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On the **Δ**-property for complex space forms

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Abstract

Inspired by the work of Lu and Tian (Duke Math J 125:351--387, 2004), Loi et al. address in (Abh Math Semin Univ Hambg 90: 99-109, 2020) the problem of studying those Kähler manifolds satisfying the Δ-property, i.e. such that on a neighborhood of each of its points the k-th power of the Kähler Laplacian is a polynomial function of the complex Euclidean Laplacian, for all positive integer k. In particular they conjectured that if a Kähler manifold satisfies the Δ-property then it is a complex space form. This paper is dedicated to the proof of the validity of this conjecture.

Keywords Kähler manifolds · Hermitian symmetric spaces · Kähler Laplacian

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1 Introduction and statement of the main result

Let Δ be the Kähler Laplacian on an n-dimensional Kähler manifold (M, g) i.e., in local coordinates $z = \{z_j\}$,

$$\Delta = \sum_{i,j=1}^{n} g^{i\bar{j}} \frac{\partial^2}{\partial z_j \partial \bar{z}_i},$$

where $g^{i\bar{j}}$ denotes the inverse matrix of the Kähler metric. We define the complex Euclidean Laplacian with respect to z as the differential operator

$$\Delta_c^z = \sum_{i=1}^{n} \frac{\partial^2}{\partial z_i \partial \bar{z}_i}.$$

A. Loi, F. Salis and F. Zuddas introduce in [5], the following notion of Δ-property:

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Definition 1 (Δ-property) For any arbitrary point $x \in M$ there exists a coordinate system z centered at x, such that every smooth function ϕ defined in a neighborhood of x fulfills the following equation for every positive integer k

$$\Delta^k \phi(0) = p_k(\Delta_c^z)\phi(0), \tag{1}$$

where p_k is a monic polynomial of degree k, independent of x, with real coefficients.

A key point in Lu and Tian's proof of the local rigidity theorem ([8] Theor. 1.2) supporting their conjecture about the characterization of the Fubini-Study metric g_{FS} on $\mathbb{C}P^n$ through the vanishing of the log-term of the universal bundle, was that the Δ-property is satisfied by $\left(\mathbb{C}P^n, g_{FS}\right)$ with respect to affine coordinates.

In [5] A. Loi, F. Salis and F. Zuddas address the problem of studying those Kähler manifolds satisfying the Δ-property. They observe that condition (1) is satisfied for any positive integer k in the center of a radial metric[1]. It arises naturally the problem to try to classify Kähler manifolds satisfying the Δ-property. In this direction their main results proved in [5] are the following two theorems:

Theorem A ([5, Theorem 1.3]). *Let (M, g) be a Kähler manifold which satisfies the Δ-property. Then its curvature tensor is parallel.*

Theorem B ([5, Theorem 1.4]). *An Hermitian symmetric space of classical type satisfying the Δ-property is a complex space form.*

By virtue of these two results the author conjectured in [5] that complex space forms can be characterized as the Kähler manifolds satisfying the Δ-property. The proof of this conjecture is indeed the main result of this paper. More precisely we prove the following result:

Theorem 1 *Let M be a Kähler manifolds satisfying the Δ-property. Then M is a complex space form.*

The proof of the theorem is based on Jordan triple system machinery.

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2 Proof of Theorem 1

The first step (see Proposition 1) is to prove that the Δ-property characterizes the complex hyperbolic space among Hermitian symmetric spaces of noncompact type (from now on HSSNCT). Let us write $\mathbb{C}H_r^1$ to denote the product of r complex hyperbolic spaces $\mathbb{C}H^1 = \left\{z \in \mathbb{C} \mid |z|^2 < 1\right\}$ equipped with the product metric $g_{hyp}^r = g_{hyp} \oplus ... \oplus g_{hyp}$, where the fundamental form associated to g_{hyp} is $\omega_{hyp} = -\frac{i}{2}\partial\bar{\partial}\log(1 - |z|^2)$. We call *affine coordinates*, the coordinates on $\mathbb{C}H_r^1$ induced by the product. The key result in our proof of

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[1] Namely a Kähler metric admitting a Kähler potential which depends only on the sum $|z|^2 = |z_1|^2 + ... + |z_n|^2$ of the moduli of a local coordinates' system z.

