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NONLINEAR FLUX THROUGH
THE BOUNDARY VERSUS
NONLINEAR HEATING

Sérgio Muniz Oliva e
Antonio Luiz Pereira

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UNIVERSIDADE DE SÃO PAULO
INSTITUTO DE MATEMÁTICA E ESTATÍSTICA

SÃO PAULO — BRASIL

Nonlinear Flux through the Boundary Versus Nonlinear Heating

Sergio Muniz Oliva*

Antonio Luiz Pereira†

1 Introduction

Let Ω be a bounded smooth domain of \mathbb{R}^2 . In this paper we consider reaction diffusion systems with dispersion of the form

$$\begin{cases} u_t = \text{Div}(a\nabla u) - \sum_{j=1}^2 B_j(x) \frac{\partial u}{\partial x_j} - \lambda u + f(u), & \text{in } \Omega, \\ \frac{\partial u}{\partial n_a} = g(u), & \text{on } \partial\Omega. \end{cases} \quad (1)$$

where $u = (u_1, \dots, u_N)^T$, $N \geq 1$, $a(x) = \text{diag}(a_1(x), \dots, a_N(x))$, $a_i \in C^1(\bar{\Omega})$, $a_i(x) > m_0 > 0$, $x \in \Omega$, $1 \leq i \leq N$, $\frac{\partial u}{\partial n_a} = \langle a\nabla u, \bar{n} \rangle$, \bar{n} is the outward normal and $B_j = \text{diag}(b_j^1, \dots, b_j^N)$ is continuous in $\bar{\Omega}$, $j = 1, 2$. Let $f = (f_1, \dots, f_N)^T : \mathbb{R}^N \rightarrow \mathbb{R}^N$, $g = (g_1, \dots, g_N)^T : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be smooth functions.

It has been shown by Pao [1978] that if f is a source of heat and if $g = 0$ then we have blow up in finite time. Our aim is to control the increase of heat by means of a dissipative flux through the boundary. To accomplish this goal we need to introduce some kind of "competition" between f and g . In fact one of the basic questions is: If g dissipates heat through the boundary,

*Departamento de Matemática Aplicada, Instituto de Matemática e Estatística, USP

†Departamento de Matemática, Instituto de Matemática e Estatística, USP

can we find a relation between the dissipation g and the source of heat f in such a way that we can assure the existence of global attractors?

This problem is not new, Pao [1978] introduced a relation between f and g and almost completely solved the problem for classical solutions working in the space of continuous functions. Later on, Alikakos [1981], following the ideas of Friedman [1964], imposing some growth conditions on f and g , showed global existence and some asymptotic behavior of the solutions. Working in \mathbb{R} , Henry [1985], solved the problem completely, and assuming a nice relation between f and g (similar to Pao [1978]), showed that the system is Morse-Smale.

Another major concern is to relax most growth conditions on f and g . In this direction Carvalho and Rodriguez-Bernal [1994a] proved, for the scalar case, $N=1$, and $\Omega \subset \mathbb{R}^n$, $n \leq 3$, that under some growth assumptions on the nonlinearity f , the problem (1), with $B_j \equiv 0$, $j = 1, 2$, has a global attractor in $H^1(\Omega)$. More specifically, for $n = 2$, f and g are required to satisfy:

$$\left. \begin{array}{l} \liminf_{|s| \rightarrow \infty} \frac{\lambda s + f(s)}{s} \leq 0 \\ \liminf_{|s| \rightarrow \infty} \frac{g(s)}{s} \leq 0 \end{array} \right\} \text{one of the inequalities being strict,} \quad (2)$$

(the dissipative conditions)

and

$$\lim_{|s| \rightarrow \infty} \frac{|f'(s)|}{e^{\eta|s|^2}} = \lim_{|s| \rightarrow \infty} \frac{|g''(s)|}{e^{\eta|s|^2}} = 0, \quad \forall \eta > 0, \quad (3)$$

(the growth conditions).

(see also, Hale [1988], Hale and Raugel [1992], Carvalho [1992] and Carvalho and Oliveira [1992], for the case $g = 0$ and more restrictive hypothesis on f).

These growth assumptions are used to obtain local existence of solutions for (1) and also play a role in obtaining energy estimates necessary to guarantee that the solution operator for (1) defines a global dynamical system which is bounded dissipative.

