{A}_i$ and $k - j > 0$. That is, $k > j$. So, P is not a tallest victorious path in $\mathcal{A}_i$; a contradiction.

Thus, $\|P_1\| > \|P\|$. Then,

$$d \leq d(\|P_1\| - \|P\|) < k - j \ .$$

This is impossible, because $k, j \in [1, d]$. Therefore, P' is a victorious path in $\mathcal{A}$.

■

Lemma 17 *Let P_i ($i \in [1,d]$) be a tallest victorious path in $\mathcal{A}_i$ with label w and let us assume that P_i terminates in $Q_{\mathcal{A}} \times \{j\}$ ($j \in [1,d]$). Let P_k ($k \in [1,d]$ and $k \neq i$) be a tallest victorious path in $\mathcal{A}_k$ with label w and let us assume that P_k terminates in $Q_{\mathcal{A}} \times \{l\}$ ($l \in [1,d]$). Then $i - j \equiv k - l$ (mod d).*

Proof. Let P_i be a tallest victorious path in $\mathcal{A}_i$ with terminus in $Q_{\mathcal{A}} \times \{j\}$ and label w. Let P_k be a tallest victorious path in $\mathcal{A}_k$ with terminus in $Q_{\mathcal{A}} \times \{l\}$ and label w.

Let us consider the projections P_i' and P_k' of P_i and P_k, respectively, in $\mathcal{A}$. From Lemma 16, it results that P_i' and P_k' are victorious path in $\mathcal{A}$. Then $\|P_i'\| = \|P_k'\|$. But, from Lemma 3,

$$\|P_i'\| = d\|P_i\| + i - j \quad \text{and} \quad \|P_k'\| = d\|P_k\| + k - l \ .$$

So, from $\|P_i'\| = \|P_k'\|$, it follows that

$$d\|P_i\| + i - j = d\|P_k\| + k - l \ .$$

Then

$$i - j = d(\|P_k\| - \|P_i\|) + k - l \ .$$

Thus, $i - j \equiv k - l \pmod{d}$.

■

Note that the previous lemma implies that $l \neq j$ and if $k = i + 1$ then $l \equiv j + 1 \pmod{d}$.

We continue the proof of Theorem 15 considering $X_i = \|\mathcal{A}_i\|$ $(1 \leq i \leq d)$. Then, X_i $(1 \leq i \leq d)$ are simple $\mathcal{Z}$-subsets of A^+. Moreover, one can verify that $\forall w \in A^+$, $wX = \infty$ if, and only if, $\forall i \in [1, d]$, $wX_i = \infty$. Hence, $wX = \infty$ iff $w \sum_{i=1}^{d} X_i = \infty$. Then, for $w \in A^+$, we can assume that $wX \neq \infty$ and $wX_i \neq \infty$ $(1 \leq i \leq d)$.

For each $i \in [1, d]$, there is a tallest victorious path P_i in $\mathcal{A}_i$, with $|P_i| = w$ and

$$\|P_i\| = w\|\mathcal{A}_i\| = wX_i \ . \tag{3}$$

Therefore, by Lemma 16, for each $i \in [1, d]$, the projection P_i' of P_i in $\mathcal{A}$ is a victorious path and

$$\|P_i'\| = w\|\mathcal{A}\| = wX \ .$$

Then

$$\sum_{i=1}^{d} \|P_i'\| = d(wX) \ . \tag{4}$$

We suppose that for each $i \in [1, d]$, P_i terminates in $Q_{\mathcal{A}} \times \{k_i\}$, for some $k_i \in [1, d]$. Then, by Lemma 17, for each pair j and $l \in [1, d]$, if $j \neq l$ it results that $k_j \neq k_l$. Therefore,

$$\sum_{i=1}^{d} k_i = \sum_{i=1}^{d} i \ . \tag{5}$$

But, by Lemma 3, for each $i \in [1, d]$,

$$\|P_i'\| = d\|P_i\| + i - k_i \ .$$

Then, using (4) and (5) we have:

$$d(wX) = \sum_{i=1}^{d} \|P_i'\| = \sum_{i=1}^{d} (d\|P_i\| + i - k_i) = \sum_{i=1}^{d} d\|P_i\| + \sum_{i=1}^{d} i - \sum_{i=1}^{d} k_i = d \sum_{i=1}^{d} \|P_i\| \ .$$

Hence, from (3),

$$wX = \sum_{i=1}^{d} \|P_i\| = \sum_{i=1}^{d} wX_i \ .$$

Thus,

$$\forall w \in A^+, \qquad wX = \sum_{i=1}^{d} wX_i = w\sum_{i=1}^{d} X_i \ .$$

Therefore,

$$X = \sum_{i=1}^{d} X_i \ .$$

The converse of this Theorem follows from the definition of simple $\mathcal{Z}$-subset and the closure of $\mathcal{Z}\,\mathrm{Rec}\,A^*$ under addition.