Theorem 1 is the following technical lemma, which extends [5, Lemma 3.1] valid for classical Hermitian symmetric spaces of compact type (from now on HSSCT) using Jordan triple system theory instead of Alekseevsky–Perelomov coordinates.

Lemma 1 *Let M be an n-dimensional HSSNCT of rank r endowed with a Kähler-Einstein metric g. Then there exists a normal global coordinates system $w = \{w_j\}_{j=1,...,n}$ such that the Kähler immersion's equations of $\left(\mathbb{C}H_r^1, g_{hyp}^r\right)$ into M read as*

$$\begin{cases} w_i = z_i \text{ for } i = 1, ..., r \\ w_i = 0 \text{ for } i = r+1, ..., n \end{cases},$$
(2)

where $z = \{z_j\}_{j=1,...,r}$ are affine coordinates on $\mathbb{C}H_r^1$.

Proof Without loss of generality we can assume that M is irreducible. Throughout the proof we use Jordan triple system theory, referring the reader to [2–4, 6, 7, 9–11] for details and further applications.

Let $(V, \{,,\})$ be the Hermitian positive Jordan triple system (from now on HPJTS) associated to M. Let $x = \lambda_1 c_1 + \cdots + \lambda_s c_s, \lambda_1 > \cdots > \lambda_s > 0$ be the spectral decomposition ([11, Definition VI.2.2]) of an element $x \in V$. By [11, Proposition VI.4.2], we can realize (M, g) as a bounded symmetric domain

$$\Omega = \{x \in V \mid \lambda_1 < 1\}$$
(3)

equipped with the Kähler-Einstein form (unique up to rescaling): $\omega(z) = -\frac{i}{2}\partial\bar{\partial}\log N(z,\bar{z})$, where N is the generic norm of V ([2, Sect. 2.2]).

Consider a frame $\mathcal{B} = \{e_1, ..., e_r\} \subset V$, namely a maximal set of mutually orthogonal, primitive tripotents ([11, Definition VI.2.1]). Let $W \subset V$ be the complex vector subspace $W = \text{span}_{\mathbb{C}}\{e_1, ..., e_r\}$. If $x = \sum_{j=1}^r x_j e_j, y = \sum_{j=1}^r y_j e_j, z = \sum_{j=1}^r z_j e_j$ are elements of W, we have (see [11, (6.11)])

$$\{x, y, z\} = 2\sum_{j=1}^s x_j \bar{y}_j z_j e_j \in W.$$

Hence any frame $\{c_1, ..., c_r\}$ of W has the form

$$c_j = e^{i\theta_j} e_{\sigma(j)}, \quad 1 \leqslant j \leqslant r$$
(4)

where $\sigma \in \mathfrak{S}_r$ is a permutation of $\{1, ..., r\}$ and W is a Hermitian positive Jordan triple subsystem of $(V, \{,,\})$. It is well know that there exists a one to one correspondence between sub-HPJTS $V' \subset V$ and complex totally geodesic sub-HSSNCT $\Omega' \subset \Omega$ (see e.g. [2, Proposition 2.1]), given by

$$V' \mapsto \Omega' = \Omega \cap V'.$$

We want to determine the HSSNCT associated to $\left(W, \{,,\}_{|W}\right)$. Let $x \in W$ and let $x = \lambda_1 c_1 + \cdots + \lambda_s c_s, \lambda_1 > \cdots > \lambda_s > 0$ be its spectral decomposition (notice that, by [11, Proposition VI.2.4], an element $x \in W$ has the same spectral decomposition in V and W). As recalled above, the associated HSSNCT realized as a bounded symmetric domain Δ^r of W is given by $\Delta^r = \{x \in W \mid \lambda_1 < 1\}$. Fixed the complex basis $\mathcal{B} = \{e_1, ..., e_r\}$ defined above, we can identify W with $\mathbb{C}^r$. With respect to this coordinates we have

$$\Delta^r = \left\{ (z_1, \ldots, z_r) \mid |z_j| < 1, j = 1, \ldots, r \right\} \subset \mathbb{C}^r$$

