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Bruce–Roberts numbers and quasihomogeneous functions on analytic varieties

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Abstract

Given a germ of an analytic variety X and a germ of a holomorphic function f with a stratified isolated singularity with respect to the logarithmic stratification of X, we show that under certain conditions on the singularity type of the pair (f, X), the following relative analog of the well-known K. Saito's theorem holds true: equality of the relative Milnor and Tjurina numbers of f with respect to X (also known as Bruce–Roberts numbers) is equivalent to the relative quasihomogeneity of the pair (f, X), i.e. to the existence of a coordinate system such that both f and X are quasihomogeneous with respect to the same positive rational weights.

Keywords: Bruce–Roberts numbers, Quasihomogeneous functions, Analytic varieties, Logarithmic vector fields, Logarithmic stratification, Stratified isolated singularities

1 Introduction-main results

Given a holomorphic function germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at the origin, its Milnor number is classically defined as

$$\mu(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta)}$$

where $\mathcal{O}_n$ is the ring of holomorphic function germs at the origin of $\mathbb{C}^n$, $\Theta \cong \mathcal{O}_n^n$ is the module of germs of vector fields (derivations), and $df(\Theta) = J(f)$ is the ideal generated by the partial derivatives of f. By Milnor's theorem (cf. [29,32]) this is exactly the rank of the middle homology group of the Milnor fiber of f, equal to the number of spheres in its bouquet decomposition.

Along with the Milnor number, one also defines the Tjurina number of f

$$\tau(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta) + \langle f \rangle}$$

(where $\langle f \rangle \subset \mathcal{O}_n$ is the ideal generated by f), which is interpreted as the dimension of the base of a semi-universal deformation of the isolated hypersurface singularity $Y = f^{-1}(0)$ defined by f.

By definition $\mu(f) \geqslant \tau(f)$, and according to a well-known theorem of Saito [33], equality $\mu(f) = \tau(f)$ is equivalent to the quasihomogeneity of f, i.e. to the existence of a coordinate

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system $x = (x_1, \ldots, x_n)$ and a vector of positive rational numbers $w = (w_1, \ldots, w_n) \in \mathbb{Q}_+^n$ (the weights), such that f can be written as

$$f(x) = \sum_{\langle w,m\rangle=1} a_m x^m$$

where $a_m \in \mathbb{C}$ and $x^m := x_1^{m_1} \ldots x_n^{m_n}$ are those monomials in the expansion of f whose exponents $m = (m_1, \ldots, m_n) \in \mathbb{N}^n$ belong to the affine hyperplane

$$\langle w, m\rangle = \sum_{i=1}^{n} w_i m_i = 1.$$

Notice that positivity of the weights w_i, $i = 1, \ldots, n$, forces f to be polynomial in the above coordinates. In fact, using a quasihomogeneous splitting lemma [33, Satz 1.3], Saito proves that the weights $w \in \mathbb{Q}_+^n$ can be always normalised, i.e. they can be chosen (in appropriate coordinates) in the half-open interval $w \in ((0, \frac{1}{2}] \cap \mathbb{Q})^n$, and as such they are uniquely defined [33, Korollar 1.7].

After Saito's proof, there have been many other characterizations of quasihomogeneity of isolated hypersurface singularities, relating it with other invariants of the singularity (cf. [40] and references therein). Moreover, Saito's result has been generalised for the case of isolated complete intersection singularities of positive dimension (icis for short) $X \subset (\mathbb{C}^n, 0)$ by Vosegaard [37], a problem of substantial difficulty and of rather long history (cf. [12,13] for curves, [38] for surfaces and [36] for purely elliptic icis of dimension $\geqslant 2$): given an icis germ $X = h^{-1}(0)$, $h \in \mathcal{O}_n^m$, $n \geqslant m + 1$, equality of its Milnor number $\mu(h)$ (i.e. the rank of the middle homology group of the Milnor fiber of h, cf. [14,16] and [26]) and its Tjurina number $\tau(h)$ (i.e. the dimension of the base of a semi-universal deformation of X, cf. [15] and [26]), is equivalent to the quasihomogeneity of X, i.e. to the existence of a coordinate system $x = (x_1, \ldots, x_n)$ and a vector of positive rational numbers $w = (w_1, \ldots, w_n) \in \mathbb{Q}_+^n$, such that the ideal of functions I_X vanishing on X admits a system of quasihomogeneous generators with respect to the weights w, that is

$$I_X = \langle h_1, \ldots, h_m\rangle$$
$$h_i(x) = \sum_{\langle w,s\rangle=d_i} b_{s,i} x^s, \quad i = 1, \ldots, m$$

where $b_{s,i} \in \mathbb{C}$, and $d_i \in \mathbb{Q}_+$ are the quasihomogeneity degrees of the h_i's.

In contrast to Saito's proof, Vosegaard's proof is highly non-trivial, since the difference $\mu(h) - \tau(h)$ of the corresponding Milnor and Tjurina numbers of an icis $X = h^{-1}(0)$ admits no simple expression, as in the hypersurface case, but involves instead several invariants coming from the mixed Hodge structure of the link $X \setminus \{0\}$, and the resolution of singularities of X (cf. [27] and [37]).

Away from the icis case, the invariant characterisation of quasihomogeneity for general analytic varieties $X \subset (\mathbb{C}^n, 0)$ is problematic, at least in terms of numerical invariants generalising the Milnor and Tjurina numbers, which typically cease to exist (i.e. they are not finite). In fact, the simplest possible characterisation of quasihomogeneity of analytic sets, involves the module of the so-called logarithmic vector fields (as defined by K. Saito [34])

$$\Theta_X := \left\{\delta \in \Theta : \delta(I_X) \subseteq I_X\right\}$$

i.e. those vector fields which are tangent to the smooth part of X. In particular, one may easily show (cf. [10] and also Theorem 3.1 and Corollary 3.4 in Sect. 3 of the present paper) that the variety X is quasihomogeneous in an appropriate coordinate system, if and only if there exists a logarithmic vector field $\delta \in \Theta_X$ which vanishes at the origin, $\delta(0) = 0$, and has positive rational eigenvalues

$$\mathrm{sp}(\delta) = w = (w_1, \ldots, w_n) \in \mathbb{Q}^n_+$$

where we denote by $\mathrm{sp}(\delta)$ the spectrum of δ, i.e. the set of eigenvalues of its linear part $j^1\delta$, viewed as a linear operator in $\mathbb{C}^n$. In fact, both Saito's and Vosegaard's proof, rely on the fact that in case where X is an isolated hypersurface singularity, or an icis respectively, then equality of the corresponding Milnor and Tjurina numbers provides exactly such a logarithmic vector field with the required positivity property on its eigenvalues.

In the present paper we give another generalisation of Saito's theorem which interpolates between the above cases and is relevant in relative singularity theory. We consider pairs (f, X) where $X \subset (\mathbb{C}^n, 0)$ is an arbitrary analytic variety, and $f \in \mathcal{O}_n$ is a function germ which has a stratified isolated singularity at the origin (in the sense of Lê [21]) with respect to the logarithmic stratification of X, as defined by K. Saito (cf. [34] for the case of hypersurfaces, and also [8] for the more general case of arbitrary analytic varieties X).

The number that naturally generalises the Milnor number in this situation is the following relative Milnor number

$$\mu_X(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_X)}$$

where $df(\Theta_X) = J_X(f)$ is the ideal generated by the (Lie) derivatives of f along logarithmic vector fields of X. This number has been consider by many authors, starting probably from the works of Arnol'd [5] in the case where X is a smooth divisor, by Lyashko [28] in the case where X is an isolated hypersurface singularity, and later on by Bruce and Roberts [8] for the more general case of arbitrary analytic varieties X (and many others which is impossible to cite). Recently (cf. [2,7,11,18,22,31]- [24,30]) it has been called the *Bruce–Roberts Milnor number* of (f, X) (or of f with respect to X). We adopt this terminology here as well.

We remark (c.f [8, pp. 64]) that finiteness of the Bruce–Roberts Milnor number $\mu_X(f) < \infty$ is equivalent to the finite $\mathcal{R}_X$-determinacy of the function f (i.e. finite determinacy under diffeomorphisms preserving the variety X), and to the existence of an $\mathcal{R}_X$-versal deformation for f as well. As it was mentioned earlier, it is also equivalent to the function f being a submersion on each logarithmic stratum of X, except possibly at the origin.

In analogy with the Bruce–Roberts Milnor number, one may also define the *Bruce–Roberts Tjurina number* of the pair (f, X) (cf. [1] and [7])

$$\tau_X(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_X) + \langle f \rangle}$$

which encodes the infinitesimal deformations of f under $\mathcal{K}_X$-equivalence, i.e. under diffeomorphisms preserving X and multiplication of f by units in $\mathcal{O}_n$. In contrast to the Bruce–Roberts Milnor number, the properties of the Bruce–Roberts Tjurina number are much less studied in the literature, with an exception being the recent work [7] of the first and third authors, where they show, inspired by a result of Liu [25], that the quotient $\mu_X(f)/\tau_X(f)$ is always bounded from above by the smallest integer r such that $f^r \in df(\Theta_X)$.

The present paper is motivated by the natural question as to characterise those pairs (f, X) for which $r = 1$, i.e. such that the equality $\mu_X(f) = \tau_X(f)$ holds. By definition, this is equivalent to $f \in df(\Theta_X)$, i.e. to the existence of a logarithmic vector field $\delta \in \Theta_X$

such that $\delta(f) = f$. Notice that by the obvious inclusion $\Theta_X \subset \Theta$, equality $\mu_X(f) = \tau_X(f)$ immediately implies $\mu(f) = \tau(f)$, and thus by Saito's theorem [33], we know that there always exist a coordinate system such that f can be reduced to a quasihomogeneous polynomial with positive rational weights. Despite this fact, we do not know if in these coordinates the variety X will also be quasihomogeneous with respect to the same weights, or if it will be quasihomogeneous at all, and this is exactly the problem that we want to address here. The following definition will be useful throughout the paper.