Later on Carvalho and Rodriguez-Bernal [1994b], proved, in \mathbb{R}^2 , that if we drop (3) and just assume that both f and g dissipate energy, then one can actually show the existence of global attractors. The key idea is to restrict the space of initial data in such a way that no growth assumptions are needed for local existence of solutions for (1).

In particular, they consider spaces of initial conditions which are embedded in $C(\bar{\Omega})$. Since all functions in this space are bounded, no growth conditions on f and g are required for local existence. They show that in such spaces the problem (1) has a global attractor and obtain some good estimates for the size of the attractor in the uniform norm. This problem has been studied in Carvalho and Ruas-Filho [1993], for the case $g = 0$

The goal of this paper is to combine the ideas of competition in Pao [1978] and Henry [1985] with the ideas of Carvalho and Rodriguez-Bernal [1994b] to prove the existence of global attractors for the problem (1), without assuming dissipation on both f and g .

Since we are going to use many of the ideas in Carvalho and Rodriguez-Bernal [1994b], let us briefly describe them. Let $X = L^2(\Omega)$ and $A : D(A) \subset X \rightarrow X$ be the operator $A = \text{diag}(A_1, \dots, A_N)$ defined by

$$D(A_i) = \{ \phi \in H^2(\Omega) : \frac{\partial \phi}{\partial n_{a_i}} = 0 \},$$

$$A_i \phi = -\text{Div}(a_i(x) \nabla \phi) + \sum_{j=1}^2 b_j^i(x) \frac{\partial \phi}{\partial x_j} + \lambda \phi,$$

where $\frac{\partial u}{\partial n_{a_i}} = a_i \langle \nabla u, \vec{n} \rangle$, $i = 1, \dots, N$.

We can define the fractional powers A^α of A (see, for example, Henry [1981]) and the fractional power spaces $X^\alpha := D(A^\alpha)$ endowed with the graph norm, $\alpha \in \mathbb{R}$, where $X^\alpha = (X^{-\alpha})'$, if $\alpha < 0$. In this case we can always view A as a sectorial operator with compact resolvent from $X^{\alpha+1}$ into X^α which is positive and self adjoint.

Hale [1986] proved the existence of a local attractor for (1), with $g = 0$ and f satisfying (2), which coincides with the embedding of the attractor for $\dot{u} + \lambda u + f(u) = 0$ into the subspace of constant functions of X^α , $\alpha > \frac{3}{4}$, if the diffusion coefficient $a(x)$ is large (see also, Hale and Rocha [1987a,b] and Hale and Sakamoto [1989]). However, the techniques employed by Hale [1986] would only apply to global attractors if some a priori bound on the size of the absorbing set could be obtained and only if the diffusion coefficient is large (see Carvalho [1992] and Carvalho and Oliveira [1992]). This a priori bounds are obtained in Carvalho and Ruas-Filho [1993] for the case $g = 0$, also working in X^α , for $\alpha > \frac{3}{4}$ and $n = 3$.

Carvalho and Rodriguez-Bernal [1994b] proved the existence of a global attractor for the problem (1) regardless of the size of the diffusion coefficient

$a(x)$, for $g \neq 0$. They also found uniform bounds on the size of the attractor allowing the application of the results of Carvalho [1992] and Carvalho and Oliveira [1992] to the case $\alpha \neq \frac{1}{2}$ as in Hale [1986], Hale and Rocha [1987a,b] and Hale and Sakamoto [1989] (see also Fusco [1987]).

The approach followed was to show that the solution operator associated to (1) is globally defined, that orbits of bounded subsets of X^α , under the flow defined by (1), are bounded subsets of X^α and that there is a bounded set that attracts points of X^α . Since the solution operator associated to (1) is compact, Theorem 3.4.6 in Hale [1989] guarantees the existence of a global attractor.

Let us mention that Hale [1986], Hale and Rocha [1987a,b], Hale and Sakamoto [1989] and Carvalho and Ruas-Filho [1993], work in X^α for $\alpha > \frac{3}{4}$ and $n = 3$. The reason for this is the embedding $X^\alpha \subset C(\bar{\Omega})$. However, for $\alpha > \frac{3}{4}$, the space X^α incorporates the boundary condition $\frac{\partial u}{\partial n_a} = 0$ and therefore, cannot be the right space to work with if $g \neq 0$. However, for $n = 2$ and $\frac{3}{4} > \alpha > \frac{1}{2}$, we have that $X^\alpha \subset C(\bar{\Omega})$ and X^α does not incorporate any boundary condition. In this case we can use a variation of constants formula for the solutions, as shown by Rodriguez-Bernal [1993] (see also Amann [1986,1988] for a somewhat different approach). Therefore, we work in this range of α .

