The Riemann localization principle fails for the Fourier series of improper Riemann integrable functions

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Abstract

More generally we prove the result above for certain functions of any subspace of the space of Kurzweil-Henstock-Denjoy-Perron integrable functions, if the subspace contains the step functions and the improper integrable functions obtained from them.

1 Fourier series

We recall that for a periodic function $f : \mathbb{R} \to \mathbb{R}$ with period 1 its Fourier coefficients are defined by

$$a_0[f] = \int_0^1 f(t)\,dt,\ a_k[f] = 2\int_0^1 f(t)\cos 2\pi kt\,dt,\ b_k[f] = 2\int_0^1 f(t)\sin 2\pi kt\,dt,\ k = 1,2,\ldots \tag{1}$$

if the integrals are defined. Its formal Fourier series is

$$a_0[f] + \sum_{k=1}^{\infty} (a_k[f]\cos 2\pi kt + b_k[f]\sin 2\pi kt)$$

and we consider the partial sums

$$s_m[f](t) = a_0[f] + \sum_{k=1}^{m} (a_k[f]\cos 2\pi kt + b_k[f]\sin 2\pi kt)\ .$$

We have

$$s_m[f](t) = \int_0^1 f(t-s)D_m(s)\,ds = \int_0^1 f(s)D_m(t-s)\,ds \tag{2}$$

where

$$D_m(t) = \frac{\sin(2m+1)\pi t}{\sin \pi t} . \tag{3}$$

See [C], chap.12; [G], chap.12; [N], §15.2; [H-1], p.224.

We denote by $E([a,b])$ the space of all step function $f : [a,b] \to \mathbb{R}$, i.e., functions $f : [a,b] \to \mathbb{R}$ such that there exists a division of $[a,b]$, $t_0 = a < t_1 < t_2 < \cdots < t_n = b$ such that f is constant in every $]t_{i-1}, t_i[$. $R([a,b])$ and $L_1([a,b])$ denote, respectively, the space of Riemann and Lebesgue integrable functions.

(2) suggests that the convergence of the Fourier series (when $m \to \infty$) at a point t_0 depends on the values of f at all $[0,1]$ but by Theorem 1.2 below this is not true for Riemann or Lebesgue integrable functions.

1.1 The Riemann-Lebesgue Lemma - (see [C], p.212; [B-B], Th.9.31; [G], Th.12.5.C; [N], p.203) *For $f \in R([0,1])$ or $f \in L_1([0,1])$ we have*

$$\lim_{m\to\infty} a_m[f] = 0 \quad \textit{and} \quad \lim_{m\to\infty} b_m[f] = 0 .$$

Proof: Since $L_1([a,b]) \supset R([a,b])$ it is enough to prove, more generally, that for $f \in L_1([a,b])$ we have

$$\lim_{\xi\to\infty} \int_a^b f(t)\cos\xi t\, dt = 0 \quad \text{and} \quad \lim_{\xi\to\infty} \int_a^b f(t)\sin\xi t\, dt = 0 .$$

[For an elementary proof for $f \in R([a,b])$ see [H-2], p. 51.]

If $f \in L_1([a,b])$, given $\varepsilon > 0$ there exists $f_\varepsilon \in E([a,b])$ such that

$$\|f - f_\varepsilon\|_1 := \int_a^b |f(t) - f_\varepsilon(t)|\, dt \le \frac{\varepsilon}{2} .$$

Hence

$$\begin{aligned} \left|\int_a^b f(t)\cos\xi t\, dt\right| &\le \int_a^b |f(t) - f_\varepsilon(t)|\, dt + \left|\int_a^b f_\varepsilon(t)\cos\xi t\, dt\right| \\ &\le \frac{\varepsilon}{2} + \left|\int_a^b f_\varepsilon(t)\cos\xi t\, dt\right| ; \end{aligned}$$

since $f_\epsilon = \sum_{j=1}^{k} \lambda_j \chi_{]c_j,d_j[}$ and since for an interval $]c,d[$ we have

$$\int_c^d \cos\xi t\, dt = \frac{1}{\xi}(\sin\xi d - \sin\xi c)\ ,$$

it follows that for sufficiently large ξ we have $\left|\int_a^b f_\epsilon(t)\cos\xi t\, dt\right| \le \frac{\epsilon}{2}$ hence the result. We proceed analogously for $\int_a^b f(t)\sin\xi t\, dt$.

