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**THE GEOMETRIC EXPONENTIAL POISSON DISTRIBUTION**

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# The geometric exponential Poisson distribution

by

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**Abstract:** Many if not most lifetime distributions are motivated only by mathematical interest. Here, a new three parameter distribution motivated mainly by lifetime issues is introduced. Some properties of the new distribution including estimation procedures, univariate generalizations and bivariate generalizations are derived. Two real data applications are described to show superior performance versus at least five of the known lifetime models.

**Keywords:** Exponential distribution; Geometric distribution; Maximum likelihood estimation; Poisson distribution.

## 1 Introduction

The statistics literature has numerous distributions for modeling lifetime data. But many if not most of these distributions lack motivation from a lifetime context. For example, there is no apparent physical motivation for the gamma distribution. It only has a more general mathematical form than the exponential distribution with one additional parameter, so it has nicer properties and provides better fits. The same arguments apply to Weibull and many other distributions.

The aim of this paper is to introduce a new three parameter lifetime distribution with strong physical motivation. As explained below, the proposed distribution encompasses the behavior of and provides better fits than many of the known lifetime distributions, including those with three parameters. We feel that that this is a remarkable feature.

We provide at least three motivations for the new distribution. We begin with one based on failures of a system.

Suppose a company has  $N$  systems functioning independently at a given time, where  $N$  is a geometric random variable with the probability mass function

$$\Pr(N = n) = (1 - \eta)\eta^{n-1} \quad (1)$$

for  $0 < \eta < 1$  and  $n = 1, 2, \dots$ . The geometric distribution is a popular model for counts.

Suppose in addition to (1) that each system is made of  $M$  parallel units, so the system will fail if all of the units fail. Assume that  $M$  is a truncated Poisson random variable independent of  $N$  with the probability mass function

$$\Pr(M = m) = \frac{\theta^m \exp(-\theta)}{m! [1 - \exp(-\theta)]} \quad (2)$$

for  $\theta > 0$  and  $m = 1, 2, \dots$ . The zero truncated Poisson distribution is also a popular model for counts. Some of its recent applications include: modeling number of illegal immigrants in four large cities in the Netherlands (van der Heijden *et al.*, 2003), models for mental health services data (Elhai *et al.*, 2008), modeling word or species frequency count data (Ginebra and Puig, 2010), and models for fertility trait phenotypes (Xu and Hu, 2011).

Assume further that the failure times of the units for the  $i$ th system, say  $Z_{i,1}, Z_{i,2}, \dots, Z_{i,M}$ , are independent and identical exponential random variables with the scale parameter  $\lambda$ . Let  $Y_i$  denote the failure time of  $i$ th system. Let  $X$  denote the time to failure of the first out of the  $N$  functioning systems. We can write  $X = \min(Y_1, Y_2, \dots, Y_N)$ . Then the cumulative distribution function of  $X$ , say  $G(x)$ , can be derived as:

$$\begin{aligned} G(x) &= \Pr[\min(Y_1, Y_2, \dots, Y_N) \leq x] \\ &= 1 - \Pr[\min(Y_1, Y_2, \dots, Y_N) > x] \\ &= 1 - \eta \sum_{n=1}^{\infty} \Pr^n(Y > x) \eta^{n-1} \\ &= 1 - \frac{(1 - \eta) \Pr(Y > x)}{1 - \eta \Pr(Y > x)} \\ &= \frac{\Pr(Y \leq x)}{1 - \eta \Pr(Y > x)} \end{aligned}$$

and

$$\begin{aligned} \Pr(Y > x) &= 1 - \Pr(Y \leq x) \\ &= 1 - \Pr[\max(Z_{i,1}, Z_{i,2}, \dots, Z_{i,M}) \leq x] \\ &= 1 - \sum_{m=1}^{\infty} [1 - \exp(-\lambda x)]^m \frac{\theta^m \exp(-\theta)}{m! [1 - \exp(-\theta)]} \\ &= 1 - \frac{\exp[\theta - \theta \exp(-\lambda x)] - 1}{\exp(\theta) - 1}, \end{aligned}$$

so

$$G(x) = \frac{\exp[-\theta \exp(-\lambda x)] - \exp(-\theta)}{1 - \exp(-\theta) - \eta \{1 - \exp[-\theta \exp(-\lambda x)]\}} \quad (3)$$

for  $x > 0$ ,  $\theta > 0$ ,  $\lambda > 0$  and  $0 < \eta < 1$ . We shall refer to the distribution given by (3) as the *geometric exponential Poisson* (GEP) distribution. The parameters,  $\theta$  and  $\eta$ , control the shape. The parameter,  $\lambda$ , controls the scale. The particular case of (3) for  $\theta \rightarrow 0$  is the *Exponential Geometric* (EG) distribution due to Adamidis and Loukas (1998). The particular case for  $\eta \rightarrow 0$  is the *exponential Poisson* (EP) distribution due to Kuş (2007).

Our second motivation is based on hazard rate function, an important characteristic for lifetime modeling. Most lifetime distributions, including the exponentiated exponential distribution, exhibit only monotonically increasing, monotonically decreasing or constant hazard rates. In addition, most lifetime distributions, including the exponentiated exponential distribution, incorporate constant hazard rate as a real particular case. These are very unrealistic features especially because there are hardly any real-life systems that have constant hazard rates.

In Section 3, we shall show that the GEP distribution exhibits monotonically increasing, monotonically decreasing and upside down bathtub hazard rates. We shall also see that the GEP distribution does not exhibit constant hazard rates. However, the GEP distribution does not also exhibit bathtub hazard rates. The latter may be a weakness of the GEP distribution. But we shall see later in Section 15 that (3) is only a particular case of the family of distributions introduced in this paper. Other members of this family may exhibit bathtub hazard rates.

Not many lifetime distributions exhibit upside down bathtub hazard rates. That the GEP distribution exhibits upside down bathtub hazard rates is an attractive feature. Upside down bathtub hazard rates are common in reliability and survival analysis. For example, such hazard rates can be observed in the course of a disease whose mortality reaches a peak after some finite period and then declines gradually (Silva *et al.*, 2010). For other practical examples yielding upside down bathtub hazard rates, see Singh and Misra (1994).