Definition 1.1 A pair (f, X) in $(\mathbb{C}^n, 0)$ will be called relatively quasihomogeneous if there exists a vector of positive rational numbers $w = (w_1, \ldots, w_n) \in \mathbb{Q}^n_+$, a system of coordinates $x = (x_1, \ldots, x_n)$ and a system of generators $\langle h_1, \ldots, h_m\rangle = I_X$ of the ideal of functions vanishing on X, such that

$$f(x) = \sum_{\langle w,k\rangle=1} a_k x^k, \quad a_k \in \mathbb{C}$$

$$h_i(x) = \sum_{\langle w,k\rangle=d_i} b_{k,i} x^k, \quad b_{k,i} \in \mathbb{C}, \quad i = 1, \ldots, m$$

where each $d_i \in \mathbb{Q}_+$ is the quasihomogeneous degree of h_i.

As it is obvious from the definition, if (f, X) is a relatively quasihomogeneous pair then there exists an Euler vector field $\chi_w = \sum_{i=1}^n w_i x_i \partial_{x_i}$ with positive rational eigenvalues, $w = (w_1, \ldots, w_n) \in \mathbb{Q}^n_+$, such that

$$\chi_w(f) = f, \quad \chi_w \in \Theta_X.$$

In particular, if $\mu_X(f) < \infty$, then equality $\mu_X(f) = \tau_X(f)$ trivially holds. So it is natural to ask about the validity of the converse implication, i.e.

"*Does* $\mu_X(f) = \tau_X(f)$ *imply the relative quasihomogeneity of the pair* (f, X)?"

The answer to this relative analog of Saito's theorem is in general negative without further assumptions on the singularity type of the pair (f, X), as one may construct several counter-examples of the following type.

Example 1.2 Let $f(x, y, z) = x$ and $X = \{xy^3z^3 + y^5 + z^5 = 0\}$. Then $\mu_X(f) = \tau_X(f) = 1$, and the pair (f, X) is not relatively quasihomogeneous in this coordinate system, and in fact in any coordinate system (since X itself cannot be made quasihomogeneous in any coordinate system). To see this, one may compute a system of generators of the module $\Theta_X \cap \Theta_Y$ and notice that the linear part of any vector field in this module is necessarily a constant multiple of the Euler vector field

$$\chi_{(-1,1,1)} = -x\partial_x + y\partial_y + z\partial_z.$$

Since the eigenvalues of any vector field are invariant under changes of coordinates, it follows that there cannot exist another Euler vector field in $\Theta_X \cap \Theta_Y$ with positive rational eigenvalues, which proves the claim (see Corollary 3.4 in Sect. 3).

Despite this fact, there is a wide class of singularities (f, X) where the relative Saito theorem does indeed hold true. Our main results in this direction can be summarised in the following result.

Theorem 1.3 *Let* (f, X) *be a pair with* $\mu_X(f) < \infty$. *Suppose also that* *(a)* $f \in \mathbf{m}^3$, *or*

(b) X *is a hypersurface with at most an isolated singularity at the origin.*

Then, equality $\mu_X(f) = \tau_X(f)$ *is equivalent to the relative quasihomogeneity of the pair* (f, X).

The proof of the theorem is a variant of Saito's original proof and will be given in Sect. 4 with a more precise statement for the range of the weights in each case, related to the multiplicity of f. In particular, part (a) of Theorem 1.3 corresponds to Theorem 4.1, and part (b) to Theorem 4.3, respectively.

The main ingredients of the proof are, apart from Saito's theorem itself, several decomposition formulas for the difference

$$\mu_X(f) - \tau_X(f) = \dim_{\mathbb{C}} \frac{df(\Theta_X) + \langle f \rangle}{df(\Theta_X)}$$

presented in Sect. 2, as well as an invariant characterisation of relative quasihomogeneity in terms of logarithmic vector fields presented in Sect. 3. The latter allows us to detect the relative quasihomogeneity of pairs (f, X) by looking solely at the possible eigenvalues of logarithmic vector fields in the intersection $\Theta_X \cap \Theta_Y$.

We remark finally that one may produce a whole class of counter-examples to the implication "$\mu_X(f) = \tau_X(f) \Longrightarrow (f, X)$ *is relatively quasihomogeneous*", generalising Example 1.2 given above; it consists of pairs (f, X) where X is a non-isolated hypersurface singularity defining an equisingular deformation of a quasihomogeneous isolated singularity $X_0 \subset (\mathbb{C}^{n-1}, 0)$, and f is a linear form corresponding to the deformation parameter. We were not able though to find any other counter-examples away from this class, and in this sense, the validity of the relative Saito theorem in the remaining cases where the multiplicity of f is $\leqslant 2$ and X is not an isolated hypersurface singularity, is still open.

2 Bruce–Roberts Milnor and Tjurina numbers

Let $X \subset (\mathbb{C}^n, 0)$ be an analytic variety and $f \in \mathcal{O}_n$ a function germ such that $\mu_X(f) < \infty$. We denote by $Y = f^{-1}(0)$ the hypersurface defined by f. Since $\mu_X(f)$ is finite, then $\mu(f)$ is also finite and thus Y is either a smooth divisor ($\mu(f) = 0$) or an isolated hypersurface singularity ($\mu(f) > 0$). We also denote by

$$\Theta_Y = \{\delta \in \Theta : \delta(f) \in \langle f \rangle\}$$

the module of logarithmic vector fields of Y, and by

$$H_Y = \ker df(\cdot) = \{\eta \in \Theta : df(\eta) = 0\}$$

the submodule of Hamiltonian (or else, Killing) vector fields of f, where we denote by $df(\cdot) : \Theta \to \mathcal{O}_n$ the corresponding evaluation map. It is easy to verify that in local coordinates $x = (x_1, \ldots, x_n)$, the latter module is generated by the derivations

$$\eta_{ij} = \partial_{x_i} f \partial_{x_j} - \partial_{x_j} f \partial_{x_i}, \quad 1 \leqslant i < j \leqslant n.$$

Finally we denote by

$$\overline{\mu}_X(f) = \dim_{\mathbb{C}} \frac{\Theta}{\Theta_X + H_Y}$$

and

$$\overline{\tau}_X(f) = \dim_{\mathbb{C}} \frac{\Theta}{\Theta_X + \Theta_Y}$$

in case where these numbers are finite. Notice that $\overline{\mu}_X(f) \geqslant \overline{\tau}_X(f)$, and

$$\overline{\mu}_X(f) - \overline{\tau}_X(f) = \dim_{\mathbb{C}} \frac{\Theta_Y}{H_Y + (\Theta_X \cap \Theta_Y)} \tag{1}$$

where we have used the isomorphism

$$\frac{\Theta_X + \Theta_Y}{\Theta_X + H_Y} \cong \frac{\Theta_Y}{(\Theta_X + H_Y) \cap \Theta_Y} = \frac{\Theta_Y}{H_Y + (\Theta_X \cap \Theta_Y)}.$$

Proposition 2.1 *Let (f, X) be a pair in $(\mathbb{C}^n, 0)$ with $\mu_X(f) < \infty$. Then the following decomposition formulas hold for the Bruce–Roberts Milnor and Tjurina numbers*

$$\mu_X(f) = \mu(f) + \overline{\mu}_X(f) \tag{2}$$

$$\tau_X(f) = \tau(f) + \overline{\tau}_X(f) \tag{3}$$

In particular

$$\mu_X(f) - \tau_X(f) = \mu(f) - \tau(f) + \overline{\mu}_X(f) - \overline{\tau}_X(f). \tag{4}$$

Proof The evaluation map $df : \Theta \to \mathcal{O}_n$ and the inclusion $df(\Theta_X) \subseteq df(\Theta)$ induce the following exact sequences of $\mathcal{O}_n$-modules:

$$0 \longrightarrow \frac{\Theta}{\Theta_X + H_Y} \overset{df}{\longrightarrow} \frac{\mathcal{O}_n}{df(\Theta_X)} \longrightarrow \frac{\mathcal{O}_n}{df(\Theta)} \longrightarrow 0 \tag{5}$$

$$0 \longrightarrow \frac{\Theta}{\Theta_X + \Theta_Y} \overset{df}{\longrightarrow} \frac{\mathcal{O}_n}{df(\Theta_X) + \langle f \rangle} \longrightarrow \frac{\mathcal{O}_n}{df(\Theta) + \langle f \rangle} \longrightarrow 0 \tag{6}$$

where the respective third morphisms of (5) and (6) are the natural projections. The exactness of the above sequences lead to relations (2), (3) and (4). □

We remark that using the 9-lemma (or by a direct argument) we obtain another short exact sequence defined by the kernels of the natural projections (5)→(6):

$$0 \longrightarrow \frac{\Theta_X + \Theta_Y}{\Theta_X + H_Y} \overset{df}{\longrightarrow} \frac{df(\Theta_X) + \langle f \rangle}{df(\Theta_X)} \longrightarrow \frac{df(\Theta) + \langle f \rangle}{df(\Theta)} \longrightarrow 0.$$

which, by (1), leads again to relation (4).

As an immediate corollary of the above we obtain the following algebraic characterisation of the equality of the Bruce–Roberts Milnor and Tjurina numbers.

Corollary 2.2 *Let (f, X) be a pair with $\mu_X(f) < \infty$. Then the following conditions are equivalent*

(1) $\mu_X(f) = \tau_X(f)$
(2) $\mu(f) = \tau(f)$ *and* $\overline{\mu}_X(f) = \overline{\tau}_X(f)$, *the latter being equivalent to*

$$\Theta_Y = \Theta_X \cap \Theta_Y + H_Y. \tag{7}$$

In the case where the variety X is also an isolated hypersurface singularity, the formulas obtained above can be substantially improved due to the following

Theorem 2.3 *([18], see also [22]) Let $h \in \mathcal{O}_n$ with an isolated singularity at the origin and let $X = h^{-1}(0)$. Let $f \in \mathcal{O}_n$ such that $\mu_X(f) < \infty$. Then*

$$\mu_X(f) = \mu(f) + \mu(f, h) + \mu(h) - \tau(h), \tag{8}$$

where $\mu(f, h)$ is the Milnor number of the icis *(f,h).*

Remark 2.4 We remark that in the case where one has equality $\mu(h) = \tau(h)$ (i.e. h is quasihomogeneous in some coordinate system), formula (8) above implies the following version of Lê-Greuel's formula [14], [20]

$$\mu_X(f) = \mu(f) + \mu(f, h)$$

and in particular:

$$df(\Theta_X) \cong \langle f \rangle + J(f, h)$$

where $J(f, h) = df(H_X) = dh(H_Y)$ is the ideal generated by the 2×2 minors of the Jacobian matrix of the icis (f, h). This will be useful in the proof of Theorem 4.3 in Sect. 4.