Theorem 1.2 **(The Riemann localization principle** - see [C], p.215; [B-B], Th.9.35; [N], p.199) *For $f \in R([0,1])$ or $f \in L_1([0,1])$ the convergence or divergence of the Fourier series of f at a point $t_0 \in [0,1]$ depends only on the values of f in an arbitrarily small neighbourhood of t_0.*

Proof: We will prove the result for $t_0 = 0$. By (2) and (3) we have $\int_0^1 D_m(t)\, dt = 1$ hence from (2) it follows that $s_m[f](0) \to s_0$ (when $m \to \infty$) iff we have

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} [f(t) - s_0] D_m(t)\, dt \to 0 \quad \text{when } m \to \infty \tag{4}$$

or, equivalently, iff $\left(\int_{-\frac{1}{2}}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\frac{1}{2}}\right)[f(t) - s_0]D_m(t)\, dt \to 0$, where $0 < \delta < \frac{1}{2}$; by 1.1 we have

$$\int_\delta^{\frac{1}{2}} [f(t) - s_0] D_m(t)\, dt = \int_\delta^{\frac{1}{2}} [f(t) - s_0]\frac{1}{\sin\pi t}\sin(2m+1)\pi t\, dt \to 0 \quad \text{when } m \to \infty$$

since the function $g(t) = [f(t) - s_0]\frac{1}{\sin\pi t}$ belongs to $L_1\left(\left[\delta, \frac{1}{2}\right]\right)$ and analogously for the integral $\int_{-\frac{1}{2}}^{-\delta}$. Hence the convergence in (4) depends only on the convergence of $\int_{-\delta}^{\delta}[f(t) - s_0]D_m(t)\, dt$ (when $m \to \infty$), i.e., on f in an arbitrarily small neighbourhood $[-\delta, \delta]$ of $t_0 = 0$. ■

Theorem 1.2 does not say that the Fourier series of $f \in L_1([0,1])$ converges to f and really, in general this is not true even if f is continuous (with $f(0) = f(1)$); using the Principle of Uniform Boundedness one can even prove that for the majority of the functions $f \in C([0,1])$ that satisfy $f(0) = f(1)$ their Fourier series diverges at the majority of the points of $[0,1]$ (majority in the sense of category of Baire); see [L], p.165; [H-1], T.11.6.

For a function $f \in L_2([0,1])$ its Fourier series converges to f in $L_2([0,1])$, i.e.,

$$\lim_{m\to\infty} \int_0^1 |f(t) - s_m[f](t)|^2 dt = 0 .$$

See [B-B], §9.8; [G], Th.12.4.C; [H-1], pp.53–55; [N], p.206.

However convergence in $L_2([0,1])$ or $L_1([0,1])$ does not imply the convergence at any point $t \in [0,1]$ as shows the sequence

$$g_1 = \chi_{[0,1]},\ g_2 = \chi_{[0,\frac{1}{2}]},\ g_3 = \chi_{]\frac{1}{2},1]},\ g_4 = \chi_{[0,\frac{1}{4}]},\ g_5 = \chi_{]\frac{1}{2},\frac{1}{4}]}, \cdots$$

and only around 1960 Carleson proved that the Fourier series of a function $f \in L_2([0,1])$ converges to f almost everywhere.

The fact that the Fourier series of a function is not necessarily convergent poses the problem of "reconstituting" a function from its Fourier series. For this purpose the processes of summability have been created. We have the

Theorem of Fejer - *For every $f \in C([0,1])$ with $f(0) = f(1)$ the sequence of the means of its Fourier series*

$$\sigma_m[f] = \frac{1}{m+1}(s_0[f] + s_1[f] + \cdots + s_m[f])$$

converges uniformly to f; see [C], p.220; [B-B], Th.9.72; [N], p.224; [H-1], p.231.

