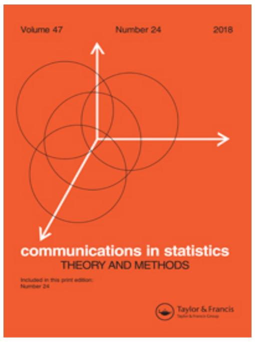

Taylor & Francis
Taylor & Francis Group

Communications in Statistics - Theory and Methods

ISSN: 0361-0926 (Print) 1532-415X (Online) Journal homepage: https://www.tandfonline.com/loi/lsta20

Improved score tests for exponential family nonlinear models

Alexsandro B. Cavalcanti, Denise A. Botter, Lúcia P. Barroso, Manoel Santos-Neto & Gauss M. Cordeiro

COMMUNICATIONS IN STATISTICS—THEORY AND METHODS
https://doi.org/10.1080/03610926.2019.1710202

Improved score tests for exponential family nonlinear models

Alexsandro B. Cavalcanti[a], Denise A. Botter[b], Lúcia P. Barroso[b], Manoel Santos-Neto[a], and Gauss M. Cordeiro[c]

[a]Departamento de Estatística, Universidade Federal de Campina Grande, Campina Grande, Brazil; [b]Departamento de Estatística, Universidade de São Paulo, São Paulo, Brazil; [c]Departamento de Estatística, Universidade Federal de Pernambuco, Recife, Brazil

ABSTRACT

This paper focuses on the corrections to the score test statistic under the exponential family nonlinear model. We use Monte Carlo simulations to compare the corrected statistics and their uncorrected versions and to examine the impact of the number of nuisance parameters on their finite-sample behaviors for the normal nonlinear regression model. The numerical results have shown that the corrected score statistic performs better than the uncorrected version. Finally, we perform a statistical analysis with real data by using the approach proposed in the article.

ARTICLE HISTORY
Received 20 June 2019
Accepted 17 December 2019

KEYWORDS
Bartlett-type correction; exponential family; Monte Carlo simulation; nonlinear model; score test

1. Introduction

Rao's score statistics (S_R) (Rao 1948; Sen 1997) are usually used in econometrics and statistics when we want to test a wide range of hypotheses in many different classes of models. In small and moderate samples size, the approximation of the true, commonly unknown, distributions of these statistics by their χ^2 asymptotic distribution may be poor. Bartlett (1937) and Bartlett-type (Cordeiro and Ferrari 1991) corrections adopt second-order likelihood asymptotic results to obtain refinements for statistics that follow more closely a χ^2 distribution under the null hyphotesis.

Bartlett-type corrections to score tests in generalized linear models for the cases of known and unknown dispersion were determined by Cordeiro, Ferrari, and Paula (1993) and Cribari-Neto and Ferrari (1995b), respectively. Similar corrections for score tests in multivariate regression models were obtained by Cribari-Neto and Zarkos (1995). Bartlett-type corrections to score tests under heteroskedasticity were determined by Cribari-Neto and Ferrari (1995a). Cordeiro and Ferrari (1996) derived improved score statistics in proper dispersion models. Botter and Cordeiro (1998) obtained corrections for the likelihood ratio statistics in generalized linear models with dispersion covariates. Other references are Cordeiro (1983, 1987), Ferrari and Uribe-Opazo (2001), Zucker, Lieberman, and Manor (2002), Uribe-Opazo, Ferrari, and Cordeiro (2008),

CONTACT Denise A. Botter botter@ime.usp.br Departamento de Estatística, Universidade de São Paulo, São Paulo, Brazil.

Lagos, Morettin, and Barroso (2010), Vargas, Ferrari, and Lemonte (2014), Lemonte, Cordeiro, and Moreno (2012), Lemonte, Cordeiro, and Moreno-Arenas (2016), Medeiros and Ferrari (2017), Medeiros, Ferrari, and Lemonte (2017), and Araújo, Cysneiros, and Montenegro (2017).

The main purpose of this paper is to derive Bartlett-type corrections (Cordeiro and Ferrari 1991) to improve the inference on the score statistic in exponential family non-linear models. This class extends the generalized linear models with varying dispersion as proposed by Smyth (1989). A Monte Carlo simulation study is performed to evaluate the performance of the corrected statistics and their uncorrected version.

The plan of the paper is as follows. In Section 2, we present the model of interest. In Section 3, we give the Bartlett-type corrected score statistic. In Section 4, we develop some Monte Carlo simulations to prove empirically that our corrected test outperforms the score test in finite samples. In Section 5, we show the potentiality of the new approach through an application with real data. Some concluding remarks are offered in Section 6.

2. Exponential family nonlinear models

Suppose $Y_1, \ldots, Y_n$ be independent random variables with each Y_ℓ having probability density function in the exponential family

$$\pi(y_\ell; \theta_\ell, \phi_\ell) = \exp\left\{\phi_\ell\left[y_\ell\theta_\ell - \mathrm{b}(\theta_\ell) - \mathrm{c}(y_\ell)\right] - \frac{1}{2}\mathrm{e}(y_\ell, \phi_\ell)\right\}, \quad \ell = 1, \ldots, n \tag{1}$$

where $\mathrm{b}(\cdot), \mathrm{c}(\cdot)$ and $\mathrm{e}(\cdot, \cdot)$ are known functions and θ_ℓ and $\phi_\ell > 0$ are the canonical and precision parameters, respectively. We assume that ϕ_ℓ is unknown and varies across observations. We also consider that $\phi_\ell^{-1} = \sigma^2 m_\ell$, where $m_\ell = \mathrm{m}(\mathbf{z}_\ell, \boldsymbol{\delta}) > 0$ is the ℓth element of the diagonal matrix $\mathbf{M}$ of dimension $n \times n$, $\mathbf{z}_\ell^\top$ is the ℓth row of the covariate matrix $\mathbf{Z}$ of dimension $n \times s$ used to construct the systematic component for the precision parameter, σ^2 is an unknown constant, finite and strictly positive, and $\boldsymbol{\delta}$ is a q-vector of unknown parameters.

For this class of models, the following relations hold: $\mathrm{E}(Y_\ell) = \mu_\ell = \mathrm{b}'(\theta_\ell) = d\mathrm{b}(\theta_\ell)/d\theta_\ell$ and $\mathrm{Var}(Y_\ell) = \phi_\ell^{-1}V_\ell$, where $V_\ell = d\mu_\ell/d\theta_\ell$ is called the variance function. The *exponential family nonlinear models* (EFNLMs) are defined by (1) and by the systematic component

$$\mathrm{h}(\mu_\ell) = \eta_\ell = \mathrm{f}(\mathbf{x}_\ell; \boldsymbol{\beta}) \tag{2}$$

where $\mathrm{h}(\cdot)$ is a link function strictly monotone and at least twice differentiable, $\boldsymbol{\beta}^\top = (\beta_1, \ldots, \beta_p)$, $p < n$, is a vector of unknown parameters to be estimated, $\mathrm{f}(\cdot; \cdot)$ is a possible nonlinear twice continuously differentiable function in the second argument and $\mathbf{x}_\ell = (x_{\ell 1}, \ldots, x_{\ell t})^\top$ is a vector that contains the values of t explanatory variables. Moreover, we suppose identifiability, i.e., different β's imply different η's, where $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_n)^\top$. Consequently, the derivative matrix $\mathbf{X}^* = \mathbf{X}^*(\boldsymbol{\beta}) = \partial\boldsymbol{\eta}/\partial\boldsymbol{\beta}^\top$ has full rank for all $\boldsymbol{\beta}$. Further, we assume valid the general regularity conditions (Cox and Hinkley 1974) for the log-likelihood function defined from (1) and (2).

Table 1. Some special models.

Model	$s(\phi_\ell)$	d_ℓ
Normal	$-\log(\phi_\ell)$	$(y_\ell-\mu_\ell)^2$
Inverse Gaussian	$-\log(\phi_\ell)$	$\frac{(y_\ell-\mu_\ell)^2}{\mu_\ell^2 y_\ell}$
Gamma	$-2[\phi_\ell\log(\phi_\ell)-\log\Gamma(\phi_\ell)]$	$2\left[\frac{y_\ell}{\mu_\ell}-\log\left(\frac{y_\ell}{\mu_\ell}\right)\right]$

If $e(y_\ell,\phi_\ell)=s(\phi_\ell)+t(y_\ell)$, we can rewrite (1) as

$$\pi(y_\ell;\theta_\ell,\phi_\ell)=\exp\{-\frac{1}{2}\left[\phi_\ell d(y_\ell,\theta_\ell)+s(\phi_\ell)+t(y_\ell)\right]\},\quad \ell=1,\dots,n \tag{3}$$

where $d_\ell=d(y_\ell,\theta_\ell)=-2[y_\ell\theta_\ell-b(\theta_\ell)-c(y_\ell)]$ corresponds to the natural exponential family with canonical parameters ϕ_ℓ and $\phi_\ell\theta_\ell$ usually called deviance. We assume that the function $s(\phi_\ell)$ has the first four derivatives. Table 1 gives the functions $s(\phi_\ell)$ and d_ℓ for the normal, inverse Gaussian and gamma distributions.

The log-likelihood function for the model defined in (3) given the vector of observations $(y_1,\dots,y_n)^\top$ is

$$\mathcal{L}(\boldsymbol{\beta},\boldsymbol{\delta},\sigma^2)=-\frac{1}{2}\sum_{\ell=1}^{n}\left[\frac{1}{\sigma^2 m_\ell}d_\ell+t(y_\ell)+s(\phi_\ell)\right],\quad \ell=1,\dots,n$$

For this parametrization, $(\boldsymbol{\delta}^\top,\sigma^2)^\top$ is orthogonal to $\boldsymbol{\beta}$, but $\boldsymbol{\delta}$ and σ^2 are not orthogonal. The orthogonal parametrization for the normal and inverse Gaussian models (Simonoff and Tsai 1994) takes the form

$$\sigma^2=\frac{\gamma}{\left(\prod_{\ell=1}^{n}m_\ell\right)^{1/n}}$$

The log-likelihood function for the reparameterized model can be expressed as

$$\mathcal{L}(\boldsymbol{\beta},\boldsymbol{\delta},\gamma)=-\frac{1}{2}\sum_{\ell=1}^{n}\left[\frac{q_\ell}{\gamma}d_\ell+t(y_\ell)+\log\left(\frac{\gamma}{q_\ell}\right)\right],\quad \ell=1,\dots,n \tag{4}$$

where $q_\ell=q_\ell(\boldsymbol{\delta})=\left(\prod_{s=1}^{n}m_s\right)^{1/n}/m_\ell$. However, for the gamma model, it is very complicated to find an orthogonal reparametrization. For the reparameterized model, the components of the score vector $\mathbf{U}=\mathbf{U}(\boldsymbol{\beta},\boldsymbol{\delta},\gamma)=(\mathbf{U}_{\boldsymbol{\beta}}^\top,\mathbf{U}_{\boldsymbol{\delta}}^\top,U_\gamma)^\top$ are

$$\mathbf{U}_{\boldsymbol{\beta}}=\tilde{\mathbf{X}}^\top\ \boldsymbol{\Phi}\ \mathbf{T}\ \mathbf{V}^{-1}\ (\mathbf{y}-\boldsymbol{\mu}),\quad \mathbf{U}_{\boldsymbol{\delta}}=-\frac{1}{2\gamma}\dot{\mathbf{Q}}^\top\mathbf{d}\quad\text{and}\quad U_\gamma=\frac{1^\top}{2\gamma}(\boldsymbol{\Phi}\mathbf{d}-1) \tag{5}$$

where $\boldsymbol{\Phi}$, $\mathbf{T}$ and $\mathbf{V}$ are $n\times n$ diagonal matrices with elements $\phi_\ell=q_\ell/\gamma, T_\ell=d\mu_\ell/d\eta_\ell$ and $V_\ell=d\mu_\ell/d\theta_\ell$, respectively, $\dot{\mathbf{Q}}$ is a $n\times q$ matrix with the ℓth row $\partial q_\ell/\partial\boldsymbol{\delta}^\top,\ell=1,\dots,n,\mathbf{d}=(d_1,\dots,d_n)^\top,\boldsymbol{\mu}=(\mu_1,\dots,\mu_n)^\top$ and $\mathbf{1}^\top$ is an n-vector of ones.

