JOURNAL OF ALGEBRA **206,** 1–16 (1998)
ARTICLE NO. JA977401

Degree Three, Four, and Five Identities of Quadratic Algebras

Irvin Roy Hentzel

Department of Mathematics, Iowa State University, 400 Carver Hall, Ames, Iowa 50011-2066

and

Luiz Antonio Peresi*

Departamento de Matemática, Universidade de São Paulo, Caixa Postal 66281, São Paulo-SP, 05315-970, Brazil

Communicated by J. Tits

Received September 25, 1995

1. INTRODUCTION

Let A be a nonassociative algebra with identity element 1 over a field F. We assume that F has characteristic $\neq 2$ and identify F with $F1$. The algebra is called a *quadratic algebra* when 1, a, a^2 are linearly dependent for all a in A, i.e.,

$$a^2 - t(a)a + n(a) = 0 \tag{1}$$

where $t: A \to F$ is a linear form and $n: A \to F$ is a quadratic form. From (1) we obtain

$$p(a,b) = ab + ba - t(a)b - t(b)a + q(a,b) = 0, \tag{2}$$

where $q(a,b) = n(a+b) - n(a) - n(b)$, and then

$$g(a,b) = t(ab) + t(ba) - 2t(a)t(b) + 2q(a,b) = 0. \tag{3}$$

* This paper was written while this author held a grant from CNPq of Brazil. Part of the research was done when this author was visiting Iowa State University on a grant from FAPESP.

0021-8693/98 $25.00

For a quadratic algebra A we have the direct sum $A = F \oplus V$, where $V = \{a \in A : t(a) = 0\}$. Let $a = \alpha + x$ and $b = \beta + y$ ($\alpha, \beta \in F$ and $x, y \in V$) be arbitrary elements in A. If $xy = -(x, y) + x.y$, where $(\, , \,) : V \times V \to F$ is a bilinear form and $x.y \in V$, then the product ab is given by

$$ab = (\alpha\beta - (x, y)) + (\beta x + \alpha y + x.y).$$

We have $t(a) = 2\alpha$ and $n(a) = \alpha^2 + (x, x)$. In general the bilinear form $(\, , \,)$ is not symmetric. It is symmetric if and only if $t(ab) = t(ba)$ for all $a, b \in A$. In this case we obtain from (3) the equation

$$h(a, b) = t(ab) - t(a)t(b) + q(a, b) = 0 \tag{4}$$

and we say that the algebra is a *symmetric quadratic algebra*.

Quadratic division algebras were studied by Osborn [2] and more recently quadratic alternative algebras by Elduque [8].

Cayley–Dickson algebras are symmetric quadratic algebras (see Zevlakov *et al.* [5]). Let V be an arbitrary vector space over F, $Q : V \times V \to F$ be a symmetric bilinear form, and $J(Q, 1) = F \oplus V$. We define a multiplication on $J(Q, 1)$ by setting

$$(\alpha + x)(\beta + y) = (\alpha\beta + Q(x, y)) + (\beta x + \alpha y).$$

The algebra $J(Q, 1)$ is a Jordan algebra (called a *Jordan algebra of degree* 2) and it is also a commutative quadratic algebra. Conversely, any commutative quadratic algebra is a Jordan algebra of degree 2.

We use the term identity *to mean polynomial identity and the term* equation *to mean any kind of identity. Let C be a class of nonassociative algebras over a field F. We say that $f(x_1, \ldots, x_n)$ is a* special central identity for all algebras in *C if there is an algebra in C for which $f(x_1, \ldots, x_n)$ is not an identity but for any A in C we have $f(a_1, \ldots, a_n) \in F$ for all $a_1, \ldots, a_n \in A$. From now on we assume that* $\mathrm{char}(F) = 0$ *or greater than the degree of the identity considered.*

The Amitsur–Levitzki theorem asserts that the minimal degree of an identity which holds for the algebra F_n of $n \times n$ matrices is $2n$, and that the multilinear homogeneous identities of degree $2n$ of this algebra are given by scalar multiples of the standard polynomial (see [1]). In [6, 7] Racine obtained Amitsur–Levitzki theorems for Cayley–Dickson algebras and Jordan algebras of degree 2. More precisely, Racine proved that minimal identities of these algebras have degree 5 and obtained all minimal identities. By a minimal identity Racine means an identity of minimal degree which is not a consequence of the alternative laws in the first case, and is a Jordan polynomial, which is not a consequence of the commutative and Jordan laws, in the second. Recently, Hentzel and Peresi [10] obtained the degree six identities and also the special central identities of degree less than or equal to six for the Cayley–Dickson algebras.

A finite set of generators for the T-ideal of identities of 2×2 matrices was determined by Yu. P. Razmyslov (1973) and V. S. Drenski (1981). S. Yu. Vasilovsky (1989) did the same for simple Jordan algebras of degree 2 when F is infinite and Q is nondegenerate. See a recent paper of Koshlukov [11] for complete references and the most recent results on weak identities of 2×2 matrices.