Our final motivation is empirical based. We show later that the proposed distribution outperforms at least five of the known two- and three-parameter distributions with respect to two real data sets. These include:

For technical reasons later on, it is useful to have a more manageable form of (3). Using the expansion

$$(1-x)^{-a} = \sum_{k=0}^{\infty} \binom{-a}{k} (-a)^k, \quad (4)$$

we can write (3) as

$$G(x) = \frac{\exp(-\theta) [\exp(-\lambda x) - 1]}{1 - \exp(-\theta) - \eta} \sum_{k=0}^{\infty} \binom{-1}{k} \left[ \frac{\eta}{1 - \exp(-\theta) - \eta} \right]^k \exp[-k\theta \exp(-\lambda x)]. \quad (5)$$

Taking the derivatives with respect to  $x$  on both sides of (5), we obtain

$$g(x) = \frac{\theta \lambda (1 - \eta) [1 - \exp(-\theta)]}{[1 - \exp(-\theta) - \eta]^2} \sum_{k=0}^{\infty} \binom{-2}{k} \left[ \frac{\eta}{1 - \exp(-\theta) - \eta} \right]^k \exp[-\lambda x - (k+1)\theta \exp(-\lambda x)]. \quad (6)$$

We can see that the GEP is a mixture of distributions having probability density functions taking the form  $C \exp[\lambda x - b \exp(-\lambda x)]$ .

The aim of this paper is to study the mathematical properties of the GEP distribution and to illustrate its applicability. The contents are organized as follows. In Section 2, we derive the probability density function corresponding to (3) and discuss its shapes. The corresponding hazard rate function and its shapes are derived in Section 3. The quantile function corresponding to (3) is stated in Section 4. The moment generating function, the characteristic function and the cumulant generating function corresponding to (3) is stated in Section 5. Expressions for the  $n$ th moment corresponding to (3) are given in Section 6. Expressions for the  $n$ th conditional moment corresponding to (3) are given in Section 7. The order statistics,  $L$  moments, and the asymptotic



distributions of the extreme order statistics are considered in Sections 8 to 10. The Rényi and Shannon entropies are derived in Section 11. The maximum likelihood estimation including the case of censoring and moments estimation are considered in Section 12. The performance of the maximum likelihood estimates with respect to sample size is assessed by simulation in Section 13. In Section 14, we propose a log-GEP regression model for lifetime censoring data. Section 15 we give applications involving two real data sets. Sections 16 and 17 describe univariate and bivariate generalizations of (3).

## 2 Probability density function

The probability density function corresponding to (3) is given by

$$g(x) = \frac{\theta\lambda(1-\eta)[1-\exp(-\theta)]\exp[-\lambda x - \theta\exp(-\lambda x)]}{[1-\exp(-\theta) - \eta\{1-\exp[-\theta\exp(-\lambda x)]\}]^2} \quad (7)$$

for  $x > 0$ ,  $\alpha > 0$ ,  $\lambda > 0$  and  $0 < \eta < 1$ . It follows from (7) that

$$\frac{d \log f(x)}{dx} = -\lambda + \lambda\theta \exp(-\lambda y) - \frac{2\theta\lambda\eta \exp[-\lambda x - \theta\exp(-\lambda x)]}{[1-\exp(-\theta) - \eta\{1-\exp[-\theta\exp(-\lambda x)]\}]}$$

and

$$\begin{aligned} \frac{d^2 \log f(x)}{dx^2} = & -\lambda^2\theta \exp(-\lambda y) + \frac{2\theta\lambda^2\eta \exp[-\lambda x - \theta\exp(-\lambda x)]}{[1-\exp(-\theta) - \eta\{1-\exp[-\theta\exp(-\lambda x)]\}]} \\ & - \frac{2\theta^2\lambda^2\eta \exp(-2\lambda x) \exp[-\theta\exp(-\lambda x)]}{[1-\exp(-\theta) - \eta\{1-\exp[-\theta\exp(-\lambda x)]\}]^2} \\ & + \frac{2\theta^2\lambda^2\eta^2 \exp(-2\lambda x) \exp[-2\theta\exp(-\lambda x)]}{[1-\exp(-\theta) - \eta\{1-\exp[-\theta\exp(-\lambda x)]\}]^3}. \end{aligned}$$

The modes of (7) are the points  $x = x_0$  satisfying  $d \log f(x)/dx = 0$ . These points correspond to a maximum, a minimum and a point of inflection if  $d^2 \log f(x)/dx^2 < 0$ ,  $d^2 \log f(x)/dx^2 > 0$  and  $d^2 \log f(x)/dx^2 = 0$ , respectively.

Note that

$$f(x) \sim \frac{\theta\lambda(1-\eta)[1-\exp(-\theta)]}{[1-\exp(-\theta)]^2} \exp(-\lambda x) \quad (8)$$

as  $x \rightarrow \infty$ ,

$$f(x) \rightarrow \theta\lambda \exp(-\theta) \quad (9)$$

as  $x \rightarrow 0$ . So, the upper tails of the probability density function decay exponentially while its lower tails approach a constant.

[Figure 1 about here.]

Figure 1 illustrates possible shapes of (7) for selected parameter values. The shape appears monotonically decreasing for  $\theta$  small and  $\eta$  not sufficiently small. The shape appears unimodal if  $\theta$  is sufficiently large and/or  $\eta$  is sufficiently small. In the latter case, the mode moves closer to zero with increasing values of  $\eta$ .

### 3 Hazard rate function

The hazard rate function of the GEP distribution is given by

$$h(x) = \frac{\theta \lambda [1 - \exp(-\theta)] \exp[-\lambda x - \theta \exp(-\lambda x)]}{[1 - \exp\{-\theta \exp(-\lambda x)\}] [1 - \exp(-\theta) \eta \{1 - \exp[-\theta \exp(-\lambda x)]\}]} \quad (10)$$

for  $x > 0$ ,  $\alpha > 0$ ,  $\lambda > 0$  and  $0 < \eta < 1$ . It follows from (10) that

$$\begin{aligned} \frac{d \log h(x)}{dx} &= -\lambda + \lambda \theta \exp(-\lambda y) - \frac{2\theta \lambda \eta \exp[-\lambda x - \theta \exp(-\lambda x)]}{[1 - \exp(-\theta) - \eta \{1 - \exp[-\theta \exp(-\lambda x)]\}]} \\ &\quad - \frac{\theta \lambda \exp[-\lambda x - \theta \exp(-\lambda x)]}{[1 - \exp\{-\theta \exp(-\lambda x)\}]} \end{aligned}$$

