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RELATIONS BETWEEN CRITICAL POINTS OF f AND ITS "RADIAL DERIVATE"

Sonia Regina L. Garcia

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INTRODUCTION.

Inspired by [2], the purpose of the first section is to show a proof of the Sard's Algebraic Theorem without the use the Tarski-Seidenberg Theorem; moreover by using this result we make a geometrical description of the following algebraic varieties:

$$V(f) = \{ x \in R^n : \text{rank } [\text{grad } f(x) \quad x] < 2 \}$$

$$V^*(f) = \{ x \in R^n : \ll \text{grad } f(x) / x \gg = 0 \}$$

where f is a polynomial of n variables.

In section 2 we define the "radial derivate" of a germ of function $f:(R^n;0) \longrightarrow (R;0)$ and we describe the relation between the behaviours of f and its "radial derivate" at the origin.

In section 3, we present a special class of germs of functions for which the behaviour of a germ and the behaviour of its "radial derivate" are the same.

In the last section we present an application to Liapunov's stability of Hamiltonian systems of two degree of freedom.

We will use the notations

$$B_r = \{ x \in R^n : \|x\| < r \},$$

$$\bar{B}_r = \{ x \in R^n : \|x\| \leq r \},$$

$$B'_r = B_r \setminus \{ 0 \},$$

$$S_r = \{ x \in R^n : \|x\| = r \}.$$

$f:(R^n,0) \longrightarrow (R^m,0)$ will denote the germ at the origin of a function $f:R^n \longrightarrow R^m$ such that $f(0) = 0$.

1. AN ELEMENTARY PROOF OF SARD'S ALGEBRAIC THEOREM.

Let $f: R^n \to R$ be a C^1-function.

We denote by $V(f)$ and $V^*(f)$ respectively the sets below:

$$V(f) = \{ x \in R^n : \text{rank} [\text{grad } f(x) \quad x] < 2 \}$$

$$= \{ x \in R^n : \langle\langle \text{grad } f(x) / x \rangle\rangle^2 = \|\text{grad } f(x)\|^2 . \|x\|^2 \}$$

$$V^*(f) = \{ x \in R^n : \langle\langle \text{grad } f(x) / x \rangle\rangle = 0 \}.$$

<u>Remark</u>:

(1) $V(f) \setminus \{ 0 \}$ is adherent to the origin, because $V(f) \cap S_\epsilon$ ($\epsilon > 0$) is precisely the set of the critical points of $f|S_\epsilon$.

(2) $V^*(f) \setminus \{ 0 \}$ can be empty. For example, if $f(x) = \|x\|^2$ then $V^*(f) = \{ 0 \}$.

(3) $(V(f) \cap V^*(f)) \setminus \{ 0 \}$ is precisely the set of the critical points of $f|(R^n \setminus \{ 0 \})$.

<u>Definition 1</u>:

A subset $V \subset R^n$ is called an algebraic variety if there is a family $\{ f_j \}_{j \in J}$ in $R[x_1,...,x_n]$ such that

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$$V = \{ x \in R^n : f_j(x) = 0, j \in J \}.$$

Definition 2:

Let $V \subset R^n$ be an algebraic variety. The ideal of $R[x_1,\dots,x_n]$ defined by

$$I[V] = \{ f \in R[x_1,\dots,x_n] : f|V = 0 \}$$

is called ideal of V.

Proposition 1:

Let $V \subset R^n$ be an algebraic variety.

There are $f_1,\dots,f_m \in R[x_1,\dots x_n]$ such that

$$V = \{ x \in R^n : f_j(x) = 0 , j = 1,\dots,m \}.$$

Proof:

It is sufficient to choose a set $\{ f_1,\dots,f_m \}$ of generators of I[V].

Remark:

(4) If $f:R^n \longrightarrow R$ is a polynomial function then V(f) and V*(f) are algebraic varieties.

(5) If $\{ f_1,\dots,f_m \}$ is a set of generators of I[V], we denote

by F the function $F:R^n \longrightarrow R^m$ which components are $f_1,\dots,f_m$.

The next proposition and the theorems 1 - 4, as their proofs are in [3].

Proposition 2:

Let $V \subset R^n$ be an algebraic variety and let $\{ f_1,\dots,f_m \}$ be a set of generators of I[V]. For any point $x \in V$, the number

$$\rho[V](x) = \text{rank } DF(x),$$

where $F=(f_1,\dots,f_m)$, is independent of the particular fixed set of generators of I[V].

Definition 3:

Let $V \subset R^n$ be an algebraic variety. The number

$$\rho = \rho[V] = \sup_{x \in V} \rho[V](x)$$

is called rank of the ideal I[V] and is denoted by rank I[V].

Definition 4:

Let $V \subset R^n$ be an algebraic variety, and ρ = rank I[V]. The set

$$M_1[V] = \{ x \in V : \rho[V](x) = \rho \}$$

is called the set of the non-critical points of V, and $V \setminus M_1[V]$ is the set of the singular points of V.

Theorem 1:

Let $V_0 \subset \mathbb{R}^n$ be an algebraic variety and consider $V_1 = V_0 \setminus M_1(V_0)$

Then V_1 is an algebraic variety.

Theorem 2:

Let $V_0 \subset \mathbb{R}^n$ be an algebraic variety. For $i \in \mathbb{N}^*$ we define by induction:

$M_i = M_1(V_{i-1})$ and $V_i = V_{i-1} \setminus M_i$.

Then:

(a) There is $k \in \mathbb{N}^*$ such that $V_k = \emptyset$; consequently $V_0 = M_1 \cup \ldots \cup M_k$.
Moreover the union is disjoint.

(b) For $i = 1,\ldots,k$, M_i is an analytic submanifold of $\mathbb{R}^n$.