In this paper we determine sets of identities which generate all degree 3, 4, and 5 identities of quadratic algebras. We obtain also the special central identities of degree 3 and 4. Furthermore, we obtain results for quadratic symmetric algebras and Jordan algebras of degree 2.

2. REPRESENTING EQUATIONS BY MATRICES

We denote by S_n the symmetric group. The group algebra FS_n is isomorphic to a direct sum of matrix algebras. A map that takes FS_n onto one of these summands is an *irreducible representation* of FS_n which can be obtained by a procedure given by Clifton [4]. The procedure associates to $\pi \in S_n$ a matrix A_π and the irreducible representation is given by $\pi \to A_I^{-1} A_\pi$, where I denotes the identity permutation. In what follows we use the isomorphisms (of modules) $FS_n \to F_t$ given by $\pi \to A_\pi$ and call them *representations* of FS_n although they are not representations in the usual sense.

Let $f = f(x_1, \ldots, x_n) = 0$ be an equation where each variable appears once in each term. Suppose that the terms of this equation can be classified in k types denoted by $T_1, \ldots, T_k$. The concept of type will be made more precise in the next section. Thus $f = f_1 \oplus \cdots \oplus f_k$ where in the expression f_i appear only terms of type T_i. Considering how the positions of the variables $x_1, \ldots, x_n$ are changed we identify f_i with an element of FS_n. Using this identification we consider $f = 0$ as an element of $FS_n \oplus \cdots \oplus FS_n$ (here we have k summands corresponding to the k types). Each representation induces a representation

$$P : FS_n \oplus \cdots \oplus FS_n \to F_t \oplus \cdots \oplus F_t.$$

Using this representation P we identify the equation $f = 0$ with a direct sum of k matrices in F_t. We call this direct sum of matrices a *matrix representation of the equation* $f = 0$. The *row space of* $f = 0$ is the row space determined by its matrix representation. Now, let $f^{(i)} = f^{(i)}(x_1, \ldots, x_n) = 0$ ($i = 1, \ldots, s$) be a set of such equations. Considering altogether the matrix representations of $f^{(1)} = 0, \ldots, f^{(s)} = 0$ given by the representation P we obtain a block matrix. The equation $f = 0$ is a consequence of these s equations if and only if the row space of $f = 0$ is contained in the row space of this block matrix, and this happens for all representations P.

This technique of representing equations by matrices was introduced by Hentzel [3] and improved in Correa, Hentzel, and Peresi [9].

3. DEGREE 3 IDENTITIES

We denote by (a, b, c) the *associator* $(ab)c - a(bc)$, by $[a, b]$ the *commutator* $ab - ba$, and by $a \circ b$ the *Jordan product* $ab + ba$.

THEOREM 1. *All degree* 3 *polynomial identities of quadratic algebras* (*or symmetric quadratic algebras*) *are consequences of* $(a, a, a) = 0$.

Proof. As clear from (1), any quadratic algebra satisfies $(a, a, a) = 0$. In what follows we prove that any other degree 3 identity is a consequence of this identity.

Using (2) and (3) we obtain the equations

$$\begin{aligned} &p(a,b)c = 0, \qquad cp(a,b) = 0, \\ &p(ab,c) = 0, \qquad t(c)p(a,b) = 0, \\ &q(p(a,b),c) = 0, \qquad t(p(a,b)c) = 0, \qquad t(cp(a,b)) = 0 \end{aligned} \tag{5}$$

and

$$g(a,b)c = 0, \qquad g(ab,c) = 0, \qquad t(c)g(a,b) = 0. \tag{6}$$

Each term of these equations can be classified as one of the following types:

$$\begin{aligned} &T_1 .\, t(R)RR, \quad T_2 .\, t(R)t(R)R, \quad T_3 .\, t(RR)R, \quad T_4 .\, q(R,R)R, \\ &T_5 .\, q(R,RR), \quad T_6 .\, t(R)q(R,R), \quad T_7 .\, t(RR.R), \quad T_8 .\, t(R.RR), \\ &T_9 .\, t(R)t(RR), \quad T_{10} .\, t(R)t(R)t(R), \quad T_{11} .\, (RR)R, \quad T_{12} .\, R(RR). \end{aligned}$$

Here R has no meaning. It is just a convenient way to say what the term looks like. We have to consider also the equations

$$\begin{aligned} &t(a)t(b)c - t(b)t(a)c = 0, \qquad q(a,b)c - q(b,a)c = 0, \\ &t(a)q(b,c) - t(a)q(c,b) = 0, \\ &t(a)t(b)t(c) - t(b)t(a)t(c) = 0, \\ &t(a)t(b)t(c) - t(c)t(a)t(b) = 0. \end{aligned} \tag{7}$$

Let $f = 0$ be a degree 3 identity which holds for all quadratic algebras. This identity involves only types T_{11} and T_{12}, and it is a consequence of Eqs. (5), (6), and (7). Thus, the row spaces of $f = 0$ are contained in the row spaces (involving only types T_{11} and T_{12}) of Eqs. (5), (6), and (7) which are

Rep. 1		Rep. 2				Rep. 3	
T_{11}	T_{12}	T_{11}		T_{12}		T_{11}	T_{12}
1	-1	0	0	0	0	0	0

Since these row spaces are those given by $(a, a, a) = 0$ it follows that $f = 0$ is a consequence of $(a, a, a) = 0$.