It can be readily shown that: 1) $E(d_\ell)=\gamma/q_\ell$; 2) $\mathrm{Var}(d_\ell)=2\gamma^2/q_\ell^2$; and 3) $E(Y_\ell d_\ell)=\mu_\ell(\gamma/q_\ell)$. These quantities are useful for calculating some cumulants of log-likelihood derivatives. If $m_\ell=\exp(\mathbf{z}_\ell^\top\boldsymbol{\delta})$, then $q_\ell=\exp\{-(\mathbf{z}_\ell-\bar{\mathbf{z}})^\top\boldsymbol{\delta}\}$ and the total Fisher information matrix can be partioned as

$$\mathbf{K} = -\mathrm{E}\left(\frac{\partial^2 \mathcal{L}}{\partial \boldsymbol{\psi} \partial \boldsymbol{\psi}^\top}\right) = \begin{pmatrix} \mathbf{X}^{*\top}\mathbf{W}\mathbf{\Phi}\mathbf{X}^* & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}(\mathbf{Z}-\bar{\mathbf{Z}})(\mathbf{Z}-\bar{\mathbf{Z}})^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{n}{2\gamma^2} \end{pmatrix} \tag{6}$$

where $\boldsymbol{\psi} = (\boldsymbol{\beta}^\top, \boldsymbol{\delta}^\top, \gamma)^\top$ is a vector of unknown parameters and $\mathbf{0}$ are matrices of zeros with appropriate dimensions.

3. Improved score tests

Our aim is to present a Bartlett-type correction to score statistics in EFNLMs. Suppose that we partition the $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$ vectors, respectively, as $(\boldsymbol{\beta}_1^\top, \boldsymbol{\beta}_2^\top)^\top$ and $(\boldsymbol{\delta}_1^\top, \boldsymbol{\delta}_2^\top)^\top$, where $\boldsymbol{\beta}_1^\top = (\beta_1, \ldots, \beta_{p_1})^\top (p_1 \leq p), \boldsymbol{\beta}_2^\top = (\beta_{p_{1+1}}, \ldots, \beta_p)^\top, \boldsymbol{\delta}_1^\top = (\delta_1, \ldots, \delta_{q_1})^\top (q_1 \leq q)$ and $\boldsymbol{\delta}_2^\top = (\delta_{q_{1+1}}, \ldots, \delta_q)^\top$, thus inducing a partition of the covariates matrices, respectively, as $\mathbf{X}^* = [\mathbf{X}_1^* \ \mathbf{X}_2^*]$ and $\mathbf{Z} = [\mathbf{Z}_1 \ \mathbf{Z}_2]$. We want to test the null hypothesis $\mathcal{H}_0 : \boldsymbol{\beta}_1 = \boldsymbol{\beta}_1^{(0)}, \boldsymbol{\delta}_1 = \boldsymbol{\delta}_1^{(0)}$ versus $\mathcal{H}_1 : \boldsymbol{\beta}_1 \neq \boldsymbol{\beta}_1^{(0)}$ or $\boldsymbol{\delta}_1 \neq \boldsymbol{\delta}_1^{(0)}$, where $\boldsymbol{\beta}_1^{(0)}$ and $\boldsymbol{\delta}_1^{(0)}$ are vectors of known constants of dimensions p_1 and q_1, respectively. The score statistic for testing $\mathcal{H}_0$ is given by $S_R = \tilde{\mathbf{U}}^\top \tilde{\mathbf{K}}^{-1} \tilde{\mathbf{U}}$, where $\mathbf{U}$ is defined in (5) and $\mathbf{K}$ in (6).

Following Cordeiro and Ferrari (1991), the Bartlett-type correction to S_R has the form

$$S_R^* = S_R\{1 - (c + bS_R + aS_R^2)\} \tag{7}$$

where the coefficients a, b, and c are

$$a = \frac{A_3}{12\nu(\nu+2)(\nu+4)}, \quad b = \frac{(A_2 - 2A_3)}{12\nu(\nu+2)}, \quad c = \frac{(A_1 - A_2 + A_3)}{12\nu} \tag{8}$$

and ν is the difference between the dimensions of the parameter spaces under the alternative and null hypotheses. The coefficients A_1, A_2 and A_3 are functions of some cumulants of log-likelihood derivatives. The general expressions for these coefficients can be found in Harris (1985) and Ferrari, Cysneiros, and Cribari-Neto (2004). In Appendix A we present some cumulants that are not usually found in the literature but are important for the calculation of the A's.

Consider the matrices $\mathbf{Z}_\beta = \mathbf{X}^*(\mathbf{X}^{*\top}\mathbf{W}\mathbf{\Phi}\mathbf{X}^*)^{-1}\mathbf{X}^{*\top}$, $\mathbf{Z}_{\beta 2} = \mathbf{X}_2^*(\mathbf{X}_2^{*\top}\mathbf{W}\mathbf{\Phi}\mathbf{X}_2^*)^{-1}\mathbf{X}_2^{*\top}$, (for $q \leq p$) and $\mathbf{P}_2 = \mathrm{diag}\{p_1, \ldots, p_n\}$ where $p_\ell = \mathrm{tr}\{\mathbf{X}_2^*(\mathbf{X}_2^{*\top}\mathbf{W}\mathbf{\Phi}\mathbf{X}_2^*)^{-1}\mathbf{D}_{22}^{(\ell)}\}$, and

$$\mathbf{D}^{(\ell)} = \left\{\frac{\partial^2 \eta_\ell}{\partial \beta_i \partial \beta_j}\right\} = \begin{pmatrix} \mathbf{D}_{11}^{(\ell)} & \mathbf{D}_{12}^{(\ell)} \\ \mathbf{D}_{21}^{(\ell)} & \mathbf{D}_{22}^{(\ell)} \end{pmatrix}$$

Further, we define $C_2 = \{c_{\ell m}\}$ and $J_2 = \{j_{\ell m}\}$, where

$$c_{\ell m} = \mathbf{x}_{2m}^*(\mathbf{X}_2^{*\top}\mathbf{W}\mathbf{\Phi}\mathbf{X}_2^*)^{-1}\mathbf{D}_{22}^{(\ell)}(\mathbf{X}_2^{*\top}\mathbf{W}\mathbf{\Phi}\mathbf{X}_2^*)^{-1}\mathbf{x}_{2m}^{*\top}$$

$$j_{\ell m} = \mathrm{tr}\{\mathbf{D}_{22}^{(\ell)}(\mathbf{X}_2^{*\top}\mathbf{W}\mathbf{\Phi}\mathbf{X}_2^*)^{-1}\mathbf{D}_{22}^{(m)}(\mathbf{X}_2^{*\top}\mathbf{W}\mathbf{\Phi}\mathbf{X}_2^*)^{-1}\}$$

and $\mathbf{x}_{2m}^*$ denotes the mth row of $\mathbf{X}_2^*$. Let $\mathbf{F} = \text{diag}\{f_1, ..., f_n\}, \mathbf{G} = \text{diag}\{g_1, ..., g_n\}, \mathbf{B} = \text{diag}\{b_1, ..., b_n\}$ and $\mathbf{H} = \text{diag}\{h_1, ..., h_n\}$ be diagonal matrices with elements f_ℓ, g_ℓ, b_ℓ and h_ℓ, respectively, given by

$$f_\ell = V_\ell^{-1}\left(\frac{d\mu_\ell}{d\eta_\ell}\right)\frac{d^2\mu_\ell}{d\eta_\ell^2}, \quad g_\ell = V_\ell^{-1}\left(\frac{d\mu_\ell}{d\eta_\ell}\right)\frac{d^2\mu_\ell}{d\eta_\ell^2} - V_\ell^{-2}\left(\frac{d\mu_\ell}{d\eta_\ell}\right)^3\frac{dV_\ell}{d\mu_\ell},$$
$$b_\ell = V_\ell^{-3}\left(\frac{d\mu_\ell}{d\eta_\ell}\right)^4\left\{\left(\frac{dV_\ell}{d\mu_\ell}\right)^2 + V_\ell\frac{d^2V_\ell}{d\mu_\ell^2}\right\} \tag{9}$$

and

$$h_\ell = V_\ell^{-2}\frac{dV_\ell}{d\mu_\ell}\left(\frac{d\mu_\ell}{d\eta_\ell}\right)^2\frac{d^2\mu_\ell}{d\eta_\ell^2} + V_\ell^{-2}\frac{d^2V_\ell}{d\mu_\ell^2}\left(\frac{d\mu_\ell}{d\eta_\ell}\right)^4 \tag{10}$$

Since $m_\ell = \exp(\mathbf{z}_\ell^\top\boldsymbol{\delta})$, we obtain $q_\ell = \exp\{-(\mathbf{z}_\ell - \bar{\mathbf{z}})^\top\boldsymbol{\delta}\}$. We define the following matrices: $\mathbf{Z}_\delta = 2(\mathbf{Z} - \bar{\mathbf{Z}})[(\mathbf{Z} - \bar{\mathbf{Z}})^\top(\mathbf{Z} - \bar{\mathbf{Z}})]^{-1}(\mathbf{Z} - \bar{\mathbf{Z}})^\top$ and $\mathbf{Z}_{\delta 2} = 2(\mathbf{Z}_2 - \bar{\mathbf{Z}}_2)[(\mathbf{Z}_2 - \bar{\mathbf{Z}}_2)^\top(\mathbf{Z}_2 - \bar{\mathbf{Z}}_2)]^{-1}(\mathbf{Z}_2 - \bar{\mathbf{Z}}_2)^\top$, where $\boldsymbol{\Phi} = (1/\gamma)\mathbf{Q}$, and $\mathbf{Q} = \text{diag}\{q_1, ..., q_n\}$. Additionally, we define $\mathbf{Z}^{(2)} = \mathbf{Z} \odot \mathbf{Z}, \mathbf{Z}^{(3)} = \mathbf{Z} \odot \mathbf{Z} \odot \mathbf{Z}$, where $\odot$ denotes the Hadamard product. The subscript d indicates that the diagonal matrix is defined from the original matrix. Therefore, we can rewrite the A's in the following forms

$$A_1 = A_{11} + A_{12} + A_{13} + A_{14}, \quad A_2 = A_{21} + A_{22} + A_{23} + A_{24} \quad \text{and} \quad A_3 = A_{31} + A_{32}$$

where all terms are defined in Appendix B.1. Now, suppose that we want to test $\mathcal{H}_0 : \boldsymbol{\delta}_1 = \boldsymbol{\delta}_1^{(0)}$ versus $\mathcal{H}_1 : \boldsymbol{\delta}_1 \neq \boldsymbol{\delta}_1^{(0)}$. Then, $\mathbf{Z}_\beta = \mathbf{Z}_{\beta 2}$ and the coefficients are presented in Appendix B.2. Finally, if we are interested in testing $\mathcal{H}_0 : \boldsymbol{\beta}_1 = \boldsymbol{\beta}_1^{(0)}$ versus $\mathcal{H}_1 : \boldsymbol{\beta}_1 \neq \boldsymbol{\beta}_1^{(0)}$, we can consider $\mathbf{Z}_\delta = \mathbf{Z}_{\delta 2}$ and then the coefficients are defined in Appendix B.3.