and

$$\begin{aligned} \frac{d^2 \log h(x)}{dx^2} &= -\lambda^2 \theta \exp(-\lambda y) + \frac{\theta \lambda^2 \eta \exp[-\lambda x - \theta \exp(-\lambda x)]}{[1 - \exp(-\theta) - \eta \{1 - \exp[-\theta \exp(-\lambda x)]\}]} \\ &\quad - \frac{\theta^2 \lambda^2 \eta \exp(-2\lambda x) \exp[-\theta \exp(-\lambda x)]}{[1 - \exp(-\theta) - \eta \{1 - \exp[-\theta \exp(-\lambda x)]\}]} \\ &\quad + \frac{2\theta^2 \lambda^2 \eta^2 \exp(-2\lambda x) \exp[-2\theta \exp(-\lambda x)]}{[1 - \exp(-\theta) - \eta \{1 - \exp[-\theta \exp(-\lambda x)]\}]^2} \\ &\quad - \frac{\theta \lambda \exp[-\lambda x - \theta \exp(-\lambda x)]}{[1 - \exp\{-\theta \exp(-\lambda x)\}]^2} - \frac{\theta^2 \lambda^2 \exp[-2\lambda x - \theta \exp(-\lambda x)]}{[1 - \exp\{-\theta \exp(-\lambda x)\}]} \\ &\quad + \frac{\theta \lambda^2 \exp[-\lambda x - \theta \exp(-\lambda x)]}{[1 - \exp\{-\theta \exp(-\lambda x)\}]} \end{aligned}$$

The modes of (10) are the points  $x = x_0$  satisfying  $d \log h(x)/dx = 0$ . These points correspond to a maximum, a minimum and a point of inflection if  $d^2 \log h(x)/dx^2 < 0$ ,  $d^2 \log h(x)/dx^2 > 0$  and  $d^2 \log h(x)/dx^2 = 0$ , respectively.

Note that

$$h(x) \rightarrow \lambda$$

as  $x \rightarrow \infty$ ,

$$f(x) \rightarrow \frac{\theta \lambda \exp(-\theta)}{[1 - \exp(-\theta)](1 - \eta)}$$

as  $x \rightarrow 0$ . So, both the initial and ultimate hazard rates are constant.

[Figure 2 about here.]

Figure 2 illustrates possible shapes of (10) for selected parameter values. The shape appears monotonically increasing for  $\eta$  sufficiently small. The shape appears monotonically decreasing if  $\eta$  is sufficiently large and  $\theta$  is not sufficient large. The shape appears upside down bathtub for both  $\theta$  and  $\eta$  sufficiently large.

## 4 Quantile function

Let  $X$  denote an GEP random variable. The cumulative distribution function of  $X$  is given by (3). Inverting  $G(x) = u$ , we obtain

$$G^{-1}(u) = -\frac{1}{\lambda} \log \left[ -\frac{1}{\theta} \log \left\{ \frac{[1 - \eta - \exp(-\theta)]p + \exp(-\theta)}{1 - \eta p} \right\} \right] \quad (11)$$

for  $0 < u < 1$ . In particular, the median of  $X$  is

$$\text{Median}(X) = -\frac{1}{\lambda} \log \left[ -\frac{1}{\theta} \log \left\{ \frac{1 - \eta - \exp(-\theta) + 2 \exp(-\theta)}{2 - \eta} \right\} \right].$$

One can use (11) for simulating GEP variates.

## 5 MGF, CHF and CGF

Let  $X$  denote an GEP random variable. By using (6), the moment generating function of  $X$ ,  $M(t) = E[\exp(tX)]$ , can be expressed as

$$M_X(t) = \frac{\theta \lambda (1 - \eta) [1 - \exp(-\theta)]}{[1 - \exp(-\theta) - \eta]^2} \sum_{k=0}^{\infty} \binom{-2}{k} \left[ \frac{\eta}{1 - \exp(-\theta) - \eta} \right]^k I_k, \quad (12)$$

where

$$I_k = \int_0^{\infty} \exp[(t - \lambda)x - (k + 1)\theta \exp(-\lambda x)] dx.$$

This integral can be calculated as

$$I_k = \frac{1}{\lambda} \int_0^1 y^{-t/\lambda} \exp\{-(k + 1)\theta y\} dy = \frac{[(k + 1)\theta]^{t/\lambda - 1}}{\lambda} \gamma(1 - t\lambda, (k + 1)\theta),$$

where  $\gamma(a, x)$  is the incomplete gamma function defined by

$$\gamma(a, x) = \int_0^a t^{a-1} \exp(-t) dt.$$

So, (12) can be expressed in the form

$$M_X(t) = \frac{\theta^{t/\lambda} (1 - \eta) [1 - \exp(-\theta)]}{[1 - \exp(-\theta) - \eta]^2} \sum_{k=0}^{\infty} \binom{-2}{k} \left[ \frac{\eta}{1 - \exp(-\theta) - \eta} \right]^k (k + 1)^{t/\lambda - 1} \gamma(1 - t\lambda, (k + 1)\theta).$$

The characteristic function of  $X$ ,  $\phi_X(t) = E[\exp(itX)]$ , and the cumulant generating function of  $X$ ,  $K_X(t) = \log \phi_X(t)$ , are given by

$$\phi_X(t) = \frac{\theta^{t/\lambda} (1 - \eta) [1 - \exp(-\theta)]}{[1 - \exp(-\theta) - \eta]^2} \sum_{k=0}^{\infty} \binom{-2}{k} \left[ \frac{\eta}{1 - \exp(-\theta) - \eta} \right]^k (k + 1)^{t/\lambda - 1} \gamma(1 + it\lambda, (k + 1)\theta)$$

and

$$\begin{aligned} K_X(t) = & \log \frac{\theta^{t/\lambda} (1 - \eta) [1 - \exp(-\theta)]}{[1 - \exp(-\theta) - \eta]^2} \\ & + \log \left\{ \sum_{k=0}^{\infty} \binom{-2}{k} \left[ \frac{\eta}{1 - \exp(-\theta) - \eta} \right]^k (k + 1)^{t/\lambda - 1} \gamma(1 + it\lambda, (k + 1)\theta) \right\}, \end{aligned}$$

respectively, where  $i = \sqrt{-1}$ .

