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Relative Asymptotic Equivalence of Evolution Equations *

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Abstract

In this paper we study the relative asymptotic equivalence between the solutions of the following two evolutions equations in a Banach space Z

$$\dot{y} = A(t)y, \qquad \dot{x} = A(t)x + f(t,x), \quad t > 0, \quad y, x \in Z,$$

where $A(t)$ generates a strongly continuous evolution operator $T(t,s)$ and the function $f : \mathbb{R}_+ \times Z \to Z$ is small in some sense. A generalized concept of dichotomy plays an important role in the proof of our main results. We provide sufficient conditions to prove that given a solution $y(t)$ of the unperturbed system, there exists a family of solutions $x(t)$ of the perturbed system such that, $\|y(t) - x(t)\| = o(\|y(t)\|)$, as $t \to \infty$. Also, under certain conditions, we prove that given a solution $x(t)$ of the perturbed system, with Liapunov number $\alpha \in \mathbb{R}$ there exists a family of solutions $y(t)$ of the unperturbed system such that $\|y(t) - x(t)\| = o(\|x(t)\|)$, as $t \to \infty$. Finally, we present examples of ordinary, partial and functional differential equations.

Key words. asymptotic equivalence, asymptotic behavior, Liapunov number, polynomial-exponential dichotomy, nonlinear evolution equation.

AMS(MOS) subject classifications. primary 34C11, 34G20, 34G10, 35B35, 35B40.

1 Introduction

Let Z be a Banach space and $\mathcal{D}$ be a dense subspace. We assume that $A(t) : \mathcal{D} \to Z$ is a family of unbounded operators which generates a strongly continuous evolution operator $T(t,s)$ (see [16]) and $f : \mathbb{R}_+ \times Z \to Z$ is a continuous function such that equation (1.2) has a unique Mild solution $x(t)$ with $x(0) = x_0$, for each $(t_0, x_0) \in \mathbb{R}_+ \times Z$.

Consider the following equations:

$$\dot{y} = A(t)y \quad t \geq 0, \quad y \in Z, \tag{1.1}$$

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$$\dot{y} = A(t)y + f(t,y), \quad t \geq 0, \quad y \in Z, \tag{1.2}$$

In order to study the asymptotic equivalence of the above equations, we consider the following problems:

- **The Direct Problem:** Given a solution $y(t)$ of (1.1), such that $y(t) \neq 0$, for all sufficient large t, does there exist a solution $x(t)$ of (1.2), such that the relative error, $\frac{\|y(t)-x(t)\|}{\|y(t)\|} \to 0$, as $t \to \infty$?

- **The Converse Problem:** Given a solution $x(t)$ of (1.2), such that $x(t) \neq 0$, for all sufficient large t, does there exist a solution $y(t)$ of (1.1), such that, $\frac{\|y(t)-x(t)\|}{\|x(t)\|} \to 0$, as $t \to \infty$?

For ordinary differential equations, results in this direction can be found in Szmidt[23], Onuchic[12], Coppel[5], Brauer and Wong[1], Rodrigues[17], etc.. For retarded differential equations, this problem was studied by Rodrigues[18]. For neutral functional-differential equations, this problem was studied by Izé-Ventura[11].

In this work, we extend the above results to abstract evolution equations and so we can also consider applications to partial differential equations and to infinite-delay partial differential equations. A generalized concept of dichotomy that is defined in this paper is very helpful to the proofs of our main results. On the solution of the converse problem, an integral inequality which was stated and proved in Rodrigues [18] (See also Pachpatte [13]) , plays an important role.

It is easy to see that if we have two functions $x(t)$, $y(t)$ defined for all sufficient large t and if $\frac{\|y(t)-x(t)\|}{\|x(t)\|} \to 0$, as $t \to \infty$ then $\frac{\|y(t)\|}{\|x(t)\|} \to 1$, as $t \to \infty$. One could compare functions $x(t)$, $y(t)$, for all sufficiently large values of t in many different ways. For example, one could require that $\|x(t) - y(t)\| \to 0$, as $t \to \infty$. For some purposes this is not satisfactory. For example, if we take $y(t) = e^t$, and $x(t) = t + e^t$ we have $\|x(t) - y(t)\| \to \infty$, but $\frac{\|y(t)-x(t)\|}{\|x(t)\|} \to 0$, as $t \to \infty$. This shows that $x(t)$ and $y(t)$ have the same behavior, or order of magnitud, near the infinity. Therefore, sometimes it is more convenient to use the relative error instead of the simple error.

In many interesting applications, when one perturbs autonomous linear equations, The Direct Problem has a positive answer. For ordinary differential equations, The Converse Problem has a positive answer, as shown in the above references. However for infinite dimensional systems the Converse Problem may not hold true. The perturbed system may have some solutions with a different order of magnitude at the infinity, namely it may have solutions that go to zero faster than any exponential. For example, $x(t) = e^{-t^2}$ is a solution of the retarded differential equation $\dot{x} = -2te^{1-2t}x(t-1)$. This equation may be considered as a perturbation of $\dot{y} = 0$. Therefore $x(t) = e^{-t^2}$ has no equivalent solution of the unperturbed equation. In a similar way, one can also verify that $x(t) = e^{-t^2}$ is a solution of the following neutral functional differential equation,

$$\frac{d}{dt}[x(t) + \frac{1}{2}x(t-1) + \frac{1}{2}x(t-2)] = h(t)x(t-4),$$

where $h(t) := -2te^{16-8t} - (t-1)d^{15-6t} - (t-2)e^{12-4t}$. The unperturbed neutral differential equation is discussed in Hale-Lunel[8], p. 285. However, it was conjectured by J. Hale and N. Onuchic that for perturbed linear retarded equations the Converse Problem has a positive answer if the chosen solution, $x(t)$, of the perturbed equation has finite Liapunov number. This was proved in Rodrigues[18]. It turns out that the same is true for quite general evolution equations, as it is proved in Section 3. We point out that, somehow the results and the proofs presented in this paper bring new ideas, simplify some procedures and extend considerably the range of applications.

