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ABSTRACT. We construct surfaces of constant mean curvature in the 2×2 real unimodular group which are invariant under the adjoint action of a circle subgroup.

0. INTRODUCTION

In 1841, C. Delaunay found a beautiful way of constructing all rotationally invariant constant mean curvature surfaces in the Euclidean space $\mathbb{R}^3$, namely, by taking the generating curve to be the trajectory of a focal point of a conic section rolling on the axis of rotation of the surface ([1]). Since then, his construction has been substantially generalized to constant curvature spaces, and more generally, to other symmetric spaces (see [6, 7] and the references therein). On the other hand, in the case of three-dimensional Riemannian geometry, the most natural class of spaces to be studied after the the spaces of constant curvature are the three-dimensional Lie groups equipped with a left-invariant metric. In particular, we mention the Heisenberg group H^3, the group of unit quaternions S^3 (which happens to be a space of constant curvature, too, so that the geometry of its submanifolds is already a classical subject) and the 2×2 real unimodular group[1], $SL(2,\mathbb{R})$. See [12, 3] for an analysis of H^3. The purpose of this paper is to initiate the study of surfaces of constant mean curvature in $SL(2,\mathbb{R})$ (equipped with its 'best' Riemannian structure). Besides its intrinsic interest, another motivation comes from the fact that, up to a double covering, $SL(2,\mathbb{R})$ is naturally (isometrically) identified with the unit tangent bundle of the real hyperbolic plane $\mathbb{R}H^2$. Generally speaking, a unit vector field on an arbitrary Riemannian manifold can be pictured as a cross-section, and hence embedded submanifold, of its unit tangent bundle T_1M. Let us agree to say that this unit vector field is *minimal* (or has *constant mean curvature*) if this submanifold is minimal (has

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[1]It is interesting to notice that from the point of view of sub-Riemannian geometry, H^3, S^3 and $SL(2,\mathbb{R})$ represent exactly those spaces of constant *sub-Riemannian* curvature in dimension 3 ([2]).

constant mean curvature, resp.) in T_1M, where the later is equipped with its natural Riemannian metric inherited from M. For compact M, this concept is closely related to the volume of unit vector fields and the volume of flows on M ([9, 4]).

A simple application of the method of equivariant differential geometry ([8]) enables us to reduce the constant mean curvature equation in $G = SL(2,\mathbb{R})$ to an ordinary differential equation on an orbit space with singularities. Namely, we consider the adjoint action of a maximally compact subgroup K on G and we study the Ad_K-invariant surfaces in G. Since G is not simply-connected, it is of interest to consider the construction for its Riemannian universal covering $\tilde{G}$, too. Let $\{e_1, e_2, e_3\}$ be the basis for the Lie algebra of G such that $[e_1, e_2] = e_3$, $[e_3, e_1] = -e_2$ and $[e_3, e_2] = e_1$ and equip G, $\tilde{G}$ with left-invariant metrics by setting that basis to be orthonormal. We state the main results as follows:

Theorem 1. *a. For each $h > 0$, there exists an embedded surface of constant mean curvature h of cylinder-type in $\tilde{G}$, and there exist infinitely many congruence classes of surfaces of constant mean curvature h of unduloid-type (embedded) and of nodoid-type (immersed) in $\tilde{G}$.*

b. There exists infinitely many congruence classes of embedded minimal surfaces of catenoid-type in $\tilde{G}$.

c. For each $h > 0$, there exists an embedded, spherical surface $\tilde{\Sigma}_h$ of constant mean curvature h in $\tilde{G}$. Moreover, the family $\{\tilde{\Sigma}_h\}_{h>0}$ forms an analytical foliation of $\tilde{G}$ punctured at the identity, and for any given value of $v > 0$, there is exactly one element $\tilde{\Sigma}_h$ in the family which bounds a region $\tilde{\Omega}_h$ of volume equal to v.

Theorem 2. *a. For each $h > 0$, there exist infinitely many congruence classes of immersed tori $S^1 \times S^1$ of constant mean curvature h in G. At least one of them is embedded.*

b. For each $h > 0$, there exists an immersed, spherical surface of constant mean curvature h in G. It is embedded if $h > 1.04$.