## 6 Moments

Let  $X$  denote an GEP random variable. By using (6), the  $n$ th moment of  $X$  can be expressed as

$$E(X^n) = \frac{\theta \lambda (1 - \eta) [1 - \exp(-\theta)]}{[1 - \exp(-\theta) - \eta]^2} \sum_{k=0}^{\infty} \binom{-2}{k} \left[ \frac{\eta}{1 - \exp(-\theta) - \eta} \right]^k I_k, \quad (13)$$

where

$$I_k = \int_0^{\infty} x^n \exp[-\lambda x - (k+1)\theta \exp(-\lambda x)] dx.$$

This integral can be calculated as

$$I_k = \frac{1}{\lambda(-\lambda)^n} \int_0^1 (\log y)^n \exp[-(k+1)\theta y] dy = \frac{n!}{\lambda^{n+1}} F_{n+1}(1, \dots, 1; 2, \dots, 2; -(k+1)\theta), \quad (14)$$

where  ${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z)$  denotes the generalized hypergeometric function defined by

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{j=0}^{\infty} \frac{(a_1)_j (a_2)_j \cdots (a_p)_j}{(b_1)_j (b_2)_j \cdots (b_q)_j} \frac{z^j}{j!},$$

where  $(c)_j = c(c+1) \cdots (c+j-1)$  denotes the ascending factorial. An equivalent expression for  $I_k$  can be obtained as

$$\begin{aligned} I_k &= \frac{1}{\lambda(-\lambda)^n} \int_0^1 (\log y)^n \exp[-(k+1)\theta y] dy \\ &= \frac{1}{\lambda(-\lambda)^n} \frac{\partial^n}{\partial s^n} \left[ (k+1)^{-s-1} \theta^{-s-1} \int_0^{(k+1)\theta} z^s \exp(-z) dz \right] \Big|_{s=0} \\ &= \frac{1}{\lambda(-\lambda)^n} \frac{\partial^n}{\partial s^n} [(k+1)^{-s-1} \theta^{-s-1} \gamma(1+s, (k+1)\theta)] \Big|_{s=0}. \end{aligned} \quad (15)$$

Combining (13) and (14), we can express the  $n$ th moment in the form

$$E(X^n) = \frac{n! \theta (1 - \eta) [1 - \exp(-\theta)]}{\lambda^n [1 - \exp(-\theta) - \eta]^2} \sum_{k=0}^{\infty} \binom{-2}{k} \frac{\eta^k {}_{n+1}F_{n+1}(1, \dots, 1; 2, \dots, 2; -(k+1)\theta)}{[1 - \exp(-\theta) - \eta]^k}. \quad (16)$$

A similar expression can be obtained by combining (13) and (15).

Some simplification of (16) is possible using special properties of hypergeometric functions (see, for example, Prudnikov *et al.* (1986, volume 3)). For instance, if  $n = 1$  then (16) can be reduced to

$$E(X) = \frac{(1 - \eta) [1 - \exp(-\theta)]}{\lambda [1 - \exp(-\theta) - \eta]^2} \sum_{k=0}^{\infty} \binom{-2}{k} \frac{\eta^k \{C + \log[(k+1)\theta] + \text{Ei}[(k+1)\theta]\}}{(k+1) [1 - \exp(-\theta) - \eta]^k},$$

where  $\text{Ei}(\cdot)$  denotes the exponential integral defined by

$$\text{Ei}(x) = \int_{-\infty}^x t^{-1} \exp(t) dt$$

and  $C$  denotes Euler's constant.

[Figure 3 about here.]

We can use (16) to compute the mean, variance, skewness and kurtosis of  $X$ . The values of these four quantities versus  $\xi$  are plotted in Figure 3 for  $\lambda = 1$ ,  $\theta = 0.5, 1, 2, 5$  and  $\eta = 0.01, 0.02, \dots, 0.99$ . Mean and variance are decreasing functions of  $\eta$  and increasing functions of  $\theta$ . Skewness and kurtosis are increasing functions of  $\eta$  and decreasing functions of  $\theta$ .

## 7 Conditional moments

Let  $X$  denote an GEP random variable. By using (6), the  $n$ th conditional moment,  $E(X^n | X > x)$ , can be expressed as

$$E(X^n | X > x) = \frac{\theta\lambda(1-\eta)[1-\exp(-\theta)]}{[1-G(x)][1-\exp(-\theta)-\eta]^2} \sum_{k=0}^{\infty} \binom{-2}{k} \left[ \frac{\eta}{1-\exp(-\theta)-\eta} \right]^k I_k, \quad (17)$$

where

$$I_k = \int_x^{\infty} y^n \exp[(t-\lambda)y - (k+1)\theta \exp(-\lambda y)] dy.$$

Letting  $t = \exp(-\lambda x)$ , this integral can be calculated as

$$\begin{aligned} I_k &= \frac{1}{\lambda(-\lambda)^n} \int_0^t (\log y)^n \exp[-(k+1)\theta y] dy \\ &= \frac{1}{\lambda(-\lambda)^n} \frac{\partial^n}{\partial s^n} \left[ (k+1)^{-s-1} \theta^{-s-1} \int_0^{(k+1)\theta t} z^s \exp(-z) dz \right] \Big|_{s=0} \\ &= \frac{1}{\lambda(-\lambda)^n} \frac{\partial^n}{\partial s^n} [(k+1)^{-s-1} \theta^{-s-1} \gamma(1+s, (k+1)\theta t)] \Big|_{s=0}. \end{aligned} \quad (18)$$

So, the conditional moment can be expressed as

$$\begin{aligned} E(X^n | X > x) &= \frac{\theta(1-\eta)[1-\exp(-\theta)]}{(-\lambda)^n [1-G(x)][1-\exp(-\theta)-\eta]^2} \sum_{k=0}^{\infty} \binom{-2}{k} \left[ \frac{\eta}{1-\exp(-\theta)-\eta} \right]^k \\ &\quad \times \frac{\partial^n}{\partial s^n} [(k+1)^{-s-1} \theta^{-s-1} \gamma(1+s, (k+1)\theta t)] \Big|_{s=0}. \end{aligned}$$

The first two conditional moments are

$$\begin{aligned} E(X | X > x) &= \frac{(1-\eta)[1-\exp(-\theta)]}{\lambda[1-G(x)][1-\exp(-\theta)-\eta]^2} \sum_{k=0}^{\infty} \frac{1}{k+1} \binom{-2}{k} \left[ \frac{\eta}{1-\exp(-\theta)-\eta} \right]^k \\ &\quad \times \{ k\theta t {}_2F_2(1, 1; 2, 2; -(k+1)\theta t) + \theta t {}_2F_2(1, 1; 2, 2; -(k+1)\theta t) \\ &\quad - \log[(k+1)\theta] \exp[-(k+1)\theta t] + \exp[-(k+1)\theta t] \log[(k+1)\theta t] \\ &\quad + \log[(k+1)\theta] + \log[(k+1)\theta t] \} \end{aligned}$$