In many perturbation problems one is often involved with perturbed linear systems and it may be necessary to obtain refined estimates of its solutions, near the infinity. For example, let us consider a nonlinear system:

$$\dot{x} = f(x)$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is sufficiently smooth. Let $\phi(t)$ be a solution such that $\sup_{t\in\mathbb{R}} \|\phi(t)\| < \infty$. This solution could be, for example, a parametrization of a homoclinic or a heteroclinic orbit. Let us suppose that one is interested in studying bounded solutions, that are close to $\phi(t)$, of a perturbed equation such as:

$$\dot{x} = f(x) + F(t, x, \lambda)$$

where λ is small, and F is a smooth function, such that $F(t, x, 0) \equiv 0$. This kind of problem can be found in Chow, Hale, Mallet-Paret[4], Chow and Hale[3], Rodrigues and Ruas [19], Rodrigues and Silveira[22], etc.. See also Rodrigues and Silveira[21], Rodrigues and Ruas-Filho[20], Palmer[14, 15], for related problems. To use Liapunov-Schmidt Method, one needs very refined estimates on the solutions of the variational system:

$$\dot{z} = f_x(\phi(t))z.$$

Under certain assumptions, the above system can be put into the form:

$$\dot{z} = f_x(0)z + B(t)z,$$

where $\|B(t)\| \to 0$, as $|t| \to \infty$, exponentially. Therefore it is important to know the behavior of the solutions of that system as $|t| \to \infty$.

This work is organized as follows. In Chapter 2 we analyze The Direct Problem. In Chapter 3 we study The Converse Problem. In Chapter 4 we present some examples that include Retarded-Functional Differential Equations and Partial Differential Equations.

2 The Direct Problem.

In this paper we shall study the relative asymptotic equivalence between the solutions of the following two evolution equations in the Banach space Z,

$$\dot{y} = A(t)y \quad t \geq 0, \quad y \in Z, \tag{2.1}$$

$$\dot{y} = A(t)y + f(t, x) \quad t \geq 0, \quad y \in Z, \tag{2.2}$$

where $A(t) : \mathcal{D} \subset Z \to Z$ is a family of unbounded operators in Z which generates a strongly continuous evolution operator $T(t,s)$ (see Pazy[16]).

In this section we shall introduce a very general concept of dichotomy that includes the usual exponential dichotomy. This new concept will play an important role in the solution of our main problems.

In this work we assume the perturbation function $f : \mathbb{R} \times Z \to Z$ is continuous and:

$$\|f(t,z)\| \leq h(t)\|z\|, \quad t \geq 0, \quad z \in Z, \tag{2.3}$$

$$\|f(t,z) - f(t,w)\| \leq h(t)\|z - w\|, \quad t \geq 0, \quad z, w \in Z, \tag{2.4}$$

where $h : \mathbb{R} \to \mathbb{R}$ is a continuous function.

Definition 2.1 (Polynomial-Exponential Dichotomy (PE-dichotomy)) We shall say that the equation (2.1) has a Polynomial-Exponential Dichotomy, with respect to $\alpha \in \mathbb{R}$ if there exist an integer $N \geq 1$, $M \geq 0$, $\varepsilon > 0$ and complementary continuous projections $S(t),\ P(t),\ Q(t),\ U(t) : Z \to Z,\ t \in \mathbb{R}$ such that

1. For $t \geq s$,
   $P(t)T(t,s) = T(t,s)P(s),\ Q(t)T(t,s) = T(t,s)Q(s),$
   $S(t)T(t,s) = T(t,s)S(s),\ U(t)T(t,s) = T(t,s)U(s)$

2. $T(t,s) : \mathcal{R}(U(s)) \to \mathcal{R}(U(t)),\ \ t \geq s$, is an isomorphism with inverse

   $$T^{-1}(t,s) = T(s,t) : \mathcal{R}(U(t)) \to \mathcal{R}(U(s)), \quad t \geq s,$$

   $T(t,s) : \mathcal{R}(Q(s)) \to \mathcal{R}(Q(t)),\ \ t \geq s$, is an isomorphism with inverse

   $$T^{-1}(t,s) = T(s,t) : \mathcal{R}(Q(t)) \to \mathcal{R}(Q(s)), \quad t \geq s,$$

3. For each ℓ, $0 \leq \ell \leq N-1$ we have:

   $$\begin{aligned} \|T(t,s)S(s)\| &\leq Me^{(\alpha-\varepsilon)(t-s)}, \quad t \geq s \\ \|T(t,s)U(s)\| &\leq Me^{(\alpha+\varepsilon)(t-s)}, \quad s \geq t \end{aligned} \tag{2.5a}$$

   $$\|T(t,s)(P(s)+Q(s))\| \leq Mt^{N-1}e^{\alpha(t-s)}, \quad t \geq s \geq \sigma \tag{2.5b}$$

   $$\begin{aligned} \|T(t,s)P(s)\| &\leq Mt^{\ell-1}s^{N-\ell}e^{\alpha(t-s)}, \quad t \geq s \geq \sigma \\ \|T(t,s)Q(s)\| &\leq Mt^{\ell}s^{N-\ell-1}e^{\alpha(t-s)}, \quad s \geq t \geq \sigma \end{aligned} \tag{2.5c}$$

   where $\sigma > 0$. If $\ell = 0$ then $P(s) \equiv 0$.

   In particular, if $P(t) = Q(t) \equiv 0$, then the above definition coincides with the usual exponential dichotomy.

Remark 2.2 *In particular, if $P(t) = Q(t) \equiv 0$, then the above definition coincides with the usual exponential dichotomy. For Ordinary Differential Equations estimates similar to (2.5c) can be found in Coddington-Levinson [2] In many interesting applications we also have that, $T(t,s) : \mathcal{R}(P(s)) \to \mathcal{R}(P(t))$, $t \geq s$, is an isomorphism with inverse $T^{-1}(t,s) = T(s,t) : \mathcal{R}(P(t)) \to \mathcal{R}(P(s))$, $t \geq s$,.*

Definition 2.3 If $\alpha \in \mathbb{R}$ and ℓ is a nonnegative integer, we shall say that a function $y : [0, \infty) \to Z$ is of order $t^\ell e^{\alpha t}$, and denote it by $y(t) \cong t^\ell e^{\alpha t}$, if,

$$0 < \liminf_{t\to\infty} \frac{\|y(t)\|}{t^\ell e^{\alpha t}} \leq \limsup_{t\to\infty} \frac{\|y(t)\|}{t^\ell e^{\alpha t}} < \infty. \tag{2.6}$$

Definition 2.4 (Mild Solution) For mild solutions $y(t)$ and $x(t)$ of equations (2.1) and (2.2) with initial conditions $y(t_0) = y_0 \in Z$, $x(t_0) = x_0 \in Z$, we mean continuous functions that satisfy:

$$y(t) = T(t, t_0)y_0, \quad t \geq t_0, \tag{2.7}$$

$$x(t) = T(t, t_0)x_0 + \int_{t_0}^{t} T(t,s) f(s, x(s))ds, \quad t \geq t_0. \tag{2.8}$$

The next theorem gives a positive answer to The Direct Problem.