Now let $f = 0$ be a degree 3 identity for symmetric quadratic algebras. Then it is a consequence of Eqs. (5), (7), $h(a, b)c = 0$, $h(ab, c) = 0$, $h(c, ab) = 0$, $t(c)h(a, b) = 0$. Since the row spaces (involving only types T_{11} and T_{12}) of these last equations coincide with those of $(a, a, a) = 0$, it follows that $f = 0$ is again a consequence of $(a, a, a) = 0$.

The *flexitor* (a, b, a) will play an important role in the rest of this paper. Since $(a, b, a) = (a \circ b)a - (ba) \circ a$ we obtain

$$\begin{aligned}(a, b, a) &= t(a)ba + t(b)a^2 - q(a, b)a - t(a)ba - t(ba)a + q(ba, a) \\ &= t(b)a^2 - q(a, b)a - t(ba)a + q(ba, a)\end{aligned}$$

by (2) and then

$$(a, b, a) = t(a)t(b)a - n(a)t(b) - q(a, b)a - t(ba)a + q(ba, a)$$

by (1). We now use (3) to obtain

$$t(a)t(b)a - q(a, b)a = \frac{1}{2}t(a \circ b)a.$$

Using this last equation we obtain

$$(a, b, a) = \frac{1}{2}t(a \circ b)a - n(a)t(b) - t(ba)a + q(ba, a),$$

i.e.,

$$(a, b, a) = \frac{1}{2}t([a, b])a - n(a)t(b) + q(ba, a). \tag{8}$$

From (4) it follows that symmetric quadratic algebras satisfy $t([a, b]) = 0$. Thus it follows from (8) that (a, b, a) is a special central identity for these algebras.

THEOREM 2. (i) *There are no special central identities of degree* 3 *for quadratic algebras*. (ii) *All degree* 3 *special central identities of symmetric quadratic algebras are consequences of* (a, b, a).

Proof. Let $f(x, y, z)$ be a special central identity of degree 3 for quadratic algebras. Let A be a quadratic algebra. For any $a, b, c \in A$ we have $f(a, b, c) \in F$. Using Eqs. (5), (6), and (7) we can express $f(a, b, c)$ as a sum of terms involving only the *field types* T_5, T_6, T_7, T_8, T_9, and T_{10}. Thus, considering the types $T_1, \ldots, T_{12}$ in a different order, i.e., putting the *polynomial types* T_{11}, T_{12} just before the field types, and representing the equations by matrices, it follows that the row spaces of $f(x, y, z)$ are contained in the row spaces of the identity $(a, a, a) = 0$. Thus (i) is proved.

Using the same argument we obtain that the row spaces of a degree 3 special central identity of symmetric quadratic algebras are contained in

Rep. 1		Rep. 2				Rep. 3	
T_{11}	T_{12}	T_{11}		T_{12}		T_{11}	T_{12}
1	−1	0	1	0	−1	0	0

Since these are the row spaces given by (a, b, a) the assertion (ii) follows.

4. DEGREE 4 IDENTITIES

We start by proving the existence of degree 4 identities.

PROPOSITION 1. *Any quadratic algebra satisfies the identities*

$$[(a, b, a), a] = 0, \qquad (a^2, a, b) - (a, a^2, b) = 0,$$

$$(b, a^2, a) - (b, a, a^2) = 0. \tag{9}$$

Furthermore, any symmetric quadratic algebra satisfies

$$[(a, b, a), c] = 0. \tag{10}$$

Proof. Identities (9) are readily obtained from (8) and (1). Since (a, b, a) is a special central identity for symmetric quadratic algebras it follows that (10) is an identity.

THEOREM 3. *All degree* 4 *polynomial identities of quadratic algebras are consequences of* $(a, a, a) = 0$, $[(a, b, a), a] = 0$, $(a^2, a, b) - (a, a^2, b) = 0$, *and* $(b, a^2, a) - (b, a, a^2) = 0$.

Proof. Equations (2) and (3) yield, respectively, 35 and 10 equations of degree 4. Each one of these equations can be expressed as a sum of terms involving 33 types. We have to consider also the 15 equations that come from the 33 types by using the fact that field elements commute. This gives us 60 equations.