Theorem 3. *a. There exists a minimal unit vector field on the real hyperbolic plane $\mathbb{R}H^2$.*

b. There are no unit vector fields of nonzero constant mean curvature on $\mathbb{R}H^2$.

Corollary 1. *There are no unit vector fields of nonzero constant mean curvature on any hyperbolic surface (compact orientable surface of genus ≥ 2 and constant curvature -1).*

1. The basic reduction

Let $G = SL(2,\mathbb{R}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R}, ad-bc=1\}$, real connected simple Lie group of dimension 3. Let α, β, γ, δ denote the canonical linear forms on the space of all 2×2 real matrices $M_2(\mathbb{R}) \cong \mathbb{R}^4$. Differentiate $ad-bc=1$ to get the defining equation for the tangent bundle TG as a subbundle of $T\mathbb{R}^4$:

$$d\alpha - c\beta - b\gamma + a\delta = 0.$$

A basis of left-invariant vector fields in G is given by

$$e_1 = \frac{1}{2}\begin{pmatrix} a & -b \\ c & -d \end{pmatrix}, \quad e_2 = \frac{1}{2}\begin{pmatrix} b & a \\ d & c \end{pmatrix}, \quad e_3 = \frac{1}{2}\begin{pmatrix} -b & a \\ -d & c \end{pmatrix},$$

with bracket relations $[e_1,e_2]=e_3$, $[e_3,e_1]=-e_2$, $[e_3,e_2]=e_1$, and the left-invariant Riemannian metric ds^2 determined by e_1, e_2, e_3 is easily computed to give

$$\frac{1}{2}ds^2 = (c^2+d^2)(\alpha^2+\beta^2) + (a^2+b^2)(\gamma^2+\delta^2) - 2(ac+bd)(\alpha\gamma+\beta\delta).$$

Consider the Iwasawa decomposition $G = KAN = SO(2)\mathbb{R}^+\mathbb{R}$. An element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ decomposes accordingly as

$$g = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}\begin{pmatrix} 1 & p^{-1}q \\ 0 & 1 \end{pmatrix}$$

where $p = \sqrt{a^2+c^2}$, $q = p^{-1}(ab+cd)$, $x = p^{-1}a$, $y = -p^{-1}c$. We want to describe the orbital geometry of the adjoint action of K on G, that is, $\mathrm{Ad}_{k_\phi} g = k_\phi g k_{-\phi}$, where

$$k_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \in K, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

Since this action is not effective, $\mathrm{Ad}_{k_\phi} = \mathrm{Ad}_{k_{\phi+\pi}} = \mathrm{Ad}_{k_{-\phi}}$, we may restrict to $\phi \in [0,\pi)$.

Lemma 1. *The orbit space $X = G/Ad_K = \{(\theta,p) : \theta \in \mathbb{R}/2\pi\mathbb{Z}, 0 < p \le 1\}$ is a topological half-cylinder $S^1 \times (0,1]$. A representative of the orbit parametrized by (θ,p) is given by $\begin{pmatrix} p\cos\theta & p^{-1}\sin\theta \\ -p\sin\theta & p^{-1}\cos\theta \end{pmatrix}$ and K is embedded into X as the subset $\{p=1\}$.*

Proof. Set $\mathrm{Ad}_{k_\phi} g = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. A simple computation gives that

$$a'b' + c'd' = (ab+cd)\cos 2\phi + \frac{1}{2}(b^2+d^2-a^2-c^2)\sin 2\phi.$$

Let $A = ab + cd = pq$, $B = \frac{1}{2}(b^2 + d^2 - a^2 - c^2) = \frac{1}{2}(q^2 + p^{-2} - p^2)$. We examine the solutions in ϕ of $A\cos 2\phi + B\sin 2\phi = 0$. Except for the case $A = B = 0$ when $g \in K$, it is easy to see that there are exactly two solutions for $\phi \in [0, \pi)$ which differ by $\pi/2$. This shows that each $g \in G$ not in K is Ad_K-conjugate to two elements in KA. Now

$$\mathrm{Ad}_{k_{\pi/2}}\begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} = \begin{pmatrix} p^{-1} & 0 \\ 0 & p \end{pmatrix},$$

therefore, exactly one of those elements in KA satisfies $0 < p < 1$. □

Next we shall compute the orbital invariants.