Theorem 3:

Let $V_0 \subset \mathbb{R}^n$ be an algebraic variety and let $V_0 = M_1 \cup \ldots \cup M_k$ be the decomposition of V_0 defined in the theorem 2. Then for $i = 1,\ldots,k$ M_i has a finite number of connected components.

Theorem 4:

If V and W are algebraic varieties of R^n, the difference $V \setminus W$ has a finite number of connected components. Moreover each one is a finite disjoint union of connected analytic submanifolds of R^n.

Definition 5:

Let $V \subset R^n$ be an algebraic variety and $g:R^n \longrightarrow R$ a C^1-function.

A point $x \in V$ is said a critical point of $g|V$ if x is a critical point of $g|M_i$, $1 \leq i \leq k$, where $V = M_1 \cup \ldots \cup M_k$ is the decomposition of V presented in the theorem 2 and $x \in M_i$. (Remember that $M_i \cap M_j = \emptyset$ if $i \neq j$).

Sard's Algebraic Theorem:

Let $V \subset R^n$ be an algebraic variety. Let $g:R^n \longrightarrow R$ be a polynomial function. Then the set of the critical values of $g|V$ is finite.

Proof:

Let $V = M_1 \cup \ldots \cup M_k$ be the decomposition of V presented in the theorem 2. We denote by Z the set of the critical points of $g|V$. Thus,

$Z = Z_1 \cup \ldots \cup Z_k$, where $Z_i = Z \cap M_i$, $i = 1,\ldots,k$.

If $\#(Z)$ and $\#(Z_i)$ denote the number of connected components of Z and Z_i respectively, $i = 1,\dots,k$, then $\#(Z) \le \#(Z_1) + \dots + \#(Z_k)$.

Let us fix $i \in \{1,\dots,j\}$.

Remember that $M_i = M_1[W]$, where $W \subset R^n$ is an algebraic variety (see theorem 2).

Let $\{f_1,\dots,f_m\}$ be a set of generators of $I[V]$.

Put $\rho = \operatorname{rank} I[V]$. Then:

$$Z_i = M_i \bigcap \left\{ x \in R^n : \frac{\partial(f_{i1},\dots,f_{i\rho},g)}{\partial(x_{j1},\dots,x_{j\rho+1})}(x) = 0,\ \begin{matrix} 1 \le i1,\dots,i\rho \le m, \\ 1 \le j1,\dots,j\rho+1 \le n \end{matrix} \right\} =$$

$$\left\{ x \in R^n : f_j(x) = 0,\ \frac{\partial(f_{i1},\dots,f_{i\rho},g)}{\partial(x_{j1},\dots,x_{j\rho+1})}(x) = 0,\ \begin{matrix} 1 \le j \le m \\ 1 \le i1,\dots,i\rho \le m \\ 1 \le j1,\dots,j\rho+1 \le n \end{matrix} \right\} \setminus W$$

Thus, Z_i is the difference of two algebraic varieties.

According to theorem 4, $\#(Z_i)$ is finite and any connected component of Z_i is a disjoint union of a finite number of connected analytic submanifolds of R^n. Thus Z is also a finite disjoint union of connected analytic submanifolds of R^n.

It is clear that g is constant in each one of those submanifolds. Then, $g|V$ has a finite number of critical values.

Proposition 3:

Let $f:R^n \longrightarrow R$ be a polynomial function. There is $r > 0$ such that if $\epsilon \in (0,r)$ then $V(f)$ is transversal to S_ϵ.

Proof:

Take $g(x) = ||x||^2$.

Since $V(f)$ is an algebraic variety, by Sard's Algebraic Theorem we see that $g|V(f)$ has a finite number of strictly positive critical values. Put $r > 0$ such that r^2 is the smallest of them (or $r = 1$ if there are no values of this kind). The result follows from theorem 2.

Using similar arguments it follows that :

Proposition 4:

Let $f:R^n \longrightarrow R$ be a polynomial function. Then there is $r > 0$ such that if $\epsilon \in (0,r)$ then $V^*(f)$ is transversal to S_ϵ.

Next we obtain a result of separation.

Proposition 5:

Let $f:R^n \longrightarrow R$ be a C^1-function such that $f(0) = 0$.

If x and y are in $V(f)$ and $\ll \operatorname{grad} f(x) / x \gg < 0$ then $V^*(f)$ separates x and y.

Proof:

For any continuous curve $c:[0,1] \longrightarrow R^n$ such that $c(0) = x$ and $c(1) = y$, the function $F_c :[0,1] \longrightarrow R$ defined by

$$F_c(t) = \langle\langle \text{grad } f(c(t)) \,/\, c(t) \rangle\rangle$$

is a continuous function and $F_c(0).F_c(1) < 0$. It follows that there is $s \in (0,1)$ such that $F_c(s) = 0$. Thus we have $c(s) \in V^*(f)$.

Corollary:

Let $f:R^n \longrightarrow R$ be a polynomial function such that $f(0) = 0$.

If there exists two components X and Y of $V(f) \cap B'_r$ (where r is given by proposition 3) such that for any $x \in X$ and $y \in Y$,

$$\langle\langle \text{grad } f(x) \,/\, x \rangle\rangle.\langle\langle \text{grad } f(y) \,/\, y \rangle\rangle < 0,$$

then $V^*(f) \setminus \{ 0 \}$ is adherent to the origin.

2. THE "RADIAL DERIVATE".

We will consider germs of functions $f:(R^n;0) \longrightarrow (R^m;0)$ and, as usual, we also denote by f any representative of the germ.

In this section we will describe the relation between the behaviour of a germ $f:(R^n;0) \longrightarrow (R;0)$ and the behaviour of the "radial derivate" of f, that we will define below.