Before we proceed, let us notice that if X is an isolated hypersurface singularity and $f \in \mathcal{O}_n$ such that $\mu_X(f) < \infty$, then we can always choose a function $h \in \mathcal{O}_n$ with an isolated singularity at the origin, such that $X = h^{-1}(0)$ and $\mu_Y(h) < \infty$ as well, where $Y = f^{-1}(0)$ (h will be nothing but a 1-parameter smoothing of the icis $X \cap Y = (f, h)^{-1}(0)$). In this case we will generally have inequalities

$$\mu_X(f) \neq \mu_Y(h), \quad \tau_X(f) \neq \tau_Y(h)$$

but the following equality

$$\overline{\tau}_X(f) = \overline{\tau}_Y(h) = \dim_{\mathbb{C}} \frac{\Theta}{\Theta_X + \Theta_Y}$$

always holds true, merely by definition (by obvious symmetry in interchanging the roles of f and h). We denote this common number by $\overline{\tau}(f, h)$

Proposition 2.5 *Let $f, h \in \mathcal{O}_n$ such that $\mu_X(f) < \infty$ and $\mu_Y(h) < \infty$ where $X = h^{-1}(0)$ and $Y = f^{-1}(0)$. Then*

$$\mu_X(f) - \tau_X(f) = \mu(f) - \tau(f) + \mu(h) - \tau(h) + \mu(f, h) - \overline{\tau}(f, h). \tag{9}$$

In particular

$$\mu_X(f) - \tau_X(f) = \mu_Y(h) - \tau_Y(h). \tag{10}$$

Proof By Theorem 2.3 and relation (3) we have

$$\begin{aligned}\mu_X(f) - \tau_X(f) &= \mu(f) + \mu(f, h) + \mu(h) - \tau(h) - \tau_X(f) \\ &= \mu(f) - \tau(f) + \mu(h) - \tau(h) + \mu(f, h) - \overline{\tau}(f, h) \\ &= \mu_Y(h) - \tau_Y(h)\end{aligned}$$

□

Let us now see how the algebraic characterisation of the equality between the Bruce–Roberts Milnor and Tjurina numbers given in Corollary 2.2, reads in the case of a pair of isolated hypersurface singularities

Corollary 2.6 *Let $f, h \in \mathcal{O}_n$ such that $\mu_X(f) < \infty$ and $\mu_Y(h) < \infty$, where $X = h^{-1}(0)$ and $Y = f^{-1}(0)$. Then, the following conditions are equivalent*

(1) $\mu_X(f) = \tau_X(f)$
(2) $\mu(f) = \tau(f)$, $\mu(h) = \tau(h)$ *and* $\mu(f, h) = \overline{\tau}(f, h)$.

Proof It suffices to check (1) ⇒ (2). By Corollary 2.2 the condition $\mu_X(f) = \tau_X(f)$ implies the following two conditions: $\mu(f) = \tau(f)$, and $\overline{\mu}_X(f) = \overline{\tau}(f, h)$. By (9) they read as

$$\mu(f) = \tau(f) \quad \text{and}$$
$$\mu(h) - \tau(h) + \mu(f, h) = \overline{\tau}(f, h).$$

By (10), condition $\mu_X(f) = \tau_X(f)$ is equivalent to $\mu_Y(h) = \tau_Y(h)$ which in turn leads, by Corollary 2.2 again, to $\mu(h) = \tau(h)$, and $\overline{\mu}_Y(h) = \overline{\tau}(f, h)$. By (9), these read as

$$\mu(h) = \tau(h) \quad \text{and}$$
$$\mu(f) - \tau(f) + \mu(f, h) = \overline{\tau}(f, h).$$

By combining these equations we obtain the required equalities: $\mu(f) = \tau(f)$, $\mu(h) = \tau(h)$ and $\mu(f, h) = \overline{\tau}(f, h)$. □

Remark 2.7 It is important to notice that the number $\overline{\tau}(f, h)$ defined above is in general not equal to the ordinary Tjurina number $\tau(f, h)$ of the icis (f, h) (cf. [26] for exact definition of the latter). Despite this fact, if $\mu_X(f) = \tau_X(f)$ then, as it will follow from the proof of Theorem 1.3, the pair (f, X) will be relatively quasihomogeneous, which immediately implies that the icis $X \cap Y = (f, h)^{-1}(0)$ will also be quasihomogeneous. Thus, as long as $\dim X \cap Y \geqslant 1$, the following equalities hold

$$\mu_X(f) = \tau_X(f) \Longrightarrow \mu(f, h) = \tau(f, h) = \overline{\tau}(f, h). \tag{11}$$

Having this, we conjecture that as long as $\dim X \cap Y \geqslant 1$, the following chain of inequalities will also hold true

$$\mu(f, h) \geqslant \tau(f, h) \geqslant \overline{\tau}(f, h),$$

where the first inequality is already well-known, due to [27].

Below we give an example which shows that the inverse implication of (11) does not hold in general, i.e. quasihomogeneity of an icis (f, h) does not imply relative quasihomogeneity of the pair $(f, X = h^{-1}(0))$ (nor of $(h, Y = f^{-1}(0))$).

Example 2.8 Let us consider the functions $f, h \in \mathcal{O}_3$ with isolated singularity at the origin given by $f(x, y, z) = x^2 + y^4 + z^5$ and $h(x, y, z) = yz + x^3$. Let $X = h^{-1}(0)$ and $Y = f^{-1}(0)$. We have that f is quasihomogeneous with respect to $(1, 1/2, 2/5)$ and h is weighted homogeneous with respect to any vector $(w_1, w_2, w_3) \in \mathbb{Q}^3_+$ such that $w_2 + w_3 = 3w_1$. We have checked that $\mu(f) = 12$, $\mu(h) = 2$. Moreover $\overline{\tau}(f, h) = 9$ and

$$\begin{array}{lll} \mu(f, h) = 10 & \mu_X(f) = 22 & \mu_Y(h) = 12 \\ \tau(f, h) = 10 & \tau_X(f) = 21 & \tau_Y(h) = 11. \end{array}$$

By the equality $\mu(f, h) = \tau(f, h) = 10$ and Vosegaard's theorem [37], it follows that the icis $X \cap Y = (f, h)^{-1}(0)$ is quasihomogeneous in an appropriate coordinate system, but the pair (f, X) (and (h, Y), respectively) is not relatively quasihomogeneous in any coordinate system, since $\mu_X(f) - \tau_X(f) = \mu_Y(h) - \tau_Y(h) = 1$. Moreover, $\tau(f, h) - \overline{\tau}(f, h) = 1$ is positive, as conjecture in Remark 2.7 above.

3 Relative quasihomogeneity and the logarithmic Poincaré-Dulac theorem

In this section we give an invariant characterisation of relative quasihomogeneity of pairs (f, X) in terms of logarithmic vector fields. This is stated in Corollary 3.4. For this we need to pass to the formal category first, where we denote by $\widehat{\mathcal{O}}_n \cong \mathbb{C}[[x]] = \mathbb{C}[[x_1, \ldots, x_n]]$ the formal completion of $\mathcal{O}_n \cong \mathbb{C}\{x\} = \mathbb{C}\{x_1, \ldots, x_n\}$. The key lemma is Theorem 3.1 below, which is a logarithmic version of the well-known Poincaré-Dulac normal form theorem (cf. [4, Ch. 5, §23]. Indeed, recall that the latter is a statement on formal normal forms of analytic vector fields $\delta \in \Theta$, vanishing at the origin, $\delta(0) = 0$ (see part (a) of Theorem 3.1 below for exact statement). In addition to this, Theorem 3.1 provides a simultaneous formal normal form, not just for vector fields $\delta \in \Theta$, $\delta(0) = 0$, but also for their invariant ideals $I \subset \mathcal{O}_n$, $\delta(I) \subseteq I$, and for their eigenfunctions $f \in \mathcal{O}_n$, $\delta(f) = \lambda f$, $\lambda \in \mathbb{C}$. It implies in particular that in appropriate formal coordinates, the whole triple (δ, I, f) can be simultaneously reduced to a Poincaré-Dulac normal form, i.e. such that the following relations simultaneously hold

$$[\delta_S, \delta] = 0, \quad \delta_S(I) \subseteq I, \quad \delta_S(f) = \lambda f,$$

where $\delta_S = \sum_{i=1}^n w_i x_i \partial_{x_i}$ is the semi-simple part of δ, i.e. the diagonal part in the Jordan decomposition of its linear part $j^1\delta$, viewed as an endomorphism of $\mathbb{C}^n$. More precisely:

Theorem 3.1 *Let $\delta \in \Theta$ be a germ of an analytic vector field at the origin of $\mathbb{C}^n$, such that $\delta(0) = 0$. Let $sp(\delta) = (w_1, \ldots, w_n) \in \mathbb{C}^n$ be the eigenvalues of the linear part $j^1\delta$ of δ. Then*

(a) there exists a formal change of coordinates (i.e. a formal diffeomorphism $\Phi \in \widehat{Diff(\mathbb{C}^n, 0)}$) such that the vector field δ is reduced to the following normal form (where we denote by the same symbol δ the transformed vector field $\Phi_*\delta \in \widehat{\Theta}$)

$$\delta = \delta_S + \delta_N, \quad [\delta_S, \delta_N] = 0, \tag{12}$$

where $\delta_S = \sum_{i=1}^n w_i x_i \partial_{x_i}$ is the semi-simple part of δ, and δ_N is a nilpotent vector field (i.e. with nilpotent linear part $j^1\delta_N$) which commutes with δ_S (and thus with δ).