Lemma 2. *a. The orbital metric is given by* $d\bar{s}^2 = 2p^4 d\theta^2 + 4p^{-2}dp^2$.
b. The length of the orbit through $(\theta, p) \in X$ *is given by:* $L(\theta, p) = L(p) = \pi\sqrt{2}(p^{-1} - p)\sqrt{p^2 + p^{-2}}$.

Proof. These assertions follow from easy calculations by making use of the following facts: a. the matrix of the orbital metric is given by the inverse of the matrix of inner products of the gradients of the basic invariants θ, p; b. the length functional is given by

$$\int_0^\pi \left|\frac{d}{d\theta}\mathrm{Ad}_{k_\phi}\begin{pmatrix} p\cos\theta & p^{-1}\sin\theta \\ -p\sin\theta & p^{-1}\cos\theta \end{pmatrix}\right| d\phi$$

. □

Let $\pi : G \to X$ be the projection, $S \subset G$ an Ad_K-invariant surface, $\gamma = \pi(S)$ its generating curve in X, $H(g)$ the mean curvature of S at a point $g \in S$ and $\bar{H}(\pi(g))$ the geodesic curvature of γ at $\pi(g)$. A standart formula of equivariant differential geometry ([7]) says that

$$H(g) = \bar{H}(\pi(g)) - \frac{d}{d\bar{n}}\log L(\pi(g)),$$

where $\bar{n}$ is the positive unit normal to γ. This formula immediately gives the following

Proposition 1. *Let* $\gamma(s) = (\theta(s), p(s))$ *be a curve in* X *parametrized by arc-length, and let* $\alpha(s)$ *be the angle between* $\partial/\partial\theta$ *and the tangent vector* $\gamma'(s)$. *Then the surface* $S = \pi^{-1}(\gamma)$ *has constant mean curvature* h *if and only if:*

$$(1)\qquad \begin{aligned} \theta' &= \frac{\cos\alpha}{\sqrt{2}p^2} \\ p' &= \frac{p\sin\alpha}{2} \\ \alpha' &= F(p)\cos\alpha + h \end{aligned}$$

where

$$F(p) = 1 + \frac{p}{2}\frac{L'(p)}{L(p)} = \frac{p^2(2p^4 - p^2 + 1)}{(p^2-1)(p^4+1)}.$$

Remark 1. The system (1) is singular on the boundary $\{p = 1\}$. Nevertheless, this kind of singularity has been studied quite thoroughly in the literature, and it is known that given any boundary point, there is a solution curve emanating from that point and perpendicular to the boundary ([7]).

2. Classification of the solution curves

The action of K on G lifts to an action on $\tilde{G}$, and the orbit space $\tilde{X} = \tilde{G}/\mathrm{Ad}_K$ is $\{(\theta, p) : \theta \in \mathbb{R}, 0 < p \leq 1\} \cong \mathbb{R} \times (0, 1]$. Now the system (1) extends from X to $\tilde{X}$, and its solution curves in $\tilde{X}$ are exactly the generating curves for constant mean curvature surfaces in $\tilde{G}$. In this section we shall classify all the solution curves of the system (1) considered defined on $\tilde{X}$. We start by listing some elementary observations about these solution curves.

Lemma 3. a. *Any translate of a solution curve in the θ-direction is also a solution curve.*

b. *Reflection of a solution curve across a line $\theta = \theta_0$ is a solution curve with opposite sign for h.*

c. *Reversal of parameter for a solution curve gives a solution curve with opposite sign for h.*

d. *Let $\gamma(s)$ be a solution curve defined for $s \in (s_0 - \epsilon, s_0]$ with $\alpha(s_0) = k\pi$, k an integer. Then $\gamma(s)$ can be continued by reflection across $\theta \equiv \theta(s_0)$.*

Proof. The first three assertions are immediate and the fourth one follows from them. □

In view of Lemma 3, without loss of generality, we henceforth assume that $h \geq 0$.

We have some explicit solutions for (1).