If $f = 0$ is a degree 4 identity for all quadratic algebras then it is a consequence of these 60 equations. Thus the row spaces of $f = 0$ are contained in the row spaces (involving only the polynomial types) given by these 60 equations. Let $u(a, b, c) = 0$ be the linearization of $(a, a, a) = 0$. The row spaces of the block matrices determined by identities (9), $u(a, b, c)d = 0$, $du(a, b, c) = 0$, and $u(ad, b, c) = 0$ match with those given by the 60 equations. It follows then that $f = 0$ is a consequence of $(a, a, a) = 0$ and identities (9), and Theorem 3 is proved.

We consider now symmetric quadratic algebras. We replace the 10 equations of degree 4 which come from (3) by those implied by (4). We have 64 equations. But now we need to add to $(a, a, a) = 0$ and identities (9) the identity (10) to obtain the row spaces. Thus we have proved the following theorem.

THEOREM 4. *All degree* 4 *polynomial identities of symmetric quadratic algebras are consequences of* $(a, a, a) = 0$, $[(a, b, a), c] = 0$, $(a^2, a, b) - (a, a^2, b) = 0$, *and* $(b, a^2, a) - (b, a, a^2) = 0$.

We denote by $\langle a, b, c\rangle$ the *Jordan associator* $(a \circ b) \circ c - a \circ (b \circ c)$.

PROPOSITION 2. *Any quadratic algebra satisfies the following special central identities*:

$$2a(a, b, b) - 2(a, b, ab) + 2b(b, a, a) - 2(b, a, ba) + a \circ (b, a, b) + b \circ (a, b, a), \tag{11}$$

$$2a^2 \circ b^2 - (a \circ b)^2 + 2a\langle a, b, b\rangle + 2b\langle b, a, a\rangle. \tag{12}$$

Furthermore, any symmetric quadratic algebra satisfies the special central identity

$$[a, b] \circ [c, d].$$

Proof. We have

$$2a(a, b, b) - 2(a, b, ab) = -2(ab)^2 - 2a(ab^2) + 2a((ab) \circ b)$$

and

$$2b(b, a, a) - 2(b, a, ba) = -2(ba)^2 - 2b(ba^2) + 2b((ba) \circ a).$$

Using (1) and (2) we obtain

$$
\begin{aligned}
-2(ab)^2 &= -2t(ab)ab + 2n(ab),\\
-2(ba)^2 &= -2t(ba)ba + 2n(ba),\\
-2a(ab^2) &= -2t(b)a.ab + 2n(b)a^2\\
&= -2t(b)a.ab + 2t(a)n(b)a - 2n(a)n(b),\\
-2b(ba^2) &= -2t(a)b.ba + 2n(a)b^2\\
&= -2t(a)b.ba + 2t(b)n(a)b - 2n(a)n(b),\\
2a((ab)\circ b) &= 2t(ab)ab + 2t(b)a.ab - 2q(ab,b)a,\\
2b((ba)\circ a) &= 2t(ba)ba + 2t(a)b.ba - 2q(ba,a)b.
\end{aligned}
$$

Thus adding these last six equations we get

$$
\begin{aligned}
&2a(a,b,b) - 2(a,b,ab) + 2b(b,a,a) - 2(b,a,ba)\\
&= 2t(a)n(b)a - 2q(ab,b)a + 2t(b)n(a)b\\
&\quad - 2q(ba,a)b + 2n(a\circ b) - 4n(a)n(b). \qquad (13)
\end{aligned}
$$

Since

$$a\circ(b,a,b) = a\circ((a\circ b)b) - a\circ((ab)\circ b)$$

and

$$b\circ(a,b,a) = b\circ((b\circ a)a) - b\circ((ba)\circ a),$$

adding the following equations (obtained by (1) and (2))

$$
\begin{aligned}
a\circ((a\circ b)b) &= t(a)a\circ b^2 + t(b)a\circ(ab) - q(a,b)a\circ b\\
&= t(a)t(b)a\circ b - 2t(a)n(b)a\\
&\quad + t(b)a\circ(ab) - q(a,b)a\circ b,\\
-a\circ((ab)\circ b) &= -t(ab)a\circ b - t(b)a\circ(ab) + 2q(ab,b)a,\\
b\circ((b\circ a)a) &= t(b)t(a)b\circ a - 2t(b)n(a)b\\
&\quad + t(a)b\circ(ba) - q(b,a)b\circ a,\\
-b\circ((ba)\circ a) &= -t(ba)b\circ a - t(a)b\circ(ba) + 2q(ba,a)b
\end{aligned}
$$

we obtain

$$a\circ(b,a,b)+b\circ(a,b,a)$$
$$=-2t(a)n(b)a+2q(ab,b)a-2t(b)n(a)b+2q(ba,a)b \quad (14)$$

by (3).

