Attractors for Parabolic Problems with Nonlinear Boundary Conditions in Fractional Power Spaces

Sérgio Muniz Oliva and Antônio Luiz Pereira[2]
Instituto de Matemática e Estatística, Universidade de São Paulo
01498 - São Paulo (SP) - Brasil

Abstract

In this work we prove existence of global attractors for reaction-diffusion problems with nonlinear boundary conditions in fractional power spaces.

Introduction

Let Ω be a bounded smooth domain of $\mathbf{R}^n$. In this paper we consider reaction diffusion systems with dispersion of the form

$$\begin{cases} u_t = \mathrm{Div}(a\nabla u) - \sum_{j=1}^{n} B_j(x)\dfrac{\partial u}{\partial x_j} - \lambda u + f(u), & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial n_a} = g(u), \qquad \text{on} \quad \partial\Omega. \end{cases} \tag{1}$$

where $u = (u_1, \cdots, u_N)^{\top}$, $N \geq 1$, $a(x) = \mathrm{diag}(a_1(x), \cdots, a_N(x))$, $a_i \in C^1(\overline{\Omega})$, $a_i(x) > m_0 > 0$, $x \in \Omega$, $1 \leq i \leq N$, $\frac{\partial u}{\partial n_a} = \langle a\nabla u, \vec{n}\rangle$, $\vec{n}$ is the outward normal, λ is a positive constant and $B_j = \mathrm{diag}(b_j^1, \cdots, b_j^N)$ is continuous in $\bar{\Omega}$, $j = 1, \cdots, n$. Let $f = (f_1, \cdots, f_N)^{\top} : \mathbf{R}^N \to \mathbf{R}^N$, $g = (g_1, \cdots, g_N)^{\top} : \mathbf{R}^N \to \mathbf{R}^N$ be smooth functions.

It has been shown by Pao [12] that if f is a source of heat and if $g = 0$ then we have blow up in finite time. Our aim is to control the increase of heat by means of a dissipative flux through the boundary. To accomplish this goal we need to introduce some kind of "competition" between f and g (see condition **H2** below for the precise condition). In fact one of the basic questions is: If g dissipates heat through the boundary, can we find a relation between the dissipation g and the source of heat f in such a way that we can assure the existence of global attractors?

In [2] the existence of the global attractor is proved for $n = 2$, assuming only dissipative properties on f and g. The key idea is to restrict the space of initial data in such a way that no growth assumptions are needed for local existence of solutions for (1)

Our goal here is to extend this result to arbitrary dimensions. To accomplish this we work in L^p spaces for a suitable choice of $1 < p < \infty$, instead of $L^2(\Omega)$, and then follow the general approach developed by Amann [1]. The main difference is that, in our approach, instead of working in the Sobolev spaces $W^{k,p}(\Omega)$, we work in the fractional power spaces associated to the operator defined by the linear part of (1) with homogeneous boundary conditions. These fractional power spaces turn out to be the so called "Lebesgue Spaces" $H_p^s(\Omega)$ (see Triebel [14], for a general discussion about these spaces and their relation with differential operators and interpolation theory). In this way we are able to use the

[2]Supported in part by CNPq Conselho Nacional de Desenvolvimento Científico e Tecnológico

well developed theory of sectorial operators as described, for example, in Henry [5]. This approach leads, in our opinion, to a considerable simplification in Amann's arguments.

We will only sketch the argument here. Detailed proofs can be found in [10].

Hypotheses

In this Section we fix the hypotheses to be used throughout this paper.

(H1)

$$\limsup_{|s|\to\infty} \tfrac{f_i(s)}{s_i} \le c_i^0; \quad \limsup_{|s|\to\infty} \tfrac{g_i(s)}{s_i} \le d_i^0 \tag{2}$$

Moreover, if f, g satisfy (H1), and given the eigenvalue problem

$$\left.\begin{array}{l} -\mathrm{Div}(a\nabla v_i) + \sum_{j=1}^{n} B_j(x)\dfrac{\partial v_i}{\partial x_j} + \lambda v_i - c_i^0 v_i = \mu_i v_i, \qquad \text{in} \quad \Omega, \\ \dfrac{\partial v_i}{\partial n_a} = d_i^0 v_i \qquad \text{on} \quad \partial\Omega \end{array}\right\} \tag{3}$$

we will assume the following,

(H2) *c_i^0 and d_i^0 are such that the first eigenvalue (in the sense defined by Remark 1), μ_1, of the problem (3) is positive.*