Definition 1:

Let $f:(R^n;0) \longrightarrow (R;0)$ be a germ of C^1-function.

We say that the origin is a strong maximum point (strong minimum point) of f if the origin is a strict maximum point (strict minimum point) of f. If the origin is a non-strict maximum point (non-strict minimum point) of f we say that it is a weak maximum point (weak minimum point) of f. If it is not a maximum or minimum point of f we say that it is a saddle point of f.

Definition 2:

We say that the germ $f:(R^n;0) \longrightarrow (R;0)$ has punctual k-jet at

$0 \in R^n$ $(k \in N)$ if there is a polynomial $P:R^n \rightarrow R$ of degree less or equal k (dg 0 = $-\infty$) such that $\lim_{x \to 0} \frac{f(x) - P(x)}{||x||^k} = 0$

Remark:

(1) If there is such P, it is unique. Let us denote it by $J^k f$.

(2) If f admits Taylor's polynomial of k-order at $0 \in R^n$, then f has k-jet at $0 \in R^n$ and $J^k f$ coincides with that polynomial.

Definition 3:

Let $f:(R^n;0) \rightarrow (R;0)$ be a germ of C^1-function.

The origin is said a critical point of f if there exists $J^1 f$ and $J^1 f$ is the null polynomial.

In what follows, $A \in Mn(R)$ will denote a symmetric positive defined matrix.

Definition 4:

Let $U \subset R^n$ be an open neighborhood of the origin, and let

$$f:U \rightarrow R$$

a C^1-function such that $f(0) = 0$.

We will call radial derivate f relative to A and we will denote by f'_A the C^1-function

$$f'_A : U \longrightarrow R$$

defined by $f'_A(x) = \langle\langle \text{grad } f(x) \,/\, Ax \rangle\rangle$.

Definition 5:

Let $f:(R^n;0) \longrightarrow (R;0)$ be a germ of C^1-function.

We whll call of radial derivate of f relative to A and we will denote by f'_A to the germ of C^1-function

$$f'_A : (R^n;0) \longrightarrow (R;0)$$

whose representatives are the radial derivates relative to A of the representatives of the germ f.

Proposition 1:

Let $f:(R^n;0) \longrightarrow (R;0)$ be a germ of C^1-function with a critical point at the origin. Then f'_A has also a critical point at the origin.

Proof:

It is sufficient to prove that $\lim_{x \to 0} \frac{f'_A(x)}{\|x\|}$ exists and is null.

If $x \neq 0$ then

$$\frac{|f'_A(x)|}{\|x\|} = \frac{|\langle\langle \text{grad } f(x) \,/\, Ax \rangle\rangle|}{\|x\|}$$

$$\leq ||\text{grad } f(x)||.\frac{||Ax||}{||x||}$$

$$\leq |A|.||\text{grad } f(x)||$$

where $|A| = \sup_{||y||=1} ||Ay||$.

Since f is a C^1-function with a critical point at the origin we have

$$\lim_{x\to 0} |A|\ ||\text{grad } f(x)|| = 0$$

and the proof is complete.

It is easy to see that if n=1 and $f:(R^n;0) \longrightarrow (R;0)$ is the germ of a polynomial function, then f and f'_A have same behaviour with respect to the critical point. This suggests the question:

"If $f:(R^n;0) \longrightarrow (R;0)$ is a germ of C^1-function, have f and f'_A the same behaviour with respect to the extreme at the origin ?"

The answer to this question is surprising because it is negative even when we consider polynomial functions and A = I (I is the identity matrix).

Example:

$P(x,y) = 36x^2 - 48xy^2 - 17y^4 = (6x - 4y^2)^2 - y^4$ has strong minimum at $0 \in R^2$.

$P'_I(x,y) = 72x^2 - 144xy^2 + 68y^4 = 72(x - y^2)^2 - 4y^4$ has saddle at $0 \in R^2$.

The knowledge of this fact suggests another question:

"What behaviour has f'_A when f has minimum (or maximum, or saddle) at the origin ?"

We will answer this question by using the following lemma:

<u>Lemma 1</u>:

Let $U \subset R^n$ be a open neighborhood of the origin and let $f:U \longrightarrow R$ be a C^1-function such that $f(0) = 0$.

If $B_\epsilon \subset U$ $(\epsilon > 0)$ and there is $x \in B'_\epsilon$ such that $f(x) > 0$ (respectively $f(x) < 0$, $f(x) = 0$) then there is $y \in B'_\epsilon$ such that $f'_A(y) > 0$ (respectively $f'_A(y) < 0$, $f'_A(y) = 0$).

Proof:

We define the C^1-curve

$c:(-\infty, 0] \longrightarrow U$

$t \longrightarrow c(t) = [\exp(tA)]x$

It is clear that $c(t) \in B'_{\|x\|} \subset B'_\epsilon$, for all $t \in (-\infty, 0)$, and the function $f \circ c$ is a C^1-function satisfying $(f \circ c)(0) = f(x)$ and also

$\lim_{t \to -\infty} (f \circ c)(t) = 0$. Then there is $s \in (-\infty, 0)$ such that $\frac{d}{dt}(f \circ c)(s)$ and

$f(x)$ have same sign.

On the other hand,

$$\frac{d}{dt}(f \circ c)(s) = \langle\langle \operatorname{grad} f(c(s)) \;/\; \frac{dc}{dt}(s) \rangle\rangle$$

$$= \langle\langle \operatorname{grad} f(c(s)) \;/\; A[\exp(sA)]x \rangle\rangle$$

$$= \langle\langle \operatorname{grad} f(c(s)) \;/\; Ac(s) \rangle\rangle$$

$$= f'_A(c(s)).$$

Thus, seting $y = c(s)$ the result follows.