4. Simulations study

In this section, we present Monte Carlo simulations to compare the performance of the score test (S_R) and its corrected version (S_R^*) in heteroscedastic exponential family nonlinear models. The simulations are based on the nonlinear regression model

$$y_\ell = \beta_1 + \exp(\beta_2 x_{\ell 2}) + \sum_{i=3}^{p}\beta_i x_{\ell i} + u_\ell, \quad \ell = 1, ..., n$$

where $u_\ell \sim N(0, \sigma^2 \exp\{\delta_1 z_{\ell 1} + \cdots + \delta_q z_{\ell q}\})$ and $\text{Cov}(u_\ell, u_m) = 0, \forall \ell \neq m$. We consider the following null hypotheses: $\mathcal{H}_0^1 : \delta_1 = \delta_2 = 0$, for $p = 2, 3, 4, 5$ and $\mathcal{H}_0^2 : \beta_1 = \beta_2 = 1$, for $q = 1, 2, 4, 5$. The response variable was generated assuming: (i) $\beta_1 = \beta_2 = \cdots = \beta_p = 1, \sigma^2 = 1$ and $\delta_1 = \delta_2 = \cdots = \delta_q = 0$ (under $\mathcal{H}_0^1$) and (ii) $\beta_1 = \beta_2 = \cdots = \beta_p = 1, \sigma^2 = 1$ and $\delta_1 = \delta_2 = \cdots = \delta_q = 1$ (under $\mathcal{H}_0^2$). The explanatory variables were generated from the uniform distribution on (0, 1) and the matrix $\mathbf{Z}$ was obtained using columns $2, ..., q$ of the matrix $\mathbf{X}$. The observable sample size n was taken equal to 10, 20, 30, 40 and 50. Under each scenario, 10,000 replications were performed and the following nominal levels (α) were considered 1%, 5% and 10%. The simulations were carried

Table 2. Null rejection rates for $\mathcal{H}_0^1$ with $p = 2, 3, 4, 5$ and $r = 2$.

		$p = 2$		$p = 3$		$p = 4$		$p = 5$	
n	α	$\Pr[S_R \geq c]$	$\Pr[S_R^* \geq c]$	$\Pr[S_R \geq c]$	$\Pr[S_R^* \geq c]$	$\Pr[S_R \geq c]$	$\Pr[S_R^* \geq c]$	$\Pr[S_R \geq c]$	$\Pr[S_R^* \geq c]$
	10%	6.11	11.15	5.39	11.08	4.30	12.23	4.49	12.87
10	5%	2.79	5.88	1.89	5.94	1.24	6.99	1.21	7.62
	1%	0.30	0.44	0.04	1.03	0.01	1.70	0.01	2.03
	10	7.96	10.33	8.37	10.48	5.96	11.06	8.66	11.30
20	5	3.42	5.32	3.73	5.39	3.94	5.52	3.81	5.89
	1	0.60	0.78	0.57	1.04	0.56	1.32	0.54	1.45
	10	8.17	10.13	8.11	10.25	8.75	10.44	7.87	10.62
30	5	4.26	5.19	4.06	5.15	4.17	5.12	3.79	5.38
	1	0.85	1.08	0.81	0.97	0.75	1.00	0.58	1.20
	10	8.61	10.07	8.40	10.16	8.96	10.34	8.32	10.37
40	5	4.31	5.06	4.22	5.17	3.75	5.06	4.03	5.17
	1	0.85	0.96	0.90	1.05	0.70	1.11	0.82	1.15
	10%	8.92	10.04	8.79	10.04	8.49	10.07	9.12	10.14
50	5%	4.43	5.09	4.36	5.06	4.32	5.03	4.35	5.07
	1%	1.02	0.91	0.89	0.98	0.95	1.08	1.07	1.04

out using the Ox matrix programing language (Doornik 2007). We shall present the null rejection rates of S_R and S_R^* for the tests of the null hypotheses $\mathcal{H}_0^1$ and $\mathcal{H}_0^2$. Thus, we vary p and q to analyze the effect of the number of nuisance parameters on the different tests under the null hypotheses $\mathcal{H}_0^1$ and $\mathcal{H}_0^2$, respectively.

Table 2 presents results for the null hypothesis $\mathcal{H}_0^1$ with $p = 2, 3, 4, 5$ and $r = 2$ (number of parameters tested). Entries are given as percentages. Note that the original score test (S_R) is conservative for all n. Compared to the S_R test, the impact of the number of nuisance parameters is much less important than for the S_R^* test. For $n < 30$, the S_R^* test is markedly more liberal as the number of parameters increases. It is noticeable that, for $n \geq 30$, the rejection rates seem not to be affected by increasing the number of parameters. Also, the corrected score statistic gives rejection rates that are closer to the nominal values than the score statistic.

The figures in Table 3 provide the rejection rates for $\mathcal{H}_0^2$ with $q = 1, 2, 4, 5$ and $r = 2$. As can be seen, for $n \geq 30$, the rejection rates are not impacted when the number of nuisance parameters increases. As before, the corrected score test is markedly liberal. On the other hand, the original score test is conservative, mainly for small sample sizes. Finally, the corrected version gives rejection rates that are closer to the nominal levels than the uncorrected version.

5. Application

The constant-elasticity-of-substitution (CES) function (Arrow et al. 1961) is a generalization of the Cobb-Douglas function. The CES production function with two inputs (L and K are the input quantities) and an output quantity (y) is given by

$$y = \alpha[\delta L^{-\rho} + (1-\delta)K^{-\rho}]^{-\frac{\tau}{\rho}} \exp(\epsilon) \tag{11}$$

where $\alpha > 0$ is the parameter that determines the productivity, $0 < \delta < 1$ is the parameter that defines the optimal distribution of the inputs, $\tau > 0$ is a parameter that measures the elasticity, $\rho > -1$ determines the elasticity of substituion and ϵ is a random component.

Table 3. Null rejection rates for $\mathcal{H}_0^2$ with $q = 1, 2, 4, 5$ and $r = 2$.

		$q = 1$		$q = 2$		$q = 4$		$q = 5$	
n	α	$\Pr[S_R \geq c]$	$\Pr[S_R^* \geq c]$	$\Pr[S_R \geq c]$	$\Pr[S_R^* \geq c]$	$\Pr[S_R \geq c]$	$\Pr[S_R^* \geq c]$	$\Pr[S_R \geq c]$	$\Pr[S_R^* \geq c]$
	10%	6.84	10.83	5.99	11.15	5.99	12.07	4.60	13.86
10	5%	1.76	4.77	2.98	5.18	1.35	6.09	1.26	7.09
	1%	0.00	0.50	0.10	1.43	0.00	0.77	0.00	1.13
	10	9.01	10.09	7.52	10.58	8.03	11.17	6.89	11.27
20	5	3.63	5.09	2.91	4.83	3.26	5.33	2.68	5.89
	1	0.23	0.72	0.22	0.89	8.35	10.57	7.88	10.49
	10	8.82	10.33	7.31	9.88	8.35	10.57	7.88	10.49
30	5	3.71	4.89	2.86	4.62	3.40	5.49	3.28	5.56
	1	0.40	0.89	0.38	0.82	0.36	1.32	0.34	1.04
	10	9.43	10.17	8.51	10.48	9.41	10.26	9.17	10.17
40	5	4.34	5.29	4.08	5.33	4.05	5.25	4.02	5.41
	1	0.54	0.96	0.48	1.05	0.43	1.12	0.40	1.16
	10%	8.99	9.81	8.28	9.80	9.02	10.13	8.85	10.49
50	5%	4.42	5.18	3.69	4.66	4.03	5.18	3.93	5.23
	1%	0.70	1.00	0.57	0.96	0.60	1.17	0.56	1.19

Table 4. Estimates for fit to the manufacturing industry data.

Coefficients	Estimate (Std. Error)
α	1.13***
	(0.08)
δ	0.34**
	(0.13)
ρ	3.02
	(2.17)
τ	1.01***
	(0.05)
σ^2	0.06
R^2	0.97
Num. obs.	30

***$p < 0{:}001$, **$p < 0{:}01$, *$p < 0{:}05$.

For a random sample of size n and the logarithmic transformation in (11), we have

$$\log(y_\ell) = \beta - \frac{\tau}{\rho}\log\{[\delta L_\ell^{-\rho} + (1-\delta)K_\ell^{-\rho}]\} + \epsilon_\ell, \quad \ell = 1, ..., n \tag{12}$$

where $\beta = \log(\alpha)$. We illustrate our methodology with a real-world data set. The data represent thirty cross-sectional observations on firms in a manufacturing industry. For more detail about the data, see Griffiths, Hill, and Judge (1993). We use the function cesEst() of the **R** package **micEconCES** (Henningsen and Henningsen 2011) to estimate the CES function defined in (12) by non-linear least-squares via the Differential Evolution algorithm (Storn and Price 1997). The **R** codes employed in this application are available from www.santosnetoce.com.br/cstm_applications. The estimates are reported in Table 4.