We now add (13) and (14) to obtain

$$\begin{aligned}2a(a,b,b)&-2(a,b,ab)+2b(b,a,a)-2(b,a,ba)\\&+a\circ(b,a,b)+b\circ(a,b,a)\\&=2n(a\circ b)-4n(a)n(b)\end{aligned}$$

and it follows that (11) is a special central identity for any quadratic algebra.

By (1) and (2) we have

$$\begin{aligned}2a^2\circ b^2&=2(t(a)a-n(a))\circ(t(b)b-n(b))\\&=2t(a)t(b)a\circ b-4t(a)n(b)a-4t(b)n(a)b+4n(a)n(b),\\-(a\circ b)^2&=-t(a\circ b)a\circ b+n(a\circ b)\\&=-2t(a)t(b)a\circ b+2q(a,b)a\circ b+n(a\circ b)\\&=-2t(a)t(b)a\circ b+2t(a)q(a,b)b+2t(b)q(a,b)a\\&\quad-2q(a,b)^2+n(a\circ b).\end{aligned}$$

Adding we obtain

$$\begin{aligned}2a^2\circ b^2-(a\circ b)^2=\{&-4t(a)n(b)+2t(b)q(a,b)\}a\\&+\{-4t(b)n(a)+2t(a)q(a,b)\}b\\&+4n(a)n(b)-2q(a,b)^2+n(a\circ b). \quad (15)\end{aligned}$$

Using (1) and (2) we get

$$\begin{aligned}a\langle a,b,b\rangle&=a\{(a\circ b)\circ b-a\circ(b\circ b)\}\\&=a\{t(a)b\circ b+t(b)a\circ b-q(a,b)\circ b-2a\circ b^2\}\\&=a\{2t(a)t(b)b-2t(a)n(b)-t(b)a\circ b\\&\qquad-2q(a,b)b+4n(b)a\}\\&=\{t(a)t(b)-2q(a,b)\}ab\\&\quad+\{-2t(a)n(b)+t(b)q(a,b)\}a-t(b)^2a^2+4n(b)a^2\\&=\{t(a)t(b)-2q(a,b)\}ab+\{2t(a)n(b)+t(b)q(a,b)\\&\qquad-t(b)^2t(a)\}a+t(b)^2n(a)-4n(a)n(b).\end{aligned}$$

Thus

$$\begin{aligned}2a\langle a,b,b\rangle &+ 2b\langle b,a,a\rangle\\ &= 2\{t(a)t(b) - 2q(a,b)\}(a\circ b)\\ &\quad + 2\Big\{\{2t(a)n(b) + t(b)q(a,b) - t(b)^2t(a)\}a\\ &\qquad\qquad + t(b)^2n(a) - 4n(a)n(b)\Big\}\\ &\quad + 2\Big\{\{2t(b)n(a) + t(a)q(a,b) - t(a)^2t(b)\}b\\ &\qquad\qquad + t(a)^2n(b) - 4n(b)n(a)\Big\}.\end{aligned}$$

Since $a\circ b = t(a)b + t(b)a - q(a,b)$ by (2), it follows that

$$\begin{aligned}2a\langle a,b,b\rangle &+ 2b\langle b,a,a\rangle\\ &= \{4t(a)n(b) - 2t(b)q(a,b)\}a\\ &\quad + \{4t(b)n(a) - 2t(a)q(a,b)\}b - 2t(a)t(b)q(a,b)\\ &\quad + 4q(a,b)^2 + 2t(b)^2n(a) + 2t(a)^2n(b) - 16n(a)n(b).\end{aligned} \tag{16}$$

Now adding (15) and (16) we see that (12) is a special central identity for quadratic algebras.

Using (2) we obtain

$$[a,b]\circ[c,d] = t([a,b])[c,d] + t([c,d])[a,b] - q([a,b],[c,d]).$$

Since $t([a,b]) = 0$ and $t([c,d]) = 0$ by (4) it follows that $[a,b]\circ[c,d]$ is a special central identity for any symmetric quadratic algebra.

THEOREM 5. (i) *All degree* 4 *special central identities of quadratic algebras are consequences of*

$$\begin{aligned}2a(a,b,b) &- 2(a,b,ab) + 2b(b,a,a)\\ &- 2(b,a,ba) + a\circ(b,a,b) + b\circ(a,b,a),\end{aligned} \tag{11}$$

$$2a^2\circ b^2 - (a\circ b)^2 + 2a\langle a,b,b\rangle + 2b\langle b,a,a\rangle. \tag{12}$$

(ii) *All degree* 4 *special central identities of symmetric quadratic algebras are consequences of*

$$(a,b,a), \qquad [a,b]\circ[c,d],$$

$$\begin{aligned}2a(a,b,b) &- 2(a,b,ab) + 2b(b,a,a)\\ &- 2(b,a,ba) + a\circ(b,a,b) + b\circ(a,b,a).\end{aligned} \tag{17}$$

Proof. (i) We determine the row spaces of the 60 equations considering the types in the following ordering: we put the field types at the end and the polynomial types just before them. We compare the part of these row spaces which are under the polynomial types with the row spaces given by $(a, a, a) = 0$ and identities (9). They coincide in representations 1, 2, 4, and 5. In representation 3 we have 2 new rows which correspond to the special central identities (11) and (12).