The propositions 2, 3, 4 and 5 follow immediately from this lemma with the help of the subsequent examples.

Proposition 2:

Let $f:(R^n;0) \longrightarrow (R;0)$ be a germ of C^1-function.

If f'_A has strong minimum (respectively strong maximum) at the origin then f has strong minimum (respectively strong maximum) at the origin.

Proposition 3:

If a germ of C^1-function $f:(R^n;0) \longrightarrow (R;0)$ has strong minimum (respectively strong maximum) at the origin then the origin can be a strong minimum point, or a weak minimum point, or a saddle point (respectively a strong maximum point, or a weak maximum point, or a saddle point) of f'_A, and each one of these behaviours is possible.

<u>Proposition 4</u>:

If a germ of C^1-function $f:(R^n;0) \longrightarrow (R;0)$ has weak minimum (respectively weak maximum) at the origin then the origin can be a weak minimum point or a saddle point (respectively a weak maximum point or a saddle point) of f'_A, and each one of these behaviours is possible.

<u>Proposition 5</u>:

If a germ of C^1-function $f:(R^n;0) \longrightarrow (R;0)$ has saddle at the origin then the origin is a saddle point of f'_A.

In what follows, $A \in Mn(R)$ will denote a symmetric positive defined matrix, $\{v_1,\ldots,v_n\} \subset R^n$ will denote a orthonormal basis of R^n such that $v_1,\ldots,v_n$ are eigenvectors of the linear transformation

$$x \in R^n \longrightarrow Ax \in R^n$$

and $\alpha_1,\ldots,\alpha_n$ are eigenvalues of this transformation that corresponds respectively to the eigenvectors $v_1,\ldots,v_n$.

Note that $\alpha_i > 0$, $1 \le i \le n$.

In the examples 1 to 5, we consider $n \ge 2$.

<u>Example 1</u>:

$$f(x) = \langle\langle x / v_1 \rangle\rangle^2$$

$$f'_A(x) = 2\,\alpha_1 \langle\langle x / v_1 \rangle\rangle^2$$

f and f'_A have weak minimum at the origin.

<u>Example 2</u>:

$$f(x) = (\langle\langle x / v_1 \rangle\rangle - \langle\langle x / v_2 \rangle\rangle^m)^2$$

where $m \in N^*$ and $m > 2\alpha_1/\alpha_2$.

$$f'_A(x) = 2(\langle\langle x / v_1 \rangle\rangle - \langle\langle x / v_2 \rangle\rangle^m)(\alpha_1 \langle\langle x / v_1 \rangle\rangle - m\alpha_2 \langle\langle x / v_2 \rangle\rangle^m)$$

f has weak minimum and f'_A has saddle at the origin.

<u>Example 3</u>:

$$f(x) = \sum_{j=1}^{n} \langle\langle x / v_j \rangle\rangle^2$$

$$f'_A(x) = 2 \sum_{j=1}^{n} \alpha_j \langle\langle x / v_j \rangle\rangle^2$$

f and f'_A have strong minimum at the origin.

Example 4:

$$f(x) = (\langle\langle x / v_1 \rangle\rangle - \langle\langle x / v_2 \rangle\rangle^m)^2 + a\langle\langle x / v_2 \rangle\rangle^{2m} + \sum_{j=3}^{n} \langle\langle x / v_j \rangle\rangle$$

where $m \in N^*$, $m = \alpha_1 / \alpha_2$ and $a = (\alpha_1 - m\alpha_2)^2 / (4m\alpha_1\alpha_2)$.

$$f'_A(x) = 2\left[\sqrt{\alpha_1} \langle\langle x / v_1 \rangle\rangle - [(\alpha_1 + m\alpha_2) / (2\sqrt{\alpha_1})] \langle\langle x / v_2 \rangle\rangle^m \right]^2 + 2\sum_{j=3}^{n} \alpha_j \langle\langle x / v_j \rangle\rangle^2$$

f has strong minimum and f'_A has weak minimum at the origin.

Example 5:

Let f be as in the last example, but with a satisfying

$$0 < a < (\alpha_1 - m\alpha_2)^2 / (4m\alpha_1\alpha_2) .$$

$$f'_A(x) = 2\left[\sqrt{\alpha_1} \langle\langle x / v_1 \rangle\rangle - [(\alpha_1 + m\alpha_2) / (2\sqrt{\alpha_1})] \langle\langle x / v_2 \rangle\rangle^m \right]^2 +$$

$$+ 2\left[m(a - 1)\alpha_2 - [(\alpha_1 + m\alpha_2)^2 / (4\alpha_1)] \langle\langle x / v_2 \rangle\rangle^{2m} \right] +$$

$$+ 2\sum_{j=1}^{n} \alpha_j \langle\langle x / v_j \rangle\rangle^2$$

f has strong minimum and f'_A has saddle at the origin.

3.A SPECIAL CLASS OF GERMS OF FUNCTIONS.

The purpose of this section is to determine a subset $\Delta \subset Mn(R)$ of symmetric positive defined matrices, and a class $\mathcal{F}$ of germs of C^1-functions $f:(R^n;0) \longrightarrow (R;0)$ such that for all $A \in \Delta$ and for all f in $\mathcal{F}$, f and f'_A have the same behaviour with respect to extremes.

It is easy to see that if n = 1, for any $A \in M1(R)$ and for any polynomial germ $P:(R;0) \longrightarrow (R;0)$, P and P'_A have the same behaviour at the origin. This is a consequence of the following facts:

Fact 1:

For all polynomial germ $P:(R;0) \longrightarrow (R;0)$ such that $J^{k-1} P = 0$ we have $J^k P'_A = kAJ^k P$, for all $A \in M1(R)$. [There, M1(R) = R++].