Figure 1 reveals that the residual plot has a pattern which indicates a non-constant dispersion. Then, based on this figure we propose to test the statistical hypothesis of constant variance. Let us assume now that $\sigma_\ell^2 = \sigma^2\exp(K_\ell\delta_1 + L_\ell\delta_2)$ for $\ell = 1, 2, ..., 30$. Therefore, our goal is to test the null hypothesis $\mathcal{H}_0 : \delta_j = 0$ for all $j \in \{1, 2\}$ versus $\mathcal{H}_1 : \delta_j \neq 0$ for at least one j. For this, we use the original score test and the corrected score test. The figures in Table 5 show that there is statistically significant evidence that the variance is non-constant. Thus, we can note that the corrected score test increases

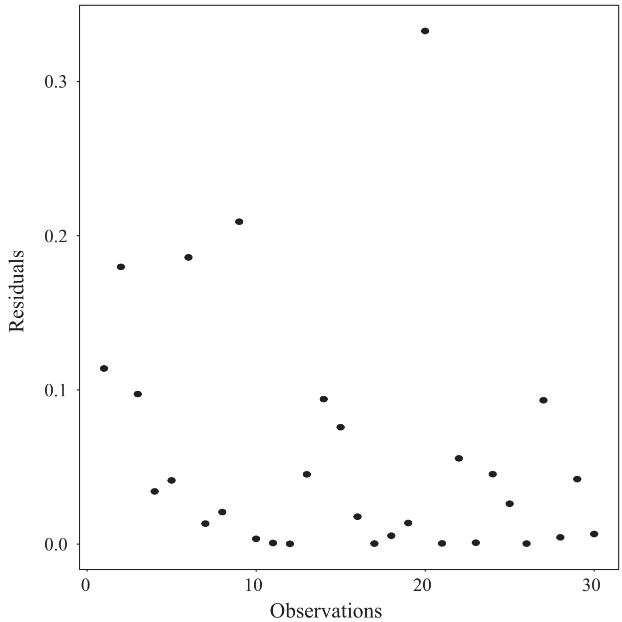

Figure 1. Plot of the squared residuals versus the index of the observations.

Table 5. Original and corrected score statistics.

Statistics	Value	p-value (%)
S_R	14.24	0.081
S_R^*	34.26	<0.001

this evidence. The p-value for the corrected score statistic is now reduced to $3.63526 \times 10^{-8} (< 0.001\%)$.

6. Concluding remarks

In this paper we present Bartlett corrections to improve hypothesis tests based on the score statistic in the exponential family nonlinear models with varying dispersion. The Bartlett-type corrections for the score statistic to test the mean and dispersion parameters have been developed for the normal and inverse Gaussian models. Numerical simulations showed that the original score test may display substantial size distortions. They also show that the corrected version displays much smaller size distortions than the original score test. Therefore, these simulations reveal that the corrected test should be preferred since it presented the best performance for testing hypotheses in this class of models. Thus, we encourage practitioners to use this test in applications.

Acknowledgments

This paper is based on part of Cavalcanti's doctoral thesis submitted to the Instituto de Matemática e Estatística, Universidade de São Paulo, under the supervision of the second author. The authors thank the Editors and referees for their constructive comments on an earlier version of this manuscript which resulted in this improved version. Also, we thank Prof. Dr. Michelli Barros (UFCG) for valuable help in simulation program.

Funding

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001 and Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) [grant number 303133/2014-7].

ORCID

Denise A. Botter http://orcid.org/0000-0002-6537-6151
Manoel Santos-Neto http://orcid.org/0000-0002-9007-0680
Gauss M. Cordeiro http://orcid.org/0000-0002-3052-6551

References

Araújo, M. C., A. H. M. Cysneiros, and L. C. Montenegro. 2017. Improved heteroskedasticity likelihood ratio tests in symmetric nonlinear regression models. *Statistical Papers*. doi:10.1007/s00362-017-0933-5.

Arrow, K. J., H. B. Chenery, B. S. Minhas, and R. M. Solow. 1961. Capital-labor substitution and economic efficiency. *The Review of Economics and Statistics* 43 (3):225. doi:10.2307/1927286.

Bartlett, M. S. 1937. Properties of sufficiency and statistical tests. *Proceedings of the Royal Society of London Series A* 160:268–82.

Botter, D. A., and G. M. Cordeiro. 1998. Improved estimates for generalized linear models with dispersion covariates. *Journal of Statistical Computation and Simulation* 62 (1-2):91–104. doi:10.1080/00949659808811926.

Cordeiro, G. M. 1983. Improved likelihood ratio statistics for generalized linear models. *Journal of the Royal Statistical Society: Series B (Methodological)* 45:404–13. doi:10.1111/j.2517-6161.1983.tb01269.x.

Cordeiro, G. M. 1987. On the corrections to the likelihood ratio statistics. *Biometrika* 74 (2):265–74. doi:10.2307/2336140.

Cordeiro, G. M., and S. L. P. Ferrari. 1991. A modified score test statistic having chi-squared distribution to order n^{-1}. *Biometrika* 78:573–82. doi:10.2307/2337026.

Cordeiro, G. M., and S. L. P. Ferrari. 1996. Bartlett-type corrections for some score tests in proper dispersion models. *Communications in Statistics - Theory and Methods* 25 (1):29–48. doi:10.1080/03610929608831678.

Cordeiro, G. M., S. L. P. Ferrari, and G. A. Paula. 1993. Improved score tests for generalized linear models. *Journal of the Royal Statistical Society: Series B (Methodological)* 55:661–74. doi:10.1111/j.2517-6161.1993.tb01931.x.

Cox, D. R., and D. V. Hinkley. 1974. *Theoretical statistics*. New York: John Wiley.

Cribari-Neto, F., and S. L. P. Ferrari. 1995a. Bartlett-corrected tests for heteroskedastic linear models. *Economics Letters* 48:113–8. doi:10.1016/0165-1765(94)00597-U.

Cribari-Neto, F., and S. L. P. Ferrari. 1995b. Second order asymptotics for score tests in generalized linear models. *Biometrika* 82 (2):426–32. doi:10.2307/2337420.

Cribari-Neto, F., and S. Zarkos. 1995. Improved test statistics for multivariate regression. *Economics Letters* 49 (2):113–20. doi:10.1016/0165-1765(95)00669-7.

Doornik, J.A. 2007. *Object-oriented matrix programming using Ox*. 3rd ed. London: Timberlake Consultants Press and Oxford.

Ferrari, S. L. P., and M. A. Uribe-Opazo. 2001. Corrected likelihood ratio tests in class of symmetric linear regression models. *Brazilian Journal of Probability and Statistics* 15:49–67.

Ferrari, S. L. P., A. H. M. A. Cysneiros, and F. Cribari-Neto. 2004. An improved test for heteroskedasticity using adjusted modified profile likelihood inference. *Journal of Statistical Planning and Inference* 124 (2):423–37. doi:10.1016/S0378-3758(03)00210-6.

Griffiths, W. E., R. C. Hill, and G. G. Judge. 1993. *Learning and practicing econometrics*. New York: John Wiley.

Harris, P. 1985. An asymptotic expansion for the null distribution of the efficiente score statistic. *Biometrika* 72 (3):653–9.
Henningsen, A., and G. Henningsen. 2011. Econometric estimation of the 'constant elasticity of substitution' function in R: Package micEconCES. FOI Working Paper 2011/9. Institute of Food and Resource Economics, University of Copenhagen.
Lagos, B. M., P. A. Morettin, and L. P. Barroso. 2010. Some corrections of the score test statistic for Gaussian ARMA models. *Brazilian Journal of Probability and Statistics* 24 (3):434–56. doi:10.1214/09-BJPS023.
Lemonte, A. J., G. M. Cordeiro, and G. Moreno. 2012. Bartlett corrections in Birnbaum-Saunders nonlinear regression models. *Journal of Statistical Computation and Simulation* 82 (6):927–35. doi:10.1080/00949655.2011.561802.
Lemonte, A. J., G. M. Cordeiro, and G. Moreno-Arenas. 2016. Improved likelihood-based inference in Birnbaum-Saunders nonlinear regression models. *Applied Mathematical Modelling* 40 (19-20):8185–200. doi:10.1016/j.apm.2016.04.007.
Medeiros, F. M. C., and S. L. P. Ferrari. 2017. Small-sample testing inference in symmetric and log-symmetric linear regression models. *Statistica Neerlandica* 71 (3):200–24. doi:10.1111/stan.12107.
Medeiros, F. M. C., S. L. P. Ferrari, and A. J. Lemonte. 2017. Improved inference in dispersion models. *Applied Mathematical Modelling* 51:317–28. doi:10.1016/j.apm.2017.06.045.
Rao, C. R. 1948. Large sample tests of statistical hypotheses concerning several parameters with applications to problems of estimation. *Mathematical Proceedings of the Cambridge Philosophical Society* 44:50–5.
Sen, P. K. 1997. Introduction to Rao (1948) large sample tests of statistical hypotheses concerning several parameters with applications to problems of estimation. In *Breakthroughs in statistics (Perspectives in Statistics)*, ed. Kotz S., N. L. Johnson. New York, NY: Springer.
Simonoff, J. S., and C. L. Tsai. 1994. Use of modified profile likelihood for improved tests of constancy of variance in regression. *Applied Statistics* 43:357–70. doi:10.2307/2986026.
Smyth, G. K. 1989. Generalized linear models with varying dispersion. *Journal of the Royal Statistical Society: Series B (Methodological)* 51:47–60. doi:10.1111/j.2517-6161.1989.tb01747.x.
Storn, R., and K. Price. 1997. Differential evolution—A simple and efficient heuristic for global optimization over continuous spaces. *Journal of Global Optimization* 11 (4):341–59.
Uribe-Opazo, M. A., S. L. P. Ferrari, and G. M. Cordeiro. 2008. Improved score tests in symmetric linear regression models. *Communications in Statistics - Theory and Methods* 37 (2):261–76. doi:10.1080/03610920701649050.
Vargas, T. M., S. L. P. Ferrari, and A. J. Lemonte. 2014. Improved likelihood inference in generalized linear models. *Computational Statistics & Data Analysis* 74:110–24. doi:10.1016/j.csda.2013.12.002.
Zucker, D. M., O. Lieberman, and O. Manor. 2002. Improved small sample inference in the mixed linear model: Bartlett correction and adjusted likelihood. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 62 (4):827–38. doi:10.1111/1467-9868.00267.