(ii) Using the same procedure with the 64 equations for the symmetric quadratic algebras and comparing with the identities given by Theorem 4, the row spaces give now the special central identities (17).

5. DEGREE 5 IDENTITIES

We denote by R_a the linear operator defined by $R_a(b) = a \circ b$ and define the linear operator $\overline{S}_3$ by

$$\overline{S}_3 = \sum_{\sigma \in S_3} (-1)^{\sigma} R_{a_{\sigma(1)}} R_{a_{\sigma(2)}} R_{a_{\sigma(3)}},$$

where $(-1)^{\sigma}$ is the sign of the permutation σ.

PROPOSITION 3. *Any quadratic algebra satisfies the identities*

$$\overline{S}_3(a^2) - \overline{S}_3(a) \circ a = 0, \tag{18}$$

$$\begin{aligned}[] [2a(a,b,b) - 2(a,b,ab) + 2b(b,a,a) - 2(b,a,ba) \\ + a \circ (b,a,b) + b \circ (a,b,a), c] = 0, \end{aligned} \tag{19}$$

$$\left[2a^2 \circ b^2 - (a \circ b)^2 + 2a\langle a,b,b\rangle + 2b\langle b,a,a\rangle, c\right] = 0, \tag{20}$$

$$[c^2, (b,a,c)] - [c, (b,a,c^2)] = 0, \tag{21}$$

$$\begin{aligned} 2((a,b,a),c,d) + 2(d,c,(a,b,a)) - ([a,b] \circ a, c, d) \\ -([a,b] \circ d, c, a) + ([a,b], c, a \circ d) = 0, \end{aligned} \tag{22}$$

$$\begin{aligned} 2((a,b,a),c,d) + 2(c,(a,b,a),d) - ([a,b] \circ a, c, d) \\ -([a,b] \circ c, a, d) + ([a,b], a \circ c, d) = 0, \end{aligned} \tag{23}$$

$$[[(b,a,c)+(c,a,b),d],a]+[([c,d],b,a)+(a,b,[c,d]),a]$$
$$+[([b,d],c,a)+(a,c,[b,d]),a]=0, \tag{24}$$
$$-(a^2\circ c,d,e)+(a^2\circ e,d,c)-(e,d,a^2\circ c)$$
$$+(e,d,(a\circ c)\circ a)-(c,d,(a\circ e)\circ a)$$
$$+(a\circ c,d,a\circ e)-(a\circ e,d,a\circ c)-((a\circ c)\circ e,d,a)$$
$$+(c,d,a^2\circ e)+((a\circ e)\circ c,d,a)=0, \tag{25}$$
$$\sum_{b,c,e}\{(a,\langle c,d,e\rangle,b)+(a,b,\langle c,d,e\rangle)\}=0, \tag{26}$$

where $\sum_{b,c,e}$ *denotes the alternating sum on the variables* b,c,e,

$$\sum_{a,b,c,d}\{10[d,[(a,b,e),c]]+10[d,[(e,b,a),c]]+8((a,b,e),c,d)$$
$$+8((e,b,a),c,d)+8(b,(a,c,e),d)$$
$$+8(b,(e,c,a),d)-8(c,b,(a,d,e))-8(c,b,(e,d,a))$$
$$-[[b,c]\circ[a,d],e]-[[e,c]\circ[a,d],b]\}=0, \tag{27}$$

where $\sum_{a,b,c,d}$ *denotes the alternating sum on the variables* a,b,c,d.

Proof. The proof that (18) is an identity is given by Racine [6, Theorem, p. 2495]. Racine states the result for Jordan algebras of degree 2, but the same proof works for quadratic algebras. The fact that (19) and (20) are identities is an immediate consequence of Proposition 2. Identity (21) is readily seen from (1). Identities (22) and (23) are straightforward from (2) and (8). Identity (24) follows from the linearized form of (8). Identity (25) follows by using (1) and (2).

We prove now identity (26). Using (2) we get

$$\begin{aligned}(c\circ d)\circ e &= t(c\circ d)e+t(e)c\circ d-q(c\circ d,e)\\ &= 2t(c)t(d)e-2q(c,d)e+t(e)t(c)d\\ &\quad +t(e)t(d)c-t(e)q(c,d)-q(c\circ d,e).\end{aligned}$$

Thus

$$\begin{aligned}\langle c,d,e\rangle &= (c\circ d)\circ e-(e\circ d)\circ c\\ &= 2t(c)t(d)e-2q(c,d)e+t(e)t(c)d+t(e)t(d)c\\ &\quad -t(e)q(c,d)-q(c\circ d,e)-2t(e)t(d)c+2q(e,d)c\\ &\quad -t(e)t(c)d-t(c)t(d)e+t(c)q(e,d)+q(e\circ d,c)\\ &= \{t(c)t(d)-2q(c,d)\}e+\{-t(e)t(d)+2q(e,d)\}c\\ &\quad -t(e)q(c,d)-q(c\circ d,e)+t(c)q(e,d)+q(e\circ d,c).\end{aligned}$$