Proposition 2. a. *For every $\theta_0 \in \mathbb{R}$, $\theta(s) \equiv \theta_0$, $p(s) = e^{-s/2}$, $\alpha(s) = -\frac{\pi}{2}$ defines a solution curve for (1) with $h = 0$.*

b. *Let $h > 0$. Then there is a unique $p_h \in (0, 1)$ such that $h = -F(p_h)$. Hence, $p(s) \equiv p_h$, $\alpha(s) = 0$ defines a solution curve for (1).*

Proof. We discuss (b). We have

$$F'(p) = \frac{-2p((p^2-1)^4 + 2p^2(p^4+1))}{(p^4+1)(p^2-1)^2} < 0,$$

and $F(0) = 0$, $\lim_{p\to 1^-} F(p) = -\infty$. Therefore, F is monotonically decreasing from $(0,1)$ onto $(0,-\infty)$. □

Lemma 4. *Let $J(p,\alpha) = p^2 L(p)\cos\alpha + hg(p)$, where $g(p) = \int_1^p 2qL(q)\,dq$. Then J is a first integral of the system (1), that is, J is constant along the solution curves of the system.*

Proof. The differential of J along a solution curve is zero. □

Lemma 5. *If $J \equiv C$ along a solution curve $\gamma(s) = (\theta(s), p(s), \alpha(s))$, then*

$$\theta(p) = \pm \int \frac{C - hg(p)}{p^3(p^4 L(p)^2 - (C - hg(p))^2)^{1/2}}\,dp.$$

Proof. (1) gives $d\theta/dp = \sqrt{2}\cot\alpha/p^3$. Now solve for α in $J \equiv C$. □

In order to study the behaviour of solutions of (1), it will be convenient to consider the following functions defined for $p \in (0,1)$:

$$\begin{aligned} P(p) &= J(p,0) = p^2 L(p) + hg(p), \\ S(p) &= -\frac{p^2 L(p)}{g(p)} \quad \text{(solving for } h \text{ in } P \equiv 0). \end{aligned}$$

Lemma 6. *a. $g'(p) > 0$, $\lim_{p\to 0^+} g(p) = -\infty$, $g(1) = 0$. Hence g is monotonically increasing from $(0,1)$ to $(-\infty, 0)$.*
b. $S'(p) > 0$, $\lim_{p\to 0^+} S(p) = 0$, $\lim_{p\to 1^-} S(p) = +\infty$. Hence g is monotonically increasing from $(0,1)$ to $(0,+\infty)$.
c. If $h > 0$, then $P'(p)$ is positive for $p \in (0, p_h)$ and negative for $p \in (p_h, 1)$ (see Proposition 2(b)), $P'(p_h) = 0$, $\lim_{p\to 0^+} P(p) = -\infty$, $P(1) = 0$ and $P(p) = 0$ if and only if $p = S^{-1}(h)$. If $h = 0$ then $\lim_{p\to 0^+} P(p) = \pi\sqrt{2}$, $P'(p) < 0$ and $P(1) = 0$.

Proof. a. $g'(p) = 2pL(p) > 0$ and $g(1) = 0$. Also, $2pL(p)$ behaves like $1/p$ as $p \to 0^+$, so the indefinite integral $\int_0^1 2pL(p)dp$ diverges.

b. A simple computation shows that $S'(p) = 2pL(p)(p^2 L(p) - g(p)F(p))/g(p)^2$. Let $Q(p) = p^2 L(p) - g(p)F(p)$. Now $Q'(p) = -g(p)F'(p) < 0$. L'Hospital rule gives $\lim_{p\to 1^-} Q(p) = \lim_{p\to 1^-} \frac{-g(p)}{1/F(p)} = \lim_{p\to 1^-} \frac{g'(p)}{F'(p)/F(p)^2} = 0$. Therefore, $Q(p) > 0$, and hence, $S'(p) > 0$. Also, $\lim_{p\to 0^+} = \pi\sqrt{2}$, so from (a) we get $\lim_{p\to 0^+} S(p) = 0$. Finally, $\lim_{p\to 1^-} S(p) = -\lim_{p\to 1^-} L(p)/g(p) = -\lim_{p\to 1^-} L'(p)/2pL(p) = -\lim_{p\to 1^-}(F(p) - 1)/p^2 = +\infty$, again by L'Hospital rule.