and

$$\begin{aligned}
E(X^2 | X > x) = & \frac{(1-\eta)[1-\exp(-\theta)]}{\lambda^2[1-G(x)][1-\exp(-\theta)-\eta]^2} \sum_{k=0}^{\infty} \frac{1}{k+1} \binom{-2}{k} \left[ \frac{\eta}{1-\exp(-\theta)-\eta} \right]^k \\
& \times \{ 2k\theta t \log[(k+1)\theta] {}_2F_2(1, 1; 2, 2; -(k+1)\theta t) \\
& + 2\theta t \log[(k+1)\theta] {}_2F_2(1, 1; 2, 2; -(k+1)\theta t) \\
& - 2k\theta t \log[(k+1)\theta t] {}_2F_2(1, 1; 2, 2; -(k+1)\theta t) \\
& - 2\theta t \log[(k+1)\theta t] {}_2F_2(1, 1; 2, 2; -(k+1)\theta t) \\
& + 2k\theta t {}_3F_3(1, 1, 1; 2, 2, 2; -(k+1)\theta t) \\
& + 2\theta t {}_3F_3(1, 1, 1; 2, 2, 2; -(k+1)\theta t) \\
& - \log^2[(k+1)\theta] \exp[-(k+1)\theta t] \\
& + 2 \log[(k+1)\theta] \exp[-(k+1)\theta t] \log[(k+1)\theta t] \\
& - \exp[-(k+1)\theta t] \log^2[(k+1)\theta t] \\
& + \log^2[(k+1)\theta t] - 2 \log[(k+1)\theta] \log[(k+1)\theta t] + \log^2[(k+1)\theta] \},
\end{aligned}$$

where  $\Gamma(a, x)$  is the complementary incomplete gamma function defined by

$$\Gamma(a, x) = \int_a^{\infty} t^{a-1} \exp(-t) dt.$$

Note that the mean residual lifetime function is  $E(X | X > x) - x$ .

## 8 Order statistics

Let  $X_1, X_2, \dots, X_n$  be a random sample from the GEP distribution. Let  $X_{i:n}$  denote the  $i$ th order statistic. The probability density function of  $X_{i:n}$  is

$$\begin{aligned}
g_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} g(x) G^{i-1}(x) \{1-G(x)\}^{n-i} \\
&= \frac{n! \theta \lambda (1-\eta)^{n-i+1} [1-\exp(-\theta)]}{(i-1)!(n-i)!} \exp[-\lambda x - \theta \exp(-\lambda x)] \\
&\quad \times \frac{\{\exp[-\theta \exp(-\lambda x)] - \exp(-\theta)\}^{i-1} \{1 - \exp[-\theta \exp(-\lambda x)]\}^{n-i}}{[1 - \exp(-\theta) - \eta \{1 - \exp[-\theta \exp(-\lambda x)]\}]^{n+1}}. \quad (19)
\end{aligned}$$

The corresponding cumulative distribution function of  $X_{i:n}$  is

$$G_{i:n}(x) = \sum_{j=i}^n \binom{n}{j} \frac{(1-\eta)^{n-j} \{\exp[-\theta \exp(-\lambda x)] - \exp(-\theta)\}^j \{1 - \exp[-\theta \exp(-\lambda x)]\}^{n-j}}{[1 - \exp(-\theta) - \eta \{1 - \exp[-\theta \exp(-\lambda x)]\}]^n}. \quad (20)$$

Using (4), we can express (19) and (20) in the series forms

$$g_{i:n}(x) = \frac{n! \theta \lambda (1-\eta)^{n-i+1} [1-\exp(-\theta)]}{(i-1)!(n-i)! [1-\exp(-\theta)-\eta]^{n+1}} \sum_{j=0}^{i-1} \sum_{k=0}^{n-i} \sum_{l=0}^{\infty} \binom{i-1}{j} \binom{n-i}{k} \binom{n+1}{l} K(j, k, l), \quad (21)$$

and

$$G_{i:n}(x) = \frac{1}{[1-\exp(-\theta)-\eta]^n} \sum_{j=i}^n \sum_{k=0}^j \sum_{l=0}^{n-j} \sum_{m=0}^{\infty} \binom{n}{j} \binom{j}{k} \binom{n-j}{l} \binom{n}{m} C(j, k, l, m), \quad (22)$$

respectively, where

$$K(j, k, l) = \frac{(-1)^{i-j+k-1} \eta^l \exp[-(i-j-1)\theta]}{[1 - \exp(-\theta) - \eta]^l} \exp[-\lambda x - (j+k+1)\theta \exp(-\lambda x)]$$

and

$$C(j, k, l, m) = \frac{(-1)^{j-k+l} \eta^m (1-\eta)^{n-j} \exp[-(j-k)\theta]}{[1 - \exp(-\theta) - \eta]^m} \exp[-(k+l+m)\theta \exp(-\lambda x)].$$

The representations, (21) and (22), are useful because the results in Sections 5 to 7 can now be applied to yield representations for moment generating function, characteristic function, cumulant generating function, moments, and conditional moments of  $X_{i:n}$ . For instance, the  $m$ th moment of  $X_{i:n}$  can be expressed as

$$E[X_{i:n}^m] = \frac{n! \theta \lambda (1-\eta)^{n-i+1} [1 - \exp(-\theta)]}{(i-1)!(n-i)! [1 - \exp(-\theta) - \eta]^{n+1}} \sum_{j=0}^{i-1} \sum_{k=0}^{n-i} \sum_{l=0}^{\infty} \binom{i-1}{j} \binom{n-i}{k} \binom{n+1}{l} E(j, k, l),$$

where

$$\begin{aligned} E(j, k, l) &= \frac{(-1)^{i-j+k-1} \eta^l \exp[-(i-j-1)\theta]}{[1 - \exp(-\theta) - \eta]^l} \int_0^{\infty} x^m \exp[-\lambda x - (j+k+1)\theta \exp(-\lambda x)] dx \\ &= \frac{m! (-1)^{i-j+k-1} \eta^l \exp[-(i-j-1)\theta]}{\lambda^{m+1} [1 - \exp(-\theta) - \eta]^l} {}_{m+1}F_{m+1}(1, \dots, 1; 2, \dots, 2; -(j+k+1)\theta). \end{aligned}$$

## 9 L moments

$L$ -moments are summary statistics for probability distributions and data samples (Hoskings, 1990). They are analogous to ordinary moments but are computed from linear functions of the ordered data values. The  $r$ th  $L$  moment is defined by

$$\lambda_r = \sum_{j=0}^{r-1} (-1)^{r-1-j} \binom{r-1}{j} \binom{r-1+j}{j} \beta_j,$$

where  $\beta_j = E\{XF(X)^j\}$ . In particular,  $\lambda_1 = \beta_0$ ,  $\lambda_2 = 2\beta_1 - \beta_0$ ,  $\lambda_3 = 6\beta_2 - 6\beta_1 + \beta_0$  and  $\lambda_4 = 20\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0$ . In general,  $\beta_r = (r+1)^{-1} E(X_{r+1:r+1})$ , so it can be computed using results in Section 8. The  $L$  moments have several advantages over ordinary moments: for example, they apply for any distribution having finite mean; no higher-order moments need be finite.