More generally for $f \in L_1([0,1])$ the sequence $\sigma_n[f]$ converges to f in the norm $\|\ \|_1$.

2 The Kurzweil-Henstock integral

The integral that we define next gives a riemannian formulation, found independently by Kurzweil (1957) and Henstock (1961), of the classical integrals of Denjoy (1912) and Perron (1914) that extend the Lebesgue integral.

We say that a function $f : [a,b] \to \mathbb{R}$ is *Kurzweil-Henstock integrable* or simply *K-integrable* (we write $f \in K([a,b])$) and that $I \in \mathbb{R}$ is its K-integral (we write ${}^{K}\!\int_a^b f(t)\,dt = I$) if for every $\varepsilon > 0$ there exists a function $\delta : [a,b] \to]0,\infty[$ (called *gauge*) such that for every tagged division (ξ_i, t_i) of $[a,b]$ [i.e., a division $t_0 = a < t_1 < t_2 < \cdots < t_n = b$ of $[a,b]$ with points, *tags*, $\xi_i \in [t_{i-1}, t_i]$] that is *δ-fine*, i.e.,

$$\xi_i \in [t_{i-1}, t_i] \subset]\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)[$$

we have

$$\left| \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) - I \right| < \varepsilon \, .$$

The book [P-S] contains a good presentation of the K-integral (also called *gauge integral*). For the K-integral we have the following properties:

2.1. $L_1([a,b]) \subset K([a,b])$; see [Lee], p.22 and [H-4], T.8.1.

2.2. $K([a,b])$ **contains its improper integrals**, i.e., if $f : [a,b] \to \mathbb{R}$ is such that

i) $f \in K([a,c])$ for every $c \in [a,b[$;

ii) There exists $\lim_{c \uparrow b} {}^{K}\!\int_a^c f(t)\,dt = I \in \mathbb{R}$

then $f \in K([a,b])$ and ${}^{K}\!\int_a^b f(t)\,dt = I$. See [Lee], coroll.7.10; [P-S], Th.13.42; [H-4], T.3.10.

From 2.2 it follows that the functions $f \in K([a,b])$ are not necessarily absolutely integrable (i.e., we may have $\int_a^b |f(t)|\,dt = \infty$). We define

$$\tilde{f}(t) = {}^{K}\!\int_a^t f(s)\,ds \, , \quad a \le t \le b \, .$$

2.3. $\tilde{f}$ is continuous for every $f \in K([a,b])$. See [Lee]. coroll.3.8; [P-S], Th.12.27; [H-4], T.3.9.

2.4. $\tilde{f} = 0 \Leftrightarrow f = 0$ a.e. (almost everywhere). See [Lee], Th.3.5 and Th.5.7; [H-4], T.4.3 and corol.4.9.

2.5. If $f \in K([a,b])$ there exists $\tilde{f}' = f$ a.e. See [Lee], Th.5.7; [H-4], T.4.8.

In $K([a,b])$ we define the Alexiewicz seminorm

$$||f||_A = \sup_{a \le t \le b} \left| {}^K\!\!\int_a^t f(s)\,ds \right|$$

and we denote by $\mathrm{K}([a,b])$ the normed quotient space (by the relation equality almost everywhere, see 2.4).

In order to emphasize that on $K([a,b])$ we consider the seminorm $||\ ||_A$. We write $K([a,b])_A$ and analogously for its subspaces and the corresponding quotient spaces.

The normed space $\mathrm{K}([a,b])_A$ is not complete, see [H-3].