Finally, as mentioned before, we will assume that

(H3) *λ is such that the linear part of the operator in (1), with Neumann Boundary condition, is a positive operator (the operator A will be defined precisely in the next section). If $B \equiv 0$, then λ can be any positive value.*

Remark 1 *From the results of Protter & Weinberger [11], and Krein & Rutman [7], we have that the first eigenvalue of (3) is always real. Here, we mean first, in the sense that all the others have greater real part. See Carvalho, Oliva, Pereira and Rodriguez-Bernal [2] for a proof of this in our case.*

Remark 2 *To avoid notational complications we will treat only the case $N = 1$, but the results remain true in higher dimensions and the same arguments apply if we assume (H1) (see [10]).*

Remark 3 *(H1) is the dissipation condition on the equation. Note that we allow either c_0 or d_0 to be positive. In other words, we allow either f or g to be a source of heat.*

Remark 4 *(H2) is a precise formulation of the "competition" between f and g that we mentioned in the Introduction. Notice that we cannot have both c_0 and d_0 positive. Moreover this condition states that our problem "behaves" as an intermediate case between the Dirichlet case ($d_0 = \infty$) and the Newmann case ($d_0 = 0$).*

Negative Fractional Power Spaces

Here we define the fractional power spaces related to the operator defined in (1), including negative powers. Let us mention that if we work in L^2, these negative fractional powers can be easily defined using duality and Fourier transforms (see for example, Rodriguez-Bernal [13]), but in L^p things are more delicate.

Denote by $\mathcal{B}$ the boundary operator $\mathcal{B}u = \dfrac{\partial u}{\partial n_a}$ and let $H^2_{p,\{\mathcal{B}\}}(\Omega)$ be the Lebesgue space with boundary condition $\mathcal{B}$ (see [14]).

Consider $A = \mathrm{diag}(A_1, \cdots, A_N)$ in $L^p(\Omega; \mathbf{C}^N)$, the operator defined by $D(A_i) = H^2_{p,\{\mathcal{B}\}}(\Omega)$; $\quad A_i u = -Div(a_i \nabla u) + \sum_{j=1}^n b^i_j(x) \frac{\partial u}{\partial x_j} + \lambda u.$

Let $A' = diag(A'_1, \cdots, A'_N)$ (the dual operator) be the operator in $L^{p'}(\Omega; \mathbf{C}^N)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, defined by $D(A'_i) = H^2_{p',\{\mathcal{C}\}}(\Omega)$; $\quad A'_i v = -Div(a_i \nabla v) - div(vB_i) + \lambda v$, where $\mathcal{C}$ is the boundary operator $\mathcal{C}v = \dfrac{\partial v}{\partial n_a} + vB \cdot \vec{n}$.

We have that (see Triebel [14], pag. 401) A' is an isomorphism from $H^2_{p',\{\mathcal{C}\}}(\Omega; \mathbf{C}^N)$ onto $L^{p'}(\Omega; \mathbf{C}^N)$; A'' (the dual operator of A') is an isomorphism from $L^p(\Omega; \mathbf{C}^N)$ onto $(H^2_{p',\{\mathcal{C}\}}(\Omega; \mathbf{C}^N))'$ and $A'' \equiv A$, in $H^2_{p,\{\mathcal{B}\}}(\Omega; \mathbf{C}^N)$.

With this, let us define the operator A_{-1} in $(H^2_{p',\{\mathcal{C}\}}(\Omega; \mathbf{C}^N))'$ by $D(A_{-1}) = L^p(\Omega; \mathbf{C}^N)$; $\quad A_{-1}u = A''u$, for all $u \in L^p(\Omega; \mathbf{C}^N)$. The following results hold (see [10]).