(compare this construction with [6, Sec. 4.5.] and [3, Example 6]). Denoted by N_W the generic norm of W, we see that the associated Kähler-Einstein form is given by

$$\omega_\Delta(z) = -\frac{i}{2}\partial\bar{\partial}\log N_W(z, \bar{z}) = -\frac{i}{2}\partial\bar{\partial}\log\left(\prod_{j=1}^r \left(1 - \left|z_j\right|^2\right)\right),$$

where we used (4) and the fact (see e.g. [11, Proposition VI.2.6]) that, with respect to spectral coordinates $x = \lambda_1 c_1 + \cdots + \lambda_s c_s$, the generic norm N of an HPJTS is given by $N(x, x) = \prod_{j=1}^s \left(1 - \lambda_j^2\right)$. We conclude that $\left(\Delta, \omega_\Delta\right) = \left(\mathbb{C}H_r^1, g_{hyp}^r\right)$. If we complete $\mathcal{B}$ to a complex basis $\left\{e_1, \ldots, e_r, f_1, \ldots, f_{n-r}\right\}$ of V, we can identify V with $\mathbb{C}^n$ obtaining coordinates $\tilde{w} = \left\{\tilde{w}_j\right\}$ for M satisfying (2). Let us choose $f_1, \ldots, f_{n-r}$ in such a way that $g_{jk}(0) = g\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}\right)(0) = \delta_{jk}$, $1 \leq j, k \leq n$. We introduce now (see e.g. [1, 4.17 Theorem]) new coordinates $w = \left\{w_j\right\}$, by solving $\tilde{w}_j = w_j + \frac{1}{2}A_{kl}^j w_k w_l$, where $A_{kl}^j = -\frac{\partial g_{l\bar{j}}}{\partial z_k}(0)$, obtaining normal coordinates which satisfy (2) (notice that $A_{kl}^j = 0$, for $1 \leq j, k, l \leq r$). The proof is complete. $\square$

We are now in a position to characterize complex hyperbolic spaces among irreducible HSSCNT. This result should be compared with [5, Theorem 3.2], where the authors characterize complex projective spaces among irreducible classical HSSCT via Δ-property.

Proposition 1 *The complex hyperbolic space is the unique HSSNCT which satisfies the Δ-property.*

Proof Let M be an n-dimensional HSSNCT endowed with its Kähler-Einstein metric g. We denote by λ the Einstein constant. Let $\tilde{z} = \left\{\tilde{z}_j\right\}$ be a holomorphic normal coordinate system centered in a point $x \in M$. We have[2]

$$\lambda g_{i\bar{j}} = \text{Ric}_{i\bar{j}} = g^{k\bar{h}}\left(-\partial_{k\bar{h}}g_{i\bar{j}} + g^{p\bar{q}}\partial_k g_{i\bar{q}}\partial_{\bar{h}}g_{p\bar{j}}\right). \tag{5}$$

Hence, if we evaluate the previous equation at 0, we get

$$\sum_h \partial_{h\bar{h}}g^{i\bar{j}}(0) = \lambda\delta^{ij}. \tag{6}$$

By (6), we get

$$\Delta^2\phi(0) = g^{h\bar{k}}\partial_{k\bar{h}}\left(g^{i\bar{j}}\partial_{\bar{j}i}\phi\right)\Big|_0 = \left((\Delta_c^{\tilde{z}})^2 + \lambda\Delta_c^{\tilde{z}}\right)\phi(0) \tag{7}$$

that is (1) is satisfied also for $k = 2$.