c. $P'(p) = 2pL(p)(h + F(p))$, $\lim_{p\to 0^+} p^2 L(p) = \pi\sqrt{2}$ and $\lim_{p\to 0^+} g(p) = -\infty$. The assertions follow. □

Now we are ready to classify the solutions of (1). We will analyse the solutions $\gamma(s) = (\theta(s), p(s), \alpha(s))$ satisfying the initial conditions

$\theta(0) = 0$, $p(s) = p_0 \in (0,1)$, $\alpha(0) = 0$ (by θ-translational invariance, this case includes all solutions that have a maximum or minimum point for the p-coordinate, that is, all solutions except the solutions $\theta \equiv \theta_0$ described in Proposition 2(a)) Let $\gamma(s)$ have $J \equiv C$. Notice that $J(p(s), \alpha(s)) \leq P(p(s))$ and $C = J(p_0, 0) = P(p_0)$.

Classification: case $h > 0$. By Lemma 6(c), we must have $C \leq P(p_h)$.

a. $C = P(p_h)$: this implies $p_0 = p_h$, so $p(s) \equiv p_h$ is the corresponding solution (cf. Proposition 2(b)). See Fig. 1.

b. $0 < C < P(p_h)$: here $P^{-1}(C)$ consists of two points, say p_C^-, p_C^+, where $p_C^- < p_h < p_C^+$. Suppose that $p_0 = p_C^-$. Since $p^2 L(p)\cos\alpha = -hg(p) + C > 0$, it follows that $\cos\alpha > 0$, hence, $t' > 0$. Also, $J(p(s), \alpha(s)) \equiv C > 0$ implies that $p(s)$ is bounded away from $p = 1$ (because $J(1, \alpha) = 0$). Since $\alpha'(0) = F(p_0) + h > 0$, the angle $\alpha(s)$ rotates in the positive direction to begin with. We claim there is s_1 such that $\alpha'(s_1) = 0$. Otherwise, since $\alpha(s) < \pi/2$, we would eventually have $p(s)$ approaching 1. Now: at a critical point $\alpha'(s) = 0$ we get, by differentiating (1), $\alpha'' = F'(p)p\sin\alpha\cos\alpha/2$, where we have $F'(p) < 0$; hence this must be a local maximum in the first quadrant and a local minimum in the fourth quadrant. We next claim that, since $\alpha(s)$ starts decreasing for $s > s_1$ and has no minimum in the first quadrant, $\alpha(s_2) = 0$ for some $s_2 > s_1$. Otherwise, $\alpha(s) \to 0^+$, $p(s) \to p_1 > p_h$; then (1) implies that $\alpha'(s) \to F(p_1) + h < 0$, so $\alpha'(s)$ is bounded away from zero, a contradiction. Finally we must have $p(s_2) = p_C^+$ and the full solution curve is obtained by reflecting successively across the lines $t \equiv kt(s_1)$ for $k : 0, \pm1, \pm2, \ldots$.

c. $C < 0$: $P^{-1}(C)$ consists of one point $p_C < S^{-1}(h)$. Again, $J \equiv C < 0$ implies that $p(s)$ is bounded away from 1, and $\alpha'(0) > 0$. We claim that there is s_1 such that $\alpha(s_1) = \pi/2$ and $\alpha'(s) > 0$ for $s \in (0, s_1)$. Otherwise, if $\alpha(s)$ reaches a maximum in $(0, \pi/2)$, by the argument in b. we must have s_1' with $\alpha(s_1') = 0$, $p(s_1') > p_C$. But then $P(p(s_1')) = C$, a contradiction. We next claim that there is s_2 such that $\alpha'(s_2) > 0$ for $s \in (s_1, s_2)$ and $\alpha(s_2) = \pi$. In fact, (1) implies that $\alpha' > 0$ for $\alpha \in [\pi/2, \pi)$, and α cannot have any maximum in this interval, so $\alpha(s)$ must reach π (because α' is bounded away from zero). Now the full solution is obtained by reflections, and it has $\alpha' > 0$. Finally, we show that $\theta(s_2) > \theta(0)$. Note that $d\theta/d\alpha = \cos\alpha/\sqrt{2}p^2\alpha'$ is positive in $(0, s_1)$ and negative in (s_1, s_2). For each $s \in (0, s_1)$, let $\bar{s} \in (s_1, s_2)$ be such that $\cos\alpha(\bar{s}) = -\cos\alpha(s)$. Then $\alpha'(\bar{s}) > \alpha'(s)$ and $p(\bar{s}) > p(s)$, so $|d\theta/d\alpha(\bar{s})| < |d\theta/d\alpha(s)|$ and hence, $\theta(s_1) - \theta(s_2) = -\int_{\pi/2}^{\pi} \frac{d\theta}{d\alpha}\, d\alpha < \int_0^{\pi/2} \frac{d\theta}{d\alpha}\, d\alpha = \theta(s_1) - \theta(0)$.