Theorem 2.6 (The integration by parts formula) - *If* $\alpha \in BV([a,b])$ *and* $f \in K([a,b])$ *then* $\alpha f \in K([a,b])$ *and*

$${}^K\!\!\int_a^b \alpha(t)f(t)\,dt = \int_a^b \alpha(t)d\tilde{f}(t) = \alpha(b)\tilde{f}(b) - \alpha(a)\tilde{f}(a) - \int_a^b d\alpha(t)\tilde{f}(t)$$

with $\left| {}^K\!\!\int_a^b \alpha(t)f(t)\,dt \right| \le [|\alpha(a)| + V[\alpha]]||f||_A$. See [Lee], coroll.12.2 or [H-3], Th.1.15.

Theorem 2.7 - *For* $f \in L_1([a,b])$ *we have* $V[\tilde{f}] = ||f||_1$. See [D-S], Th.IV 12.3 or [H-4], T.5.1.

3 Barrelled and ultrabornological spaces

In the theory of Banach spaces the uniform boundedness principle and the closed graph theorem are specially important since they have many applications; see [H], [H-1], [L], [T]. In the theory of locally convex spaces, the classes of barrelled and of ultrabornological spaces are, respectively, the ones where those theorems are still valid. We will consider these classes only in the normed (not necessarily complete) spaces. For references see [J] and [B]. The proofs of the results we bring next are simple and direct. The main examples are the space $\mathrm{K}([a,b])_A$ of §2 and the spaces $R^1([a,b])_A$, $E^1([a,b])_A$ and others of §5.

We say that a closed subset T of a normed space E is a *barrel* if it is convex, symmetric and absorbing (i.e., $E = \bigcup_{n\in\mathbf{N}} nT$); we say that E is *barrelled* if every barrel is a neighbourhood of zero.

3.1. Every Banach space, or more generally, every normed space that is a Baire space is barrelled.

Theorem 3.2 - *In a barrelled space we have the* **uniform boundedness principle,** *i.e., given a family $A_i \in L(E,F)$, $i \in I$, of linear continuous mappings of a barrelled space E into a normed space F such that for every $x \in E$ we have $\sup_{i\in I} \|A_i x\| < \infty$ then*

$$\sup_{i\in I} \|A_i\| < \infty \ .$$

3.3. If $(E_i)_{i\in I}$ and E are normed spaces with linear mappings $f_i : E_i \to E$ such that the norm of E is the finest one for which the f_i are continuous, then E is barrelled if the E_i are barrelled.

3.4. If E is a barrelled dense subspace of a normed space F, then any subspace G, $E \subset G \subset F$, is also barrelled.

We say that a normed space E is *ultrabornological* if there exists a family $(E_i)_{i\in I}$ of Banach spaces and linear mappings $f_i : E_i \to E$ such that the norm of E is the finest one that makes the f_i continuous. From 3.1 and 3.3 follows immediately

3.5. Every ultrabornological space is barrelled.

4 Integral representation theorems

We denote by $BV_b^+([a,b])$ the space of all functions of bounded variation $\alpha : [a,b] \to \mathbb{R}$ that are right continuous in $[a,b[$ with $\alpha(b) = 0$. We recall that $V_{[a,b]}[\alpha]$ or simply $V[\alpha]$ denotes the variation of α in $[a,b]$. $C_a([a,b])$ denotes the space of all continuous functions $f : [a,b] \to \mathbb{R}$ such that $f(a) = 0$.

From the Riesz representation theorem [see [H], Th.3.2.5; [T], Th.4.32-C; [H-1], pp. 204–206] follows

4.1. The mapping $\alpha \in BV_b^+([a,b]) \mapsto S_\alpha \in C_a([a,b])'$ [where S_α is the Stieltjes integral operator $S_\alpha(g) = \int_a^b d\alpha(t)g(t)$ for every $g \in C_a([a,b])$] is an isometry (i.e., $\|S_\alpha\| = V(\alpha)$) of the first space onto the second.

4.2. The mapping $f \in E([a,b])_A \mapsto \tilde{f} \in C_a([a,b])$ is an isometry, i.e., $\|\tilde{f}\| = \|f\|_A$, of the first space onto a dense subspace of the second. The same still holds if we replace $E([a,b])$ by any subspace of $K([a,b])$ that contains $E([a,b])$.