(b) Any eigenfunction $f \in \widehat{\mathcal{O}}_n$ of δ with eigenvalue $\lambda \in \mathbb{C}$ is also an eigenfunction of its semi-simple part δ_S with the same eigenvalue, in the coordinates of the normal form (12)

$$\delta(f) = \lambda f \iff \delta_S(f) = \lambda f, \quad \delta_N(f) = 0.$$

(c) Any δ-invariant ideal $I \subset \widehat{\mathcal{O}}_n$ is also δ_S and δ_N-invariant in the coordinates of the normal form (12)

$$\delta(I) \subseteq I \iff \delta_S(I) \subseteq I, \quad \delta_N(I) \subseteq I.$$

(d) Any δ_S-invariant ideal $I \subset \widehat{\mathcal{O}}_n$ is weighted homogeneous with respect to the weights $w = sp(\delta) \in \mathbb{C}^n$ given by the eigenvalues of δ_S, i.e. there exist formal series $h_i \in \widehat{\mathcal{O}}_n$,

$i = 1, \ldots, m$, such that

$$I = \langle h_1, \ldots, h_m \rangle$$
$$\delta_S(h_i) = d_i h_i, \quad d_i \in \mathbb{C}, \quad i = 1, \ldots, m.$$

(e) *Any weighted homogeneous ideal $I = \langle h_1, \ldots, h_m \rangle \subset \widehat{\mathcal{O}}_n$ with complex weights $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ and complex weighted degree $d = (d_1, \ldots, d_m) \in \mathbb{C}^m$, is also weighted homogeneous with respect to a system of rational weights $w' = (w'_1, \ldots, w'_n) \in \mathbb{Q}^n$, and with rational weighted degree $d' = (d'_1, \ldots, d'_m) \in \mathbb{Q}^m$.*

Proof All the parts of the theorem are well-known, but they are scattered across the literature, so we give below the exact references for each case.

(a) This is exactly the Poincaré-Dulac normal form theorem referred to above, usually included in standard courses of differential equations cf. [4, Ch. 5, §23], see also [33, Satz 3.1] in terms of derivations.
(b) This can be easily proved by using the decomposition of a function $f \in \widehat{\mathcal{O}}_n$ in its weighted homogeneous components and the commutativity of the semi-simple with the nilpotent parts of δ, cf. [33, Satz 3.2].
(c) This is slightly more complicated than (b), cf. [19, Theorem 3.2] for a recent analytic proof.
(d) This is a combination of [35, (1.3), (2.2)-(2.4)]. Indeed, in our notation, [35, (2.2)-(2.3)] imply that any semi-simple vector field $\delta_S \in \widehat{\Theta}$ induces a complex $(\mathbb{C}, +)$-grading of $\widehat{\mathcal{O}}_n$ determined by its eigenvalues $sp(\delta_S) = w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ (and conversely). By [35, (1.3)] any δ_S-invariant ideal $I \subset \widehat{\mathcal{O}}_n$ is just a graded submodule with respect to this complex grading. Part [35, (2.4)] implies finally that any graded submodule $I \subset \widehat{\mathcal{O}}_n$ as above, can be generated by weighted homogeneous formal series as in the statement of the theorem. For an alternative proof of this latter fact, see also [9, Lemma 3.2], where one has to replace the corresponding $\mathbb{Z}$-grading by a complex one.
(e) This is a straightforward modification of [33, Lemma 1.4] proved there for a single weighted homogeneous function.

□

Remark 3.2 The notion of weighted homogeneity as it appears in parts (c)-(d) above, i.e. with respect to a system of (in general) complex weights $w \in \mathbb{C}^n$, is what Saito called "weak quasihomogeneity" in [33, Definition 1.1.(ii)]. Here, and throughout the rest of the paper, we will not use this terminology and we will refer to such formal series simply as "weighted homogeneous", whereas we reserve the term "quasihomogeneous" for weighted homogeneous formal series with positive rational weights $w \in \mathbb{Q}^n_+$ (caution is needed here, as many authors use the terms "weighted homogeneous" and "quasihomogeneous" for the same object, i.e. with positive rational weights).

Remark 3.3 We will call the coordinates of the Poincaré-Dulac normal form (12) of δ, Poincaré-Dulac coordinates. These are a priori only formal, but in case where the eigenvalues $\mathrm{sp}(\delta) = (w_1, \ldots, w_n) \in \mathbb{C}^n$ satisfy certain arithmetic conditions (e.g. they belong in the Poincaré domain, cf. [4, Ch. 5, §24]), then they can be chosen to be analytic. In such a case, if $X = V(I)$ is an analytic variety and $f \in \mathcal{O}_n$ is an analytic function germ, then the theorem above says that for any vector field $\delta \in \Theta$, $\delta(0) = 0$, the following equivalences hold in analytic Poincaré-Dulac coordinates

$$\delta(f) = f \iff \delta_S(f) = f, \quad \delta_N(f) = 0$$
$$\delta \in \Theta_X \iff \delta_S \in \Theta_X, \quad \delta_N \in \Theta_X.$$

As an immediate corollary of the above we obtain the following invariant characterization of relative quasihomogeneity for pairs (f, X).

Corollary 3.4 *A pair (f, X) is relatively quasihomogeneous if and only if there exists a logarithmic vector field $\delta \in \Theta_X$, $\delta(0) = 0$, with positive rational eigenvalues, which admits f as an eigenfunction (we can always choose the eigenvalue equal to 1):*

$$\delta \in \Theta_X, \quad \delta(f) = f$$
$$sp(\delta) = (w_1, \dots, w_n) \in \mathbb{Q}_+^n.$$

Proof Since the eigenvalues of δ are positive, all parts (a)-(d) in Theorem 3.1 hold in the analytic category as well. Thus, passing to Poincaré-Dulac coordinates $\delta = \delta_S + \delta_N$, where $\delta_S = \sum_{i=1}^n w_i x_i \partial_{x_i}$, we obtain

$$\delta_S \in \Theta_X, \quad \delta_S(f) = f$$

which is what we wanted to prove. Indeed, by part (d) of Theorem 3.1 we can find a system of quasihomogeneous generators $\langle h_1, \dots, h_m \rangle = I_X$ of the ideal of functions vanishing on X, so that

$$\delta_S(f) = f \quad \text{and}$$
$$\delta_S(h_i) = d_i h_i, \quad d_i \in \mathbb{Q}_+, \quad i = 1, \dots, m$$

which is equivalent to the relative quasihomogeneity of the pair (f, X) in terms of Definition 1.1. □

4 Relative Saito Theorems

In this section we prove Theorem 1.3. Throughout the section we denote by $\mathbf{m} \subset \mathcal{O}_n$ the maximal ideal of the ring of holomorphic function germs. We start with Part (a) of Theorem 1.3 with a more precise statement on the range of the weights.

Theorem 4.1 *Let (f, X) be a pair with $\mu_X(f) < \infty$, where $X \subset (\mathbb{C}^n, 0)$ is an arbitrary analytic set. Suppose also that $f \in \mathbf{m}^3$. Then, condition $\mu_X(f) = \tau_X(f)$ implies the relative quasihomogeneity of the pair (f, X) with respect to a system of weights $w \in ((0, \frac{1}{2}) \cap \mathbb{Q})^n$.*

Proof From condition (2) in Corollary 2.2 and Saito's theorem [33, Satz 4.1] applied to f, we know that there exists a coordinate system and an Euler vector field $\chi_w \in \Theta_Y$, $\chi_w(f) = f$, with positive rational eigenvalues $w \in \mathbb{Q}_+^n$. In fact, since $f \in \mathbf{m}^3$ we can choose the coordinates such that $w \in ((0, \frac{1}{2}) \cap \mathbb{Q})^n$ (see [33, Satz 1.3]). Suppose that χ_w is not tangent to X in these coordinates (or else there is nothing to prove). It follows then by the relation (7) in Corollary 2.2 that there exists a Hamiltonian vector field $\eta \in H_Y$ of f, such that the new vector field $\delta = \chi_w - \eta$ is tangent to X, and remains Euler for f:

$$\delta = \chi_w - \eta \in \Theta_X \cap \Theta_Y, \quad \delta(f) = \chi_w(f) = f.$$

Denote now by $\mathrm{sp}(\delta) = \mathrm{sp}(\chi_w - \eta) = w'$ the eigenvalues of the linear part of δ. Since $f \in \mathbf{m}^3$ we have that $j^1\eta = 0$ for any $\eta \in H_Y = \langle \partial_{x_i} f \partial_{x_j} - \partial_{x_j} f \partial_{x_i} \rangle_{1 \leqslant i<j \leqslant n}$ (i.e. all Hamiltonian vector fields have zero 1-jet), and thus we obtain the following equality of the corresponding eigenvalues

$$w' = w \in ((0, \frac{1}{2}) \cap \mathbb{Q})^n.$$

The proof is concluded now by Corollary 3.4. □

Remark 4.2 An alternative proof of Theorem 4.1 can be given without using the decomposition $\Theta_Y = \Theta_X \cap \Theta_Y + H_Y$ obtained by the algebraic formula $\overline{\mu}_X(f) = \overline{\tau}_X(f)$, but using instead Alexandrov-Kersken's theorem [3,17], according to which (cf. [39, Proposition 1.2])

$$\Theta_Y = \langle \chi_w \rangle + H_Y,$$

in appropriate quasihomogeneous coordinates of f, where χ_w is an Euler vector field for f, with positive rational eigenvalues $w = (w_1, \dots, w_n) \in ((0, \frac{1}{2}) \cap \mathbb{Q})^n$. Then, since $\mu_X(f) = \tau_X(f)$ implies by definition that there exists $\delta \in \Theta_X \cap \Theta_Y \subset \Theta_Y$, with $\delta(f) = f$, it follows that

$$\delta = \chi_w + \eta$$

for some $\eta \in H_Y$. The rest of the proof follows exactly the same lines.

Let us prove now Part (b) of Theorem 1.3 under a more precise statement for the weights.