Fact 2:

For all non-null polynomial germ $P:(R;0) \longrightarrow (R;0)$, the first non-zero jet of P detects the behaviour of P at the origin.

It is clear that this is false if n > 1.

To have the fact 1 satisfied for n > 1 we consider

$$\Delta = \{ A \in Mn(R) : A = aI,\ a \in R{+}{+} \} \subset Mn(R).$$

Thus, if $P:(R^n;0) \longrightarrow (R;0)$ is a polynomial germ with null (k-1) -jet, then $J^k P'_A = kaJ^k P$, for all $A = aI \in \mathbb{A}$.

On the other hand, to determine a class Σ of germs of C^1 -functions for which the fact 2 extending, we need the concept of k-decidablety.

Definition 1:

Let $f:(R^n;0) \longrightarrow (R;0)$ be a germ of function such that there is punctual r-jet of f at $0 \in R^n$.

We say that f is k-decidable ($k \leq r$) if it is possible, by using only $J^k f$, to decide if the origin is a maximum, a minimum or a saddle point of f.

Note that if f is k-decidable and $k \leq m \leq r$, then f is m-decidable.

Proposition 1:

If $P:(R^n;0) \longrightarrow (R;0)$ is a k-decidable polynomial germ with null (k-1)-jet, for any matrix $A = aI \in \mathbb{A}$, P and P'_A have the same behaviour at the origin.

Proof:

$J^k_A P' = ka J^k P$, and thus, as P is k-decidable, the thesis holds.

Proposition 2:

Let $P:(R^n;0) \longrightarrow (R;0)$ be a germ of C^1-function and $A \in Mn(R)$ a symmetric positive defined matrix. If there exists J^{k-1} (grad f) then there exists $J^k_A f'$ and $J^k_A f'(x) = \langle\langle J^{k-1}(\text{grad } f)(x) / Ax \rangle\rangle$.

Proof:

It is sufficient to note that for $x \neq 0$ we have:

$$\frac{| f'_A(x) - \langle\langle J^{k-1}(\text{grad } f)(x) / Ax \rangle\rangle |}{\|x\|^k}$$

$$= \left| \left\langle\left\langle \frac{\text{grad } f(x) - J^{k-1}(\text{grad } f)(x)}{\|x\|^{k-1}} \Big/ \frac{Ax}{\|x\|} \right\rangle\right\rangle \right|$$

$$\leq |A| \frac{\| \text{grad } f(x) - J^{k-1}(\text{grad } f)(x) \|}{\|x\|^{k-1}}$$

Proposition 3:

Let $f:(R^n;0) \longrightarrow (R;0)$ be a germ of C^1-function such that there is J^{k-1} (grad f). Then there is $J^k f$ and $J^{k-1}(\text{grad } f) = \text{grad } J^k f$.

Proof:

Let $P:(\mathbb{R}^n;0) \longrightarrow (\mathbb{R};0)$ be a polynomial germ defined by

$$P(x) = \int_0^1 \langle\langle J^{k-1}(\mathrm{grad}\, f)(tx) \,/\, x \rangle\rangle \, dt .$$

It is clear that P has degree less then or equal k.

Moreover, for $x \neq 0$ we have:

$$\frac{| f(x) - P(x) |}{||x||^k}$$

$$= \left| \int_0^1 \frac{\langle\langle \mathrm{grad}\, f(tx) \,/\, x \rangle\rangle - \langle\langle J^{k-1}(\mathrm{grad}\, f)(tx) \,/ x \rangle\rangle}{||x||^k} \, dt \right|$$

$$\leq \int_0^1 t^{k-1} \frac{|| \mathrm{grad}\, f(tx) - J^{k-1}(\mathrm{grad}\, f)(tx) ||}{||tx||^{k-1}} \, dt$$

$$\leq \frac{1}{k} \sup_{0<||y||\leq||x||} \frac{|| \mathrm{grad}\, f(y) - J^{k-1}(\mathrm{grad}\, f)(y) ||}{||y||^{k-1}}$$

It is a consequence of the definition of k-jet that

$$\sup_{0<||y||\leq||x||} \frac{|| \mathrm{grad}\, f(y) - J^{k-1}(\mathrm{grad}\, f)(y) ||}{||y||^{k-1}} \xrightarrow[x \to 0]{} 0$$

Thus, there exists $J^k f$ and $J^k f = P$.

Now, define R, Q_j and S_j, $1 \le j \le n$, by

$$Q_j = J^{k-1} \frac{\partial f}{\partial x_j} \quad , \; f = P + R \quad , \; \frac{\partial f}{\partial x_j} = Q_j + S_j$$

Fix j ($1 \le j \le k$). Then, for x sufficient small:

$$f(x) = f(x - x_j e_j) + \int_0^{x_j} \frac{\partial f}{\partial x_j} (x - x_j e_j + s e_j) \, ds$$

$$= P(x - x_j e_j) + R(x - x_j e_j) +$$

$$+ \int_0^{x_j} Q_j (x - x_j e_j + s e_j) \, ds + \int_0^{x_j} S_j (x - x_j e_j + s e_j) \, ds$$

when $x = \sum_{i=1}^{n} x_j e_j$.

We have:

a) $P(x - x_j e_j)$ is a polynomial of degree less than or equal k.

b) $\lim_{x \to 0} \dfrac{R(x - x_j e_j)}{||x||^k} = 0$

c) $\int_0^{x_j} Q_j (x - x_j e_j + s e_j) \, ds$ has degree less than or equal k.

d) $G_j(x) = \int_0^{x_j} S_j(x - x_j e_j + se_j)\, ds$ has k-jet and $J^k G_j = 0$

(because $||x - x_j e_j + se_j|| \leq ||x||$ if $0 \leq s \leq x_j$, and $J^{k-1} S_j = 0$).