Proposition 1 *A_{-1} is a sectorial operator, with $\rho(A) = \rho(A_{-1})$. Moreover, given $\theta \geq 0$, if we define $X^\theta_{-1} = D(A^\theta_{-1})$ then A_{-1} is also a sectorial operator in X^θ_{-1}, which we denote by $A_{\theta-1}$.*

Theorem 1 *Write $X^\theta = D(A^\theta)$ Then, if $0 \leq \theta \leq 1$ $X^{\theta+1}_{-1} = X^\theta = H^{2\theta}_{p,\{\mathcal{B}\}}$*

Notation 1 *Having this result in mind we will define, for all $0 \leq s \leq 1$, $X^{-s} = X^{1-s}_{-1}$*

Local Existence

We observe that we can always consider the Lebesgue spaces as real Banach spaces, even though our functions are taking complex values. With this in mind, if (H3) holds then it follows from Proposition 1 and the results of Henry [5] that $A_{-\beta}$ generates an analytic semigroup in $X^{-\beta}$ for $0 < \beta < 1$ which satisfies, for $-\beta < \alpha < 1 - \beta$

$$\left\| e^{-A_{-\beta}t} u_0 \right\|_{X^\alpha} \leq M e^{-\epsilon t} \|u_0\|_{X^\alpha}, \quad t \geq 0, \ \left\| e^{-A_{-\beta}t} u_0 \right\|_{X^\alpha} \leq M e^{-\epsilon t} t^{-(\alpha+\beta)} \|u_0\|_{X^{-\beta}}, \ t > 0. \tag{4}$$

for some $\epsilon > 0, M > 0$.

Now, in order to use comparison arguments we need to choose α big enough so as to have $X^\alpha \subset \bar{C}(\Omega)$. We also need β small enough to ensure that the boundary conditions are not incorporated in $X^{1-\beta}$, and finally we require $\alpha + \beta < 1$, to be able to apply standard arguments of semilinear parabolic theory as developed for instance in ([5]). This can indeed be done if p is big enough.

Since we are going to use the linear operator A with homogeneous boundary conditions to define the abstract problem, we need to include the nonlinear boundary conditions in the equation. Furthermore since we are working with complex valued functions, we will need to complexify f and g. This is done as follows.

Notation 2 *We denote by* $f_{\mathbf{C}}: \mathbf{C}^N \to \mathbf{C}^N$, $g_{\mathbf{C}}: \mathbf{C}^N \to \mathbf{C}^N$ *the complexifications of* f *and* g, *respectively, where* $F_{\mathbf{C}}(\zeta) := F(\zeta_R) + iF(\zeta_I)$, $\zeta = \zeta_R + i\zeta_I \in \mathbf{R}^N + i\mathbf{R}^N = \mathbf{C}^N$, *for any function* $F: \mathbf{R}^N \to \mathbf{R}^N$. *Now, let us consider the map* $g_\gamma: X^\alpha \to X^{-\beta}$ *defined by* $\langle g_\gamma(u), \phi\rangle := \int_{\partial\Omega} \gamma(g_{\mathbf{C}}(u))\gamma(\phi)$, *for all* $\phi \in H^{2\beta}_{p'}(\Omega)$, *where* γ *denotes the trace operator. Similarly, we define* $f_\Omega: X^\alpha \to X^{-\beta}$ *by* $\langle f_\Omega(u), \phi\rangle := \int_\Omega f_{\mathbf{C}}(u)\phi$, *for all* $\phi \in H^{2\beta}_{p'}(\Omega)$.

We will also denote by $h := f_\Omega + g_\gamma$.

It is easy to show that f_Ω and g_γ are well defined, and h is Lipschitz continuous in bounded sets of X^α. From results in ([5]), the following existence result follows readily.

Theorem 2 *Suppose that (H1) and (H3) hold and* $\frac{n}{2p} < \alpha < 1 - \beta < 1 - \frac{1}{2p'} = \frac{1}{2} + \frac{1}{2p}$. *Then the abstract parabolic problem*

$$\begin{cases} \dfrac{du}{dt} + A_{-\beta}u = h(u) \\ u(0) = u_0 \in X^\alpha \end{cases} \tag{5}$$

has an unique solution for any $u_0 \in X^\alpha$, *which is given by the variation of constants formula*

$$T(t)u_0 = e^{-A_{-\beta}t}u_0 + \int_0^t e^{-A_{-\beta}(t-s)}h(T(s)u_0)ds. \tag{6}$$

Moreover, if the maximal interval of existence of the solution $T(t)u_0$ *is* $[0, t_{max}[$ *then either* $t_{max} = +\infty$ *or* $\|T(t)u_0\|_{X^\alpha} \to \infty$ *as* $t \to t_{max}$.

Regularity Result

The solutions provided by theorem 2 turn out to be more regular. In fact, using results from [9] and [8], we obtain the following result.