Proof: We still have to show that the images of the step functions are dense in $C_a([a,b])$ and this is immediate since they are the continuous piecewise linear functions that are zero at a; their density follows from the uniform continuity of the continuous functions on $[a,b]$. ■

If we denote by $\tilde{E}([a,b])$ the image of $E([a,b])$ in $C_a([a,b])$ (by the mapping $f \mapsto \tilde{f}$) it follows that any linear continuous functional $F \in E([a,b])'_A$ defines one and only one $\tilde{F} \in \tilde{E}([a,b])'$ given by $\tilde{F}(\tilde{f}) := F(f)$ for every $f \in E([a,b])$; from the density of $\tilde{E}([a,b])$ in $C_a([a,b])$ it follows that $\tilde{F}$ has one and only one extension as a linear continuous functional on $C_a([a,b])$. Hence we proved

4.3. Any linear continuous functional $F \in E([a,b])'_A$ defines one and only one linear continuous functional $\tilde{F} \in C_a([a,b])'$; we have $\tilde{F}(\tilde{f}) = F(f)$ for every $f \in E([a,b])$.

The same holds if we replace $E([a,b])$ by any subspace of $K([a,b])$ that contains $E([a,b])$.

Given $F \in E([a,b])'_A$, by 4.3 and 4.1 there exists one and only one $\alpha \in BV_b^+([a,b])$ such that $\tilde{F}(g) = S_\alpha(g) = \int_a^b d\alpha(t)g(t)$ for every $g \in C_a([a,b])$. In particular for $f \in E([a,b])$ we have $\tilde{F}(\tilde{f}) = \int_a^b d\alpha(t)\tilde{f}(t)$. From Th. 2.6 it follows that

$$\tilde{F}(\tilde{f}) = \int_a^b d\alpha(t)\tilde{f}(t) = \alpha(b)\tilde{f}(b) - \alpha(a)\tilde{f}(a) - \int_a^b \alpha(t)d\tilde{f}(t) = -{}^K\!\!\int_a^b \alpha(t)f(t)\,dt$$

since $\alpha(b) = 0$ and $\tilde{f}(a) = 0$; if we define the multiplication operator $M_\alpha(f) = {}^K\!\!\int_a^b \alpha(t)f(t)dt$ we have $M_\alpha(f) = -S_\alpha(\tilde{f})$ hence by 4.1 we have $||M_\alpha|| = V[\alpha]$, i.e., we proved

4.4. The mapping $\alpha \in BV_b^+([a,b]) \mapsto M_\alpha \in E([a,b])'_A$ (where $M_\alpha(f) = {}^K\!\!\int_a^b \alpha(t)f(t)\,dt$) is an isometry (i.e., $||M_\alpha|| = V[\alpha]$) of the first space onto the second and $E([a,b])'_A = K([a,b])'_A$. The same holds if we replace $E([a,b])$ by any subspace of $K([a,b])$ that contains $E([a,b])$.

5 The subspace $R^1([a,b])$ of improper Riemann integrable functions

The construction that follows, starting with $R([a,b])$, can be repeated in an analogous way starting with $E([a,b])$ or $L_1([a,b])$.

Given a function $f : [a,b] \to \mathbb{R}$ we write $f \in R^1([a,b])$ if there exists a division $t_0 = a < t_1 < t_2 < \cdots < t_n = b$ of $[a,b]$ such that

i) For every $[c,d] \subset]t_{i-1}, t_i[$ we have $f \in R([c,d])$.

ii) Given $c_i \in]t_{i-1}, t_i[$ the improper integrals

$$\lim_{x \downarrow t_{i-1}} \int_x^{c_i} f(t)\,dt \in \mathbb{R} \quad \text{and} \quad \lim_{y \uparrow t_i} \int_{c_i}^{y} f(t)\,dt \in \mathbb{R}$$

exist. We still denote them, respectively, by $\int_{t_{i-1}}^{c_i} f(t)\,dt$ and $\int_{c_i}^{t_i} f(t)\,dt$. More generally for $t \in]t_i, t_{i+1}]$ we define the improper integral

$$\int_a^t f(s)\,ds = \int_a^{c_1} f(s)\,ds + \int_{c_1}^{t_1} f(s)\,ds + \int_{t_1}^{c_2} f(s)\,ds + \cdots + \int_{c_i}^{t_i} f(s)\,ds + \int_{t_i}^{t} f(s)\,ds \,.$$