Observe that for $\Omega \subset \mathbb{R}$ we can take $\alpha = \frac{1}{2}$ and all the results hold true without any changes. Therefore, we restrict the presentation to the two dimensional case.

The paper will proceed as follows: in Section 2 we introduce the hypotheses that are going to be used in this paper, in Section 3 we define the spaces that are going to be used and show local existence of solutions for (1). In Section 4 we use the notion of sub- and super-solutions to show that the semigroup is bounded and that the solutions are defined for all time. Finally in Section 5 we show existence of global attractors and a bound for such attractors.

2 Hypotheses

In this Section we fix the hypotheses to be used throughout this paper.

(H1) $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and $g \in C^2(\mathbb{R}^N, \mathbb{R}^N)$ are such that, there exist $c_0, d_0 \in \mathbb{R}$ in such a way that the following condition is satisfied, for all

$i = 1, \dots, N,$

$$\left. \begin{aligned} \liminf_{s_i \rightarrow \pm\infty} \frac{f_i(s)}{s_i} &\leq c_0 \\ \liminf_{s_i \rightarrow \pm\infty} \frac{g_i(s)}{s_i} &\leq d_0 \end{aligned} \right\} \quad (4)$$

Moreover, if f, g satisfy (H1), and given the eigenvalue problem

$$\left. \begin{aligned} \operatorname{Div}(a \nabla v_i) - \sum_{j=1}^2 B_j(x) \frac{\partial v_i}{\partial x_j} - \lambda v_i + c_0 v_i &= \mu_i v_i, \quad \text{in } \Omega, \\ \frac{\partial v_i}{\partial n_a} &= d_0 v_i \quad \text{on } \partial\Omega \end{aligned} \right\} \quad (5)$$

we will assume the following,

(H2) All eigenvalues of (5) are negative.

Remark 2.1 To avoid notational complications we will treat only the case $N = 1$, but it will be clear from the proofs that the results remain true in more dimensions and the same arguments apply if we assume (H1).

Remark 2.2 (H1) is the dissipation condition on the equation, however we allow either c_0 or d_0 to be positive. In other words, we allow either f or g to be a source of heat.

Remark 2.3 (H2) is a precise formulation of the “competition” between f and g that we mentioned in the Introduction. Notice that we cannot have both c_0 and d_0 positive. Moreover this condition states that our problem “behaves” as an intermediate case between the Dirichlet case ($d_0 = \infty$) and the Neumann case ($d_0 = 0$).

3 Local Existence

Let us start by fixing the state spaces. First of all, it is well known that A defined by

$$D(A) = \left\{ \phi \in H^2(\Omega), \text{ such that } \frac{\partial \phi}{\partial n_a} = 0 \right\} \quad (6)$$

$$A\phi = -\text{Div}(a\nabla\phi) + \sum_{j=1}^2 B_j(x) \frac{\partial \phi}{\partial x_j} + \lambda\phi$$

where $\frac{\partial u}{\partial n_a} = a(\nabla u, \vec{n})$, generates an analytic semigroup on $X^\alpha = D(A^\alpha)$, for $0 < \alpha < 1$ which satisfies, for suitable λ

$$\|e^{-At}u_0\|_{X^\alpha} \leq Me^{-\epsilon t}\|u_0\|_{X^\alpha}, \quad t \geq 0 \quad (7)$$

$$\|e^{-At}u_0\|_{X^\alpha} \leq Me^{-\epsilon t}t^{-\alpha}\|u_0\|_X, \quad t > 0.$$

for some $\epsilon > 0, M > 0$. In particular, if $B_j \equiv 0, j = 1, 2, \lambda$ can be any positive number.

Let \tilde{A} be the extension of A to the space $X^\beta = H^{2\beta}$ with $\beta < 0$. \tilde{A} is also a sectorial operator and generates an analytic semigroup.