Theorem 3 *Suppose that (H1) holds and that* α, β *and* p *as in Theorem (2). Let* $u_0 \in X^\alpha$ *and let* u *be the solution of (5). Then, there exists* $\epsilon > 0$ *such that* $u(t, \cdot) \in C^{2+\epsilon}(\bar{\Omega})$, *for all* $t > 0$. *Moreover,* $Re(u(t, \cdot))$ *is a classical solution of (1), for any* $t > 0$.

Remark 5 *Now that we have local existence for (5) and since all functions and coefficients in the equation are real, we can take the real part of the solution, and we still have a solution. Thus from now on we will suppose that* X^α *is the real part of functions in* $H^{2\alpha}_p(\Omega)$.

Existence of Global Attractors

Hypothesis (**H2**) allows to use a comparison argument to obtain estimates of solutions in the uniform norm. Then, with the help of a 'bootstrap' argument based on the variation of constants formula we can prove:

Lemma 1 *If* V *is a bounded subset of* X^α *then* $\bigcup_{t\geq 0} T(t)V$ *is also a bounded subset of* X^α.

Now, taking into account the regularization properties of the semigroup the existence of a global compact attractor follows from results in ([4]), once point dissipativiness is established. This last property is also achieved by using comparison and bootstrap arguments.

To state our main result, we introduce some notation. Let φ be the first eigenfunction of (3), $m = \min_{x\in\bar{\Omega}} \varphi(x)$. and ξ as in 2. Let also $\Sigma_\theta = \{u \in X^\alpha : |u(x)| \leq \theta\varphi(x),$ for all $x \in \bar{\Omega}\}$. Then we have

Theorem 4 *The problem (1) has a global attractor $\mathcal{A}$ in X^α. Furthermore $\mathcal{A} \subset \Sigma_\theta$ if $\theta m \geq \xi$.*

References

[1] **Amann, H.**, *Parabolic Evolution Equations and Nonlinear Boundary Conditions*, Journal of Diff Eq. **72** (1988), 201-269.

[2] **Carvalho, A.N., Oliva, S.M., Pereira, A.L. and Rodriguez-Bernal, A.**, *Parabolic Problems with Nonlinear Boundary Conditions*, submitted.

[3] **Carvalho, A.N. and Rodriguez-Bernal, A.**, *Global attractors for Parabolic Problems with Nonlinear Boundary Conditions in Fractional Power Spaces*, ICSMS-USP Preprint Series (to appear) (1994).

[4] **Hale, J.K.**, Asymptotic Behavior of Dissipative Systems, Mathematical Surveys and Monographs, 25, AMS, 1988.

[5] **Henry, D.**, Geometric Theory of Semilinear Parabolic Equations, Lectures Notes in Mathematics, **840**, Springer-Verlag, 1981.

[6] **Henry, D.**,*Some Infinite Dimensional Morse-Smale Systems Defined by Parabolic Differential Equations*, journal of Differential Equations **59** (1985), 165-205.

[7] **Krein and Rutman**, "Linear Operators Leaving Invariant a Cone in a Banach Space", Amer. Math. Soc. Trans. (1) 10 [1962], 3-95.

[8] **Ladyženskaja, O.A. and Ural'ceva, N.N.**, Équations aux dérivées partielles de type elliptique, Monographies Universitaires de Mathématiques, Dunod Paris, 1968.

[9] **Lions J.L. and Magenes, E.**, *Problemi ai Limiti Non Omogenei V*, Ann. Scuola Norm. Sup. Pisa **16** (1962), 1-44.

[10] **Oliva, S.M. and Pereira, A.L.**, *Attractors for Parabolic Problems with Nonlinear Boundary Conditions in Fractional Power Spaces*, RT-MAT 95-11, Trabalhos do Departamento de Matemática - IME-USP.

[11] **Protter and Weinberger**, Maximum Principles in Differential Equations, Prentice Hall, inc. [1967].

[12] **Pao, C.V.**, *Asymptotic Behavior and Nonexistence of Global Solutions for a Class of Nonlinear Boundary Value Problems of Parabolic Type*, Journal of Math. Anal. and Appl. **65** (1978), 616-637.

[13] **Rodriguez-Bernal, A.**, *Some Remarks on PDEs with Nonlinear Newman Boundary Conditions*, CDSNS Preprint Series #93-145 (1993).

[14] **Triebel, H.**, Interpolation Theory, Function Spaces, Differential Operators, North-Holland, Amsterdan, 1978.

Electronic Mail: smo@ime.usp.br, alpereir@ime.usp.br