We also write $\tilde{f}(t) = \int_a^t f(s)\,ds$ and define the seminorm $\|f\|_A = \sup_{a \le t \le b} \left| \int_a^b f(s)\,ds \right|$

Remark: If we start with $E([a,b])$ we obtain a space that we denote by $E^1([a,b])$. If we start with $L_1([a,b])$ we obtain the classical space $L_1{}^1([a,b])$ of Lebesgue-Cauchy integrable functions. We could repeat the process and obtain spaces $R^2([a,b])$, $E^2([a,b])$, $L_1{}^2([a,b])$ etc. Since $E([a,b])$, $R([a,b])$, $L_1([a,b]) \subset K([a,b])$ it follows from 2.2 that $E^1([a,b])$, $R^1([a,b])$, $L_1{}^1([a,b]) \subset K([a,b])$ and that $K^1([a,b]) = K([a,b])$.

Theorem 5.1 (Gilioli) - *The normed quotient spaces associated to $E^1([a,b])_A$, $R^1([a,b])_A$, $L_1{}^1([a,b])_A$ and $K([a,b])_A$ are ultrabornological.* See [G-1], Th.4.4 and Th.4.6; [G-2], Th.2.7 and coroll.3.2.

Theorem 5.2 (Gilioli) - *All subspaces of $K([a,b])_A$ that contain $E^1([a,b])_A$ are barrelled, hence for them the uniform boundedness principle is valid.*

Proof: Since $E([a,b])$ is dense in $K([a,b])_A$ the result follows from Th. 5.1, 3.5, 3.4 and Th. 3.2. ■

6 The main theorem

From Th. 2.6 it follows that

$$a_n[\,] : f \in K([0,1])_A \mapsto a_n[f] = {}^{K}\!\!\int_0^1 f(t)\cos 2\pi n t \in \mathbb{R}$$

and

$$b_n[\,] : f \in K([0,1])_A \mapsto b_n[f] = {}^{K}\!\!\int_0^1 f(t)\sin 2\pi n t \in \mathbb{R}$$

are linear continuous functionals.

6.1. We have $a_n[\ ], b_n[\ ] \in E([0,1])'_A$ with $\lim_{n\to\infty} ||a_n[\ ]|| = \infty$, where, obviously,

$$||a_n[\ ]|| = \sup\left\{|a_n[f]| \,\middle|\, f \in E([0,1])_A\ ,\ \ ||f||_A \leq 1\right\}.$$

and analogously for $b_n[\ ]$. The same still holds if we replace $E([0,1])$ by any subspace of $K([0,1])$ that contains $E([0,1])$.

Proof: By 4.4 and Th. 2.7 we have

$$\begin{aligned} ||a_n[\ ]|| &= V[\cos 2\pi nt] = ||2\pi n \sin 2\pi nt||_1 = \int_0^1 |2\pi n \sin 2\pi nt|\, dt \\ &\geq 2\pi n \int_0^1 \sin^2 2\pi nt = \pi n \end{aligned}$$

and analogously for $||b_n[\ ]||$. ■

In contrast to the Lemma of Riemann-Lebesgue (1.1) we have

Theorem 6.2 - *There exists $f \in R^1([0,1])$ such that $\overline{\lim}_{n\to\infty} a_n[f] = \infty$ and analogously for the $b_n[\ ]$. The same still holds if we replace $R^1([0,1])$ by any subspace of $K([0,1])$ that contains $E^1([0,1])$.*