Theorem 4.3 *Let (f, X) be a pair with $\mu_X(f) < \infty$, where $X = h^{-1}(0) \subset (\mathbb{C}^n, 0)$ is a hypersurface having at most an isolated singularity at the origin. Then, condition $\mu_X(f) = \tau_X(f)$ implies the relative quasihomogeneity of the pair (f, X) with respect to a system of weights $w \in ((0, 1] \cap \mathbb{Q})^n$.*

Proof The proof splits into two cases with respect to the multiplicity of h. We show first that in each case separately, the pair (f, X) is formally relatively quasihomogeneous with respect to the weights $w \in ((0, 1] \cap \mathbb{Q})^n$ indicated in the theorem, and then we pass to the analytic category as in [33], using Artin's approximation theorem [6].

(1) Case $h \in \mathbf{m} \setminus \mathbf{m}^2$.

In this case $X = h^{-1}(0)$ is a smooth hypersurface, so we can always choose coordinates such that it is weighted homogeneous for any weight system; in particular

$$h(x, y) = x$$

so that $X = h^{-1}(0) = \{x = 0\}$. In these coordinates the function f has the form

$$f(x, y) = xf_1(x, y) + f_0(y),$$

where either both f and $f_0 = f|_X = f|_{\{x=0\}}$ have an isolated singularity at the origin, or f is regular ($\partial_x f(0) = f_1(0) \neq 0$) and $f_0 = f|_{\{x=0\}}$ has an isolated singularity at the origin. It is easy to see now that condition $\mu_X(f) = \tau_X(f)$ implies $\mu(f) = \tau(f)$ and $\mu(f_0) = \tau(f_0)$, or equivalently, that there exists a vector field $\delta \in \Theta$ and a function $u \in \mathcal{O}_n$, such that

$$\delta(x) = ux, \quad \delta(f) = f \quad \text{and} \quad \delta|_{\{x=0\}}(f_0) = f_0.$$

Let $\mathrm{sp}(\delta) = w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ be the eigenvalues of the linear part of δ. Passing to formal Poincaré-Dulac coordinates if necessary (notice that this can be easily achieved by formal diffeomorphisms preserving $X = \{x = 0\}$), we can suppose by Theorem 3.1 items (b)-(e), that the pair

$$f(x, y) = xf_1(x, y) + f_0(y), \quad X = \{x = 0\} \tag{13}$$

is already weighted homogeneous with rational weights (which we denote by the same symbol) $w \in \mathbb{Q}^n$, i.e.

$$\chi_w(x) = w_1 x, \quad \chi_w(f) = f \;\text{ and }\; \chi_w|_{\{x=0\}}(f_0) = f_0,$$

where $\chi_w = w_1 x\partial_x + \sum_{i=2}^n w_i y_i \partial_{y_i}$ is the semi-simple part of δ. Notice that in this coordinate system, the function f_1 appearing as a factor of f in (13), is also weighted homogeneous of degree $\deg_w f_1 = 1 - w_1$, i.e. $\chi_w(f_1) = (1 - w_1)f_1$ holds true.

Below we will show that there always exists a further formal change of coordinates which reduces the weighted homogeneous pair (f, X)=(13) above to a relatively quasihomogeneous pair with respect to positive rational weights $w \in ((0, 1] \cap \mathbb{Q})^n$, as in the statement of the theorem. For this we will need first the following auxiliary result.

Lemma 4.4 *Let $(f, X = \{x = 0\})$ be a weighted homogeneous pair as in (13) above, of corresponding weighted degree $\deg_w f = 1$ and $\deg_w x = w_1 \in \mathbb{Q}$, respectively. Since the function $f \in \mathbf{m}$ has at most an isolated singularity at the origin, the weighted degree $w_1 = \deg_w x \in \mathbb{Q}$ of the variable x is always positive, and in particular it lies within the half-open unit interval $w_1 \in (0, 1] \cap \mathbb{Q}$.*

Proof Indeed, since the function f has at most an isolated singularity at the origin, it follows by [33, Korollar 1.6] that only the following possibilities may occur for the monomials appearing in the expansion of f (i.e. of xf_1)

(1.a) the monomial x appears in the expansion of f (in which case f is regular, $\partial_x f(0) = f_1(0) \neq 0$) and then

$$w_1 = \deg_w f = 1,$$

(1.b) a monomial of the form x^{k+1} appears in the expansion of f for some $k \geqslant 1$ (equivalently x^k appears in the expansion of f_1) and in this case

$$w_1 = \frac{1}{k+1} \in (0, \frac{1}{2}] \cap \mathbb{Q},$$

(1.c) or a monomial of the form $x^{k+1}y_j$ appears in the expansion of f for some $j = 2, \ldots, n$ and $k \geqslant 0$ (or equivalently $x^k y_j$ appears in f_1), and in this case

$$w_1 = \frac{1 - w_j}{k+1} \in \mathbb{Q}, \quad k \geqslant 0.$$

Thus, it suffices to check only the case (1.c), where we claim that $w_1 \in (0, 1) \cap \mathbb{Q}$, otherwise the function f will necessarily have non-isolated singularities.

Indeed, suppose on the contrary that $w_1 \leqslant 0$ is non-positive. Without loss of generality we may suppose also that $j = 2$, i.e. the monomial $x^{k+1}y_2$ appears in the expansion of f. Then $w_2 = 1 - (k + 1)w_1 \geqslant 1$ and since the function $f_0 = f|_{\{x=0\}}$ has an isolated singularity at the origin, it follows by [33, Korollar 1.9] that there exists some $j' \neq 2$, say $j' = 3$, such that $w_3 \leqslant 0$. By Saito's lemma [33, Lemma 1.10] applied to the function

f_0, there exists a weighted homogeneous change of coordinates in the y-variables only (and thus trivially preserving the coordinate function $h(x, y) = x$ as well as its defining hypersurface $X = \{x = 0\}$), which sends f_0 to a quasihomogeneous polynomial

$$f_0(y) = y_2y_3 + \widetilde{f_0}(y_4, \dots, y_n),$$

where $\widetilde{f_0}$ is a quasihomogeneous polynomial with respect to the positive rational weights $(w_4, \dots, w_n) \in ((0, 1) \cap \mathbb{Q})^{n-3}$, and with an isolated singularity at the origin (when viewed as a function in the $(y_4, \dots, y_n)$-variables). Since the above change of coordinates in the y-variables is weighted homogeneous and preserves the coordinate function x, the function f is also sent to a (new) weighted homogeneous function

$$f(x, y) = xf_1(x, y) + y_2y_3 + \widetilde{f_0}(y_4, \dots, y_n)$$

for some (new) weighted homogeneous function f_1 of positive degree $\deg_w f_1 = 1 - w_1 \geqslant 1$. As a result, no monomial of the form $x^l y_3^{m_3}$ (of non-positive weighted degree) may appear in the expansion of f_1, and thus $f_1 \in \langle y_2, y_4, \dots, y_n\rangle$. It follows from this that $\partial_x f_1 \in \langle y_2, y_4, \dots, y_n\rangle$, and $\partial_{y_3} f_1 \in \langle y_2, y_4, \dots, y_n\rangle$ as well.

Computing now the weighted degrees of the partial derivatives of the function f_1 with respect to the remaining $(y_2, y_4, \dots, y_n)$-variables, we obtain

$$\deg_w \partial_{y_2} f_1 = 1 - w_1 - w_2 = kw_1 \leqslant 0,$$

is non-positive, whereas

$$\deg_w \partial_{y_j} f_1 = 1 - w_1 - w_j > 0, \quad \forall j = 4, \dots, n$$

are all positive (because $1 - w_1 \geqslant 1$ and $w_j \in (0, 1)$ for all $j = 4, \dots, n$). The non-positivity of the weighted degree of the partial derivative $\partial_{y_2} f_1$, implies that no monomial of the form $y_2^{m_2} y_4^{m_4} \dots y_n^{m_n}$ (of positive weighted degree) may appear in its expansion, and thus $\partial_{y_2} f_1 \in \langle x, y_3\rangle$. On the contrary, the positivity of the weighted degrees of the rest of the partial derivatives $\partial_{y_j} f_1$, $j = 4, \dots, n$, implies that no monomial of the form $x^l y_3^{m_3}$ (of non-positive weights) may appear in their expansion, and thus $\partial_{y_j} f_1 \in \langle y_2, y_4, \dots, y_n\rangle$, for all $j = 4, \dots, n$.

Summing up, it follows from the above that the partial derivatives of the function f with respect to the $(x, y_3, y_4, \dots, y_n)$-variables (i.e. all except $\partial_{y_2} f = x\partial_{y_2} f_1 + y_3$), vanish simultaneously upon restriction on the $y_2 = y_4 = \dots = y_n = 0$ plane

$$\partial_x f = f_1 + x\partial_x f_1 \in \langle y_2, y_4, \dots, y_n\rangle, \quad \partial_{y_i} f = x\partial_{y_i} f_1 + \partial_{y_i} f_0 \in \langle y_2, y_4, \dots, y_n\rangle, \quad \forall i = 3, \dots, n.$$

Thus, the singular locus $\Sigma(f)$ of f contains the variety

$$\{y_2 = y_4 = \dots = y_n = y_3 + \partial_{y_2} f_1(x, 0, y_3, 0, \dots, 0) = 0\} \subseteq \Sigma(f),$$

which is at most of codimension $n-1$. This is a contradiction and thus $w_1 > 0$ is necessarily positive.