It follows that

$$\begin{aligned}(a,\langle c,d,e\rangle,b)+(a,b,\langle c,d,e\rangle)\\ &=\{t(c)t(d)-2q(c,d)\}\{(a,e,b)+(a,b,e)\}\\ &\quad+\{-t(e)t(d)+2q(e,d)\}\{(a,c,b)+(a,b,c)\}\end{aligned}$$

and now it is clear that

$$\sum_{b,c,e}(a,\langle c,d,e\rangle,b)+(a,b,\langle c,d,e\rangle)=0.$$

Finally we prove identity (27). Using the linearized form of (8) we obtain

$$\begin{aligned}10\sum_{a,b,c,d}&\{[d,[(a,b,e),c]]+[d,[(e,b,a),c]]\}\\ &=5\sum_{a,b,c,d}\{t([a,b])[d,[e,c]]+t([e,b])[d,[a,c]]\},\end{aligned}$$

$$\begin{aligned}8\sum_{a,b,c,d}&\{((a,b,e),c,d)+((e,b,a),c,d)\}\\ &=4\sum_{a,b,c,d}\{t([a,b])(e,c,d)+t([e,b])(a,c,d)\},\end{aligned}$$

$$\begin{aligned}8\sum_{a,b,c,d}&\{(b,(a,c,e),d)+(b,(e,c,a),d)\}\\ &=4\sum_{a,b,c,d}\{t([a,c])(b,e,d)+t([e,c])(b,a,d)\}\\ &=-4\sum_{a,b,c,d}\{t([a,b])(c,e,d)+t([e,b])(c,a,d)\},\end{aligned}$$

$$\begin{aligned}-8\sum_{a,b,c,d}&\{(c,b,(a,d,e))+(c,b,(e,d,a))\}\\ &=-4\sum_{a,b,c,d}\{t([a,d])(c,b,e)+t([e,d])(c,b,a)\}\\ &=4\sum_{a,b,c,d}\{t([a,b])(c,d,e)+t([e,b])(c,d,a)\}.\end{aligned}$$

Equation (2) yields

$$\begin{aligned}-\sum_{a,b,c,d}&[[b,c]\circ[a,d],e]\\ &=-\sum_{a,b,c,d}\{t([b,c])[[a,d],e]+t([a,d])[[b,c],e]\}\\ &=\sum_{a,b,c,d}\{t([a,b])[[d,c],e]+t([a,b])[[d,c],e]\},\end{aligned}$$

$$
\begin{aligned}
&- \sum_{a,b,c,d} [[e,c]\circ[a,d],b] \\
&\quad = - \sum_{a,b,c,d} \{t([e,c])[[a,d],b] + t([a,d])[[e,c],b]\} \\
&\quad = \sum_{a,b,c,d} \{t([e,b])[[a,d],c] + t([a,b])[[e,c],d]\}.
\end{aligned}
$$

Adding these identities we obtain

$$
\begin{aligned}
\sum_{a,b,c,d} \{&10[d,[(a,b,e),c]] + 10[d,[(e,b,a),c]] \\
&+8((a,b,e),c,d) + 8((e,b,a),c,d) \\
&+8(b,(a,c,e),d) + 8(b,(e,c,a),d) \\
&-8(c,b,(a,d,e)) - 8(c,b,(e,d,a)) \\
&-[[b,c]\circ[a,d],e] - [[e,c]\circ[a,d],b]\} \\
= \sum_{a,b,c,d} t([a,b])\{&4[d,[e,c]] + 4(e,c,d) \\
&-4(c,e,d) + 4(c,d,e) + 2[[d,c],e]\} \\
+ \sum_{a,b,c,d} t([e,b])\{&5[d,[a,c]] + 4(a,c,d) \\
&-4(c,a,d) + 4(c,d,a) + [[a,d],c]\} = 0
\end{aligned}
$$

since

$$
\sum_{c,d} \{4[d,[e,c]] + 4(e,c,d) - 4(c,e,d) + 4(c,d,e) + 2[[d,c],e]\} = 0
$$

and

$$
\begin{aligned}
\sum_{a,c,d} \{&5[d,[a,c]] + 4(a,c,d) - 4(c,a,d) \\
&+4(c,d,a) + [[a,d],c]\} = 0.
\end{aligned}
$$

THEOREM 6. *All degree* 5 *polynomial identities of quadratic algebras are consequences of* $(a,a,a) = 0$, $[(a,b,a),a] = 0$, $(a^2,a,b) - (a,a^2,b) = 0$, $(b,a^2,a) - (b,a,a^2) = 0$, *and identities* (18), ..., (27).