Thus,

$$P(x) = P(x - x_j e_j) + \int_0^{x_j} Q_j(x - x_j e_j + se_j)\, ds ,$$

and as a consequence $\dfrac{\partial f}{\partial x_j} = Q_j$.

It follows that $J^{k-1}(\text{grad } f) = \text{grad } J^k f$.

Note that we have not a reciprocal proposition of the proposition 3. We can see this with the following

__Example:__ Let $f:(R;0) \longrightarrow (R;0)$ be the function defined by

$$f(x) = \begin{cases} x^3 \sin(1/x) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

Then $j^2 f = 0$ and $\dfrac{df(x)}{dx} = \begin{cases} 3x^2 \sin(1/x) - x \cos(1/x), & x \neq 0 \\ 0 & , x = 0 \end{cases}$

Note that there is not $J^1(\text{grad } f)$.

__Proposition 4__:

Let $f:(R^n;0) \longrightarrow (R;0)$ be a germ of C^1-function. If f is k-decidable, $J^{k-1} f = 0$ and there is J^{k-1} (grad f) then f and f'_A have the same behaviour with respect to extremes, for any $A = aI \in \mathbf{A}$.

Proof:

As f is k-decidable, $P = J^k f$ is k-decidable, and as $J^{k-1} f = 0$ we have $J^{k-1} P = 0$.

It is a consequence of the proposition 1 that P and P'_A have same behaviour with respect to extremes, and the same is true for f and f'_A.

With this proposition we consider that we cath the goal.It is sufficient to consider

$$D_k = \{f:(R^n;0) \longrightarrow (R;0) : f \text{ is k-decidable}, J^{k-1} f = 0, J^{k-1} (\text{grad } f)\}$$

$E = \bigcup_{k\geq 1} D_k$ and

$\mathbf{A} = \{ A \in Mn(R) : A = aI, a \in R{+}{+} \}$

for that if $f \in E$ and $A \in \mathbf{A}$ then f and f'_A have the same behaviour with respect to extremes at the origin.

4.AN APPLICATION TO THE LIAPUNOV'S STABILITY OF HAMILTONIAN SYSTEMS OF TWO DEGREE OF FREEDOM.

Our intention in this section is to prove a particular case of the Liapunov's Instability Theorem by using two auxiliary functions: one, the mechanical energy, which will guarantee the invariance of a region in the phase space, and another which will show that a certain region in this space is repulsive.

The follow notations will be used in this section:

$U \subset R^n$ will denote an open neighborhood of the origin.

Potential Energy (with critical point at $0 \in R^n$)

$$\text{A } C^2\text{-function } \Pi : U \longrightarrow R, \quad q \longrightarrow \Pi(q) \qquad , \qquad J^1 \Pi = 0$$

Kinetic Energy

$$\text{A } C^2\text{-function } T: U \times R^n \longrightarrow R, \quad (q,p) \longrightarrow T(q,p) = (1/2) \, . \, p^t B(q) \, p$$

where $B \in C^2(U, Mn(R))$ and $B(q)$ is a symmetric positive defined matrix, for all $q \in U$.

Hamiltonian Function (Mechanical Energy)

$$\text{The } C^2\text{-function } H: U \times R^n \longrightarrow R, \quad (q,p) \longrightarrow H(q,p) = T(q,p) + \Pi(q)$$

Hamilton's Equations

$$(1)\qquad \begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial q} \end{cases}$$

The following theorem is a result of Liapunov, of 1897.

Liapunov's Instability Theorem

If Π have not minimum at $q = 0 \in U$ and this fact is 2-decidable then the origin of the mechanical system (1) is unstable.

(see [4]).

The next result is a generalization of the Chetaev's Theorem.

(to see [4]).

Chetaev's Theorem (generalized)

Let $G = \{ (q,p) \in U\times R^n : H(q,p) < 0 \}$

If there are an open set $\Theta \subset G$ adherent to the origin and a C^1-function $W: U\times R^n \longrightarrow R$ such that, for some $\epsilon > 0$ (with $\bar{B}_\epsilon \subset U\times R^n$),

(i) $W > 0$ in $\Theta_\epsilon = \Theta \cap \bar{B}_\epsilon$,

(ii) $\dot{W} = \langle\langle \operatorname{grad} W / (\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}) \rangle\rangle > 0$ in $G_\epsilon = G \cap \bar{B}_\epsilon$,

then the origin of the mechanical system (1) is unstable.

<u>Proof</u>:

(a) For any $\delta > 0$, the set

$$G^{\delta} = \{ (q,p) \in G : H(q,p) \leq -\delta \}$$

is invariant, because H is constant in each trajectory of the mechanical system (1).

(b) $W > 0$ in Θ_{ϵ}.

(c) If $G^{\delta}_{\epsilon} = G^{\delta} \cap \bar{B}_{\epsilon}$ is not empty, then $\dot{W} > 0$ in G^{δ}_{ϵ} and G^{δ}_{ϵ} is a compact set.

Thus, if a motion has initial condition $(q,p) \in \Theta$, and we take $\delta = W(q,p)/2 > 0$, it follows from (a), (b) and (c) that this motion must leaves the compact set G^{δ}_{ϵ} by a point of its boundary that was in S_{ϵ}.

We are going to present now a weaker version of Liapunov's Instability Theorem, that we want prove using Chetaev's Theorem.

<u>Theorem 1</u>:

Let $U \subset R^2$ be an open neighborhood of $0 \in R^2$.