Analogous arguments as above show that we can neither have $w_1 \geqslant 1$. Indeed, suppose on the contrary that $w_1 \geqslant 1$ and that without loss of generality, $j = 2$ again, i.e. a monomial of the form $x^{k+1}y_2$ appears in the expansion of f. Then $\deg_w y_2 = w_2 = 1 - (k+1)w_1 \leqslant 0$ is non-positive, and by [33, Korollar 1.9] applied to $f_0 = f|_{\{x=0\}}$ there exists a $j' \neq 2$, say again $j' = 3$, such that $w_3 \geqslant 1$. Exactly in the same way as before, Saito's lemma [33, Lemma 1.10] applied to f_0, implies that there exists a weighted homogeneous change of coordinates in the y-variables only (and thus trivially preserving the coordinate function $h(x, y) = x$, as well as its defining hypersurface $X = \{x = 0\}$) which sends f to a (new) weighted homogeneous function

$$f(x, y) = xf_1(x, y) + y_2y_3 + \widetilde{f}_0(y_4, \dots, y_n).$$

Here, the function $\widetilde{f}_0$ is again a quasihomogeneous polynomial with respect to the weights $(w_2, \dots, w_n)$
$\in ((0, 1) \cap \mathbb{Q})^{n-3}$, and with an isolated singularity at the origin, whereas the function f_1 is a weighted homogeneous function, which is now of non-positive degree $\deg_w f_1 = 1 - w_1 \leqslant 0$. This implies in turn that no monomial of the form $x^l y_3^{m_3} \dots y_n^{m_n}$ (of positive weighted degree) may appear in its expansion, and thus $f_1 \in \langle y_2 \rangle$. As a result $\partial_x f_1 \in \langle y_2 \rangle$, and $\partial_{y_j} f_1 \in \langle y_2 \rangle$, for all $j = 3, \dots, n$, as well.

It follows from the above that the partial derivatives of the function f with respect to the $(x, y_3, y_4, \dots, y_n)$-variables (i.e. all except $\partial_{y_2} f = x\partial_{y_2} f_1 + y_3$ again), vanish simultaneously upon restriction on the $y_2 = y_4 = \dots = y_n = 0$ plane

$$\partial_x f = f_1 + x\partial_x f_1 \in \langle y_2 \rangle, \quad \partial_{y_i} f = x\partial_{y_i} f_1 + \partial_{y_i} f_0 \in \langle y_2, y_4, \dots, y_n \rangle, \quad \forall i = 3, \dots, n.$$

Thus, the singular locus $\Sigma(f)$ of f contains again the variety

$$\{y_2 = y_4 = \dots = y_n = y_3 + \partial_{y_2} f_1(x, 0, y_3, 0, \dots, 0) = 0\} \subseteq \Sigma(f),$$

which is at most of codimension $n - 1$. This is a contradiction and thus necessarily $w_1 \in (0, 1) \cap \mathbb{Q}$. This finishes the proof of the lemma. □

Thus, we have shown that in all possible cases (1.a)-(1.c) treated above, the weight w_1 always lies within the half-open unit interval $w_1 \in (0, 1] \cap \mathbb{Q}$. Now we will show that the rest of the weights $w' = (w_2, \dots, w_n) \in \mathbb{Q}^{n-1}$ can be simultaneously chosen positive, and in particular within the open unit interval $w' \in ((0, 1) \cap \mathbb{Q})^{n-1}$, which will finish the proof for the case (1).

The sub-case (1.a) is trivial. Since $f_1(0) \neq 0$, the change of coordinates $x \mapsto x(f_1(x, y))^{-1}$ sends the function f to

$$f(x, y) = x + f_0(y),$$

where $f_0 \in \mathbf{m}^2$ is a weighted homogeneous function with an isolated singularity at the origin (when viewed as a function of the y-variables only). By Saito's theorem [33, Satz 1.3], there exists a further change of coordinates in the y-variables which sends f_0 to a quasihomogeneous polynomial with normalised weights $w' \in ((0, \frac{1}{2}] \cap \mathbb{Q})^{n-1}$. This finishes the proof for the sub-case (1.a).

For the rest of the sub-cases (1.b)-(1.c) where $f \in \mathbf{m}^2$ has an isolated singularity at the origin, the positivity of the weights $w' \in \mathbb{Q}^n$ follows now by a direct application of Saito's lemma [33, Lemma 1.10] to the function f. Indeed, suppose on the contrary that there exists r weights which are non-positive, say $w_{2i} \leqslant 0$, $i = 1, \dots, r$ (corresponding to the variables y_{2i}, $i = 1, \dots, r$). Then, by [33, Korollar 1.9] applied to the function f, there exist exactly r more weights which are bigger or equal to one, say $w_{2i+1} \geqslant 1$, $i = 1, \dots, r$ (corresponding to the variables y_{2i+1}, $i = 1, \dots, r$), and the rest $n - 2r$ weights corresponding to the remaining $(x, y_{2r+2}, \dots, y_n)$-variables (including w_1 by Lemma 4.4 above) belong in the open unit interval, $(w_1, w_{2r+2}, \dots, w_n) \in ((0, 1) \cap \mathbb{Q})^{n-2r}$. Applying now Saito's lemma [33, Lemma 1.10] to the function f, there exists a weighted homogeneous change of coordinates in the y-variables only (and thus trivially preserving the coordinate function $h(x, y) = x$, as well as its defining hypersurface $X = \{x = 0\}$), which sends f to a quasihomogeneous polynomial

$$f(x, y) = \sum_{i=1}^{r} y_{2i}y_{2i+1} + \widetilde{f}(x, y_{2r+2}, \dots, y_n),$$

where $\widetilde{f}$ is a quasihomogeneous polynomial with respect to the weights $(w_1, w_{2r+2}, \dots, w_n) \in ((0, 1) \cap \mathbb{Q})^{n-2r}$, which has an isolated singularity at the origin (when viewed as a function of the $(x, y_{2r+2}, \dots, y_n)$-variables). Notice that in these coordinates, the restriction $f_0 = f|_{\{x=0\}}$ of f on $X = \{x = 0\}$ is also a quasihomogeneous polynomial

$$f_0(y) = \sum_{i=1}^{n} y_{2i}y_{2i+1} + \widetilde{f_0}(y_{2r+2}, \dots, y_n),$$

where $\widetilde{f_0} = \widetilde{f}|_{\{x=0\}}$ has an isolated singularity at the origin (when viewed as a function of the $(y_{2r+2}, \dots, y_n)$-variables). Decomposing now $\widetilde{f}(x, y_{2r+2}, \dots, y_n) = x\widetilde{f_1}(x, y_{2r+2}, \dots, y_n) + \widetilde{f_0}(y_{2r+2}, \dots, y_n)$, where $\widetilde{f_1}$ is an appropriate quasihomogeneous polynomial of degree $\deg_w \widetilde{f_1} = 1 - w_1$, we conclude that the pair (f, X)=(13) has been formally reduced to a relatively quasihomogeneous pair

$$f(x, y) = x\widetilde{f_1}(x, y_{2r+2}, \dots, y_n) + \sum_{i=1}^{r} y_{2i}y_{2i+1} + \widetilde{f_0}(y_{2r+2}, \dots, y_n), \quad X = \{x = 0\}$$

with respect to the weights $w \in ((0, 1) \cap \mathbb{Q})^n$. This finishes the proof for the case (1).

(2) Case $h \in \mathbf{m}^2$.

In this case $X = h^{-1}(0)$ also has an isolated singularity at the origin. The proof splits now into the following sub-cases with respect to the multiplicity of f.

(2.1) Sub-case $f \in \mathbf{m}^3$. This reduces to Theorem 4.1 and there is nothing to prove.

(2.2) Sub-case $f \in \mathbf{m}^2 \setminus \mathbf{m}^3$. Working as in the proof of Theorem 4.1 we have by Saito's theorem applied to f that there exist coordinates and an Euler vector field $\chi_w \in \Theta_Y$, $\chi_w(f) = f$, with positive rational eigenvalues $w \in ((0, 1/2] \cap \mathbb{Q})^n$ (see again [33, Satz 1.3]). Suppose that in these coordinates the vector field χ_w is not tangent to X (else there is nothing to prove). Then, by relation (7) in Corollary 2.2 again, there exists a Hamiltonian vector field $\eta \in H_Y$ for f such that the new vector field $\delta = \chi_w - \eta$ is tangent to X and remains Euler for f

$$\delta = \chi_w - \eta \in \Theta_X \cap \Theta_Y, \quad \delta(f) = \chi_w(f) = f.$$

Let $\mathrm{sp}(\delta) = \mathrm{sp}(\chi_w - \eta) = w' = (w'_1, \dots, w'_n) \in \mathbb{C}^n$ be the eigenvalues of δ. Since $f \in \mathbf{m}^2 \backslash \mathbf{m}^3$, we have in general $j^1\eta \neq 0$ for $\eta \in H_Y$ (i.e. the set of Hamiltonian vector fields with nonzero linear part is non-empty), and so we may suppose that $w' \neq w$ (else there is again nothing to prove). A priori the eigenvalues $w'_i \in \mathbb{C}$ are complex numbers, but passing to Poincaré-Dulac coordinates if necessary, we may suppose by Theorem 3.1 (b)-(e) that the pair $(f, X = h^{-1}(0))$ is already formally weighted homogeneous with respect to a vector of rational weights (which we denote by the same symbol) $w' \in \mathbb{Q}^n$, where h is a weighted homogeneous generator of the (formal) ideal of functions vanishing on X, of rational degree $\deg_{w'} h = d \in \mathbb{Q}$. It suffices to show now that the weights $w'_i \in \mathbb{Q}$ can be chosen positive, and in particular $w' \in ((0, 1) \cap \mathbb{Q})^n$. In particular we will prove the following result.

Claim 4.5 *There exists a weighted homogeneous change of coordinates, such that the pair* (f, X) *is relatively quasihomogeneous with respect to positive rational weights* $w' \in ((0,1)\cap \mathbb{Q})^n$.

