Extensions of Immersions in Codimension One and Characterization of Stable Maps*

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1. INTRODUCTION

Let $f = \{f_1, \ldots, f_\rho\}$ be a normal family of oriented closed curves in an oriented surface N (regular maps of S^1 into N such that the set of its intersection and self-intersection points is a finite set of transverse double points). Let V denote an oriented connected bordered surface with border ∂V and let $F: V \to N$ be a sense-preserving immersion. We say that F extends f if $F|\partial V = f$. Let M denote an oriented connected surface and let $g: M \to N$ be a stable map. We denote S(g) for the singular set of g. If f is regular, except for a finite set K of cusp points, we say that g extends f if $g|S(g) = f$.

In this note we state some theorems about the following problems:

I. To determine the immersions $F: V \to N$ that extend f.

II. To determine the stable maps $g: M \to N$ that extend f.

The problem of extension of curves has been originally formulated about 1948 by H. Hopf & C. Lowener. They searched for analytic extensions of a given curve in complex plane [7, 12]. S. J. Blank [2] gave us the first contribution to its solution. Blank's methods turned out to be very fruitful in that kind of question. He solved problem I when $\rho = 1$, $N = \mathbb{R}^2$ and V is the disc D^2. Further contributions were given by G. K. Francis [5, 6], M. L. Marx [4, 10] (and others). They refined Blank's methods and they applied them to Hopf and Lowener's problem.

Proofs of the theorems below will appear elsewhere.

2. STATEMENT OF RESULTS

For the nomenclature that we use we refer to the bibliography. Besides these we introduce some definitions.

For every f as in problem I we consider the Gaussian circles [3, 4], which can be positive or negative.

Definition 1 – We call central circles (positive or negative) the Gaussian circles with interior not containing any other Gaussian circle.

Let $A_f = \{R, W, S, P\}$ be an assemblage on f. We obtain A_f from a finite set W of small circles around an arbitrary prescribed point ∞ in N, a finite set R of rays in N which crosses f and W and two permutations S and P on this set of crossings [4]. We denote the crossings by assigned letters (a different letter for each ray). A positively (negatively) assigned letter corresponds to a positive (negative) crossing. We assume that two assigned letters are distinct if either the letters or the signs are different. Let us suppose that S is in a reduced from, that is, not with a factor like $(a^+ \ldots a^-)$, $(a^- \ldots a^+)$, $(\ldots a^+ \, a^- \ldots)$ or $(\ldots a^- \, a^+ \ldots)$.

We change the conditions of sufficiency on R [4] to the following:

i) every curve of f is crossed and

ii) at least one ray is taken for each central circle of f.

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DEFINITION 2 – An assemblage on f is simple if each cycle of SP contains only distinct assigned letters.

Let ξ be the number of cycles in SP, let ν be the number of negative crossings and let β be the cardinality of W. Let V and N be compact.

THEOREM 1 – f has an extension $F: V \to N$ if and only if:

i) f has GPC property [4] and

ii) f has an effective transitive [4] simple assemblage. Moreover, V is determined by β, $\chi(V) = \xi - \nu + \beta$ and f has as many classes of extension [4] as there are effective transitive simple assemblages.

For every f as in problem II let Γ be a connected assigned by cromatic graph associated to f with vertices V^i and let $\phi^i = \text{star}(V^i)$, $i = 1,\ldots,m$ as in [6]. Let $\theta\colon K \to \{-1,1\}$ assign K. In order to get an assemblage on ϕ^i, for each i, associated to Γ and θ we include a ray from each cusp point $x \in K$ [6]. The crossing on χ is counted negatively and only if x and V^i are equally assigned [11]. An assemblage on f is collection $A = \{A_i\}_{i=1}^m$ of such assemblages [6]. A is simple when A_i is simple for each i.

Let ξ^i be the number of cycles of S^iP^i, let $\xi = \sum_{i=1}^{m} \xi^i$ and let $\beta = \sum_{i=1}^{m} \beta^i$. Let k be the cardinality of K.

Suppose that M and N are compact.

THEOREM 2 – f extends to a stable map g: $M \to N$ if and only if:

i) f has GPC property and

ii) f has an effective transitive simple assemblage. Moreover, M is determined by (Γ,θ,β), $\chi(M) = \xi - 2\nu + k + \beta$ and f has as many classes of extensions as there are effetive transitive simple assemblages.

Now we extend problem I to the case when f is a normal family of proper curves in $\mathbb{R}^2$ (regular maps of S' and regular proper maps of $\mathbb{R}$ into $\mathbb{R}^2$ with a finite set of crossings). We consider a disc D in $\mathbb{R}^2$ with the closed curves in its interior and such that its oriented boundary C crosses each of the other curves twice and transversely. Let ∞ belong to C and include C as a ray, as well as rays from the central circles of $f \cup C$. We do not reduce the cycles in S of the form $(a^+ \ldots a^-)$ with in C. A simple assemblage include cycles in SP with equally assigned letters corresponding to crossings on C.

DEFINITION 3 – A negative fan is a cycle with only negatively assigned letters.

Let δ be the number of negative fans of crossing on C. Let ρ_0 be the number of closed curves of f and let ν_0 be the number of negative crossings outside C.