Theorem 2.5 *Let $y(t)$ be a solution of (2.1) with $y(t) \cong t^\ell e^{\alpha t}$. Suppose that (2.3), (2.4) are satisfied and that equation (2.1) has PE-dichotomy with respect to α, in the sense given by Definition 2.1 and the continuous function $h(t)$ satisfies*

$$\int_0^\infty s^{N-1} h(s)ds < \infty. \tag{2.9}$$

Then there exists a solution $x(t)$ of (2.2) such that

$$\lim_{t\to\infty} \frac{\|x(t) - y(t)\|}{\|y(t)\|} = 0. \tag{2.10}$$

Proof: If we let $z := e^{-\alpha t}(x - y(t))$ in equation (2.2), we obtain the following equation for z,

$$z' = A_\alpha(t)z + F(t,z), \quad t > 0, \tag{2.11}$$

where

$$A_\alpha(t) = (A(t) - \alpha I) \quad \text{and} \quad F(t,z) = e^{-\alpha t} f(t, y(t) + e^{\alpha t} z). \tag{2.12}$$

Therefore, the evolution operator $T_\alpha(t,s)$ generated by the family of operators $A_\alpha(t)$ is given by

$$T_\alpha(t,s) = e^{-\alpha(t-s)} T(t,s), \quad t \geq s. \tag{2.13}$$

Hence, redefining the projections $P(s)$, $Q(s)$, if necessary, in such a way they include the projections $U(s)$, $S(s)$, we obtain the following conditions for $T_\alpha(t,s)$ and $F(t,z)$

$$\begin{aligned}
\|T_\alpha(t,s)P(s)\| &\leq Mt^{\ell-1}s^{N-\ell}, \quad t \geq s \geq \sigma > 0, &(2.14)\\
\|T_\alpha(t,s)Q(s)\| &\leq Mt^{\ell}s^{N-\ell-1}, \quad s \geq t \geq \sigma > 0, &(2.15)\\
\|F(t,z)\| &\leq h(t)e^{-\alpha t}\|y(t)\| + h(t)\|z\|, \quad t \geq 0, \quad z \in Z, &(2.16)\\
\|F(t,z_1) - F(t,z_2)\| &\leq h(t)\|z_1 - z_2\|, \quad t \geq 0, \quad z_1, z_2 \in Z. &(2.17)
\end{aligned}$$

Now, the problem is reduced to prove the existence of a mild solution of (2.11) such that

$$\lim_{t\to\infty} \frac{\|z(t)\|}{t^\ell} = 0. \tag{2.18}$$

If $z(t)$ is a mild solution of (2.11) such that (2.18) holds, then

$$\begin{aligned}
z(t) &= T_\alpha(t,\sigma)z(\sigma) + \int_\sigma^t T_\alpha(t,s)F(s,z(s))ds\\
&= T_\alpha(t,\sigma)\left[Q(\sigma)z(\sigma) + \int_\sigma^\infty T_\alpha(\sigma,s)Q(s)F(s,z(s))ds\right]\\
&+ T_\alpha(t,\sigma)P(\sigma)z(\sigma) + \int_\sigma^t T_\alpha(t,s)P(s)F(s,z(s))ds\\
&- \int_t^\infty T_\alpha(t,s)Q(\sigma)F(s,z(s))ds, \quad t \geq \sigma.
\end{aligned}$$

As it will become clear in the next calculation, in order to make sure that condition (2.18) is satisfied we impose that:

$$Q(\sigma)z(\sigma) + \int_\sigma^\infty T_\alpha(\sigma,s)Q(s)F(s,z(s))ds = 0. \tag{2.19}$$

From (2.9) and (2.16) it follows that the above improper integrals are convergent.

Therefore, we look for a function $z(t)$ that satisfies the following integral equation

$$\begin{aligned}
z(t) &= T_\alpha(t,\sigma)P(\sigma)z(\sigma) + \int_\sigma^t T_\alpha(t,s)P(s)F(s,z(s))ds &(2.20)\\
&- \int_t^\infty T_\alpha(t,s)Q(s)F(s,z(s))ds, \quad t \geq \sigma.
\end{aligned}$$

Conversely, if $z(t)$, with $t^{-\ell}z(t)$ bounded, satisfies the integral equation (2.20), then z is a mild solution of (2.11). In fact,

$$z(t) = T_\alpha(t,\sigma)\left[P(\sigma)z(\sigma) - \int_\sigma^\infty T_\alpha(\sigma,s)Q(s)F(s,z(s))ds\right] + \int_\sigma^t T_\alpha(t,s)F(s,z(s))ds, \quad t \geq \sigma.$$

Therefore,

$$z(t) = T_\alpha(t,\sigma)w + \int_\sigma^t T_\alpha(t,s)F(s,z(s))ds, \quad t \geq \sigma,$$

where

$$w = P(\sigma)z(\sigma) - \int_{\sigma}^{\infty} T_{\alpha}(\sigma, s)Q(s)F(s, z(s))ds, \tag{2.21}$$

Now, to complete the proof we shall show that the integral equation (2.20) has a solution in the Banach space

$$Z_{\ell} = \{z \in C([\sigma, \infty), Z) : \|z(t)\|_{\ell} := \sup_{t \geq \sigma} t^{-\ell}\|z(t)\| < \infty\}. \tag{2.22}$$

We define the operator $\Gamma : Z_{\ell} \to Z_{\ell}$, by:

$$(\Gamma z)(t) := T_{\alpha}(t, \sigma)P(\sigma)w + \int_{\sigma}^{t} T_{\alpha}(t, s)P(s)F(s, z(s))ds - \int_{t}^{\infty} T_{\alpha}(t, s)Q(s)F(s, z(s))ds$$

for $t \geq \sigma$ and $z \in Z_{\ell}$.