By combining (5), (6) and (7) above, we get that every smooth function ϕ defined in a neighborhood V of the origin fulfills the following

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[2] We are going to use the notation ∂_i to denote $\frac{\partial}{\partial z_i}$ and a similar notation for higher order derivatives. We are also going to use Einstein's summation convention for repeated indices.

$$\Delta^3\phi(0) = \left((\Delta_c^{\tilde{z}})^3 + 3\lambda(\Delta_c^{\tilde{z}})^2 + \lambda^2\Delta_c^{\tilde{z}}\right)\phi(0) + 2\sum_{l,h=1}^n \partial_{l\bar{h}}g^{i\bar{j}}\partial_{jh\bar{l}\bar{i}}\phi\Big|_0 +$$

$$+ \sum_{l,h=1}^n \partial_{lh}g^{i\bar{j}}\partial_{j\bar{h}\bar{l}\bar{i}}\phi\Big|_0 + \sum_{l,h=1}^n \partial_{\bar{l}\bar{h}}g^{i\bar{j}}\partial_{jhl\bar{i}}\phi\Big|_0 + \sum_{l,h=1}^n \partial_{lh\bar{l}\bar{h}}g^{i\bar{j}}\partial_{j\bar{i}}\phi\Big|_0, \tag{8}$$

where we use that by differentiating (5) and evaluating in the origin, the coefficients of the third order derivatives of ϕ vanish.

Let $\{\tilde{z}_j = w_j\}$ be the system of normal coordinates given in Lemma 1 and assume (up to automorphism of M) that they are centered at x. If $\{z_i\}_{i=1,\dots,r}$ are affine coordinates on $(\mathbb{C}H_r^1, g_{hyp}^r)$, by taking into account Lemma 1, we can compute

$$\Delta^3\left(|z_1|^4\right)\Big|_0 = 3\lambda\left(\Delta_c^w\right)^2(|z_1|^4)\Big|_0 + 8\frac{\partial^2 g^{1\bar{1}}}{\partial w_1 \partial \overline{w}_1}\Big|_0 = 12\lambda + 16. \tag{9}$$

Furthermore, if $r \neq 1$, namely if M is different from a complex hyperbolic space, we also compute

$$\Delta^3\left(|z_1 z_2|^2\right)\Big|_0 = 3\lambda\left(\Delta_c^w\right)^2(|z_1 z_2|^2)\Big|_0$$

$$+ 4\left(\frac{\partial^2 g^{2\bar{2}}}{\partial w_1 \partial \overline{w}_1} + \frac{\partial^2 g^{1\bar{1}}}{\partial w_2 \partial \overline{w}_2} + \frac{\partial^2 g^{1\bar{2}}}{\partial w_2 \partial \overline{w}_1} + \frac{\partial^2 g^{2\bar{1}}}{\partial w_1 \partial \overline{w}_2}\right)\Big|_0 \tag{10}$$

$$= 6\lambda.$$

If M has rank greater than 1, let us assume by contradiction that the Δ-property is valid, in particular around each point of M there exists a local coordinate system with respect to which (1) is satisfied for $k = 1, 2, 3$. Let us denote such coordinate system by $f = (f_1, \dots, f_n)$. Since in [5, Theorem 2.1] is showed that every second order derivative of the holomorphic change of coordinates sending f to $\tilde{z}$ vanish at $f = 0$, we get

$$\Delta^3\phi(0) = \left((\Delta_c^f)^3 + \sum_{i=1}^2 a_i(\Delta_c^f)^i\right)\phi(0) =$$

$$= \left(\sum_{i=1}^2 a_i(\Delta_c^{\tilde{z}})^i\right)\phi\Big|_0 + \sum_{i_1,i_2,i_3,\alpha,\beta} \frac{\partial^3\tilde{z}_\alpha}{\partial f_{i_1}\partial f_{i_2}\partial f_{i_3}}\overline{\frac{\partial^3\tilde{z}_\beta}{\partial f_{i_1}\partial f_{i_2}\partial f_{i_3}}}\frac{\partial^2\phi}{\partial\tilde{z}_\alpha\partial\overline{\tilde{z}}_\beta}\Big|_0 +$$