Suppose $\frac{1}{2} < \alpha < \frac{3}{4}$. Following Rodriguez-Bernal [1993] we consider, for $u \in X^\alpha$, the functional $h(u)$ which acts on test functions in the following way

$$\langle h(u), \phi \rangle = \langle f_\Omega(u), \phi \rangle + \langle g_\Gamma(u), \gamma(\phi) \rangle,$$

where γ denotes the trace operator, and $f_\Omega : X^\alpha \rightarrow L^2(\Omega)$ and $g_\Gamma : X^\alpha \rightarrow H^{\frac{1}{2}}(\Omega)$ are the maps defined by

$$\begin{aligned} f_\Omega(u)(x) &= f(u(x)), \text{ and} \\ g_\Gamma(u) &= \gamma(g_\Omega(u)), \text{ with} \\ g_\Omega(u)(x) &= g(u(x)). \end{aligned}$$

With this, we can state the following

Theorem 3.1 Assume that $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ are C^1 and C^2 functions respectively,

$$\frac{1}{2} < \alpha < \frac{3}{4}, \quad -\frac{1}{2} < \beta < -\frac{1}{4} \text{ and } \alpha < \beta + 1.$$

Then the abstract parabolic problem

$$\begin{cases} \frac{du}{dt} = \bar{A}u + h(u) \\ u(0) = u_0 \in X^\alpha \end{cases} \quad (8)$$

has an unique solution for any $u_0 \in X^\alpha$, which is given by the variation of constants formula

$$T(t)u_0 = e^{\bar{A}t}u_0 + \int_0^t e^{\bar{A}(t-s)}h(T(s)u_0)ds. \quad (9)$$

Moreover, if the maximal interval of existence of the solution $T(t)u_0$ is $[0, t_{\max}[$ then either $t_{\max} = +\infty$ or $\|T(t)u_0\|_{X^\alpha} \rightarrow \infty$ as $t \rightarrow t_{\max}$.

Proof:

The result follows from Henry's result [1981], once we prove that $h : X^\alpha \rightarrow X^\beta$ is Lipschitz continuous in bounded sets of X^α . This can be done using the continuity of the imbedding $X^\alpha \hookrightarrow C^0(\Omega)$. Details can be found in Carvalho and Rodriguez-Bernal [1994b].

Remark 3.1 *With the same hypotheses on f and g it can be proved (Rodriguez-Bernal [1993]) that any solution u of (8) satisfies*

$$t \rightarrow u \in H^2(\Omega), \text{ and } \frac{\partial u}{\partial n_\alpha} = g \text{ in } H^{\frac{1}{2}}(\Gamma).$$

So u can be considered a solution of our original problem (1).

4 Boundedness of the Semigroup

In this section we prove that solutions of (1) with initial data in X^α , are globally defined and orbits of bounded subsets of X^α , under the flow determined by (1), are also bounded in X^α .

To accomplish this goal, we use comparison results. We start by defining the concepts of sub- and super-solutions.

Definition 4.1 A C^2 function $\bar{u} : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ (\underline{u} respectively) is a super-solution (sub-solution) of the problem

$$\begin{cases} u_t = \text{Div}(a \nabla u) - \sum_{j=1}^2 B_j(x) \frac{\partial u}{\partial x_j} - \lambda u + f(u), & \text{in } \Omega, \\ \frac{\partial u}{\partial n_a} = g(u), & \text{on } \partial\Omega \\ u(0) = u_0. \end{cases} \quad (10)$$

if it satisfies

$$\begin{aligned} \bar{u}_t &\geq \text{Div}(a \nabla \bar{u}) - \sum_{j=1}^2 B_j(x) \frac{\partial \bar{u}}{\partial x_j} - \lambda \bar{u} + f(\bar{u}), & \text{in } \Omega, \\ \frac{\partial \bar{u}}{\partial n_a} &\geq g(\bar{u}), & \text{on } \partial\Omega \\ \bar{u}(0) &\geq u_0. \end{aligned} \quad (11)$$

(and respectively with the \geq sign replaced by the \leq sign).

A basic result for our arguments is the following

Theorem 4.1 (Pao [1978])

If f is locally Lipschitz and \bar{u} and \underline{u} are respectively a super- and sub-solution of the problem (10), satisfying

$$\underline{u} \leq \bar{u}, \text{ in } \Omega \times (0, T),$$

then, there exists a solution u of (10) such that

$$\underline{u} \leq u \leq \bar{u}, \text{ in } \Omega \times (0, T).$$

Let φ be the first positive normalized eigenfunction of (5) and $m = \min_{x \in \Omega} \varphi(x)$. We know that $m > 0$. For each $\theta \in \mathbb{R}_+$, define

$$\Sigma_\theta = \left\{ u \in X^\alpha : |u(x)| \leq \theta \varphi(x), \text{ for all } x \in \bar{\Omega} \right\}.$$

From the dissipative hypothesis (H1) on f and g , we know that there exists

$\xi \in \mathbb{R}$, such that

$$\frac{f(s)}{s} \leq c_0 \text{ and } \frac{g(s)}{s} \leq d_0,$$

for all s with $|s| \geq \xi$.