Appendix

A. Cumulants of log-likelihood derivatives in exponential family nonlinear models

Let $\mathcal{L} = \mathcal{L}(\boldsymbol{\beta}, \boldsymbol{\delta}, \gamma)$ be the log-likelihood (4). We hall use the standard notation for joint cumulants of log-likelihood derivatives, i.e., $\kappa_{rs} = \mathrm{E}(\partial^2\mathcal{L}/\partial\beta_r\partial\beta_s)$, $\kappa_{r,s} = \mathrm{E}(\partial\mathcal{L}/\partial\beta_r\ \partial\mathcal{L}/\partial\beta_s)$, $\kappa_{rst} = \mathrm{E}(\partial^3\mathcal{L}/\partial\beta_r\partial\beta_s\partial\beta_t)$, $\kappa_{r,s,t} = \mathrm{E}(\partial\mathcal{L}/\partial\beta_r\ \partial\mathcal{L}/\partial\beta_s\ \partial\mathcal{L}/\partial\beta_t)$, $\kappa_{r,st} = \mathrm{E}(\partial\mathcal{L}/\partial\beta_r\ \partial^2\mathcal{L}/\partial\beta_s\partial\beta_t)$, $\kappa_{rs}^{(t)} = \partial\kappa_{rs}/\partial\beta_t$, and so on. We obtain the following cumulants:

$$\kappa_{rs} = -\frac{1}{\gamma}\sum_{\ell=1}^{n} x^*_{\ell r} q_\ell w_\ell x^*_{\ell s}, \quad \kappa_{RS} = -\frac{1}{2}\sum_{\ell=1}^{n}\frac{q_{\ell_{RS}}}{q_\ell}, \quad \kappa_{\gamma\gamma} = -\frac{n}{2\gamma^2}, \quad \kappa_{rS} = \kappa_{r\gamma} = \kappa_{R\gamma} = 0$$

where $x^*_{\ell r} = \partial\eta_\ell/\partial\beta^r$ and $q_{\ell_{RS}} = \partial^2 q_\ell/\partial\delta^R\partial\delta^S$, for $r = 1, \ldots, p$ and $R, S = 1, \ldots, q$. The third-order cumulants are

$$\kappa_{rst} = -\frac{1}{\gamma}\sum_{\ell=1}^{n}\{q_\ell(f_\ell+2g_\ell)x^*_{\ell r}x^*_{\ell s}x^*_{\ell t} + q_\ell w_\ell(x^{**}_{\ell rs}x^*_{\ell t} + x^{**}_{\ell rt}x^*_{\ell s} + x^{**}_{\ell st}x^*_{\ell r})\}, \quad \kappa_{\gamma\gamma\gamma} = \frac{2n}{\gamma^3},$$

$$\kappa_{RST} = -\frac{1}{2}\sum_{\ell=1}^{n}\frac{q_{\ell_{RST}}}{q_\ell}, \quad \kappa_{rsT} = -\frac{1}{\gamma}\sum_{\ell=1}^{n}x^*_{\ell r}q_{\ell_T}w_\ell x^*_{\ell s}, \quad \kappa_{rs\gamma} = \frac{1}{\gamma^2}\sum_{\ell=1}^{n}x^*_{\ell r}q_\ell w_\ell x^*_{\ell s}$$

$$\kappa_{RS\gamma} = \frac{1}{2\gamma}\sum_{\ell=1}^{n}\frac{q_{\ell_{RS}}}{q_\ell}, \quad \kappa_{\gamma,\gamma\gamma} = -\frac{n}{\gamma^3}, \quad \kappa_{\gamma,\gamma,\gamma} = \frac{n}{\gamma^3}, \quad \kappa_{\gamma,RS} = -\frac{1}{2\gamma}\sum_{\ell=1}^{n}\frac{q_{\ell_{RS}}}{q_\ell}$$

$$\kappa_{R\gamma\gamma} = \kappa_{r\gamma\gamma} = \kappa_{RSt} = \kappa_{rS\gamma} = 0, \quad \kappa_{r,s,t} = \frac{1}{\gamma}\sum_{\ell=1}^{n}q_\ell(f_\ell - g_\ell)x^*_{\ell r}x^*_{\ell s}x^*_{\ell t},$$

$$\kappa_{R,S,T} = \frac{1}{2}\sum_{\ell=1}^{n}\left\{\frac{q_{\ell_{RST}}}{q_\ell} - \frac{q_{\ell_{RS}}q_{\ell_T}}{q_\ell^2} - \frac{q_{\ell_{RT}}q_{\ell_S}}{q_\ell^2} - \frac{q_{\ell_{ST}}q_{\ell_R}}{q_\ell^2}\right\}, \quad \kappa_{r,s,T} = -\frac{1}{\gamma}\sum_{\ell=1}^{n}q_{\ell_T}x^*_{\ell r}x^*_{\ell s}w_\ell,$$

$$\kappa_{r,s,\gamma} = \frac{1}{\gamma^2}\sum_{\ell=1}^{n}q_\ell x^*_{\ell r}x^*_{\ell s}w_\ell, \quad \kappa_{R,S,\gamma} = \frac{1}{\gamma}\sum_{\ell=1}^{n}\frac{q_{\ell_{RS}}}{q_\ell}, \quad \kappa_{r,S,\gamma} = \kappa_{r,\gamma,\gamma} = \kappa_{r,S,T} = \kappa_{R,\gamma,\gamma} = 0,$$

$$\kappa_{r,st} = \frac{1}{\gamma}\sum_{\ell=1}^{n}q_\ell(g_\ell x^*_{\ell r}x^*_{\ell s}x^*_{\ell t} + w_\ell x^*_{\ell_r}x^{**}_{\ell_{st}}), \quad \kappa_{R,ST} = \frac{1}{2}\sum_{\ell=1}^{n}\frac{q_{\ell_R}q_{\ell_{ST}}}{q_\ell^2},$$

$$\kappa_{r,sT} = \frac{1}{\gamma}\sum_{\ell=1}^{n}x^*_{\ell r}q_{\ell_T}w_\ell x^*_{\ell s}, \quad \kappa_{r,s\gamma} = -\frac{1}{\gamma^2}\sum_{\ell=1}^{n}q_\ell x^*_{\ell r}x^*_{\ell s}w_\ell, \quad \kappa_{R,S\gamma} = -\frac{1}{2\gamma}\sum_{\ell=1}^{n}\frac{q_{\ell_R}q_{\ell_S}}{q_\ell^2},$$

$$\kappa_{r,ST} = \kappa_{r,S\gamma} = \kappa_{r,\gamma\gamma} = \kappa_{R,st} = \kappa_{R,St} = \kappa_{R,s\gamma} = \kappa_{R,\gamma\gamma} = \kappa_{\gamma,rs} = \kappa_{\gamma,rS} = \kappa_{\gamma,\gamma r} = \kappa_{\gamma,\gamma R} = 0$$

where $x^{**}_{\ell rs} = \partial^2\eta_\ell/\partial\beta^r\partial\beta^s, q_{\ell_{RST}} = \partial^3 q_\ell/\partial\delta^R\partial\delta^S\partial\delta^T$, for $r,s,t = 1,\ldots,p$ and $R,S,T = 1\ldots,q$ and the quantities q_ℓ, f_ℓ and g_ℓ, $\ell = 1,\ldots,n$, are defined by Equations (4) and (9), respectively. Further, the fourth-order cumulants are given by

$$\begin{aligned}\kappa_{rstu} = -\frac{1}{\gamma}\sum_{\ell=1}^{n}q_\ell\Bigg\{&\left[\left(\frac{\partial^2 w}{\partial\eta^2}\right) + \left(\frac{\partial^2\mu}{\partial\eta^2}\right)\left(\frac{\partial^2\theta}{\partial\eta^2}\right) + 2\left(\frac{\partial\mu}{\partial\eta}\right)\left(\frac{\partial^3\theta}{\partial\eta^3}\right)\right]_\ell x^*_{\ell_r}x^*_{\ell_s}x^*_{\ell_t}x^*_{\ell_u}\\
&+ w_\ell(x^{***}_{\ell_{rst}}x^*_{\ell_u} + x^{***}_{\ell_{rsu}}x^*_{\ell_t} + x^{***}_{\ell_{rut}}x^*_{\ell_s} + x^{***}_{\ell_{ust}}x^*_{\ell_r} + x^{**}_{\ell_{rs}}x^{**}_{\ell_{tu}} + x^{**}_{\ell_{rt}}x^{**}_{\ell_{su}} + x^{**}_{\ell_{ru}}x^{**}_{\ell_{st}})\\
&+ (x^{**}_{\ell_{rs}}x^*_{\ell_t}x^*_{\ell_u} + x^{**}_{\ell_{ts}}x^*_{\ell_r}x^*_{\ell_u} + x^{**}_{\ell_{us}}x^*_{\ell_r}x^*_{\ell_t} + x^{**}_{\ell_{ru}}x^*_{\ell_s}x^*_{\ell_t} + x^{**}_{\ell_{tu}}x^*_{\ell_r}x^*_{\ell_s} + x^{**}_{\ell_{rt}}x^*_{\ell_s}x^*_{\ell_u})\\
&\times\left[2\left(\frac{\partial\mu}{\partial\eta}\right)\left(\frac{\partial^2\theta}{\partial\eta^2}\right) + \left(\frac{\partial\theta}{\partial\eta}\right)\left(\frac{\partial^2\mu}{\partial\eta^2}\right)\right]_\ell\Bigg\},\end{aligned}$$

$$\kappa_{RSTU} = -\frac{1}{2}\sum_{\ell=1}^{n}\frac{q_{\ell_{RSTU}}}{q_\ell}, \quad \kappa_{RS\gamma\gamma} = -\frac{1}{\gamma^2}\sum_{\ell=1}^{n}\frac{q_{\ell_{RS}}}{q_\ell} \quad \kappa_{r,s,t,u} = \frac{1}{\gamma}\sum_{\ell=1}^{n}q_\ell b_\ell x^*_{\ell_r}x^*_{\ell_s}x^*_{\ell_t}x^*_{\ell_u},$$

$$\begin{aligned}\kappa_{R,S,T,U} = \frac{1}{2}\sum_{\ell=1}^{n}\Bigg(&\frac{q_{\ell_{RSTU}}}{q_\ell} - \frac{q_{\ell_{RST}}q_{\ell_U}}{q_\ell^2} - \frac{q_{\ell_{RSU}}q_{\ell_T}}{q_\ell^2} - \frac{q_{\ell_{RTU}}q_{\ell_S}}{q_\ell^2} - \frac{q_{\ell_{STU}}q_{\ell_R}}{q_\ell^2}\\
&+ \frac{2q_{\ell_R}q_{\ell_S}q_{\ell_{TU}}}{q_\ell^3} + \frac{2q_{\ell_R}q_{\ell_T}q_{\ell_{SU}}}{q_\ell^3} + \frac{2q_{\ell_R}q_{\ell_U}q_{\ell_{ST}}}{q_\ell^3} + \frac{2q_{\ell_S}q_{\ell_T}q_{\ell_{RU}}}{q_\ell^3}\\
&+ \frac{2q_{\ell_S}q_{\ell_U}q_{\ell_{RT}}}{q_\ell^3} + \frac{2q_{\ell_T}q_{\ell_U}q_{\ell_{RS}}}{q_\ell^3} - \frac{q_{\ell_{RS}}q_{\ell_{TU}}}{q_\ell^2} - \frac{q_{\ell_{RT}}q_{\ell_{SU}}}{q_\ell^2} - \frac{q_{\ell_{RU}}q_{\ell_{ST}}}{q_\ell^2}\Bigg),\end{aligned}$$

$$\kappa_{r,s,T,U}=\frac{2}{\gamma}\sum_{\ell=1}^{n}q_\ell w_\ell x_{\ell_r}^{*}x_{\ell_s}^{*}\frac{q_{\ell_T}q_{\ell_U}}{q_\ell^2},\quad \kappa_{r,s,\gamma,\gamma}=\frac{2}{\gamma^3}\sum_{\ell=1}^{n}q_\ell w_\ell x_{\ell_r}^{*}x_{\ell_s}^{*},\quad \kappa_{R,S,\gamma,\gamma}=\frac{3}{\gamma^2}\sum_{\ell=1}^{n}\frac{q_{\ell_{RS}}}{q_\ell},$$

$$\kappa_{r,s,tu}=\frac{1}{\gamma}\sum_{\ell=1}^{n}q_\ell\{(h_\ell-b_\ell)x_{\ell_r}^{*}x_{\ell_s}^{*}x_{\ell_t}^{*}x_{\ell_u}^{*}+(f_\ell-g_\ell)x_{\ell_r}^{*}x_{\ell_s}^{*}x_{\ell_{tu}}^{**}\},\quad \kappa_{R,S,\gamma\gamma}=-\frac{2}{\gamma^2}\sum_{\ell=1}^{n}\frac{q_{\ell_{RS}}}{q_\ell},$$

$$\kappa_{R,S,TU}=-\sum_{\ell=1}^{n}\frac{q_{\ell_R}q_{\ell_S}q_{\ell_{TU}}}{q_\ell^3},\quad \kappa_{r,s,TU}=-\frac{1}{\gamma}\sum_{\ell=1}^{n}w_\ell x_{\ell_r}^{*}x_{\ell_s}^{*}q_{\ell_{TU}},\quad \kappa_{r,s,\gamma\gamma}=-\frac{2}{\gamma^3}\sum_{\ell=1}^{n}q_\ell w_\ell x_{\ell_r}^{*}x_{\ell_s}^{*}$$

where $x_{\ell rst}^{***}=\partial^3\eta_\ell/\partial\beta^r\partial\beta^s\beta^t$, $q_{\ell_{RSTU}}=\partial^4 q_\ell/\partial\delta^R\partial\delta^S\partial\delta^T\partial\delta^U$, for $r,s,t,u=1,\ldots,p$ and $R,S,T,U=1,\ldots,q$, and the quantities b_ℓ and h_ℓ defined by Equations (9) and (10), respectively.