Proof: By Th. 5.1 $E^1([0,1])_A$ is ultrabornological, hence barrelled by 3.5 [By 3.4 the same still holds for any subspace of $K([0,1])$ that contains $E^1([0,1])$ since $E([0,1])$ and hence, $E^1([0,1])$ is dense in $K([0,1])_A$]. If we had $\sup_n |a_n[f]| < \infty$ for every $f \in E^1([0,1])$ from the uniform boundedness principle (Th. 3.2) it would follow that $\sup_n ||a_n[\ ]|| < \infty$ in contradiction to 6.1. The proof for the $b_n[\ ]$ is analogous. ■

Theorem 6.3 - **In $R^1([0,1])$ the Riemann localization principle fails:** *given $t_0 \in [0,1]$ and $[c,d] \subset [0,1]$ such that $t_0 \notin [c,d]$ there exists $f \in R^1([0,1])$ that is zero outside $[c,d]$ and such that*

$$s_m[f](t_0) \not\to 0 \text{ when } m \to \infty$$

and even

$$\overline{\lim_m} |s_m[f](t_0)| = \infty .$$

The same still holds if we replace $R^1([0,1])$ by any subspace of $K([0,1])$ that contains $E^1([0,1])$.

Proof: In order to simplify the notation let us take $t_0 = 0$; if $\varphi \in E^1([0,1])$ is zero outside $[c,d]$ (where $t_0 = 0 \notin [c,d]$) by (2) and (3) of §1 we have

$$s_m[\varphi](0) = \int_0^1 \varphi(t) D_m(t)\,dt = \int_c^d \varphi(t) D_m(t)\,dt = \int_c^d \varphi(t) \frac{\sin(2m+1)\pi t}{\sin \pi t}\,dt .$$

If we define $F_m(\varphi) = \int_c^d \varphi(t) D_m(t)\,dt$ by 4.4 we have $||F_m|| \geq \bar{V}_{[c,d]}[D_m]$ ($\geq$ since we do not have necessarily $D_m(d) = 0$).

a) We have $\lim_{m\to\infty} V_{[c,d]}[D_m] = \infty$. Indeed: in order to simplify the calculations we suppose that $[c,d] \subset]0, \frac{1}{2}]$. By Th. 2.7 we have

$$\begin{aligned} V_{[c,d]}[D_m] &= \int_c^d |D'_m(t)|\,dt \\ &= \int_c^d \left| \frac{1}{\sin \pi t}(2m+1)\pi \cos(2m+1)\pi t - \frac{1}{\sin^2 \pi t}\pi \sin(2m+1)\pi t \cos \pi t \right| dt \\ &\geq \frac{1}{\sin \pi d}(2m+1)\pi \int_c^d |\cos(2m+1)\pi t|\,dt - \frac{\pi}{\sin^2 \pi c} \int_c^d |\sin(2m+1)\pi t|\,dt \end{aligned}$$

(since $d \leq \frac{1}{2}$ it follows that in $[c,d]$ we have $\sin \pi t \leq \sin \pi d$) that tends evidently to ∞ with m since the second integral is $\leq (d-c)$.

b) Since $E^1([c,d])$ is barrelled (by Th. 5.2) from $||F_m|| \geq V_{[c,d]}[D_m]$, a) and the uniform boundedness principle it follows that there exists $\varphi \in E^1([c,d])$ such that

$$\overline{\lim}_m |F_m(\varphi)| = \infty .$$

c) If we define $f(t) = \begin{cases} \varphi(t) & \text{for } c \leq t \leq d \\ 0 & \text{on } [0,1] \setminus [c,d] \end{cases}$ we have $f \in E^1([0,1])$ and

$$\overline{\lim}_m |s_m[f](0)| = \overline{\lim}_m |F_m(\varphi)| = \infty . \qquad \blacksquare$$

Remarks

1. Working with the Denjoy integral and its methods the result of Theorem 6.3 was known for the space $D^*([0,1])$ of functions that are Denjoy integrable in the strict sense; see Th. III.2.6 of [C-D].

2. It would be interesting to find *explicitly* a function $f \in R^1([0,1])$ for which the Riemann localization principle fails.

3. And to find the smallest number of singularities (points of improper integrability) that such an example must have.

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