Lemma 4.1 *If $\theta m \geq \xi$ then Σ_θ is a positively invariant set for the local solution of (1).*

Proof:

Let

$$\Sigma_\theta^1 = \{u \in X^\alpha : u(x) \leq \theta\varphi(x), \text{ for all } x \in \bar{\Omega}\}$$

$$\Sigma_\theta^2 = \{u \in X^\alpha : u(x) \geq -\theta\varphi(x), \text{ for all } x \in \bar{\Omega}\}$$

Since $\Sigma_\theta = \Sigma_\theta^1 \cap \Sigma_\theta^2$ it is enough to show that Σ_θ^1 and Σ_θ^2 are positively invariant.

Let $u_0 \in \Sigma_\theta^1$, and suppose, for contradiction, that there exists $t_0 \in [0, t_{\max}[$ and $x_0 \in \bar{\Omega}$ such that

$$T(t_0)u_0(x_0) \geq \theta\varphi(x_0).$$

Consider $\bar{v}(t) = e^{\mu(t-t_0)}\theta\varphi$, where μ is the eigenvalue associated with φ . We have that

$$\frac{\partial \bar{v}}{\partial t} = \text{Div}(a\nabla \bar{v}) - \sum_{j=1}^2 B_j(x) \frac{\partial \bar{v}}{\partial x_j} - \lambda \bar{v} + c_0 \bar{v} \geq$$

$$\text{Div}(a\nabla \bar{v}) - \sum_{j=1}^2 B_j(x) \frac{\partial \bar{v}}{\partial x_j} - \lambda \bar{v} + f(\bar{v})$$

$$\frac{\partial \bar{v}}{\partial n_a} = d_0 \bar{v} \geq g(\bar{v}),$$

for all $t \in]0, t_0]$.

Thus \bar{v} is a super-solution for the problem (1). It follows from Theorem 4.1 that

$$T(t)u_0 \leq \bar{v}(t), \text{ in } \bar{\Omega} \text{ for all } t \in [0, t_0[.$$

In particular, $T(t_0)u_0(x_0) \leq \theta\varphi(x_0)$ and we reach a contradiction.

To prove that Σ_θ^2 is positively invariant we proceed in a similar way, using now that $\underline{v} = -\bar{v}$ is a sub-solution for the problem (1).

Lemma 4.2 *If B is a bounded subset of X^α , with $\frac{1}{2} < \alpha < \frac{3}{4}$, then $\cup_{t \geq 0} T(t)B$ is also a bounded subset of X^α .*

Proof:

Since the inclusion map $i : X^\alpha \hookrightarrow C^0(\bar{\Omega})$ is continuous, there exists $\theta \in \mathbb{R}$ such that $B \subset \Sigma_\theta$. We can, of course assume that $\theta m \geq \xi$. Lemma 4.1 implies that $T(t)u_0 \in \Sigma_\theta$, for all $t \in [0, t_{max}]$ so

$$\|T(t)u_0\|_\infty \leq \theta \|\varphi\|_\infty.$$

Applying the variation of constants formula, we obtain

$$\|T(t)u_0\|_{X^\alpha} \leq M e^{-\epsilon t} \|u_0\|_{X^\alpha} + M \int_0^t (t-s)^{-\alpha+\beta} e^{-\epsilon(t-s)} \|h(T(s)u_0)\|_{X^\beta} ds,$$

where $M, \epsilon > 0$ are constants depending only on the semigroup e^{At} and $-\frac{1}{2} < \beta < -\frac{1}{4}$, $\alpha < \beta + 1$.