The equation $m_\ell=\exp(z_\ell^\top\delta)$ gives $q_\ell=\exp\{-(z_\ell-\bar{z})^\top\delta\}, q_{\ell_R}=-(z_\ell-\bar{z})_R\exp\{-(z_\ell-\bar{z})^\top\delta\}=-(z_\ell-\bar{z})_R q_\ell, q_{\ell_{RS}}=(z_\ell-\bar{z})_{RS}q_\ell, q_{\ell_{RST}}=-(z_\ell-\bar{z})_{RST}q_\ell$ and $q_{\ell_{RSTU}}=(z_\ell-\bar{z})_{RSTU}q_\ell$, where $(z_\ell-\bar{z})_R=z_{\ell_R}-\bar{z}_R$, $(z_\ell-\bar{z})_{RS}=(z_{\ell_R}-\bar{z}_R)(z_{\ell_S}-\bar{z}_S)$ and so on, gives $\bar{z}=(\bar{z}_1,\ldots,\bar{z}_q)^\top$ and $\bar{z}_R=(1/n)\sum_{\ell=1}^{n}z_{\ell_R}$, for $R=1,\ldots,q$. Thus we obtain the following cumulants

$$\kappa_{RS}=-\frac{1}{2}\sum_{\ell=1}^{n}(z_\ell-\bar{z})_{RS}\quad \kappa_{RST}=\frac{1}{2}\sum_{\ell=1}^{n}(z_\ell-\bar{z})_{RST},$$

$$\kappa_{rsT}=\frac{1}{\gamma}\sum_{\ell=1}^{n}x_{\ell r}^{*}w_\ell x_{\ell s}^{*}(z_\ell-\bar{z})_T q_\ell,\quad \kappa_{RS\gamma}=\frac{1}{2\gamma}\sum_{\ell=1}^{n}(z_\ell-\bar{z})_{RS},$$

$$\kappa_{R,S,T}=\sum_{\ell=1}^{n}(z_\ell-\bar{z})_{RST},\quad \kappa_{r,s,T}=\frac{1}{\gamma}\sum_{\ell=1}^{n}(z_\ell-\bar{z})_R q_\ell x_{\ell r}^{*}x_{\ell s}^{*}w_\ell,$$

$$\kappa_{R,S,\gamma}=\frac{1}{\gamma}\sum_{\ell=1}^{n}(z_\ell-\bar{z})_{RS},\quad \kappa_{r,sT}=-\frac{1}{\gamma}\sum_{\ell=1}^{n}x_{\ell r}^{*}w_\ell x_{\ell s}^{*}(z_\ell-\bar{z})_T q_\ell,$$

$$\kappa_{R,ST}=-\frac{1}{2}\sum_{\ell=1}^{n}(z_\ell-\bar{z})_{RST},\quad \kappa_{R,S\gamma}=-\frac{1}{2\gamma}\sum_{\ell=1}^{n}(z_\ell-\bar{z})_{RS},$$

$$\kappa_{\gamma,RS}=-\frac{1}{2\gamma}\sum_{\ell=1}^{n}(z_\ell-\bar{z})_{RS},\quad \kappa_{\gamma,RS}=-\frac{1}{2\gamma}\sum_{\ell=1}^{n}(z_\ell-\bar{z})_{RS},$$

$$\kappa_{RSTU}=-\frac{1}{2}\sum_{\ell=1}^{n}(z_\ell-\bar{z})_{RSTU},\quad \kappa_{RS\gamma\gamma}=-\frac{1}{\gamma^2}\sum_{\ell=1}^{n}(z_\ell-\bar{z})_{RS},$$

$$\kappa_{R,S,T,U}=3\sum_{\ell=1}^{n}(z_\ell-\bar{z})_{RSTU},\quad \kappa_{r,s,T,U}=\frac{2}{\gamma}\sum_{\ell=1}^{n}q_\ell w_\ell x_{\ell_r}^{*}x_{\ell_s}^{*}(z_\ell-\bar{z})_{TU},$$

$$\kappa_{R,S,\gamma,\gamma}=\frac{3}{\gamma^2}\sum_{\ell=1}^{n}(z_\ell-\bar{z})_{RS},\quad \kappa_{R,S,TU}=-\sum_{\ell=1}^{n}(z_\ell-\bar{z})_{RSTU},$$

$$\kappa_{r,s,TU}=-\frac{1}{\gamma}\sum_{\ell=1}^{n}w_\ell x_{\ell_r}^{*}x_{\ell_s}^{*}(z_\ell-\bar{z})_{TU}q_\ell,\quad \kappa_{R,S,\gamma\gamma}=-\frac{2}{\gamma^2}\sum_{\ell=1}^{n}(z_\ell-\bar{z})_{RS}$$

B. Cumulants for the Bartlett correction factor

B.1. *Tests for components of β and δ*

Suppose that we want to test the null hypothesis $\mathcal{H}_0:\boldsymbol{\beta}_1=\boldsymbol{\beta}_1^{(0)},\boldsymbol{\delta}_1=\boldsymbol{\delta}_1^{(0)}$ versus $\mathcal{H}_1:\boldsymbol{\beta}_1\neq\boldsymbol{\beta}_1^{(0)}$ or $\boldsymbol{\delta}_1\neq\boldsymbol{\delta}_1^{(0)}$. Then, the coefficients for the Bartlett-type corrections can be expressed terms

$$
\begin{aligned}
A_{11} &= 3\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{F}\mathbf{Z}_{\beta 2d}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})\mathbf{Z}_{\beta 2d}\mathbf{F}\boldsymbol{\Phi}\mathbf{1} + 6\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}\mathbf{P}_2(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})\mathbf{Z}_{\beta 2d}\mathbf{F}\boldsymbol{\Phi}\mathbf{1} \\
&\quad + 3\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}\mathbf{P}_2(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})\mathbf{P}_2\mathbf{W}\boldsymbol{\Phi}\mathbf{1} + 3\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}\mathbf{Z}_{\beta 2d}(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})\mathbf{Z}_{\delta 2d}\mathbf{1} \\
&\quad + 3\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}\mathbf{Z}_{\beta 2d}(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})\mathbf{Z}_{\beta 2d}\mathbf{W}\boldsymbol{\Phi}\mathbf{1} + \frac{3}{4}\mathbf{1}^\top\mathbf{Z}_{\delta 2d}(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})\mathbf{Z}_{\delta 2d}\mathbf{1}, \\
A_{12} &= 6\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{F}\mathbf{Z}_{\beta 2d}\mathbf{Z}_{\beta 2}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d(\mathbf{F} - \mathbf{G})\boldsymbol{\Phi}\mathbf{1} + 6\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}\mathbf{P}_2\mathbf{Z}_{\beta 2}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d(\mathbf{F} - \mathbf{G})\boldsymbol{\Phi}\mathbf{1} \\
&\quad + 6\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}\mathbf{Z}_{\beta 2d}\mathbf{Z}_{\delta 2}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\mathbf{W}\boldsymbol{\Phi}\mathbf{1} + 6\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}\mathbf{Z}_{\beta 2d}\mathbf{Z}_{\delta 2}(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})_d\mathbf{1} \\
&\quad + \frac{12}{n}\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}\mathbf{Z}_{\beta 2d}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\mathbf{W}\boldsymbol{\Phi}\mathbf{1} + \frac{12}{n}\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}\mathbf{Z}_{\beta 2d}(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})_d\mathbf{1} \\
&\quad + 3\mathbf{1}^\top\mathbf{Z}_{\delta 2d}\mathbf{Z}_{\delta 2}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\mathbf{W}\boldsymbol{\Phi}\mathbf{1} + 3\mathbf{1}^\top\mathbf{Z}_{\delta 2d}\mathbf{Z}_{\delta 2}(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})_d\mathbf{1} \\
&\quad + \frac{6}{n}\mathbf{1}^\top\mathbf{Z}_{\delta 2d}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\mathbf{W}\boldsymbol{\Phi}\mathbf{1} + \frac{6}{n}\mathbf{1}^\top\mathbf{Z}_{\delta 2d}(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})_d\mathbf{1}, \\
A_{13} &= -6\mathbf{1}^\top\boldsymbol{\Phi}(2\mathbf{G} - \mathbf{F})[\mathbf{Z}_{\beta 2}^{(2)} \odot (\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})]\mathbf{F}\boldsymbol{\Phi}\mathbf{1} - 6\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}\left[(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2}) \odot \mathbf{J}_2\right]\mathbf{W}\boldsymbol{\Phi}\mathbf{1} \\
&\quad - 6\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}[(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2}) \odot \mathbf{C}_2^\top]\mathbf{F}\boldsymbol{\Phi}\mathbf{1} + \frac{9}{2}\mathbf{1}^\top[\mathbf{Z}_{\delta 2}^{(2)} \odot (\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})]\mathbf{1} \\
&\quad - 6\mathbf{1}^\top\boldsymbol{\Phi}(2\mathbf{G} - \mathbf{F})\left[(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2}) \odot \mathbf{C}_2\right]\mathbf{W}\boldsymbol{\Phi}\mathbf{1} + \frac{18}{n}\mathbf{1}^\top[\mathbf{Z}_{\delta 2} \odot (\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})]\mathbf{1} \\
&\quad + 6\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}[(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2}) \odot \mathbf{Z}_{\beta 2}^{(2)}]\mathbf{W}\boldsymbol{\Phi}\mathbf{1} + 12\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}\left[(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2}) \odot \mathbf{Z}_{\beta 2}\mathbf{Z}_{\delta 2}\right]\mathbf{W}\boldsymbol{\Phi}\mathbf{1}, \\
A_{14} &= -12\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}\mathbf{Z}_{\beta 2d}(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})_d\mathbf{1} - 6\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{H}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d Z_{\beta 2d}\mathbf{1} \\
&\quad - 12\mathbf{1}^\top(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})_d\mathbf{Z}_{\delta 2d}\mathbf{1} - 6\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\mathbf{Z}_{\delta 2d}\mathbf{1} \\
&\quad - 6\mathbf{1}^\top\boldsymbol{\Phi}(\mathbf{F} - \mathbf{G})\mathbf{P}_2(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\mathbf{1} - \frac{12}{n}\mathbf{1}^\top(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})_d\mathbf{1}, \\
A_{21} &= -3\mathbf{1}^\top\boldsymbol{\Phi}(\mathbf{F} - \mathbf{G})(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\mathbf{Z}_{\beta 2}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d(\mathbf{F} - \mathbf{G})\boldsymbol{\Phi}\mathbf{1} - \frac{6}{n}\mathbf{1}^\top\left[(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})_d\right]^2\mathbf{1} \\
&\quad - 3\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\mathbf{Z}_{\delta 2}(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})_d\mathbf{1} - 3\mathbf{1}^\top(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})_d\mathbf{Z}_{\delta 2}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\mathbf{W}\boldsymbol{\Phi}\mathbf{1} \\
&\quad - \frac{12}{n}\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})_d\mathbf{1} - 3\mathbf{1}^\top(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})_d\mathbf{Z}_{\delta 2}(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})_d\mathbf{1} \\
&\quad - 3\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\mathbf{Z}_{\delta 2}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\mathbf{1} - \frac{6}{n}\mathbf{1}^\top\left[\boldsymbol{\Phi}\mathbf{W}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\right]^2\mathbf{1}, \\
A_{22} &= -6\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{F}\mathbf{Z}_{\beta 2d}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d(\mathbf{F} - \mathbf{G})\boldsymbol{\Phi}\mathbf{1} \\
&\quad - 6\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}\mathbf{Z}_{\beta 2d}(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})(\mathbf{Z}_\beta - Z_{\beta 2})_d\mathbf{W}\boldsymbol{\Phi}\mathbf{1} + 6\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}\mathbf{Z}_{\beta 2d}(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})_d\mathbf{1} \\
&\quad - 3\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}\mathbf{Z}_{\delta 2d}(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\mathbf{1} - 3\mathbf{1}^\top\mathbf{Z}_{\delta 2d}(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})_d\mathbf{1} \\
&\quad - 6\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}\mathbf{P}_2(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d(\mathbf{F} - \mathbf{G})\boldsymbol{\Phi}\mathbf{1}, \\
A_{23} &= -6\mathbf{1}^\top\boldsymbol{\Phi}(\mathbf{F} - \mathbf{G})[\mathbf{Z}_{\beta 2} \odot (\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})^{(2)}](\mathbf{F} - \mathbf{G})\boldsymbol{\Phi}\mathbf{1} - \frac{12}{n}\mathbf{1}^\top[(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})^{(2)}]\mathbf{1} \\
&\quad - 6\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}[\mathbf{Z}_{\delta 2} \odot (\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})^{(2)}]\mathbf{W}\boldsymbol{\Phi}\mathbf{1} - 6\mathbf{1}^\top[\mathbf{Z}_{\delta 2} \odot (\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})^{(2)}]\mathbf{1} \\
&\quad - 12\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}\left[\mathbf{Z}_{\beta 2} \odot (\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2}) \odot (\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})\right]\mathbf{W}\boldsymbol{\Phi}\mathbf{1} - \frac{12}{n}\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}[(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})^{(2)}]\mathbf{W}\boldsymbol{\Phi}\mathbf{1}, \\
A_{24} &= 3\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{B}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d^2\mathbf{1} + 12\mathbf{1}^\top\boldsymbol{\Phi}\mathbf{W}(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})_d(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\mathbf{1} + 9\mathbf{1}^\top(\mathbf{Z}_\delta - \mathbf{Z}_{\delta 2})_d^2\mathbf{1},
\end{aligned}
$$