Next, we shall prove that Γ maps Z_{ℓ} into Z_{ℓ}. In fact, for $z \in Z_{\ell}$ we have the following estimate:

$$\begin{aligned}
t^{-\ell}\|(\Gamma z)(t)\| &\leq M\frac{1}{t}\sigma^{N}\sigma^{-\ell}\|w\| + \frac{M}{t}\int_{\sigma}^{t} s^{N}s^{-\ell}e^{-\alpha s}\|y(s)\|h(s)ds \\
&+ \frac{M}{t}\int_{\sigma}^{t} s^{N}s^{-\ell}\|z(s)\|h(s)ds + M\int_{t}^{\infty} s^{N-1}s^{-\ell}e^{-\alpha s}\|y(s)\|h(s)ds \\
&+ M\int_{t}^{\infty} s^{N-1}s^{-\ell}\|z(s)\|h(s)ds \\
&\leq \frac{M\sigma^{N-\ell}}{t}\|w\| + \frac{M\|y\|_{\alpha,\ell}}{t}\int_{\sigma}^{t} s^{N}h(s)ds + \frac{M\|z\|_{\ell}}{t}\int_{\sigma}^{t} s^{N}h(s)ds \\
&+ M\|y\|_{\alpha,l}\int_{t}^{\infty} s^{N-1}h(s)ds + M\|z\|_{\ell}\int_{t}^{\infty} s^{N-1}h(s)ds,
\end{aligned}$$

where $\|y\|_{\alpha,l} := \sup_{t \geq \sigma} t^{-\ell}e^{-\alpha t}\|y(t)\|$. Integrating by parts, we obtain

$$\frac{1}{t}\int_{\sigma}^{t} s^{N}h(s)ds = \frac{1}{t}\int_{\sigma}^{t} s\frac{d}{ds}\left(\int_{\infty}^{s} \tau^{N-1}h(\tau)\, d\tau\right)\, ds =$$

$$\int_{\infty}^{t} \tau^{N-1}h(\tau)d\tau - \frac{\sigma}{t}\int_{\infty}^{\sigma} s^{N-1}h(s)ds - \frac{1}{t}\int_{\sigma}^{t}\left(\int_{\infty}^{s} \tau^{N-1}h(\tau)d\tau\right)ds,$$

and

$$\lim_{t\to\infty}\frac{1}{t}\int_{\sigma}^{t}\left(\int_{\infty}^{s} \tau^{N-1}h(\tau)d\tau\right)ds = \lim_{t\to\infty}\int_{\infty}^{t} s^{N-1}h(s)ds = 0.$$

Therefore, $\lim_{t\to\infty} t^{-\ell}\|(\Gamma z)(t)\| = 0$ and so, $\Gamma z \in Z_{\ell}$, $\forall z \in Z_{\ell}$.

Next, we shall prove that Γ is a contraction. Let $z_1, z_2 \in Z_\ell$ and consider

$$\begin{aligned} t^{-\ell}\|(\Gamma z_1)(t) - (\Gamma z_2)(t)\| &\leq \frac{M}{t}\int_\sigma^t s^N h(s) s^{-\ell}\|z_1(s) - z_2(s)\|ds \\ &+ M\int_t^\infty s^{N-1}h(s)s^{-\ell}\|z_1(s) - z_2(s)\|ds \\ &\leq M\left\{\frac{1}{t}\int_\sigma^t s^N h(s)ds + \int_t^\infty s^{N-1}h(s)ds\right\}\|z_1 - z_2\|_\ell \\ &\leq M\left(\int_\sigma^\infty s^{N-1}h(s)ds\right)\|z_1 - z_2\|_\ell. \end{aligned}$$

From condition (2.9) we can choose σ sufficiently large such that

$$\int_\sigma^\infty s^{N-1}h(s)ds < \frac{1}{M}.$$

Therefore, Γ is a contraction mapping from Z_ℓ to Z_ℓ. From the Contraction Principle, we conclude that it has a unique fixed point which depends on $w \in Z$. The solution $x(t)$ of (2.2) that solves The Direct Problem, is given by:

$$x(t) = e^{\alpha t}z(t) + y(t)$$

where $z(\cdot)$ is the fixed point of Γ. The above estimates also imply that $t^{-\ell}z(t) \to 0$, as $t \to \infty$.

Remark 2.6 From the proof of Theorem 2.5 we conclude that for each $\xi \in P(\sigma)Z$ the corresponding solution $z(t)$ of the integral equation (2.20) satisfies

$$z(\sigma) = \xi - \int_\sigma^\infty T_\alpha(\sigma, s)Q(s)F(s, z(s))ds. \tag{2.23}$$

and we have $P(\sigma)z(\sigma) = \xi$. So, if we define the function: $\Phi : P(\sigma)Z \to Z$ by:

$$\Phi(\xi) := \xi - \int_\sigma^\infty T_\alpha(\sigma, s)Q(s)F(s, z(s))ds$$

and if we consider the set:

$$\Sigma = \{\Phi(\xi) : \xi \in P(\sigma)Z, \text{ and } z(t) \text{ is solution of (2.20)}\}, \tag{2.24}$$

then we can prove the following result:

Corollary 2.7 *The set Σ given by (2.24) is a continuous manifold. Moreover, there exists a continuous function $\Phi : P(\sigma)Z \to Z$, such that $P(\sigma)\Phi(\xi) = \xi$, $\forall\ \xi \in P(\sigma)Z$.*

In particular we conclude that Σ is a graph and for each initial data in Σ we obtain a solution $x(t) = e^{\alpha t}z(t) + y(t)$ of (2.2), such that $\|x(t) - y(t)\| = o(\|y(t)\|)$.

3 The Converse Problem.

The object of this section is to give a positive answer to The Converse Problem, in a special case. In fact, given a solution $x(t)$ of (2.2), with finite Liapunov number, we will prove that there exists a solution $y(t)$ of (2.1), such that $\|x(t) - y(t)\| = o(\|x(t)\|)$.

For a proof of the following lemma, see Rodrigues [18] and Pachpatte [13].

Lemma 3.1 *Let $\rho, g \in L_1([0,\infty), \mathbb{R})$ be nonnegative continuous functions. Let $\gamma(t) > 0$ be a decreasing continuous function for $t \geq \sigma$ and σ sufficiently large, in such a way that, $\beta := \int_\sigma^\infty g(s)ds + \int_\sigma^\infty \rho(s)ds < 1$. Suppose that $u(t)$ is a nonnegative continuous function such that $\gamma(t)u(t)$ is bounded and*

$$u(t) \leq K + \int_\sigma^t u(s)\rho(s)ds + \frac{1}{\gamma(t)} \int_t^\infty \gamma(s)u(s)g(s)ds$$

for $t \geq \sigma$, where K is a constant. Then,

$$u(t) \leq \frac{K}{1-\beta} e^{\left(\frac{1}{1-\beta}\right)\int_t^\infty g(s)ds}$$

for $t \geq \sigma$. In particular $u(t)$ is bounded for $t \geq \sigma$.