If f_k is a non-closed curve of f, we can still apply Gauss decomposition to f_k. We obtain a set $\{C_i\}_{i=1}^r$ (or $\emptyset$) of circles and a simple oriented non-closed curve ℓ. Each one of these circles has a rotation number $\tau(C_i) = \pm 1$. We associate to ℓ the number $\tau(\ell) = 1/2\ (-1/2)$ if the first crossing of C and ℓ (as we run over C, starting at ∞) is negative (positive).

DEFINITION 4 – The generalized rotation number of f_k is the rational number $\tau_k = \tau(f_k, \infty)$ given by

$$\tau_k = \sum_{i=1}^{r} \tau(C_i) + \tau(\ell) = (\Sigma \pm 1) \pm 1/2$$

(or $\pm 1/2$, in case the set of circles is empty).

If $1 \leq \rho_0 < \rho$, let $f = \{f_1,\ldots,f_\rho\}$ be indexed so that f_i is closed for $i = 1,\ldots,\rho_0$ and f_i is non-closed for $i = \rho_0 + 1,\ldots,\rho$. Let τ_i denote the rotation number of f_i, for $i = 1,\ldots,\rho_0$.

DEFINITION 5 – The generalized rotation number of f is the rational number

$$\tau = \tau(f,\infty) = \sum_{i=1}^{\rho} \tau_i.$$

Let V denote a bordered non-compact connected 2-dimensional properly imbedded submanifold of $\mathbb{R}^2$.

THEOREM 3 – f extends to proper $F: V \to \mathbb{R}^2$ if and only if it has an effective transitive simple assemblage with $\rho_0 = 2 + \nu_0 - \xi$ and

$\delta = 1$. If f extends then V is unique and $\beta = 1 - \frac{\rho + \rho_0}{2} - \tau$. Moreover f has as many classes of extension as there are effective transitive simple assemblages as above.

NOTE — In case $\rho = \rho_0$, the extensions of f to $F: V \to \mathbb{R}^2$ with V compact correspond to effective transitive simple assemblages with $\rho = 2 + \nu - \xi$ and $\delta = 0$.

Now let f be proper and regular except for a finite set K of cusp points. Let $\rho^i, \rho^i{}_0, \delta^i$ and τ^i be defined for each $i = 1, 2\ldots, m$.

THEOREM 4 — f extends to a proper stable map $g: \mathbb{R}^2 \to \mathbb{R}^2$ if and only if it has an effective transitive simple assemblage associated to a simply-connected graph Γ such that

i) $\rho_0^i = 2 + \nu_0^i - \xi^i$, for any $i \in \{1,2\ldots, \ldots, m\}$ and

ii) $\delta^k = 1$ for a unique $k \in \{1,2\ldots, m\}$ and $\delta^i = 0$ for $i \neq k$ when $\rho = \rho_0$.

$\delta^j = 1$ for any j with $\rho^j \neq \rho_0^j$ and $\delta^i = 0$ for any i with $\rho^i = \rho_0^i$ when $\rho \neq \rho_0$.

If f extends then $\beta^j = 1 - \frac{\rho^j + \rho_0^j}{2} - \tau^j$ for any j with $\delta^j = 1$ and $\beta^i = 0$ otherwise. Moreover, f has as many classes of extension as there are effective transitive simple assemblages of the obove type.

3. REFERENCES

[1] BAILEY, K. D., (1975), Extending closed plane curves to immersions of a disc with n handles, *Trans. Amer. Math. Soc.*, **206**, 1-24.

[2] BLANK, S. J., (1967), *Extending immersions of the circle*, Dissertation, Brandeis Univ. 1967 (Poénaru, Exposé, 342, *Sem. Bourbaki*, N. Y., (1969).

[3] CHILLINGWORTH, D. R. J., (1972), Winding numbers on surfaces I, *Math. Ann.*, **196**, 218-249.

[4] EZELL, C. L. & MARX, M. L., (1980), Branched extensions of curves in orientable surfaces, *Trans. Amer. Math. Soc.*, **259**, n.º 2, June.

[5] FRANCIS, G. K., (1976), Assembling compact Riemann surfaces with given boundary curves and branck points on the sphere, *Illinois J. Math.*, **20**, 198-217.

[6] FRANCIS, G. K. & TROYER, S. F., (1977), Excellent maps with given folds and cusps, *Houston J. of Math.*, **3**, n.º 2.

[7] HAEFLIGER, A., (1960), Quelques rémarques sur les applications différentiables d'une surface dans le plan, *Ann. Inst. Fourier* (Grenoble) **10**, 47-60.

[8] LEVINE, H., (1966), Mappings of manifolds into the plane, *Amer. J. Math.*, **88**, n.º 2, 357-365.

[9] LEVINE, H., (1976), Stable maps: an introduction with low dimensional examples, *Bol. Soc. Bras. Mat.*, **7**, n.º 2, 145-184.

[10] MARX, M. L., (1974), Extensions of normal immersions of S^1 onto $\mathbb{R}^2$, *Trans. Amer. Math. Soc.*, **187**, 302-326.

[11] QUINE, J. R., (1978), A global theorem for singularities of maps between oriented 2-manifolds, *Trans. Amer. Math. Soc.*, **236**, 307-314.

[12] TITUS, C. J., (1973), Extensions through codimension one and to sense-preserving mappings, *Ann. Inst. Fourier* (Grenoble), **23**, 215-227.