If $\Pi : U \longrightarrow R$ (of C^2-class) has 2-decidable saddle at 0, then the origin of the dinamical system (1) is unstable.

The two next lemmas will permit to use Chetaev's Theorem:

Lemma 1:

Let $f: R^2 \longrightarrow R$ be a C^2-function such that $f(0) = 0$.

If:

(i) f has 2-decidable saddle at the origin,

(ii) $J^2 f$ is null exactly on the straight lines $y = \pm ax$ $(a > 0)$,

(iii) $J^2 f(0,y) < 0$ if $y \neq 0$,

then arbitrarily near of the identity matrix $I \in M2(R)$ there is a diagonal matrix $A \in M2(R)$ and there is $\epsilon' > 0$ such that

$$\left.\begin{matrix} (x,y) \in B'_{\epsilon'} \\ f(x,y) \leq 0 \end{matrix}\right\} \Longrightarrow f'_A(x,y) < 0.$$

Proof:

It follows of (ii) and (iii) that $P = J^2 f$ is defined by

$P(x,y) = (ax - y)(ax + y) = a^2 x^2 - y^2$.

Let $A = \begin{bmatrix} \alpha^2 & 0 \\ 0 & 1 \end{bmatrix}$ where $\alpha > 0$.

Then

$P'_A = \langle\langle \operatorname{grad} P(x,y) \,/\, (\alpha^2 x, y) \rangle\rangle = 2(a^2 \alpha^2 x^2 - y^2)$.

Now we observe that there exists $J^1(\text{grad } f)$ (because f is a C^1-function), and as a consequence, $P'_A = J^2 f'_A$.

Thus, $f = P + R$, $f'_A = P'_A + S$, $J^2 R = J^2 S = 0$.

Let be $\alpha \in (0,1)$. Let us fix $b \in (a\alpha, a)$.

(a) There is $\epsilon'' > 0$ such that

$$\left.\begin{array}{l}(x,y) \in B'_{\epsilon''} \\ b^2x^2 - y^2 \geq 0\end{array}\right\} \Longrightarrow f(x,y) > 0$$

Let us show it.

If $(x,y) \neq 0 \in R^2$ and $b^2x^2 - y^2 \geq 0$ we have

$$\frac{b^2x^2 - y^2}{||(x,y)||^2} \geq 0 \quad \text{and} \quad ||(x,y)||^2 \leq (b^2 + 1)\,x^2 .$$

Then, since $0 < b < a$ it follow that

$$\frac{f(x,y)}{||(x,y)||^2} = \frac{b^2x^2 - y^2}{||(x,y)||^2} + \frac{(a^2 - b^2)\,x^2}{||(x,y)||^2} + \frac{R(x,y)}{||(x,y)||^2} \geq$$

$$\geq \frac{a^2 - b^2}{b^2 + 1} + \frac{R(x,y)}{||(x,y)||^2} .$$

Thus, it is enough to take $\epsilon'' > 0$ such that

$$\frac{|R(x,y)|}{\|(x,y)\|^2} < \frac{a^2 - b^2}{2(b^2+1)} \quad \text{for all } (x,y) \in B'_{\varepsilon''}$$

and we have (a).

In the same way we show that

(b) There is $\mathcal{E}'' > 0$ such that

$$\left.\begin{array}{l} (x,y) \in B'_{\mathcal{E}''} \\ b^2x^2 - y^2 \leq 0 \end{array}\right\} \Longrightarrow f'_A(x,y) < 0$$

Take $\varepsilon' = \min\{\varepsilon'', \mathcal{E}''\}$

For $(x,y) \in B'_{\varepsilon'}$ we have:

$$f(x,y) \leq 0 \overset{(a)}{\Longrightarrow} b^2x^2 - y^2 < 0 \overset{(b)}{\Longrightarrow} f'_A(x,y) < 0$$

Lemma 2:

Let $U \subset R^2$ be an open neighborhood of $0 \in R^2$, and let $\Pi : U \longrightarrow R$ be a potential energy of C^1-class with 2-decidable saddle at the origin. Suppose that $J^2\Pi$ is null exactly at the straight lines $q_2 = \pm a q_1$ $(a > 0)$, and that $J^2\Pi(0,q_2) < 0$ if $q_2 \neq 0$.

Let be $G = \{ (q,p) \in U \times R^2 : H(q,p) < 0 \}$

Then there are a symmetric positive defined matrix $A \in M2(R)$, a non empty open set $\Theta \in G$ adherent to the origin and $\epsilon > 0$ (with $\bar{B}_\epsilon \subset U\times R^2$) such that the C^1-function

$$W: U\times R^2 \longrightarrow R$$
$$(q,p) \longrightarrow W(q,p) = -H \ll q \,/\, Ap \gg$$

satisfies

(i) $W > 0$ in $\Theta_\epsilon = \Theta \cap \bar{B}_\epsilon$,

(ii) $\dot{W} > 0$ in $G_\epsilon = G \cap \bar{B}_\epsilon$.

<u>Proof</u>:

It follows from lemma 1 that arbitrarily near of the identity matrix $I \in M2(R)$ there is a diagonal matrix

$$A = \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix} \in M2(R)$$

and $\epsilon' > 0$ (we can suppose that $\bar{B}_{\epsilon'} \subset U$) such that

$$\left.\begin{matrix} q \in B'_{\epsilon'} \\ \Pi(q) \le 0 \end{matrix}\right\} \Longrightarrow \Pi'_A(q) < 0 .$$

Let be $\Theta = \{ (q,p) \in U\times R^n : W(q,p) > 0 \} \cap G$

(a) Θ is an open set adherent to the origin.