Proof. We determine all degree 5 equations which are consequences of Eqs. (2) and (3). We find the types needed to express these equations. We represent these equations by matrices and determine the row spaces which involve only the polynomial types. Since these row spaces coincide with the rows spaces representing identities $(a,a,a) = 0$, $[(a,b,a),a] = 0$, $(a^2,a,b) - (a,a^2,b) = 0$, $(b,a^2,a) - (b,a,a^2) = 0$, (18), ..., (27) the theorem is proved.

PROPOSITION 4. *Any symmetric quadratic algebra satisfies the identities*

$$((a,b,a),c,d) = 0, \qquad (c,(a,b,a),d) = 0,$$
$$(c,d,(a,b,a)) = 0, \qquad [[a,b]\circ[c,d],e] = 0, \tag{28}$$

$$([a,b]\circ c,c,d) - ([a,b],c^2,d) = 0,$$
$$([a,b]\circ c,d,c) - ([a,b],d,c^2) = 0. \tag{29}$$

Proof. We know from Theorem 2 that (a,b,a) is a special central identity for symmetric quadratic algebras. Thus (28) are identities for these algebras. From (4) we obtain that $t([a,b]) = 0$. Thus identities (29) are readily obtained from (1) and (2).

THEOREM 7. *All degree* 5 *polynomial identities of symmetric quadratic algebras are consequences of* $(a,a,a) = 0$, $[(a,b,a),c] = 0$, $(a^2,a,b) - (a,a^2,b) = 0$, $(b,a^2,a) - (b,a,a^2) = 0$, *and identities* (18), (20), (25), (26), (28), *and* (29).

Proof. We start with Eqs. (2) and (4) and then use the same argument as in the proof of Theorem 6.

6. JORDAN ALGEBRAS OF DEGREE 2

Jordan algebras of degree 2 are the commutative quadratic algebras. Thus, to obtain the identities and central identities, we have to add to the set of equations those equations implied by commutativity and do again the calculations we did in Sections 3, 4, and 5. We obtain the following results.

THEOREM 8. *For Jordan algebras of degree* 2 *we have*:

(i) *All degree* 3 *polynomial identities are consequences of* $ab - ba = 0$.

(ii) *All degree* 4 *polynomial identities are consequences of* $ab - ba = 0$ *and* $(a^2,b,a) = 0$.

(iii) *All degree* 5 *polynomial identities are consequences of* $ab - ba = 0$, $(a^2,b,a) = 0$, *and* $\overline{S}_3(a^2) - \overline{S}_3(a)\circ a = 0$.

(iv) *There are no special central identities of degree* 3.

(v) *The special central identities of degree* 4 *and* 5 *are consequences of* $a(a,b,b) - (a,b,ab) + b(b,a,a) - (b,a,ba)$.

The result stated in (iii) agrees with the result obtained by Racine in [6].

7. REMARKS

The set of identities in the statement of each theorem is independent in the sense that each one of the identities is not a consequence of the other ones.

Behind the proofs of the theorems there are extensive calculations on matrices which were done by a computer program named Crunch. We used the computer facilities at Iowa State University (Project Vincent) and at the University of São Paulo.

REFERENCES

1. A. S. Amitsur and J. Levitzki, Minimal identities for algebras, *Proc. Amer. Math. Soc.* **1** (1950), 449–463.
2. J. M. Osborn, Quadratic division algebras, *Trans. Amer. Math. Soc.* **105** (1962), 202–221.
3. I. R. Hentzel, Processing identities by group representation, *in* "Computers in Nonassociative Rings and Algebras" (R. E. Beck and B. Kolman, Eds.), Academic Press, New York, 1977.
4. J. M. Clifton, A simplification of the computation of the natural representation of the symmetric group S_n, *Proc. Amer. Math. Soc.* **83** (1981), 248–250.
5. K. A. Zevlakov, A. M. Slin'ko, I. P. Shestakov, and A. I. Shirshov, "Rings That Are Nearly Associative," Academic Press, New York, 1982.
6. M. L. Racine, Minimal identities for Jordan algebras of degree 2, *Comm. Algebra* **13** (1985), 2493–2506.
7. M. L. Racine, Minimal identities of octonion algebras, *J. Algebra* **115** (1988), 251–260.
8. A. Elduque, Quadratic alternative algebras, *J. Math. Phys.* **31** (1990), 1–5.
9. I. Correa, I. R. Hentzel, and L. A. Peresi, Minimal identities of Bernstein algebras, *Algebra Groups Geom.* **11** (1994), 181–199.
10. I. R. Hentzel and L. A. Peresi, Identities of Cayley–Dickson algebras, *J. Algebra* **188** (1997), 292–309.
11. P. Koshlukov, Weak polynomial identities for the matrix algebra of order two, *J. Algebra* **188** (1997), 610–625.