To compute the norm $\|h(T(s)u_0)\|_{X^\beta}$, let $\phi \in X^{-\beta} = H^{-2\beta}(\Omega)$. We have

$$\begin{aligned} \langle h(T(s)u_0), \phi \rangle_{-2\beta, -2\beta} &= \int_\Omega f_\Omega(T(s)u_0) \phi(x) dx + \int_\Gamma g_\Gamma(T(s)u_0) \gamma(\phi(x)) dx \\ &\leq \|f_\Omega(T(s)u_0)\|_{L^2(\Omega)} \|\phi\|_{L^2(\Omega)} + \|g_\Gamma(T(s)u_0)\|_{L^2(\Gamma)} \|\gamma(\phi)\|_{L^2(\Gamma)} \\ &\leq (\|f_\Omega(T(s)u_0)\|_{L^2(\Omega)} + K \|g_\Gamma(T(s)u_0)\|_{L^2(\Gamma)}) \|\phi\|_{H^{-2\beta}}, \end{aligned}$$

where K is a bound for the continuous linear map $\gamma : H^{-2\beta}(\Omega) \rightarrow L^2(\Gamma)$.

Thus,

$$\begin{aligned}
\|T(t)u_0\|_{X^\alpha} &\leq Me^{-\alpha t}\|u_0\|_{X^\alpha} + M \int_0^{t_{\max}} \left(K\|g_\Gamma(T(s)u_0)\|_{L^2(\Gamma)} \right. \\
&\quad \left. + \|f_\Omega(T(s)u_0)\|_{L^2(\Omega)} \right) (t-s)^{-\alpha+\beta} e^{-\alpha(t-s)} ds \\
&\leq Me^{-\alpha t}\|u_0\|_{X^\alpha} + M \int_0^{t_{\max}} \left[K\|g_\Gamma(T(s)u_0)\|_\infty |\Gamma|^{\frac{1}{2}} \right. \\
&\quad \left. + \|f_\Omega(T(s)u_0)\|_\infty |\Omega|^{\frac{1}{2}} \right] (t-s)^{-\alpha+\beta} e^{-\alpha(t-s)} ds \\
&\leq Me^{-\alpha t}\|u_0\|_{X^\alpha} + M \left[\sup_{|\tau| \leq \theta \|v\|_\infty} |g(\tau)| |\Gamma|^{\frac{1}{2}} \right. \\
&\quad \left. + \sup_{|\tau| \leq \theta \|v\|_\infty} |f(\tau)| |\Omega|^{\frac{1}{2}} \right] \int_0^\infty (t-s)^{-\alpha+\beta} e^{-\alpha(t-s)} ds,
\end{aligned}$$

for all $t \in [0, t_{\max}[$. Therefore, $\|T(t)u_0\|_{X^\alpha}$ is bounded by a constant depending only on B . In particular, $t_{\max} = \infty$.

5 Existence of Global Attractors

The first step to show the existence of global attractors will be to show a “contraction property” of the sets Σ_θ , similar to the property for rectangles, considered by Carvalho [1993]. It is interesting to notice that we cannot use rectangles here since they are not invariant (unless f and g are both negative). In fact if f is positive at some point $x_0 \in \Omega$ and a constant function u_0 is chosen as an initial condition at time t_0 then $\frac{\partial u}{\partial t}(u_0, t_0, x_0) > 0$, so $T(t)u_0$ grows at the point x_0 for some time. A similar argument can be used for a point on $\partial\Omega$ if $g > 0$. We show in Fig 1 below the result of a simulation (for $N=1$) taking $c_0 > 0$ and $d_0 < 0$.

Lemma 5.1 *Suppose $\bar{\theta} \in \mathbb{R}$ satisfy $\bar{\theta}m > \xi$. Then, for any θ there exists a \bar{t} such that*

$$T(t)\Sigma_\theta \subset \Sigma_{\bar{\theta}},$$

for all $t \geq \bar{t}$.

Proof:

Let $u \in \Sigma_\theta$. We can suppose without loss of generality that $\theta \geq \bar{\theta}$. Let $\bar{v} = e^{\mu t} \theta \varphi$, $\underline{v} = -\bar{v}$. As in Lemma 4.1, we can prove that \bar{v} and \underline{v} are super- and sub-solutions respectively. Thus, using Theorem 4.1 and the uniqueness of solution, we have that

$$\underline{v} \leq T(t)u \leq \bar{v},$$

as long as $e^{\mu t} \theta \geq \bar{\theta}$.

So, $T(t)u$ enters $\Sigma_{\bar{\theta}}$ eventually. Since $\Sigma_{\bar{\theta}}$ is positively invariant, the result follows.