Lemma 3.2 *Let $A(t) : \mathcal{D} \subset Z \to Z$ be a generator of an evolution operator $T(t,s)$. Assume that there exist complementary projections $P(s)$, $Q(s)$, $\alpha < \beta$ and $K > 0$, such that:*

$$\|T(t,s)P(s)\| \leq Ke^{\alpha(t-s)}, \ t \geq s$$

For $s > t$ we suppose that $T(s,t) : \mathcal{R}(Q(t)) \to \mathcal{R}(Q(s))$ is an isomorphism and $T(t,s) := (T(s,t))^{-1} : \mathcal{R}(Q(s)) \to \mathcal{R}(Q(t))$ satisfies:

$$\|T(t,s)Q(s)\| \leq Ke^{\beta(t-s)}, \ t \leq s.$$

We also assume that (2.3), (2.4) are satisfied with $\int_0^\infty h(s)ds < \infty$

Then

- $\|T(t,s)Q(s)x\| \geq \frac{1}{K}e^{\beta(t-s)}\|Q(s)x\|, \ t \geq s, \ \forall x \in Z.$
- *There is no solution $x(t)$ of (2.2) with Liapunov number μ, with $\alpha < \mu < \beta$.*

Proof: For $t \geq s$, we have:

$$\|Q(s)x\| = \|T(s,t)Q(t)T(t,s)Q(s)x\| \leq Ke^{\beta(s-t)}\|T(t,s)Q(s)x\|,$$

which proves the first part.

Let us suppose now that $\limsup_{t\to\infty} \frac{1}{t} \ln \|x(t)\| = \mu$, where $\alpha < \mu < \beta$. Then we have for $t \geq \sigma > 0$,

$x(t) = T(t,\sigma)x_0 + \int_\sigma^t T(t,s)f(s,x(s))ds$

$= T(t,\sigma)x_0 + \int_\sigma^t T(t,s)P(s)f(s,x(s))ds + \int_\sigma^t T(t,s)Q(s)f(s,x(s))ds$

$= T(t,\sigma)x_0 + \int_\sigma^t T(t,s)P(s)f(s,x(s))ds + \int_\sigma^\infty T(t,s)Q(s)f(s,x(s))ds$
$- \int_t^\infty T(t,s)Q(s)f(s,x(s))ds$
$= T(t,\sigma)[x_0 + \int_\sigma^\infty T(\sigma,s)Q(s)f(s,x(s))ds] + \int_\sigma^t T(t,s)P(s)f(s,x(s))ds$
$- \int_t^\infty T(t,s)Q(s)f(s,x(s))ds =$
$T(t,\sigma)w + \int_\sigma^t T(t,s)P(s)f(s,x(s))ds - \int_t^\infty T(t,s)Q(s)f(s,x(s))ds,$
where,

$$w := x_0 + \int_\sigma^\infty T(\sigma,s)Q(s)f(s,x(s))ds$$

If $\delta > 0$ is such that $\mu + \delta < \beta, \mu - \delta > \alpha$, we have $\|x(t)\| \leq Le^{(\mu+\delta)t},\ t \geq \sigma$, for a suitable $L \geq 0$. Moreover,

$$\|T(t,s)Q(s)f(s,x(s))\| \leq Ke^{\beta(t-s)}h(s)\|x(s)\| \leq Ke^{\beta(t-s)}Le^{(\mu+\delta)s}h(s) \leq$$

$$Ke^{\beta t}e^{-[\beta-(\mu+\delta)]s}Lh(s).$$

Therefore the above integrals are convergent. We also have

$$e^{-(\mu+\delta)t}\int_t^\infty T(t,s)Q(s)f(s,x(s))ds \leq KLe^{[\beta-(\mu+\delta)]t}\int_t^\infty e^{-[\beta-(\mu+\delta)]s}h(s)ds \leq$$

$KL\int_t^\infty h(s)ds,$

Therefore $e^{-(\mu+\delta)t}\int_t^\infty T(t,s)Q(s)f(s,x(s))ds \to 0$, as $t \to \infty$. Also,

$$e^{-(\mu+\delta)t}\int_\sigma^t \|T(t,s)P(s)f(s,x(s))\|ds \leq e^{-(\mu+\delta)t}\int_\sigma^t Ke^{\alpha(t-s)}h(s)\|x(s)\|ds \leq$$

$$KLe^{-[(\mu+\delta)-\alpha]t}\int_\sigma^t e^{\alpha(t-s)}h(s)e^{(\mu+\delta)s}ds \leq KLe^{-[(\mu+\delta)-\alpha]t}\int_\sigma^t e^{[(\mu+\delta)-\alpha]s}h(s)ds.$$

Therefore $e^{-(\mu+\delta)t}\int_\sigma^t T(t,s)P(s)f(s,x(s))ds \to 0$, as $t \to \infty$.
We claim that $Q(\sigma)w = 0$. Otherwise we would have:

$$e^{-(\mu+\delta)t}\|T(t,\sigma)Q(\sigma)w\| \geq \frac{1}{K}e^{[\beta-(\mu+\delta)](t-\sigma)}e^{-(\mu+\delta)\sigma}\|Q(\sigma)w\| \to \infty,$$

as $t \to \infty$.
Then we have,

$$x(t) = T(t,\sigma)P(\sigma)w + \int_\sigma^t T(t,s)P(s)f(s,x(s))ds - \int_t^\infty T(t,s)Q(s)f(s,x(s))ds.$$

Since

$$\|T(t,s)P(s)\| \leq Ke^{\alpha(t-s)} \leq Ke^{(\mu-\delta)(t-s)},\ t \geq s,$$
$$\|T(t,s)Q(s)\| \leq Ke^{\beta(t-s)} \leq Ke^{(\mu+\delta)(t-s)},\ t \leq s,$$

we have:

$\|x(t)\| \leq Ke^{(\mu-\delta)(t-\sigma)}\|P(\sigma)w\| + \int_\sigma^t Ke^{(\mu-\delta)(t-s)}h(s)\|x(s)\|ds+$

$\int_t^\infty Ke^{(\mu+\delta)(t-s)}h(s)\|x(s)\|ds$

which implies:

$e^{-(\mu-\delta)t}\|x(t)\| \leq$

$$Ke^{-(\mu-\delta)\sigma}\|P(\sigma)w\| + K\int_\sigma^t h(s)e^{-(\mu-\delta)s}\|x(s)\|ds + Ke^{2\delta t}\int_t^\infty e^{-2\delta s}h(s)e^{-(\mu-\delta)s}\|x(s)\|ds$$

If we let $u(t) := e^{-(\mu-\delta)t}\|x(t)\|$, we obtain:

$$u(t) \leq Ke^{-(\mu-\delta)\sigma}\|P(\sigma)w\| + K\int_\sigma^t h(s)u(s)ds + Ke^{2\delta t}\int_t^\infty e^{-2\delta s}h(s)u(s)ds.$$

From Lemma (3.1) it follows that $u(t) := e^{-(\mu-\delta)t}\|x(t)\|$ is bounded for $t \geq \sigma$ and this contradicts the hypothesis that the Liapunov number of $x(t)$ is μ. ■

Lemma 3.3 *Let $A(t) : \mathcal{D} \subset Z \to Z$ be a generator of an evolution operator $T(t,s)$. Assume that there exists complementary projections $P(s)$, $Q(s)$, $\alpha < \beta$, $K > 0$ and a positive integer n, such that:*

$$\|T(t,s)P(s)\| \leq Kt^n e^{\alpha(t-s)}, \; t \geq s \geq \sigma > 0$$

For $s > t$ we assume that $T(s,t) : \mathcal{R}(Q(t)) \to \mathcal{R}(Q(s))$ is an isomorphism and $T(t,s) := (T(s,t))^{-1} : \mathcal{R}(Q(s)) \to \mathcal{R}(Q(t))$ satisfies:

$$\|T(t,s)Q(s)\| \leq Ke^{\beta(t-s)}, \; t \leq s.$$

Assume that $\int_0^\infty s^n h(s)ds < \infty$. If $x(t)$ is a solution of (2.2) with Liapunov number α then $\frac{1}{e^{\alpha t}t^n}\|x(t)\|$ is bounded for $t \geq \sigma$.

Proof: We have that,

$$x(t) = T(t,\sigma)x_0 + \int_\sigma^t T(t,s)f(s,x(s))ds =$$

$$T(t,\sigma)w + \int_\sigma^t T(t,s)P(s)f(s,x(s))ds - \int_t^\infty T(t,s)Q(s)f(s,x(s))ds,$$

where $w := x_0 + \int_\sigma^\infty T(\sigma,s)Q(s)f(s,x(s))ds$.

Let ε such that $\alpha < \alpha+\varepsilon < \beta$. If we let $0 < \delta < \varepsilon$, we obtain:

$$\|T(t,s)Q(s)\| \leq Ke^{(\alpha+\varepsilon)(t-s)} \leq Ke^{(\alpha+\delta)(t-s)}, \; t \leq s.$$

Proceeding as in the previous lemma we obtain:

$$\|\int_t^\infty T(t,s)Q(s)f(s,x(s))ds\| \leq Ke^{(\alpha+\delta)t}\int_t^\infty h(s)\frac{\|x(s)\|}{e^{(\alpha+\delta)s}}ds$$

and so the above integrals are convergent. Also,

$\| \int_\sigma^t T(t,s)P(s)f(s,x(s))ds\| \leq K\int_\sigma^t t^n e^{\alpha(t-s)}h(s)\|x(s)\|ds \leq$

$Ke^{\alpha t}t^n \int_\sigma^t e^{-\alpha s}h(s)\|x(s)\|ds$

Therefore,

$$\frac{1}{t^n e^{(\alpha+\delta)t}}\|T(t,\sigma)w\| \leq \frac{\|x(t)\|}{t^n e^{(\alpha+\delta)t}} + Ke^{-\delta t}\int_\sigma^t e^{\delta s}h(s)\frac{\|x(s)\|}{e^{(\alpha+\delta)s}}ds + \frac{K}{t^n}\int_t^\infty h(s)\frac{\|x(s)\|}{e^{(\alpha+\delta)s}}ds$$

If $Q(\sigma)w \neq 0$, we have

$$\frac{1}{t^n e^{(\alpha+\delta)t}}\|T(t,\sigma)Q(\sigma)w\| \geq \frac{1}{Kt^n}e^{-(\alpha+\delta)t}e^{(\alpha+\varepsilon)(t-\sigma)} = \frac{e^{-(\alpha+\varepsilon)\sigma}}{Kt^n}e^{(\varepsilon-\delta)t} \to \infty,$$

as $t \to \infty$.

This implies that $Q(\sigma)w = 0$ and so $P(\sigma)w = w$. Then

$x(t) = T(t,\sigma)P(\sigma)w + \int_\sigma^t T(t,s)P(s)f(s,x(s))ds - \int_t^\infty T(t,s)Q(s)f(s,x(s))ds.$

Thus,

$$\frac{\|x(t)\|}{t^n e^{\alpha t}} \leq Kt^n e^{-\alpha\sigma} + M\int_\sigma^t s^n h(s)\frac{\|x(s)\|}{s^n e^{\alpha s}}ds + \frac{K}{t^n e^{-\varepsilon t}}\int_t^\infty e^{-\varepsilon s}s^n h(s)\frac{\|x(s)\|}{s^n e^{\alpha s}}ds$$

If we let $u(t) := \frac{\|x(t)\|}{t^n e^{\alpha t}}$ and $\gamma(t) := t^n e^{-\varepsilon t}$ and apply Lemma (3.1), we obtain that $u(t) := \frac{\|x(t)\|}{t^n e^{\alpha t}}$ is bounded for $t \geq \sigma$. ■

Theorem 3.4 *Let $A(t) : \mathcal{D} \subset Z \to Z$ be a generator of an evolution operator $T(t,s)$. For a fixed $\alpha \in \mathbb{R}$ suppose that the system (2.1) has solutions with Liapunov number α and there exists integer $N \geq 1$, such that for each solution $y(t)$ of (2.1)with this property, there exists integer $0 \leq \ell \leq N-1$, such that $\|y(t)\| \cong t^\ell e^{\alpha t}$. For each $0 \leq \ell \leq N-1$ we assume that we have a Polynomial-Exponential Dichotomy as in Definition 2.1.*

Then if $x(t)$ is a solution of (2.2) with Liapunov number α then there exists solution $y(t)$ of (2.1), such that

$$\lim_{t\to\infty}\frac{\|x(t)-y(t)\|}{\|y(t)\|} = 0.$$

Moreover there exists ℓ, $0 \leq \ell \leq N-1$, such that $\|x(t)\| \cong t^\ell e^{\alpha t}$.