$$+ (\Delta_c^{\tilde{z}})^3\phi\Big|_0 + \sum_{\substack{i_1,i_2,i_3 \\ \alpha_1,\dots,\alpha_4}} \frac{\partial^3\tilde{z}_{\alpha_4}}{\partial f_{i_1}\partial f_{i_2}\partial f_{i_3}}\prod_{l=1}^3\overline{\frac{\partial\tilde{z}_{\alpha_l}}{\partial f_{i_l}}}\frac{\partial^4\phi}{\partial\tilde{z}_{\alpha_1}\partial\overline{\tilde{z}}_{\alpha_2}\partial\overline{\tilde{z}}_{\alpha_3}\partial\overline{\tilde{z}}_{\alpha_4}}\Big|_0 +$$

$$+ \sum_{\substack{i_1,i_2,i_3 \\ \alpha_1,\dots,\alpha_4}} \overline{\frac{\partial^3\tilde{z}_{\alpha_4}}{\partial f_{i_1}\partial f_{i_2}\partial f_{i_3}}}\prod_{l=1}^3\frac{\partial\tilde{z}_{\alpha_l}}{\partial f_{i_l}}\frac{\partial^4\phi}{\partial\tilde{z}_{\alpha_1}\partial\tilde{z}_{\alpha_2}\partial\tilde{z}_{\alpha_3}\partial\overline{\tilde{z}}_{\alpha_4}}\Big|_0.$$

The previous formula implies the relation

$$\Delta^3\left(|z_1|^4\right)(0) = 2\Delta^3\left(|z_1 z_2|^2\right)(0),$$

therefore we have a contradiction from the comparison with (9) and (10). The proof is complete. $\qquad\square$

Remark 1 Notice that (see [5, Remark 3]) we have proved the stronger statement that the complex hyperbolic space is the unique HSSNCT such that around any point there exists a global coordinate system with respect to which (1) is satisfied for $k = 1, 2, 3$.

Finally, we can prove our main result.

Proof of Theorem 1 By Theorem A in the introduction, a Kähler manifold (M, g) satisfying the Δ-property is an Hermitian symmetric space. Therefore (M, g) can be decomposed as a Kähler product

$$(\mathbb{C}^n, g_0) \times (C_1, g_1) \times ... \times (C_h, g_h) \times (N_1, \hat{g}_1) \times ... \times (N_l, \hat{g}_l),$$

where $(\mathbb{C}^n, g_0)$ is the flat Euclidean space, (C_i, g_i) are irreducible HSSCT and $(N_i, \hat{g}_i)$ are irreducible HSSNCT.

By [5, Theorem 2.1], a Hermitian symmetric space where (1) is fulfilled for $k = 1, 2$, is the flat Euclidean space otherwise it is a Kähler product of Hermitian symmetric space of either compact or noncompact type. Hence, we are going to prove our statement by characterizing the complex projective space form among HSSCT in analogy with what we have done for hyperbolic spaces in Proposition 1.

Let (C, g) be an HSSCT, let (C^*, g^*) the non compact dual and $(V, \{,,\})$ be the associated HPJTS. Let us identify V with $\mathbb{C}^n$ by fixing any complex basis of V. Then (see e.g. [2, Sect. 2.4]) V equipped with the Kähler form

$$\omega_{FS} = \frac{i}{2}\partial\bar{\partial}\log N(z, -z)$$

is holomorfically isometric (up to homotheties) to an open dense subset of (C^*, g^*), therefore we can consider the coordinate system given in Lemma 1 as local coordinates for (C, g). With respect this coordinates, the Kähler potential $\Phi = \log N(z, -z)$ for the metric g satisfies

$$\Phi(z, \bar{z}) = -\Phi^*(z, -\bar{z})_{|C^*} \tag{11}$$

where $\Phi^*(z, \bar{z}) = -\log N(x, x)$ is the Kähler potential for g^* given in (3).

By (11) and (8), we get

$$\Delta_C^3\left(|z_i \bar{z}_j|^2\right)(0) = -\Delta_{C^*}^3\left(|z_i \bar{z}_j|^2\right)(0)$$

for every $1 \le i, j \le \dim(C)$. Hence, if C^* is not the hyperbolic space, namely if C is not a complex projective space, (1) for $k = 3$ cannot be satisfied as proved in Proposition 1. The proof is complete. $\qquad\square$

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