Indeed there is a sequence $\{q_n\}_{n\in N}$ converging to the origin of R^2 such that $\Pi(q_n) < 0$. Choosing small enough $s_n \in R$ ($n \in N$), we have

$H(q_n, s_n q_n) < 0$ and as a consequence $W(q_n, s_n q_n) > 0$ (because A is near of the identity matrix).

(b) For a convenient choice of A, there is C > 0 such that W satisfies (i) and (ii).

In fact, let $U: U \times R^2 \times R^2 \longrightarrow R$ be defined by

$$\Psi(q,p,r,s) = \langle\langle\, p \,/\, B(q)Ap \,\rangle\rangle - \langle\langle\, q \,/\, A \frac{\partial T}{\partial q} \,\rangle\rangle$$

where $A = \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix} \in M2(R)$.

Ψ is a continuous function and for any $\underline{p} \in S^1$, $\Psi(0,\underline{p},1,1) > 0$.

It follows from the compacity of S^1 and of the continuity of Ψ that there is $\delta > 0$ (with $B_\delta \subset U$) such that if

$\|q\| < \delta$, $p \in S^1$, $|r - 1| < \delta$ and $|s - 1| < \delta$

then $\Psi(q,p,r,s) > 0$.

Fix $A = \begin{bmatrix} \underline{r} & 0 \\ 0 & \underline{s} \end{bmatrix} \in M2(R)$ with the extra conditions $|\underline{r} - 1| < \delta$

$|\underline{s} - 1| < \delta$, and consider the correspondent $\epsilon' > 0$.

Then $\Psi(q,p,\underline{r},\underline{s}) > 0$ on $B_\delta \times S^1 \subset U \times R^2$.

Now observe that $\Psi(q,tp,\underline{r},\underline{s}) = t^2\, \Psi(q,p,\underline{r},\underline{s})$ for all $t \in R$. Thus,

$\Psi(q,p,\underline{r},\underline{s}) > 0$ for all $q \in B_\delta$ and for all non-null $p \in R^2$.

Let $\epsilon > 0$ be smaller than $\min\{\delta, \epsilon'\}$.

Put $G_\epsilon = G \cap \bar{B}_\epsilon$ and $\Theta_\epsilon = \Theta \cap \bar{B}_\epsilon$. Then:

(i) $W(q,p) > 0$ for all $(q,p) \in \Theta_\epsilon$.

(This is a consequence of:

$$H < 0 \text{ in } G,\ \langle\langle q / Ap \rangle\rangle > 0 \text{ in } \Theta \text{ and } \Theta_\epsilon \subset G \cap \Theta.)$$

(ii) $\dot{W}(q,p) > 0$ for all $(q,p) \in G_\epsilon$.

It follows that

$$\dot{W}(q,p) = -H\,[\,\langle\langle \dot{q} / Ap \rangle\rangle + \langle\langle q / A\dot{p} \rangle\rangle\,]$$

$$= -H\,[\,\langle\langle \frac{\partial H}{\partial p} / Ap \rangle\rangle - \langle\langle \frac{\partial H}{\partial q} / Aq \rangle\rangle\,]$$

$$= -H\,[\,\langle\langle B(q)p / Ap \rangle\rangle - \langle\langle \frac{\partial T}{\partial q} / Aq \rangle\rangle - \langle\langle \frac{\partial \Pi}{\partial q} / Aq \rangle\rangle$$

$$= -H\,[\,\Psi(q,p,r,s) - \Pi'_A(q)\,].$$

Now, we have $\Pi(q) \leq H(q,p) < 0$ for all $(q,p) \in G_\epsilon$, of where $\Pi'_A(q) < 0$ for all $(q,p) \in G_\epsilon$. Moreover, $\Psi(q,p,r,s) \geq 0$ for all (q,p) in G_ϵ. Thus, $\dot{W} > 0$ in G_ϵ.

<u>Proof of the theorem 1</u>:

If the C^2-function $\Pi : U \to R,\ q \to \Pi(q)$ have 2-decidable saddle at the origin ($U \subset R^2$), choosing a convenient system of coordinates, we can assume that $J^2\Pi$ is null exactly at the straight lines $q_2 = \pm a q_1$, $a > 0$

and that $J^2 \widetilde{\Pi}(0,q_2) < 0$ if $q_2 \neq 0$.

Using the lemma 2 and Chetaev's Theorem it follows that the origin is unstable.

<u>Comments</u>:

We could not extend this proof of the theorem 1 for $n \geq 3$ degrees of freedom.

This occur because the result of the lemma 1, which is the basis of our proof, is false if $n \geq 3$.

The next example will justify the last assertion.

<u>Example</u>:

Let be $f: R^3 \longrightarrow R$ defined by $f(x,y,z) = x^2 - y^2 + z^4$.

f has 2-decidable saddle at the origin.

For $A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & g \end{bmatrix} \in M3(R)$ positive defined matrix we have

$$f'_A(x,y,z) = 2x\,(ax + by + cz) - 2y\,(bx + dy + ez) + 4z^3\,(cx + ey + gz)$$

As a consequence:

$f(0,z^2,z) = 0$ and

$$f'_A(0,z^2,z) = -2dz^4 - 2ez^3 + 4ez^5 + 4gz^4$$

$$= -2ez^3 + 2(2g - d)z^4 + 4ez^5 .$$

To make possible the next implication

$$\text{"}\left.\begin{array}{l} f(x,y,z) \leq 0 \\ 0 < \|(x,y,z)\| < o \end{array}\right\} \Longrightarrow f'_A(x,y,z) < 0\text{"}$$

we need

(a) e = 0 and

(b) 2g - d = 0.

But the condition (b) would imply that it is not possible to choose A arbitrarily close to the identity matrix I ∈ M3(R).

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