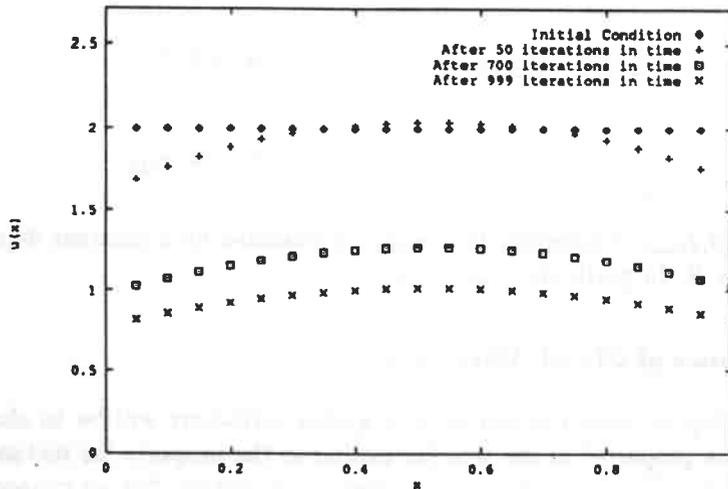


Figure 1: Simulation for $N = 1$, and with initial condition $u_0 \equiv 2$. We also used $c_0 > 0$ and $d_0 < 0$ (defined in (H1)).

Theorem 5.1 *The problem (1) has a global attractor \mathcal{A} in X^α . Furthermore $u(x) \in \Sigma_\theta$, for all $x \in \bar{\Omega}$, $u \in \mathcal{A}$ if $\theta m \geq \xi$.*

Proof:

Since, by Lemma 4.2, $T(t)$ takes bounded sets of X^α into bounded sets of X^α for any $t \geq 0$, and the semigroup regularizes the solutions, only point dissipativeness remains to be proved (see Hale [1988]).

Let $\bar{\theta} \in \mathbb{R}$ be such that $\bar{\theta}m \geq \xi$. If u is any element of X^α , it follows from the continuity of the imbedding $X^\alpha \hookrightarrow C^0(\bar{\Omega})$ that $u \in \Sigma_\theta$, for some θ and then, applying Lemma 5.1, we conclude that $T(t)u \in \Sigma_{\bar{\theta}}$, for t big enough. Let $v = T(t_0)u \in \Sigma_{\bar{\theta}}$.

Applying the variation of constants formula, as in Lemma 4.2, we obtain

$$\begin{aligned} \|T(t)v\|_{X^\alpha} &\leq Me^{-\epsilon t} \|v\|_{X^\alpha} + M \int_0^\infty \left[K \|g_\Gamma(T(s)v)\|_\infty |\Gamma|^{\frac{1}{2}} \right. \\ &\quad \left. + \|f_\Omega(T(s)v)\|_\infty |\Omega|^{\frac{1}{2}} \right] (t-s)^{-\alpha+\beta} e^{-\epsilon(t-s)} ds, \end{aligned}$$

where M and K are independent of v .

Observing that $T(s)v \in \Sigma_{\bar{\theta}}$, for any $s \geq 0$, we conclude that, for t sufficiently large,

$$\begin{aligned} \|T(t_0+t)u\|_{X^\alpha} &= \|T(t)v\|_{X^\alpha} \leq M + \\ &M \left[\sup_{|\tau| \leq \bar{\theta} \|v\|_\infty} |g(\tau)| |\Gamma|^{\frac{1}{2}} + \sup_{|\tau| \leq \bar{\theta} \|v\|_\infty} |f(\tau)| |\Omega|^{\frac{1}{2}} \right] \int_0^\infty (t-s)^{-\alpha+\beta} e^{-\epsilon(t-s)} ds, \end{aligned}$$

for t large.

Thus, the set in X^α bounded by the right-hand side above, attracts points. This proves point dissipativeness. Furthermore, since $\mathcal{A} \subset \Sigma_\theta$, for some θ , it follows from Lemma 5.1, taking t large, that

$$\mathcal{A} = T(t)\mathcal{A} \subset \Sigma_{\bar{\theta}}, \quad \text{if } \bar{\theta} > \xi$$

so

$$\mathcal{A} \subset \Sigma_{\bar{\theta}} = \bigcap_{\bar{\theta}m > \xi} \Sigma_{\bar{\theta}},$$

which proves the second part of the thesis.

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Electronic Mail: smo@ime.usp.br, alpereir@ime.usp.br