$$
\begin{aligned}
A_{31} &= 3\mathbf{1}^\top\mathbf{\Phi}(\mathbf{F}-\mathbf{G})(\mathbf{Z}_\beta-\mathbf{Z}_{\beta 2})_d(\mathbf{Z}_\beta-\mathbf{Z}_{\beta 2})(\mathbf{Z}_\beta-\mathbf{Z}_{\beta 2})_d(\mathbf{F}-\mathbf{G})\mathbf{\Phi}\mathbf{1}\\
&\quad + 3\mathbf{1}^\top(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})_d(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})_d\mathbf{1} + 6\mathbf{1}^\top\mathbf{\Phi}\mathbf{W}(\mathbf{Z}_\beta-\mathbf{Z}_{\beta 2})_d(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})_d\mathbf{1}\\
&\quad + 3\mathbf{1}^\top\mathbf{\Phi}\mathbf{W}(\mathbf{Z}_\beta-\mathbf{Z}_{\beta 2})_d(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})(\mathbf{Z}_\beta-\mathbf{Z}_{\beta 2})_d\mathbf{W}\mathbf{\Phi}\mathbf{1},\\
A_{32} &= 2\mathbf{1}^\top\mathbf{\Phi}(\mathbf{F}-\mathbf{G})[(\mathbf{Z}_\beta-\mathbf{Z}_{\beta 2})^{(3)}](\mathbf{F}-\mathbf{G})\mathbf{\Phi}\mathbf{1} + 2\mathbf{1}^\top[(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})^{(3)}]\mathbf{1}\\
&\quad + 6\mathbf{1}^\top\mathbf{\Phi}\mathbf{W}[(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})\odot(\mathbf{Z}_\beta-\mathbf{Z}_{\beta 2})^{(2)}]\mathbf{W}\mathbf{\Phi}\mathbf{1}
\end{aligned}
$$

B.2. *Tests for components of δ*

Here, suppose that we want to test the null hypothesis $\mathcal{H}_0 : \boldsymbol{\delta}_1 = \boldsymbol{\delta}_1^{(0)}$, versus $\mathcal{H}_1 : \boldsymbol{\delta}_1 \neq \boldsymbol{\delta}_1^{(0)}$. In this situation, just consider $\mathbf{Z}_\beta = \mathbf{Z}_{\beta 2}$ and then the coefficients of the corrected score reduce to

$$
\begin{aligned}
A_{11} &= 3\mathbf{1}^\top\mathbf{\Phi}\mathbf{W}\mathbf{Z}_{\beta d}(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})\mathbf{Z}_{\delta 2d}\mathbf{1} + 3\mathbf{1}^\top\mathbf{\Phi}\mathbf{W}\mathbf{Z}_{\beta d}(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})\mathbf{Z}_{\beta d}\mathbf{W}\mathbf{\Phi}\mathbf{1}\\
&\quad + \frac{3}{4}\mathbf{1}^\top\mathbf{Z}_{\delta 2d}(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})\mathbf{Z}_{\delta 2d}\mathbf{1},\\
A_{12} &= 6\mathbf{1}^\top\mathbf{\Phi}\mathbf{W}\mathbf{Z}_{\beta d}\mathbf{Z}_{\delta 2}(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})_d\mathbf{1} + \frac{12}{n}\mathbf{1}^\top\mathbf{\Phi}\mathbf{W}\mathbf{Z}_{\beta d}(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})_d\mathbf{1}\\
&\quad + 3\mathbf{1}^\top\mathbf{Z}_{\delta 2d}\mathbf{Z}_{\delta 2}(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})_d\mathbf{1} + \frac{6}{n}\mathbf{1}^\top\mathbf{Z}_{\delta 2d}(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})_d\mathbf{1},\\
A_{13} &= \frac{9}{2}\mathbf{1}^\top[\mathbf{Z}_{\delta 2}^{(2)}\odot(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})]\mathbf{1} + \frac{18}{n}\mathbf{1}^\top[\mathbf{Z}_{\delta 2}\odot(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})]\mathbf{1}\\
&\quad + 6\mathbf{1}^\top\mathbf{\Phi}\mathbf{W}[(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})\odot\mathbf{Z}_\beta{}^{(2)}]\mathbf{W}\mathbf{\Phi}\mathbf{1},\\
A_{14} &= -12\mathbf{1}^\top\mathbf{\Phi}\mathbf{W}\mathbf{Z}_{\beta d}(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})_d\mathbf{1} - 12\mathbf{1}^\top(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})_d Z_{\delta 2d}\mathbf{1} - \frac{12}{n}\mathbf{1}^\top(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})_d\mathbf{1},\\
A_{21} &= -\frac{6}{n}\mathbf{1}^\top\left[(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})_d\right]^2\mathbf{1} - 3\mathbf{1}^\top(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})_d\mathbf{Z}_{\delta 2}(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})_d\mathbf{1},\\
A_{22} &= 6\mathbf{1}^\top\mathbf{\Phi}\mathbf{W}\mathbf{Z}_{\beta d}(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})_d\mathbf{1} - 3\mathbf{1}^\top\mathbf{Z}_{\delta 2d}(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})_d\mathbf{1},\\
A_{23} &= -\frac{12}{n}\mathbf{1}^\top[(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})^{(2)}]\mathbf{1} - 6\mathbf{1}^\top[\mathbf{Z}_{\delta 2}\odot(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})^{(2)}]\mathbf{1},\\
A_{24} &= 9\mathbf{1}^\top(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})_d^2\mathbf{1},\\
A_{31} &= 3\mathbf{1}^\top(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})_d(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})_d\mathbf{1},\\
A_{32} &= 2\mathbf{1}^\top[(\mathbf{Z}_\delta-\mathbf{Z}_{\delta 2})^{(3)}]\mathbf{1}
\end{aligned}
$$

Further, for testing all the components of $\boldsymbol{\delta}$, we have $\mathbf{Z}_{\delta 2} = \mathbf{0}$. Hence,

$$
\begin{aligned}
&A_{11} = 3\mathbf{1}^\top\mathbf{\Phi}\mathbf{W}\mathbf{Z}_{\beta d}\mathbf{Z}_\delta\mathbf{Z}_{\beta d}\mathbf{W}\mathbf{\Phi}\mathbf{1}, \quad A_{12} = \frac{12}{n}\mathbf{1}^\top\mathbf{\Phi}\mathbf{W}\mathbf{Z}_{\beta d}\mathbf{Z}_{\delta d}\mathbf{1},\\
&A_{13} = 6\mathbf{1}^\top\mathbf{\Phi}\mathbf{W}[\mathbf{Z}_\delta\odot\mathbf{Z}_\beta{}^{(2)}]\mathbf{W}\mathbf{\Phi}\mathbf{1}, \quad A_{14} = -12\mathbf{1}^\top\mathbf{\Phi}\mathbf{W}\mathbf{Z}_{\beta d}\mathbf{Z}_{\delta d}\mathbf{1} - \frac{12}{n}\mathbf{1}^\top\mathbf{Z}_{\delta d}\mathbf{1},\\
&A_{21} = -\frac{6}{n}\mathbf{1}^\top[\mathbf{Z}_{\delta d}]^2\mathbf{1}, \quad A_{22} = 6\mathbf{1}^\top\mathbf{\Phi}\mathbf{W}\mathbf{Z}_{\beta d}\mathbf{Z}_\delta\mathbf{Z}_{\delta d}\mathbf{1}, \quad A_{23} = -\frac{12}{n}\mathbf{1}^\top[\mathbf{Z}_\delta{}^{(2)}]\mathbf{1},\\
&A_{24} = 9\mathbf{1}^\top\mathbf{Z}_{\delta d}^2\mathbf{1}, \quad A_{31} = 3\mathbf{1}^\top\mathbf{Z}_{\delta d}\mathbf{Z}_\delta\mathbf{Z}_{\delta d}\mathbf{1}, \quad A_{32} = 2\mathbf{1}^\top[\mathbf{Z}_\delta{}^{(3)}]\mathbf{1}
\end{aligned}
$$