## 10 Extreme values

If  $\bar{X} = (X_1 + \dots + X_n)/n$  denotes the sample mean then by the usual central limit theorem  $\sqrt{n}(\bar{X} - E(X))/\sqrt{\text{Var}(X)}$  approaches the standard normal distribution as  $n \rightarrow \infty$ . Sometimes one would be interested in the asymptotics of the extreme values  $M_n = \max(X_1, \dots, X_n)$  and  $m_n = \min(X_1, \dots, X_n)$ .



Let  $g(t) = 1/\lambda$ . Take the cumulative distribution function and the probability density function as specified by (3) and (7), respectively. Note from (8) and (9) that

$$\lim_{t \rightarrow \infty} \frac{1 - F(t + xg(t))}{1 - F(t)} = \lim_{t \rightarrow \infty} \frac{f(t + x/\lambda)}{f(t)} = \lim_{t \rightarrow \infty} \frac{\exp(-\lambda t - x)}{\exp(-\lambda t)} = \exp(-x)$$

as  $t \rightarrow \infty$  and

$$\lim_{t \rightarrow 0} \frac{F(tx)}{F(t)} = \lim_{t \rightarrow \infty} \frac{xf(tx)}{f(t)} = x$$

as  $t \rightarrow 0$ . Hence, it follows from Theorem 1.6.2 in Leadbetter *et al.* (1987) that there must be norming constants  $a_n > 0$ ,  $b_n$ ,  $c_n > 0$  and  $d_n$  such that

$$\Pr \{a_n (M_n - b_n) \leq x\} \rightarrow \exp \{-\exp(-x)\}$$

and

$$\Pr \{c_n (m_n - d_n) \leq x\} \rightarrow 1 - \exp(-x)$$

as  $n \rightarrow \infty$ . The form of the norming constants can also be determined. For instance, using Corollary 1.6.3 in Leadbetter *et al.* (1987), one can see that  $b_n = F^{-1}(1 - 1/n)$  and  $a_n = \lambda$ , where  $F^{-1}(\cdot)$  denotes the inverse function of  $F(\cdot)$ .

## 11 Entropies

An entropy is a measure of variation or uncertainty of a random variable  $X$ . Two popular entropy measures are the Rényi and Shannon entropies (Shannon, 1951; Rényi, 1961). The Rényi entropy of a random variable with probability density function  $g(\cdot)$  is defined as

$$I_R(\gamma) = \frac{1}{1-\gamma} \log \int_0^\infty g^\gamma(x) dx$$

for  $\gamma > 0$  and  $\gamma \neq 1$ . The Shannon entropy of a random variable  $X$  is defined by  $E[-\log g(X)]$ . It is the particular case of the Rényi entropy for  $\gamma \uparrow 1$ .

Here, we derive expressions for the Rényi and Shannon entropies when  $X$  has the GEP distribution. By using the series representation, (4), we can write

$$\begin{aligned} \int_0^\infty g^\gamma(x) dx &= \theta^\gamma \lambda^\gamma (1-\eta)^\gamma [1 - \exp(-\theta)]^\gamma \int_0^\infty \frac{\exp[-\gamma \lambda x - \gamma \theta \exp(-\lambda x)]}{[1 - \exp(-\theta) - \eta \{1 - \exp[-\theta \exp(-\lambda x)]\}]^{2\gamma}} dx \\ &= \frac{\theta^\gamma \lambda^\gamma (1-\eta)^\gamma [1 - \exp(-\theta)]^\gamma}{[1 - \exp(-\theta) - \eta]^{2\gamma}} \sum_{i=0}^\infty \binom{-2\gamma}{i} \left[ \frac{\eta}{1 - \exp(-\theta) - \eta} \right]^i \\ &\quad \times \int_0^\infty \exp[-\gamma \lambda x - (\gamma + i)\theta \exp(-\lambda x)] dx \\ &= \frac{\lambda^{\gamma-1} (1-\eta)^\gamma [1 - \exp(-\theta)]^\gamma}{[1 - \exp(-\theta) - \eta]^{2\gamma}} \sum_{i=0}^\infty \binom{-2\gamma}{i} \left[ \frac{\eta}{1 - \exp(-\theta) - \eta} \right]^i \frac{\gamma(\gamma, (\gamma + i)\theta)}{(\gamma + i)^\gamma}. \end{aligned}$$

So, the Rényi entropy for the GEP distribution is given by

$$I_R(\gamma) = \frac{1}{1-\gamma} \log \left\{ \frac{\lambda^{\gamma-1} (1-\eta)^\gamma [1 - \exp(-\theta)]^\gamma}{[1 - \exp(-\theta) - \eta]^{2\gamma}} \right\} + \frac{1}{1-\gamma} \log \left\{ \sum_{i=0}^{\infty} \binom{-2\gamma}{i} \left[ \frac{\eta}{1 - \exp(-\theta) - \eta} \right]^i \frac{\gamma(\gamma, (\gamma+i)\theta)}{(\gamma+i)^\gamma} \right\}. \quad (23)$$

The Shannon entropy can be obtained by limiting  $\gamma \uparrow 1$  in (23). However, it is easier to derive an expression for it from first principles. Using the series expansion for  $\log(1-z)$ , we can write

$$\begin{aligned} E[-\log g(X)] &= -\log \{ \lambda \theta (1-\eta) [1 - \exp(-\theta)] \} + \lambda E(X) + \theta E[\exp(-\lambda X)] \\ &\quad - 2E \{ \log [1 - \exp(-\theta) - \eta \{1 - \exp[-\theta \exp(-\lambda x)]\}] \} \\ &= -\log \{ \lambda \theta (1-\eta) [1 - \exp(-\theta)] \} + \lambda E(X) + \theta E[\exp(-\lambda X)] \\ &\quad - 2 \log [1 - \exp(-\theta) - \eta] \\ &\quad - 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left[ \frac{\eta}{1 - \exp(-\theta) - \eta} \right]^k E \{ \exp[-k\theta \exp(-\lambda X)] \} \\ &= -\log \{ \lambda \theta (1-\eta) [1 - \exp(-\theta)] \} + \lambda E(X) + \theta M_X(-\lambda) \\ &\quad - 2 \log [1 - \exp(-\theta) - \eta] \\ &\quad - 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left[ \frac{\eta}{1 - \exp(-\theta) - \eta} \right]^k A(k), \end{aligned}$$

where  $M_X(\cdot)$  is given by Section 5,  $E(X)$  is given by Section 6, and  $A(k)$  is given by Lemma 1 in the appendix.