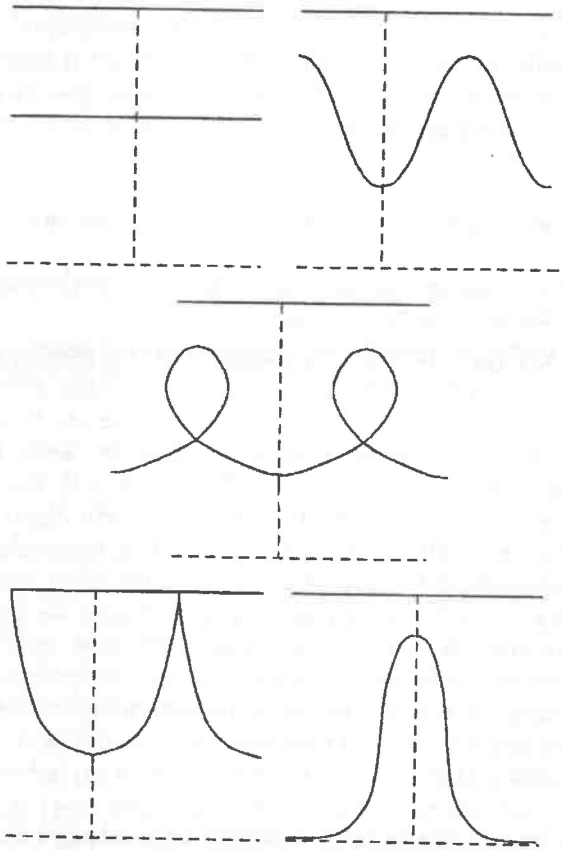

FIGURE 1. In order: solution curves of cylinder-type, unduloid type, nodoid-type, sphere-type and catenoid-type.

d. $C = 0$: here $P^{-1}(0) = S^{-1}(h)$, so $p_0 = S^{-1}(h)$. It follows from the above that there is s_1 such that $\alpha'(s) > 0$ for $s \in (0, s_1)$ and $\alpha(s_1) = \pi/2$, $p(s_1) = 1$. In this case, reflection across the line $t \equiv 0$ gives a solution curve starting and terminating at the singular boundary $p = 1$.

Classification: case $h = 0$. By Lemma 6(c), we must have $0 < C < \pi/\sqrt{2}$, and $p_0 = P^{-1}(C)$ is the unique possible value of $p(s)$ for which we can have $\alpha(s) = 0$. Also, $C > 0$ implies that $t' > 0$. Now $\alpha(s) \in (-\pi/2, 0)$ for $s > 0$ and then (1) implies that $\alpha'(s) < 0$ for $s > 0$. Hence, $p(s) \to 0$ and then $\cos\alpha(s) = C/P(p(s)) \to C/\pi\sqrt{2}$. Thus, the solution is *not* periodic in this case.