B.3. *Tests for components of β*

Finally, suppose that we want to test the null hypothesis $\mathcal{H}_0 : \boldsymbol{\beta}_1 = \boldsymbol{\beta}_1^{(0)}$ versus $\mathcal{H}_1 : \boldsymbol{\beta}_1 \neq \boldsymbol{\beta}_1^{(0)}$. If we consider $\mathbf{Z}_\delta = \mathbf{Z}_{\delta 2}$, then the coefficients of the corrected score statistic are

$$\begin{aligned}
A_{11} &= 3\mathbf{1}^\top \boldsymbol{\Phi}\mathbf{F}\mathbf{Z}_{\beta 2d}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})\mathbf{Z}_{\beta 2d}\mathbf{F}\boldsymbol{\Phi}\mathbf{1} + 6\mathbf{1}^\top \boldsymbol{\Phi}\mathbf{W}\mathbf{P}_2(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})\mathbf{Z}_{\beta 2d}\mathbf{F}\boldsymbol{\Phi}\mathbf{1} \\
&\quad + 3\mathbf{1}^\top \boldsymbol{\Phi}\mathbf{W}\mathbf{P}_2(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})\mathbf{P}_2\mathbf{W}\boldsymbol{\Phi}\mathbf{1}, \\
A_{12} &= 6\mathbf{1}^\top \boldsymbol{\Phi}\mathbf{F}\mathbf{Z}_{\beta 2d}\mathbf{Z}_{\beta 2}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d(\mathbf{F} - \mathbf{G})\boldsymbol{\Phi}\mathbf{1} + 6\mathbf{1}^\top \boldsymbol{\Phi}\mathbf{W}\mathbf{P}_2\mathbf{Z}_{\beta 2}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d(\mathbf{F} - \mathbf{G})\boldsymbol{\Phi}\mathbf{1} \\
&\quad + 6\mathbf{1}^\top \boldsymbol{\Phi}\mathbf{W}\mathbf{Z}_{\beta 2d}\mathbf{Z}_\delta(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\mathbf{W}\boldsymbol{\Phi}\mathbf{1} + \frac{\mathbf{12}}{n}\mathbf{1}^\top \boldsymbol{\Phi}\mathbf{W}\mathbf{Z}_{\beta 2d}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\mathbf{W}\boldsymbol{\Phi}\mathbf{1} \\
&\quad + 3\mathbf{1}^\top \mathbf{Z}_{\delta d}\mathbf{Z}_\delta(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\mathbf{W}\boldsymbol{\Phi}\mathbf{1} + \frac{6}{n}\mathbf{1}^\top \mathbf{Z}_{\delta d}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\mathbf{W}\boldsymbol{\Phi}\mathbf{1}, \\
A_{13} &= -6\mathbf{1}^\top \boldsymbol{\Phi}(2\mathbf{G} - \mathbf{F})\left[\mathbf{Z}_{\beta_2}^{(2)} \odot (\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})\right]\mathbf{F}\boldsymbol{\Phi}\mathbf{1} - 6\mathbf{1}^\top \boldsymbol{\Phi}\mathbf{W}\left[(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2}) \odot \mathbf{J}_2\right]\mathbf{W}\boldsymbol{\Phi}\mathbf{1} \\
&\quad - 6\mathbf{1}^\top \boldsymbol{\Phi}\mathbf{W}\left[(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2}) \odot \mathbf{C}_2^\top\right]\mathbf{F}\boldsymbol{\Phi}\mathbf{1} - 6\mathbf{1}^\top \boldsymbol{\Phi}(2\mathbf{G} - \mathbf{F})\left[(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2}) \odot \mathbf{C}_2\right]\mathbf{W}\boldsymbol{\Phi}\mathbf{1} \\
&\quad + \mathbf{12}\mathbf{1}^\top \boldsymbol{\Phi}\mathbf{W}\left[(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2}) \odot \mathbf{Z}_{\beta 2}\mathbf{Z}_\delta\right]\mathbf{W}\boldsymbol{\Phi}\mathbf{1}, \\
A_{14} &= -6\mathbf{1}^\top \boldsymbol{\Phi}\mathbf{H}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\mathbf{Z}_{\beta 2d}\mathbf{1} - 6\mathbf{1}^\top \boldsymbol{\Phi}\mathbf{W}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\mathbf{Z}_{\delta d}\mathbf{1} \\
&\quad - 6\mathbf{1}^\top \boldsymbol{\Phi}(\mathbf{F} - \mathbf{G})\mathbf{P}_2(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\mathbf{1}, \\
A_{21} &= -3\mathbf{1}^\top \boldsymbol{\Phi}(\mathbf{F} - \mathbf{G})(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\mathbf{Z}_{\beta 2}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d(\mathbf{F} - \mathbf{G})\boldsymbol{\Phi}\mathbf{1} \\
&\quad - 3\mathbf{1}^\top \boldsymbol{\Phi}\mathbf{W}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\mathbf{Z}_\delta(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\mathbf{1} - \frac{6}{n}\mathbf{1}^\top\left[\boldsymbol{\Phi}\mathbf{W}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d\right]^2\mathbf{1}, \\
A_{22} &= -6\mathbf{1}^\top \boldsymbol{\Phi}\mathbf{F}\mathbf{Z}_{\beta 2d}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d(\mathbf{F} - \mathbf{G})\boldsymbol{\Phi}\mathbf{1} \\
&\quad - 6\mathbf{1}^\top \boldsymbol{\Phi}\mathbf{W}\mathbf{P}_2(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d(\mathbf{F} - \mathbf{G})\Phi\mathbf{1}, \\
A_{23} &= -6\mathbf{1}^\top \boldsymbol{\Phi}(\mathbf{F} - \mathbf{G})[\mathbf{Z}_{\beta 2} \odot (\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})^{(2)}](\mathbf{F} - \mathbf{G})\Phi\mathbf{1} \\
&\quad - 6\mathbf{1}^\top \boldsymbol{\Phi}\mathbf{W}\left[\mathbf{Z}_\delta \odot (\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})^{(2)}\right]\mathbf{W}\boldsymbol{\Phi}\mathbf{1} \\
&\quad - \frac{\mathbf{12}}{n}\mathbf{1}^\top \boldsymbol{\Phi}\mathbf{W}[(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})^{(2)}]\mathbf{W}\boldsymbol{\Phi}\mathbf{1}, \\
A_{24} &= 3\mathbf{1}^\top \boldsymbol{\Phi}\mathbf{B}(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d^2\mathbf{1}, \\
A_{31} &= 3\mathbf{1}^\top \boldsymbol{\Phi}(\mathbf{F} - \mathbf{G})(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})_d(F - G)\boldsymbol{\Phi}\mathbf{1}, \\
A_{32} &= 2\mathbf{1}^\top \boldsymbol{\Phi}(\mathbf{F} - \mathbf{G})\left[(\mathbf{Z}_\beta - \mathbf{Z}_{\beta 2})^{(3)}\right](\mathbf{F} - \mathbf{G})\boldsymbol{\Phi}\mathbf{1}
\end{aligned}$$

Finally, for testing all the components of $\boldsymbol{\beta}$, we have $\mathbf{Z}_{\beta 2} = \mathbf{P}_2 = \mathbf{C}_2 = \mathbf{J}_2 = 0$. Hence,

$$A_{11} = A_{13} = A_{22} = 0, \quad A_{12} = 3\ \mathbf{1}^\top\ \mathbf{Z}_{\delta d}\ \mathbf{Z}_\delta\ \mathbf{Z}_{\beta d}\ \mathbf{W}\ \boldsymbol{\Phi}\ \mathbf{1} + \frac{6}{n}\ \mathbf{1}^\top\ \mathbf{Z}_{\delta d}\ \mathbf{Z}_{\beta d}\ \mathbf{W}\ \boldsymbol{\Phi}\ \mathbf{1},$$

$$A_{14} = -6\ \mathbf{1}^\top\ \boldsymbol{\Phi}\ \mathbf{W}\ \mathbf{Z}_{\beta d}\ \mathbf{Z}_{\delta d}\ \mathbf{1}, \quad A_{21} = -3\ \mathbf{1}^\top\ \boldsymbol{\Phi}\ \mathbf{W}\ \mathbf{Z}_{\beta d}\ \mathbf{Z}_\delta\ \mathbf{Z}_{\beta d}\ \mathbf{1} - \frac{6}{n}\ \mathbf{1}^\top\ [\boldsymbol{\Phi}\ \mathbf{W}\ \mathbf{Z}_{\beta d}]^2\ \mathbf{1},$$

$$A_{23} = -6\ \mathbf{1}^\top\ \boldsymbol{\Phi}\ \mathbf{W}\ [\mathbf{Z}_\delta \odot \mathbf{Z}_\beta{}^{(2)}]\ \mathbf{W}\ \boldsymbol{\Phi}\ \mathbf{1} - \frac{\mathbf{12}}{n}\ \mathbf{1}^\top\ \boldsymbol{\Phi}\ \mathbf{W}\ \left[\mathbf{Z}_\beta{}^{(2)}\right]\ \mathbf{W}\ \boldsymbol{\Phi}\ \mathbf{1}, \quad A_{24} = 3\ \mathbf{1}^\top\ \boldsymbol{\Phi}\ \mathbf{B}\ \mathbf{Z}_{\beta_d}^2\ \mathbf{1},$$

$$A_{31} = 3\ \mathbf{1}^\top\ \boldsymbol{\Phi}\ (\mathbf{F} - \mathbf{G})\ \mathbf{Z}_{\beta d}\ \mathbf{Z}_\beta\ \mathbf{Z}_{\beta d}\ (\mathbf{F} - \mathbf{G})\ \boldsymbol{\Phi}\ \mathbf{1}, \quad A_{32} = 2\ \mathbf{1}^\top\ \boldsymbol{\Phi}\ (\mathbf{F} - \mathbf{G})\ \mathbf{Z}_\beta{}^{(3)}\ (\mathbf{F} - \mathbf{G})\ \boldsymbol{\Phi}\ \mathbf{1}$$