Proof: From the previous lemma it follows that $\frac{\|x(t)\|}{t^{N-1}e^{\alpha t}}$ is bounded for $t \geq \sigma$. Let $\ell := \min\{m \in \{0,\cdots,N-1\} : \frac{\|x(t)\|}{t^m e^{\alpha t}}\text{is bounded for } t \geq \sigma\}$.

As in Lemma (3.2), we obtain

$$x(t) = y(t) + \int_\sigma^t T(t,s)[P(s)+S(s)]f(s,x(s))ds - \int_t^\infty T(t,s)[Q(s)+U(s)]f(s,x(s))ds, \tag{3.1}$$

for $t \geq \sigma$, where $y(t) := T(t,\sigma)[x(\sigma) + \int_\sigma^\infty T(\sigma,s)Q(s)f(s,x(s))ds]$

Thus,

$$\|x(t)-y(t)\| \leq M\int_\sigma^t t^{l-1}s^N e^{\alpha t}h(s)s^{-\ell}e^{-\alpha s}\|x(s)\|ds + M\int_t^\infty t^\ell s^{N-1}e^{\alpha t}h(s)s^{-\ell}e^{-\alpha s}\|x(s)\|ds.$$

Therefore,

$$\frac{\|x(t)-y(t)\|}{t^\ell e^{\alpha t}} \leq M\|x\|_{\alpha l}\left\{\frac{1}{t}\int_\sigma^t s^N h(s)ds + \int_t^\infty s^{N-1}h(s)ds\right\}.$$

As in Theorem 2.5 we obtain,

$$\lim_{t\to\infty}\frac{\|x(t)-y(t)\|}{t^\ell e^{\alpha t}} = 0.$$

This implies that $\frac{\|y(t)\|}{t^\ell e^{\alpha t}}$ is bounded, for $t \geq \sigma > 0$.

Suppose first that $\ell = 0$. In this case, we have that $e^{-\alpha t}\|x(t)\|$ is bounded for $t \geq \sigma$. From (3.1) it follows:

$$x(t) = T(t,\sigma)w + \int_\sigma^t T(t,s)S(s)f(s,x(s))ds - \int_t^\infty T(t,s)[Q(s)+U(s)]f(s,x(s))ds, \quad t \geq \sigma.$$

where $w := T(t,\sigma)[x(\sigma) + \int_\sigma^\infty T(\sigma,s)[Q(s)+U(s)]f(s,x(s))ds]$

We claim that $[Q(\sigma)+U(\sigma)]w \neq 0$. If this is not the case then we have $S(\sigma)w = w$ and so:

$$\|T(t,\sigma)w\| = \|T(t,\sigma)S(\sigma)w\| \leq Ke^{(\alpha-\varepsilon)(t-\sigma)}\|w\|, \quad t \geq \sigma$$

Then we have:

$$\|x(t)\| \leq \|T(t,\sigma)w\| + \int_\sigma^t \|T(t,s)S(s)\|h(s)\|x(s)\|ds + \int_t^\infty \|T(t,s)Q(s)\|h(s)\|x(s)\|ds, \quad t \geq \sigma$$

$$\|x(t)\| \leq Ke^{(\alpha-\varepsilon)(t-\sigma)}\|w\| + \int_\sigma^t Ke^{(\alpha-\varepsilon)(t-s)}h(s)\|x(s)\|ds + \int_t^\infty Ks^{N-1}e^{\alpha(t-s)}h(s)\|x(s)\|ds, \quad t \geq \sigma.$$

Therefore, for $t \geq \sigma$, we have:

$$e^{-(\alpha-\varepsilon)t}\|x(t)\| \leq Ke^{-(\alpha-\varepsilon)\sigma}\|w\| + K\int_\sigma^t e^{-(\alpha-\varepsilon)s}\|x(s)\|ds + e^{\varepsilon t}\int_t^\infty e^{-\varepsilon s}s^{N-1}h(s)e^{-(\alpha-\varepsilon)s}\|x(s)\|ds.$$

From Lemma 3.1 it follows that $e^{-(\alpha-\varepsilon)t}\|x(t)\|$ is bounded for $t \geq \sigma$, which contradicts the fact that $x(t)$ has Liapunov number α.

Since $\dfrac{\|y(t)\|}{e^{\alpha t}}$ is bounded for $t \geq \sigma$, we have that $U(\sigma)w = 0$.

Then have that $0 < \liminf_{t\to\infty} \frac{\|y(t)\|}{e^{\alpha t}} \leq \limsup_{t\to\infty} \frac{\|y(t)\|}{e^{\alpha t}} < \infty$, which implies that $0 < \liminf_{t\to\infty} \frac{\|x(t)\|}{e^{\alpha t}} \leq \limsup_{t\to\infty} \frac{\|x(t)\|}{e^{\alpha t}} < \infty$.

If $\ell \geq 1$, from (3.1) we obtain:

$$\|x(t)\| \leq \|y(t)\| + M\int_\sigma^t t^{l-1}s^N e^{\alpha t}h(s)s^{-\ell}e^{-\alpha s}\|x(s)\|ds + M\int_t^\infty t^\ell s^{N-1}e^{\alpha t}h(s)s^{-\ell}e^{-\alpha s}\|x(s)\|ds.$$

We claim that $\frac{\|y(t)\|}{t^{\ell-1}e^{\alpha t}}$ is not bounded in $[\sigma,\infty)$. Let us suppose that $\frac{\|y(t)\|}{t^{\ell-1}e^{\alpha t}} \leq L$. Then we have,

$$\frac{\|x(t)\|}{t^{\ell-1}e^{\alpha t}} \leq \frac{\|y(t)\|}{t^{\ell-1}e^{\alpha t}} + M\int_\sigma^t s^N e^{\alpha t}h(s)s^{-\ell}e^{-\alpha s}\|x(s)\|ds + Mt\int_t^\infty s^{N-1}e^{\alpha t}h(s)s^{-\ell}e^{-\alpha s}\|x(s)\|ds.$$

Therefore,

$$\frac{\|x(t)\|}{t^{\ell-1}e^{\alpha t}} \leq L + M\int_\sigma^t s^{N-1}h(s)\frac{\|x(s)\|}{s^{\ell-1}e^{\alpha s}}ds + Mt\int_t^\infty \frac{1}{s}s^{N-1}h(s)\frac{\|x(s)\|}{s^{\ell-1}e^{\alpha s}}ds.$$

From Lemma (3.1) it follows that $\dfrac{\|x(t)\|}{t^{\ell-1}e^{\alpha t}}$ is bounded, which contradicts the definition of ℓ.