3. The proof of Theorem 1

We refer to the classification of solution curves of the system (1) in the previous section.

a, b. The solution curves of (1) generate surfaces of constant mean curvature h in $\tilde{G}$. If $h > 0$, these are of cylinder-type for $C = P(p_h)$, unduloid-type for $0 < C < P(p_h)$ and of nodoid-type for $C < 0$. If $h = 0$, these are of catenoid-type for $0 < C < \pi\sqrt{2}$.

c. Let $h > 0$ and take the solution curve with $C = 0$. This solution, say γ_h, generates an embedded, spherical surface $\tilde{\Sigma}_h$ of constant mean curvature h in $\tilde{G}$. Use θ-translational invariance to make γ_h start at the point $(\theta, p) = (0, 1)$. Set $u = \cos\alpha$. Then $0 \le u \le 1$ parametrizes the left half of γ_h up to its minimal value of p. Let $(\theta(h,u), p(h,u))$ be the parametric representation of the left half of γ_h. Then

$$\begin{aligned}
\theta(h,a) &= \int_0^a \frac{\partial\theta}{\partial u}\,du, \quad \frac{\partial\theta}{\partial h}(h,a) = \int_0^a \frac{\partial^2\theta}{\partial h\partial u}\,du \quad \text{and} \\
p(h,a) &= 1 + \int_0^a \frac{\partial p}{\partial u}\,du, \quad \frac{\partial p}{\partial h}(h,a) = \int_0^a \frac{\partial^2 p}{\partial h\partial u}\,du.
\end{aligned}$$

Now, from $J \equiv 0$ we get that $u = h/S(p)$, so

$$\begin{aligned}
\frac{\partial u}{\partial p} &= -\frac{hS'(p)}{S(p)^2} < 0, \quad \frac{\partial^2 u}{\partial h\partial p} = -\frac{S'(p)}{S(p)^2} < 0, \\
\frac{\partial^2 p}{\partial h\partial u} &= \frac{\partial}{\partial h}\left(\frac{1}{\partial u/\partial p}\right) = -\left(\frac{\partial u}{\partial p}\right)^{-2}\frac{\partial^2 u}{\partial h\partial p} > 0 \quad \text{and} \\
\frac{\partial\theta}{\partial u} &= \frac{\partial\theta}{\partial p}\frac{\partial p}{\partial u} = -\frac{\sqrt{2}}{p^3}\frac{u}{\sqrt{1-u^2}}\frac{\partial p}{\partial u}, \quad \frac{\partial^2\theta}{\partial h\partial u} = -\frac{\sqrt{2}u}{p^3\sqrt{1-u^2}}\frac{\partial^2 p}{\partial h\partial u} < 0.
\end{aligned}$$

Hence, for each fixed a, $p(h,a)$ is a monotonically increasing function of h but $\theta(h,a)$ is a monotonically decreasing function of h. Combining this with the monotonicity of $u \mapsto p(h,u)$ and $u \mapsto \theta(h,u)$, we see that for $h' > h$, the region $R_{h'}$ bounded by $\gamma_{h'}$ must be a proper subset of the region R_h bounded by γ_h and therefore $\mathrm{vol}(\tilde{\Omega}_{h'}) < \mathrm{vol}(\tilde{\Omega}_h)$ where $\tilde{\Omega}_{h'}$, $\tilde{\Omega}_h$ are the inverse images of $R_{h'}$, R_h, respectively. This shows that $h \in (0, +\infty) \mapsto \mathrm{vol}(\tilde{\Omega}_h) \in (0, +\infty)$ is strictly decreasing. It is also surjective, as $\lim_{h\to 0^+} \mathrm{vol}(\tilde{\Omega}_h) = +\infty$.

4. The proof of Theorem 2

Again, we refer to the classification of solution curves of (1).

a. If $h > 0$, the solution curves with $C \neq 0$ are all periodic. They lift to closed surfaces of constant mean curvature in G as long as their periods are rational multiples of 2π. These surfaces will be immersed tori. The torus corresponding to $C = P(p_h)$ is embedded.

b. For each $h > 0$, the solution curve with $C = 0$ lifts to an immersed, spherical surface of constant mean curvature in G. It will be embedded if the period $T(h)$ given by Lemma 5 satisfies

$$T(h) = 2\sqrt{2}h \int_{S^{-1}(h)}^{1} \frac{dp}{p^3\sqrt{S(p)^2 - h^2}} < 2\pi.$$

Computer estimates show that this holds as long as $h > 1.04$ (or $p_0 > 0.485$).