## 12 Maximum likelihood estimation

Here, we consider estimation of the unknown parameters of the GEP distribution by the method of maximum likelihood. Let  $x_1, x_2, \dots, x_n$  be a random sample from (7). Then the log-likelihood function is

$$\begin{aligned} \log L(\theta, \lambda, \eta) &= n \log [\theta \lambda (1-\eta) [1 - \exp(-\theta)]] - \lambda \sum_{i=1}^n x_i - \theta \sum_{i=1}^n \exp(-\lambda x_i) \\ &\quad - 2 \sum_{i=1}^n \log [1 - \exp(-\theta) - \eta \{1 - \exp[-\theta \exp(-\lambda x_i)]\}]. \end{aligned} \quad (24)$$

The first derivatives of the log-likelihood function with respect to the parameters  $\theta$ ,  $\lambda$  and  $\eta$  are:

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} + \frac{n}{\exp(\theta) - 1} - 2 \sum_{i=1}^n \frac{\exp(-\theta) - \eta \exp[-\lambda x_i - \theta \exp(-\lambda x_i)]}{1 - \exp(-\theta) - \eta \{1 - \exp[-\theta \exp(-\lambda x_i)]\}}, \quad (25)$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^n x_i - 2\eta \theta \sum_{i=1}^n \frac{x_i \exp[-\lambda x_i - \theta \exp(-\lambda x_i)]}{1 - \exp(-\theta) - \eta \{1 - \exp[-\theta \exp(-\lambda x_i)]\}}, \quad (26)$$

$$\frac{\partial \log L}{\partial \eta} = -\frac{n}{1-\eta} + 2 \sum_{i=1}^n \frac{1 - \exp[-\theta \exp(-\lambda x_i)]}{1 - \exp(-\theta) - \eta \{1 - \exp[-\theta \exp(-\lambda x_i)]\}}. \quad (27)$$

The maximum likelihood estimates of  $(\theta, \lambda, \eta)$ , say  $(\hat{\theta}, \hat{\lambda}, \hat{\eta})$ , are the simultaneous solutions of the equations  $\partial \log L / \partial \theta = 0$ ,  $\partial \log L / \partial \lambda = 0$ , and  $\partial \log L / \partial \eta = 0$ .

Maximization of (24) can be performed by using well established routines like `nlm` or `optimize` in the R statistical package. Our numerical calculations showed that the surface of (24) was reasonably smooth. The routines were able to locate the maximum in all cases and for different starting values. However, to ease the computations it is useful to have reasonable starting values. These can be obtained, for example, by the method of moments. Let  $m_1 = (1/n) \sum_{i=1}^n x_i$ ,  $m_2 = (1/n) \sum_{i=1}^n x_i^2$  and  $m_3 = (1/n) \sum_{i=1}^n x_i^3$  denote the first three sample moments. Equating these moments with the theoretical versions given by Section 6, we have  $m_1 = E(X)$ ,  $m_2 = E(X^2)$  and  $m_3 = E(X^3)$ . These equations can be solved simultaneously to obtain the moments estimates.

For interval estimation of  $(\theta, \lambda, \eta)$  and tests of hypothesis, one requires the Fisher information matrix. Applying Lemmas 2 and 3 in the appendix, we can express the Fisher information matrix for the estimators of  $(\theta, \lambda, \eta)$  as

$$\mathbf{I} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix},$$

where

$$\begin{aligned} I_{11} &= E \left( -\frac{\partial^2 \log L}{\partial \theta^2} \right) = \frac{n}{\theta^2} + \frac{n \exp(\theta)}{[\exp(\theta) - 1]^2} - 2n \exp(-\theta) B(0, 0, 1) + 2n\eta B(2, 1, 1) \\ &\quad - 2n \exp(-2\theta) B(0, 0, 2) + 4n\eta \exp(-\theta) B(1, 1, 2) - 2n\eta^2 B(2, 2, 2), \\ I_{12} &= E \left( -\frac{\partial^2 \log L}{\partial \theta \partial \lambda} \right) = -nD(1, 0, 0, 1) + 2n\eta D(1, 1, 1, 1) - 2n\eta\theta D(2, 1, 1, 1) \\ &\quad - 2n\eta\theta \exp(-\theta) D(1, 1, 1, 2) + 2n\eta^2\theta D(2, 2, 2, 1), \\ I_{13} &= E \left( -\frac{\partial^2 \log L}{\partial \theta \partial \eta} \right) = -nB(1, 1, 1) + 2n \exp(-\theta) B(0, 0, 2) - 2n \exp(-\theta) B(0, 1, 2) \\ &\quad - 2n\eta B(1, 1, 2) + 2n\eta B(1, 2, 2), \\ I_{22} &= E \left( -\frac{\partial^2 \log L}{\partial \lambda^2} \right) = \frac{n}{\lambda^2} - 2n\eta\theta D(1, 1, 1, 2) + 2n\eta\theta^2 D(2, 1, 1, 2) - 2n\eta^2\theta^2 D(2, 2, 2, 2), \\ I_{23} &= E \left( -\frac{\partial^2 \log L}{\partial \lambda \partial \eta} \right) = 2n\theta D(1, 1, 1, 1) + 2n\eta\theta D(1, 1, 2, 1) - 2n\eta\theta D(1, 2, 2, 1), \\ I_{33} &= E \left( -\frac{\partial^2 \log L}{\partial \eta^2} \right) = \frac{n}{(1-\eta)^2} - 2nB(0, 0, 2) + 4nB(0, 1, 2) - 2nB(0, 2, 2), \end{aligned}$$

where explicit expressions for  $B(0, 0, 1)$ ,  $B(2, 1, 1)$ ,  $B(0, 0, 2)$ ,  $B(1, 1, 2)$ ,  $B(2, 2, 2)$ ,  $D(1, 0, 0, 1)$ ,  $D(1, 1, 1, 1)$ ,  $D(2, 1, 1, 1)$ ,  $D(1, 1, 1, 2)$ ,  $D(2, 2, 2, 1)$ ,  $B(1, 1, 1)$ ,  $B(0, 1, 2)$ ,  $B(1, 2, 2)$ ,  $D(2, 1, 1, 2)$ ,  $D(2, 2, 2, 2)$ ,  $D(1, 1, 2, 1)$ ,  $D(1, 2, 2, 1)$ ,  $B(0, 1, 2)$  and  $B(0, 2, 2)$  are given by Lemmas 2 and 3 in the appendix.