Therefore,

$$0 < \liminf_{t\to\infty} \frac{\|y(t)\|}{t^\ell e^{\alpha t}} \leq \limsup_{t\to\infty} \frac{\|y(t)\|}{t^\ell e^{\alpha t}} < \infty$$

and so

$$0 < \liminf_{t\to\infty} \frac{\|x(t)\|}{t^\ell e^{\alpha t}} \leq \limsup_{t\to\infty} \frac{\|x(t)\|}{t^\ell e^{\alpha t}} < \infty.$$

■

Remark 3.5 All the examples to be included in the next chapter are special cases of the class that will be described below.

Let Z be a Banach space and $A : \mathcal{D} \subset \mathcal{Z} \to \mathcal{Z}$ be a generator of a C_0-semigroup $T(t)$. Let $\beta \in \mathbb{R}$ be such that the set $\Sigma_\beta := \{\lambda \in \sigma(A) : Re(\lambda) > \beta\}$ is finite and every element of this set an eigenvalue of A with finite dimensional generalized eingenspace. Let $\lambda \in \Sigma_\beta$ and $\alpha := Re(\lambda)$. let $m+1 \in \mathbb{N}$ be the least integer k such that $\mathcal{N}(A-\lambda I)^{k+1} = \mathcal{N}(A-\lambda I)^k$.

Let $f : \mathbb{R} \times Z \to Z$ be a continuous function such that $\|f(t,x)\| \leq h(t)\|x\|$, where $h : \mathbb{R}_+ \to \mathbb{R}$ is a continuous function such that $\int_0^\infty t^m h(t)\, dt < \infty$. Consider the equations:

$$\dot{y} = Ay \tag{3.2}$$

$$\dot{x} = Ax + f(t,x) \tag{3.3}$$

The conclusions of Theorem (2.5) hold for any solution $y(t) \cong t^\ell e^{\alpha t}$ of (4.1) with $0 \leq \ell \leq m$. The conclusions of Theorem (3.4) hold for any solution $x(t)$ of (4.2) with Liapunov number greater than β.

4 Examples.

Example 4.1 Consider the following matrix given by a Jordan block of dimension $m+1$:

$$J_\alpha := \begin{pmatrix} \alpha & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & \alpha & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & \alpha \end{pmatrix}$$

For each $0 \le \ell \le m$, the projections of (2.5) or (3.4) are given by:

$$P(s) := e^{Js} \begin{pmatrix} 0 & 0 \\ 0 & I_\ell \end{pmatrix} e^{-Js}$$

and

$$Q(s) := e^{Js} \begin{pmatrix} I_{m+1-\ell} & 0 \\ 0 & 0 \end{pmatrix} e^{-Js}$$

In a natural way one can extend the above definition to general linear differential equations with constant coefficients defined in finite dimensional spaces.

Example 4.2 Consider the retarded functional differential equations:

$$\dot{y} = L(y_t) \tag{4.1}$$

$$\dot{x} = L(x_t) + f(t, x_t) \tag{4.2}$$

with appropriate conditions on L and f. This example is discussed in [18]

Example 4.3 Let $Z := L^2(0, \pi)^N$. Consider the equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + J_\gamma u \tag{4.3}$$

with Dirichlet boundary conditions, where J_γ is defined as above, $\gamma \in \mathbb{R}$ and

$$u := \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}.$$

If we indicate by $T_1(t)$ the analytic semigroup associated to the operator $A := \frac{\partial^2}{\partial x^2} : (H^2(0,\pi) \cap H_0^1(0,\pi))^N \to (L^2(0,\pi))^N$ and by $T(t)$ the semigroup associated to $A + J_\gamma$ then $T(t) = e^{J_\gamma t} T_1(t)$ and the spectrum of $A + J_\gamma$ is given by $\{\gamma - n^2, n \in \mathbb{N}\}$. Each eigenvalue has multiplicity N and the generalized eigenspace associated to it has dimension N.

In fact, one can show that the semigroup has the form:

$$T(t)z = \sum_1^\infty e^{-n^2 t} e^{J_\gamma t} P_n z$$

where P_n is the projection in the direction of the eigenfunction $\begin{pmatrix} \sin(nx) \\ \vdots \\ \sin(nx) \end{pmatrix}$.

For a fixed n_0 and $0 \leq \ell \leq N-1$, one can define the projections $P(s)$, $Q(s)$ given in (2.5b) and (2.5c) by:

$$P(s) := e^{J_\gamma s} \begin{pmatrix} 0 & 0 \\ 0 & I_\ell \end{pmatrix} e^{-J_\gamma s} P_{n_o}, \quad Q(s) := e^{J_\gamma s} \begin{pmatrix} I_{N-\ell} & 0 \\ 0 & 0 \end{pmatrix} e^{-J_\gamma s} P_{n_0}$$

In this case the projections $S(s)$, $U(s)$ given in (2.5a) can be defined as:

$$U(s) := \sum_{1}^{n_o - 1} P_n, \quad S(s) := I - \sum_{1}^{n_o} P_n$$

The estimates given in (2.5a) can be obtained in a natural way.

Remark 4.4 With some adaptations, the above analysis can be extended to a more general problem, given by:

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + Bu \tag{4.4}$$

with Dirichlet boundary conditions on a bounded domain $\Omega \subset \mathbb{R}^n$, where B, D are $N \times N$ matrices, such that $BD = DB$.

Under certain conditions, using results presented in Hale-Lunel [8] and Wu [25], one can obtain similar results for Partial-Functional Differential equations, including the case of infinite delay.

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Resumo

Neste trabalho discutimos a equivalência assintótica entre uma equação de evolução linear e uma perturbada. Discutimos as dificuldades que aparecem no caso de dimensão infinita. Apresentamos o conceito de Dicotomia Polinômio-Exponencial, que teve um papel importante na resolução do problema. Analisamos aplicações a Equações Diferenciais Ordinárias, Parciais, com Retardamento e Neutras.

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