Proof We argue by contradiction by showing that if in the situation above there exist $k \geq 1$, weights which are non-positive, say $w' \leq 0, i = 1, \ldots, k$, then $\mu_X(f) = \infty$. Indeed, since f has isolated singularity, we have by [33, Korollar 1.9] that there exist k more weights which are greater or equal to 1, $w'_{k+i} \geqslant 1, i = 1, \ldots, k$, and the rest $(n-2k)$-weights belong in the open unit interval $w'_{2k+j} \in (0,1), j = 1, \ldots, n-2k$. In particular, working as in the proof of [33, Lemma 1.10], we can find a system of weighted homogeneous coordinates $(x, y, z) = (x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_{n-2k})$ of corresponding degrees

$$\begin{aligned} &\deg_{w'} x_i = w'_i, \quad i = 1, \ldots, k \\ &\deg_{w'} y_i = 1 - w'_i, \quad i = 1, \ldots, k \\ &\deg_{w'} z_j = w'_{2k+j}, \quad j = 1, \ldots, n-2k \end{aligned}$$

such that:

$$f(x, y, z) = \sum_{i=1}^{k} x_i y_i + f_0(z),$$

is weighted homogeneous with respect to the weights

$$w' = (w'_1, \ldots, w'_k, 1 - w'_1, \ldots, 1 - w'_k, w'_{2k+1}, \ldots, w'_n),$$

of degree $\deg_{w'} f = 1$, where $f_0 \in \mathbf{m}^2$ is a quasihomogeneous polynomial with respect to the weights $(w'_{2k+1}, \ldots, w'_n) \in ((0,1) \cap \mathbb{Q})^{n-2k}$. In these coordinates, the function $h = h(x, y, z)$ defining the hypersurface $X = h^{-1}(0)$, is also weighted homogeneous of some degree $\deg_{w'} h = d \in \mathbb{Q}$. Now we will show the following auxiliary result concerning the possible range of the weighted degree d of the function h.

Lemma 4.6 *Since the function* $h \in \mathbf{m}^2$ *has an isolated singularity at the origin, then* $d = 1$ *and the restriction* $j^2 h(x, y, 0)$ *of the* 2*-jet of* h *on the* $z = 0$ *plane, is a non-degenerate quadratic form.*

Proof We remark that the implication "$d = 1 \Longrightarrow$ *the 2-jet* $j^2 h(x, y, 0)$ *is a non-degenerate quadratic form*", is contained explicitly in [33, Proof of Lemma 1.10], so we will only show in details that $d = 1$. We argue by contradiction by showing that if $d \neq 1$, then the function h will necessarily have non-isolated singularities.

Indeed, in order to do so let us first fix an ordering of the weights which, up to renumbering, we can take as

$$0 \geqslant w'_1 \geqslant w'_2 \geqslant \ldots \geqslant w'_k, \tag{14}$$

and let us suppose also that $d < 1$ to start with. This means that no monomial of the form y^l (of weighted degree bigger or equal to one) may appear in the expansion of h, and thus $h \in \langle x, z \rangle$. So, we can decompose h as

$$h(x, y, z) = \sum_{i=1}^{k} x_i h_i(x, y, z) + h_0(z),$$

where h_0 is a quasihomogeneous polynomial with respect to the weights $(w'_{2k+1}, \dots, w'_n) \in ((0,1)\cap\mathbb{Q})^{n-2k}$, and the functions h_i are weighted homogeneous of corresponding degrees $\deg_{w'} h_i = d - w'_i$, $i = 1, \dots, k$. Computing now the singular locus $\Sigma(h)$ of h, we see that it contains the variety

$$\{x = z = h_1(0, y, 0) = \cdots = h_k(0, y, 0) = 0\} \subseteq \Sigma(h),$$

and as $\Sigma(h) = \{0\}$, this variety must define the origin, i.e. none of the functions h_i can vanish identically upon restriction on the $x = z = 0$ plane. But this implies in turn that $\deg_{w'} h_i = d - w'_i \geqslant 1$, for all $i = 1, \dots, k$, for if not, say if $\deg_{w'} h_1 = d - w'_1 < 1$, then no monomial of the form y^l may appear in the expansion of h_1, and thus $h_1 \in \langle x, z\rangle$, which is a contradiction. By the positivity now of the weighted degrees of the functions h_i, it follows that no monomial of the form x^l may appear in their expansion, and thus $h_i \in \langle y, z\rangle$, for all $i = 1, \dots, k$.

Computing now the weighted degrees of the partial derivatives of the functions $h_i \in \langle y, z\rangle$ with respect to the y-variables, we obtain

$$\deg_{w'} \partial_{y_j} h_i = d - w'_i - (1 - w'_j) = d - 1 + (w'_j - w'_i), \quad i, j = 1, \dots, k.$$

Since by assumption $d - 1 < 0$, and since for all $j \geqslant i$ the differences $w'_j - w'_i \leqslant 0$ are non-positive with respect to the fixed ordering (14), it follows that

$$\deg_{w'} \partial_{y_j} h_i = d - 1 + (w'_j - w'_i) < 0, \quad \forall j \geqslant i.$$

The negativity now of the above weighted degrees, implies that no monomials of the form $y^l z^m$ (of positive weighted degree) may appear in the expansion of the partial derivatives $\partial_{y_j} h_i$, for all $j \geqslant i$, and thus $\partial_{y_j} h_i \in \langle x\rangle$, for all $j \geqslant i$. For $i = 1$ in particular, we obtain that all the partial derivatives of the function h_1 with respect to the y-variables, vanish identically upon restriction on the $x = 0$ plane, i.e. $\partial_{y_j} h_1 \in \langle x\rangle$, for all $j = 1, \dots, k$. But since $h_1 \in \langle y, z\rangle$, this implies that $h_1 \in \langle x, z\rangle$ as well. Indeed, in order to verify this we can write $h_1(x, y, z) = g_1(y, z) + \widetilde{h}_1(x, y, z)$, where $g_1(0) = 0$, and $\widetilde{h}(0, y, z) = 0$ (and also $\widetilde{h}_1(x, 0, 0) = 0$). Then $\partial_{y_j} h_1|_{x=0} = \partial_{y_j} g = 0$ implies $g_1 = g_1(z)$ and thus $h(x, y, z) = g_1(z) + \widetilde{h}_1(x, y, z)$ vanishes identically on $x = z = 0$, i.e. $h_1 \in \langle x, z\rangle$, as claimed. But this means that the singular locus $\Sigma(h)$ of h contains the variety

$$\{x = z = h_2(0, y, 0) = \cdots = h_k(0, y, 0) = 0\} \subseteq \Sigma(h),$$

which is at most of codimension $n - 1$. This is a contradiction and thus $d \geqslant 1$.

Analogous arguments as above show that we can neither have $d > 1$ strictly. Indeed, if we suppose on the contrary that $d > 1$, then no monomial of the form x^l (of non-positive weighted degree) may appear in the expansion of h, and thus $h \in \langle y, z\rangle$. So, we can decompose this time h as

$$h(x, y, z) = \sum_{i=1}^{k} y_i h_i(x, y, z) + h_0(z),$$

where h_0 is a quasihomogeneous polynomial with respect to the weights $(w'_{2k+1}, \dots, w'_n) \in ((0,1)\cap\mathbb{Q})^{n-2k}$ as before, and the functions h_i are weighted homogeneous of corresponding degrees $\deg_{w'} h_i = d - (1 - w'_i)$, $i = 1, \dots, k$. Computing again the singular locus $\Sigma(h)$ of h, we see that it contains the variety

$$\{y = z = h_1(x, 0, 0) = \cdots = h_k(x, 0, 0) = 0\} \subseteq \Sigma(h),$$

and since $\Sigma(h) = \{0\}$, this variety must define the origin, i.e. none of the functions h_i can vanish identically upon restriction on the $y = z = 0$ plane. This implies in turn that the weighted degrees of the functions h_i must be non-positive, $\deg_{w'} h_i = d - (1 - w'_i) \leqslant 0$, $i = 1, \ldots, k$, for if not, say if $\deg_{w'} h_1 = d - (1 - w'_1) > 0$, then no monomial of the form x^l (of non-positive weighted degree) may appear in the expansion of h_1, and thus $h_1 \in \langle y, z\rangle$, which is a contradiction. Thus indeed, $\deg_{w'} h_i = d - (1 - w'_i) \leqslant 0$, for all $i = 1, \ldots, k$. The non-positivity of the above weighted degree implies in turn that no monomial of the form $y^l z^m$ may appear in the expansion of the h_i's, and thus $h_i \in \langle x\rangle$, for all $i = 1, \ldots, k$.

Computing now the weighted degrees of the partial derivatives of the functions $h_i \in \langle x\rangle$ with respect to the x-variables, we obtain

$$\deg_{w'} \partial_{x_j} h_i = d - (1 - w'_i) - w'_j = d - 1 + (w'_i - w'_j), \quad i, j = 1, \ldots, k.$$

Since by assumption $d - 1 > 0$, and since for all $i \leqslant j$ the differences $w'_i - w'_j \geqslant 0$ are non-negative with respect to the fixed ordering (14), it follows that

$$\deg_{w'} \partial_{x_j} h_i = d - 1 + (w'_i - w'_j) > 0, \quad \forall i \leqslant j.$$

The positivity now of the above weighted degrees implies in turn that no monomial of the form x^l (of non-positive weighted degree) may appear in the expansion of the partial derivatives $\partial_{x_j} h_i$, for all $i \leqslant j$, and thus $\partial_{x_j} h_i \in \langle y, z\rangle$, for all $i \leqslant j$. In particular, for $i = 1$, this implies that all the partial derivatives along the x-variables of the function $h_1 \in \langle x\rangle$ vanish identically upon restriction on the $y = z = 0$ plane, and thus $h_1 \in \langle y, z\rangle$ as well. Indeed, in order to verify this we can write $h_1(x, y, z) = g_1(x) + \widetilde{h}_1(x, y, z)$, where $g_1(0) = 0$ and $h_1(x, 0, 0) = 0$ (and also $h_1(0, y, z) = 0$). Then $\partial_{x_j} h_1|_{y=z=0} = \partial_{x_j} g_1 = 0$, implies $g_1(x) = g_1(0) = 0$, and thus $h = \widetilde{h}_1 \in \langle y, z\rangle$ as claimed. But then, the singular locus $\Sigma(h)$ of h contains the variety

$$\{y = z = h_2(x, 0, 0) = \ldots, h_k(x, 0, 0) = 0\} \subseteq \Sigma(h),$$

which is of codimension at most $n - 1$. This is a contradiction, and thus $d = 1$ as claimed.