5. The proof of Theorem 3

a. This is essentially the surface S_0 generated by the solution curve described in Proposition 2(a). In fact, take $\theta_0 = 0$ and let v be any unit vector tangent to the real hyperbolic plane $\mathbb{R}H^2 = G/K$ at the basepoint $x_0 = 1K$. Extend v to a unit vector field ξ on $\mathbb{R}H^2$ by parallel translating along all radial geodesics emanating from x_0 (ξ is well-defined). Under the isometric identification of $G/\{\pm 1\}$ with the unit tangent bundle of $\mathbb{R}H^2$ (sending 1 to v), we claim that $S_0 \equiv S_0/\{\pm 1\}$ corresponds to the image of ξ as a cross-section of that bundle. In fact: $S_0 = \{\mathrm{Ad}_k a : k \in K/\{\pm 1\}, a \in A\}$; any $x \in \mathbb{R}H^2$, $x \neq x_0$, can be written uniquely as $x = kaK$ for some $k \in K/\{\pm 1\}$, $a \in A$ ("polar decomposition" of $\mathbb{R}H^2$); $A \subset G$ induces the one-parameter subgroup of transvections of $\mathbb{R}H^2$ along the geodesic with initial speed v and $\mathrm{Ad}_k A \subset G$ induces the one-parameter subgroup of transvections of $\mathbb{R}H^2$ along the geodesic with initial speed $k.v$, for each $k \in K$. Hence, $\mathrm{Ad}_k a \in S_0$ identifies with the tangent vector $\xi(x)$.

b. A unit vector field of constant mean curvature $h > 0$ on $\mathbb{R}H^2$ would lift to a surface S of constant mean curvature h in $\tilde{G}$. Consider the compact spherical surface $\tilde{\Sigma}_h$ of Theorem 1. Note that $\tilde{G} \to \mathbb{R}H^2$ is a trivial line bundle. Since right-translation by K is an isometry in $\tilde{G}$, we may move $\tilde{\Sigma}_h$ along that direction until it touches S in such a way that the tangent planes of $\tilde{\Sigma}_h$ and S at the touching point coincide. By Höpf's maximum principle ([5]), $\tilde{\Sigma}_h$ and S must coincide, a contradiction.

6. Final remarks

1. An *isoperimetric region* in a fixed Riemannian manifold M^n is by definition a compact set with given volume v which minimizes the $(n-1)$-dimensional measure of its boundary among all possible regions with the same volume. It follows easily form the formulae of first variation

of volume that the regular part of the boundary of an isoperimetric region is a hypersurface of constant mean curvature. In our case $n = 3$, so J. Simons codimension theorem ([11]) implies that boundaries of isoperimetric regions are always regular, and a result by B. Kleiner ([12], p. 493) implies that isoperimetric regions of sufficiently small volumes in $\tilde{G}$ (resp. G) are spherical and must be exactly the regions $\tilde{\Omega}_h$ (resp. their projections Ω_h in G) of Theorem 2. It is tempting to conjecture, based on [7], that *any* isoperimetric region in G (resp. $\tilde{G}$) must be Ad_K-invariant. The trueness of this conjecture implies that:

a. In $\tilde{G}$, the solution to the isoperimetric problem for volume v is the region $\tilde{\Omega}_h$ with volume v.
b. There is v_0 such that the solution to the isoperimetric problem in G is the region Ω_h with $\mathrm{vol}(\Omega_h) = v$ if $v \leq v_0$ and it is the region Φ_h with $\mathrm{vol}(\Phi_h) = v$ if $v \geq v_0$, where Φ_h is the Ad_K-invariant solid torus generated by the subset of X given by $p_h \leq p \leq 1$. Observe that G is of the topological type of $S^1 \times \mathbb{R}^2$, and this situation mimics the behaviour of isoperimetric regions for cylinders studied in [10].

2. The isometry group of G is of dimension 4, its connected component being isomorphic to the direct product $K \times G$, where the G-factor corresponds to the left-translations by elements of K, A, N and the K-factor correponds to the right translations by elements of K. Therefore, besides the *rotational* constant mean curvature surfaces studied in this paper, G may accomodate also *translational* and *helicoidal* constant mean cuvature surfaces. An investigation in this direction will appear elsewhere.

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