■

In the proof of Theorem 15, if $\mathcal{A}$ is an $\mathcal{M}$-A-automaton (resp. $\mathcal{Z}^-$-A-automaton), from Corollary 5 it follows that each $\mathcal{A}_i$ ($1 \le i \le d$) is an $\mathcal{M}$-A-automaton (resp. $\mathcal{Z}^-$-A-automaton). Moreover, Lemmas 16 and 17 stay valid when each $\mathcal{A}_i$ is an $\mathcal{M}$-A-automaton (resp. $\mathcal{Z}^-$-A-automaton). Thus, the characterization given in Theorem 15 is also valid to $\mathcal{M}$-subsets (resp. $\mathcal{Z}^-$-subsets).

Corollary 18 *An $\mathcal{M}$-subset (resp. $\mathcal{Z}^-$-subset) of A^+ is recognizable if and only if it is the sum of a finite number of simple $\mathcal{M}$-subsets (resp. $\mathcal{Z}^-$-subsets) of A^+.*

■

The following corollaries consider the general case of recognizable $\mathcal{Z}$-subsets, $\mathcal{M}$-subsets and $\mathcal{Z}^-$-subsets of A^*.

Corollary 19 *Let X be a recognizable $\mathcal{Z}$-subset (resp. $\mathcal{M}$-subset, $\mathcal{Z}^-$-subset) of A^*. Then, X is the sum of a finite number of simple $\mathcal{Z}$-subsets (resp. $\mathcal{M}$-subsets, $\mathcal{Z}^-$-subsets) of A^* if and only if $1X \in \{0, \infty\}$.*

Proof. Let X be a recognizable $\mathcal{Z}$-subset of A^* such that $1X \in \{0,\infty\}$. By Theorem 15, it is enough to consider the case in which $1X = 0$. Let X_i's ($1 \le i \le d$) be the simple $\mathcal{Z}$-subsets of A^+, obtained from Theorem 15 for $X + \mathbf{A^+}$. For each $i \in [1,d]$, let us consider the $\mathcal{Z}$-subset $Y_i = \min(X_i, 1)$. It is clear that Y_i is simple and $1Y_i = 0$. Then, $X = \sum_{i=1}^{d} Y_i$.

The converse of this corollary follows immediately from the definitions of simple $\mathcal{Z}$-subsets and of the $\mathcal{Z}$-subsets addition operation.

The proof for X in $\mathcal{M}\,\mathrm{Rec}\,A^*$ or $\mathcal{Z}^-\,\mathrm{Rec}\,A^*$ is similar. ■

Corollary 20 *Let X be a recognizable $\mathcal{Z}$-subset (resp. $\mathcal{M}$-subset, $\mathcal{Z}^-$-subset) of A^* such that $1X \notin \{0,\infty\}$. Then, there is a positive integer d and there are d simple $\mathcal{Z}$-subsets (resp. $\mathcal{M}$-subsets, $\mathcal{Z}^-$-subsets) of A^*, $X_1,\ldots,X_d$, and a recognizable $\mathcal{Z}$-subset (resp. $\mathcal{M}$-subset, $\mathcal{Z}^-$-subset) Y of A^* such that*

$$X = \min(\sum_{i=1}^{d} X_i, Y) .$$

Proof. It is enough to consider $X = \min(\sum_{i=1}^{d} X_i, 1X + 1)$, where the $\mathcal{Z}$-subsets (resp. $\mathcal{M}$-subsets, $\mathcal{Z}^-$-subsets) X_i's $(1 \leq i \leq d)$ are obtained from Theorem 15 (resp. Corollary 18) for $X + \mathbf{A}^+$. ■

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