It follows now from the above that since $d = 1$, no monomial of the form x^l (of non-positive weighted degree) may appear in the expansion of h, and thus $h \in \langle y, z\rangle$. So, we can decompose again h as

$$h(x, y, z) = \sum_{i=1}^{k} y_i h_i(x, y, z) + h_0(z),$$

where h_0 is a quasihomogeneous polynomial with respect to the weights $(w'_{2k+1}, \ldots, w'_n) \in ((0, 1) \cap \mathbb{Q})^{n-2k}$ as before, and the functions h_i are weighted homogeneous of corresponding degrees $\deg_{w'} h_i = w'_i = \deg_{w'} x_i \leqslant 0$, $i = 1, \ldots, k$. In particular, by the same arguments as above, we obtain $h_i \in \langle x\rangle$, for all $i = 1, \ldots, k$, as no monomials of the form $y^l z^m$ (of positive weighted degree) may appear in their expansion. The non-degeneracy now of the 2-jet $j^2 h(x, y, 0) = \sum_{i,j=1}^{k} \lambda_{ij} y_i x_j$, $\lambda_{ij} = \partial_{x_j} h_i(0)$, follows by [33, Proof of Lemma 1.10], as it was indicated in the beginning of the proof of the lemma. □

It follows from the above that in the presence of negative weights, the pair $(f, X = h^{-1}(0))$ is reduced to a weighted homogeneous pair necessarily of the same degree $d = 1$

$$f(x, y, z) = \sum_{i=1}^{k} x_i y_i + f_0(z)$$
$$h(x, y, z) = \sum_{i=1}^{k} y_i h_i(x, y, z) + h_0(z)$$

where $h_i \in \langle x \rangle$, $i = 1, \ldots, k$, and the 2-jet $j^2 h_i(x, y, 0) = \sum_{j=1}^{k} \lambda_{ij} y_i x_j$ is a non-degenerate quadratic form. We can now readily verify that $\mu_X(f) = \infty$, which will finish the proof of the claim.

Indeed, this follows by direct computation of the zero locus $V(df(\Theta_X))$ of the relative Jacobian ideal $df(\Theta_X)$ which, by Remark 2.4 and the equality $\mu_X(f) = \tau_X(f)$, becomes

$$df(\Theta_X) = \langle f \rangle + J(f, h)$$

where $J(f, h)$ is the ideal generated by the 2×2 minors of the Jacobian matrix of the icis (f, h). A simple calculation shows now that $V(df(\Theta_X)) = V(\langle f \rangle + J(f, h))$ contains the set

$$\{y = z = x_i h_j(x, 0, 0) - x_j h_i(x, 0, 0) = 0, \quad 1 \leqslant i < j \leqslant k\} \subseteq V(df(\Theta_X))$$

which is a determinantal variety defining a curve in the $y = z = 0$ space. This proves the claim. □

(2.3) Sub-case $f \in \mathbf{m} \setminus \mathbf{m}^2$. This is analogous (dual) to the case (1) presented in the beginning of the proof of the theorem, after interchanging the roles of f and h. Briefly, since f is a regular function, we can choose coordinates such that

$$f(x, y) = x,$$

is weighted homogeneous for any weight system. In these coordinates $Y = f^{-1}(0) = \{x = 0\}$ and we can write the function h defining $X = h^{-1}(0)$ as

$$h(x, y) = xh_1(x, y) + h_0(y),$$

where $\mu_Y(h) < \infty$. Since $h \in \mathbf{m}^2$, this implies that both h and $h_0 = h|_Y = h|_{\{x=0\}}$ have an isolated singularity at the origin. By Corollary 2.6, condition $\mu_X(f) = \tau_X(f)$ is equivalent to $\mu_Y(h) = \tau_Y(h)$, which is in turn equivalent to $\mu(h) = \tau(h)$ and $\mu(h_0) = \tau(h_0)$. These all imply that there exists a logarithmic vector field $\delta \in \Theta_Y$ and a unit $u \in \mathcal{O}_n$, $u(0) \neq 0$, such that

$$\delta(x) = x, \quad \delta(h) = uh \text{ and } \delta|_{\{x=0\}}(h_0) = u|_{\{x=0\}} h_0.$$

Let $\mathrm{sp}(\delta) = w = (1, w') \in \{1\} \times \mathbb{C}^{n-1}$ denote the eigenvalues of the linear part of the vector field δ, where $w' = (w_2, \ldots, w_n) \in \mathbb{C}^{n-1}$ are the eigenvalues of the linear part of its restriction $\delta|_Y = \delta_{\{x=0\}}$ on $Y = f^{-1}(0) = \{x = 0\}$. After passing to Poincaré-Dulac coordinates (notice again that this can easily be achieved by formal diffeomorphisms preserving the function $f(x, y) = x$), we may suppose by Theorem 3.1 (b)-(e), that the eigenvalues w' are in fact rational numbers $w' \in \mathbb{Q}^{n-1}$, and that the pair

$$f(x, y) = x, \quad X = h^{-1}(0) = \{xh_1(x, y) + h_0(y) = 0\} \tag{15}$$

is already weighted homogeneous with respect to the weights $w = (1, w')$, of corresponding degrees $\deg_w f = \deg_w x = 1$ and $\deg_w h = \deg_w h_0 = d \in \mathbb{Q}$, respectively, where $d \neq 0$.

After dividing all the weights by d and denoting by $\widetilde{w} = (w_1, \widetilde{w}') = (w_1 = \frac{1}{d}, \frac{w'}{d}) \in \mathbb{Q}^n$ the new vector of weights, we obtain that the pair (f, X)=(15) above remains weighted homogeneous, but now of corresponding degrees $\deg_{\widetilde{w}} f = w_1 \in \mathbb{Q}$ and $\deg_{\widetilde{w}} h = 1$, respectively. Now we can show in exactly the same way as in case (1) that the following result is true.

Lemma 4.7 *Let $(f(x, y) = x, X = h^{-1}(0))$ be a weighted homogeneous pair as in (15) above, of corresponding weighted degrees $\deg_{\widetilde{w}} f = w_1 \in \mathbb{Q}$ and $\deg_{\widetilde{w}} h = 1$, respectively. Since the function $h \in \mathbf{m}^2$ has an isolated singularity at the origin, the weighted degree $w_1 = \deg_{\widetilde{w}} x \in \mathbb{Q}$ of the variable x is always positive, and in particular it lies within the open unit interval $w_1 \in (0, 1) \cap \mathbb{Q}$.*

Proof The proof is completely the same with the proof of Lemma 4.4 in case (1) with the roles of f and h interchanged, and we do not need to repeat it here. For clarity, we remark that the changes of coordinates used there (obtained by applying now Saito's lemma [33, Lemma 1.10] to h in place of f) involve only changes in the y-variables, and thus they trivially preserve both the coordinate function $f(x, y) = x$ and its defining hypersurface $Y = \{x = 0\}$. □

It remains to show now that the rest of the weights $\widetilde{w}' = (w'_2, \dots, w'_n) \in \mathbb{Q}^{n-1}$ can be simultaneously chosen positive, and in particular within the open unit interval $\widetilde{w}' \in ((0, 1) \cap \mathbb{Q})^{n-1}$. But this follows again in exactly the same way as in case (1), after interchanging the roles of f and h. Briefly, suppose on the contrary that there exist r non-positive weights, say $w'_{2i} \leqslant 0$ (corresponding to the variables y_{2i}, $i = 1, \dots, r$). Then, by exactly the same arguments as in case (1), we can apply Saito's lemma [33, Lemma 1.10] to the function h, which provides a weighted homogeneous change of coordinates in the y-variables only (and thus trivially preserving the coordinate function $f(x, y) = x$), sending h to a quasihomogeneous polynomial

$$h(x, y) = \sum_{i=1}^{r} y_{2i} y_{2i+1} + \widetilde{h}(x, y_{2r+2}, \dots, y_n),$$

where $\widetilde{h}$ is a quasihomogeneous polynomial with respect to the weights $(w_1, w'_{2r+2}, \dots, w'_n) \in ((0, 1) \cap \mathbb{Q})^{n-2r}$, which has an isolated singularity at the origin (when viewed as a function of the $(x, y_{2r+2}, \dots, y_n)$-variables). Arguing again as in the proof of case (1), we can decompose further the quasihomogeneous polynomial $\widetilde{h}$ as $\widetilde{h}(x, y_{2r+2}, \dots, y_n) = x\widetilde{h}_1(x, y_{2r+2}, \dots, y_n) + \widetilde{h}_0(y_{2r+2}, \dots, y_n)$ for some appropriate quasihomogeneous polynomial $\widetilde{h}_1$ of degree $\deg_{\widetilde{w}} \widetilde{h}_1 = 1 - w_1$, and a quasihomogeneous polynomial $\widetilde{h}_0 = \widetilde{h}|_{\{x=0\}}$, which also has an isolated singularity at the origin (when viewed as a function of the $(y_{2r+2}, \dots, y_n)$-variables). We conclude that the pair (f, X)=(15) has been formally reduced to a relatively quasihomogeneous pair

$$f(x, y) = x, \quad X = h^{-1}(0) = \{x\widetilde{h}_1(x, y_{2r+2}, \dots, y_n) + \sum_{i=1}^{r} y_{2i} y_{2i+1} + \widetilde{h}(x, y_{2r+2}, \dots, y_n) = 0\}$$

with respect to the weights $\widetilde{w} \in ((0, 1) \cap \mathbb{Q})^n$. This finishes the proof of the sub-case (2.3), and thus of the case (2) as well.

Thus we have proved that in all possible cases (1)-(2) treated above, there exists a formal change of coordinates $y = \Phi(x)$ sending the pair (f, X) (in fact $(f, \langle h \rangle)$) to a relatively quasihomogeneous (and thus polynomial) pair, say $(\widetilde{f}, \widetilde{X})$ (in fact $(\widetilde{f}, \langle\widetilde{h}\rangle)$), with respect to positive rational weights $w \in ((0,1] \cap \mathbb{Q})^n$. Since polynomial functions are obviously analytic, this means that the system of analytic equations

$$\widetilde{f}(y) = f(x),$$
$$\widetilde{h}(y) = u(x)h(x),$$

where $u(0) \neq 0$ is a unit, always admits a formal solution $y = \Phi(x)$. By Artin's approximation theorem [6], the system admits an analytic solution as well, and this finishes the proof of the theorem. □

Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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