Then, as  $n \rightarrow \infty$ ,  $\sqrt{n}(\hat{\theta} - \theta, \hat{\lambda} - \lambda, \hat{\eta} - \eta)$  approaches a trivariate normal vector with zero means and variance-covariance matrix  $\mathbf{I}^{-1}$ . The properties of  $(\hat{\theta}, \hat{\lambda}, \hat{\eta})$  can be derived based on this normal approximation. For example,  $\text{Var}(\hat{\theta}) = (I_{33}I_{22} - I_{32}I_{23})/\Delta$ ,  $\text{Cov}(\hat{\theta}, \hat{\lambda}) = -(I_{33}I_{12} - I_{32}I_{13})/\Delta$ ,  $\text{Cov}$

$(\hat{\theta}, \hat{\eta}) = (I_{23}I_{12} - I_{22}I_{13})/\Delta$ ,  $\text{Var}(\hat{\lambda}) = (I_{33}I_{11} - I_{31}I_{13})/\Delta$ ,  $\text{Cov}(\hat{\lambda}, \hat{\eta}) = -(I_{23}I_{11} - I_{21}I_{13})/\Delta$  and  $\text{Var}(\hat{\eta}) = (I_{22}I_{11} - I_{21}I_{12})/\Delta$ , where  $\Delta = I_{11}(I_{33}I_{22} - I_{32}I_{23}) - I_{21}(I_{33}I_{12} - I_{32}I_{13}) + I_{31}(I_{23}I_{12} - I_{22}I_{13})$ .

Often with lifetime data, one encounters censoring. There are different forms of censoring: type I censoring, type II censoring, etc. Here, we consider the general case of multicensored data: there are  $n$  subjects of which

- $n_0$  are known to have the values  $t_1, \dots, t_{n_0}$ .
- $n_1$  are known to belong to the interval  $[s_{i-1}, s_i]$ ,  $i = 1, \dots, n_1$ .
- $n_2$  are known to have exceeded  $r_i$ ,  $i = 1, \dots, n_2$  but not observed any longer.

Note that  $n = n_0 + n_1 + n_2$ . Note too that type I censoring and type II censoring are contained as particular cases of multicensoring.

In the case of multicensoring, the log-likelihood function is:

$$\log L(\theta, \lambda, \eta) = \sum_{i=1}^{n_0} \log g(t_i) + \sum_{i=1}^{n_1} \log [G(s_i) - G(s_{i-1})] + \sum_{i=1}^{n_2} \log [1 - G(r_i)], \quad (28)$$

where  $g(\cdot)$  and  $G(\cdot)$  are given by (7) and (3), respectively. The first derivatives of the log-likelihood function with respect to the parameters  $\theta$ ,  $\lambda$  and  $\eta$  are:

$$\begin{aligned} \frac{\partial \log L}{\partial \theta} &= \sum_{i=1}^{n_0} \frac{1}{g(t_i)} \frac{\partial g(t_i)}{\partial \theta} + \sum_{i=1}^{n_1} \frac{1}{G(s_i) - G(s_{i-1})} \left[ \frac{\partial G(s_i)}{\partial \theta} - \frac{\partial G(s_{i-1})}{\partial \theta} \right] \\ &\quad - \sum_{i=1}^{n_2} \frac{1}{1 - G(r_i)} \frac{\partial G(r_i)}{\partial \theta}, \end{aligned} \quad (29)$$

$$\begin{aligned} \frac{\partial \log L}{\partial \lambda} &= \sum_{i=1}^{n_0} \frac{1}{g(t_i)} \frac{\partial g(t_i)}{\partial \lambda} + \sum_{i=1}^{n_1} \frac{1}{G(s_i) - G(s_{i-1})} \left[ \frac{\partial G(s_i)}{\partial \lambda} - \frac{\partial G(s_{i-1})}{\partial \lambda} \right] \\ &\quad - \sum_{i=1}^{n_2} \frac{1}{1 - G(r_i)} \frac{\partial G(r_i)}{\partial \lambda}, \end{aligned} \quad (30)$$

$$\begin{aligned} \frac{\partial \log L}{\partial \eta} &= \sum_{i=1}^{n_0} \frac{1}{g(t_i)} \frac{\partial g(t_i)}{\partial \eta} + \sum_{i=1}^{n_1} \frac{1}{G(s_i) - G(s_{i-1})} \left[ \frac{\partial G(s_i)}{\partial \eta} - \frac{\partial G(s_{i-1})}{\partial \eta} \right] \\ &\quad - \sum_{i=1}^{n_2} \frac{1}{1 - G(r_i)} \frac{\partial G(r_i)}{\partial \eta}. \end{aligned} \quad (31)$$

The first term in (28) is the same as (24) with  $(x_i, n)$  replaced by  $(t_i, n_0)$ . Also the first terms in (29)-(31) are the same as (25)-(27) with  $(x_i, n)$  replaced by  $(t_i, n_0)$ . So, it is sufficient to find explicit expressions for the partial derivatives in (29)-(31). They are

$$\begin{aligned} \frac{\partial G(x)}{\partial \theta} &= \frac{\exp(-\theta) - \exp[-\lambda x - \theta \exp(-\lambda x)]}{1 - \exp(-\theta) - \eta \{1 - \exp[-\theta \exp(-\lambda x)]\}} \\ &\quad - \frac{\{\exp[-\theta \exp(-\lambda x)] - \exp(-\theta)\} \{\exp(-\theta) - \eta \exp[-\lambda x - \theta \exp(-\lambda x)]\}}{[1 - \exp(-\theta) - \eta \{1 - \exp[-\theta \exp(-\lambda x)]\}]^2}, \end{aligned}$$

$$\frac{\partial G(x)}{\partial \lambda} = \frac{\theta x \exp[-\lambda x - \theta \exp(-\lambda x)]}{1 - \exp(-\theta) - \eta \{1 - \exp[-\theta \exp(-\lambda x)]\}} - \frac{\eta \theta x \{\exp[-\theta \exp(-\lambda x)] - \exp(-\theta)\} \exp[-\lambda x - \theta \exp(-\lambda x)]}{[1 - \exp(-\theta) - \eta \{1 - \exp[-\theta \exp(-\lambda x)]\}]^2},$$

$$\frac{\partial G(x)}{\partial \eta} = \frac{\{\exp[-\theta \exp(-\lambda x)] - \exp(-\theta)\} \{1 - \exp[-\theta \exp(-\lambda x)]\}}{[1 - \exp(-\theta) - \eta \{1 - \exp[-\theta \exp(-\lambda x)]\}]^2}.$$

The maximum likelihood estimates of  $(\theta, \lambda, \eta)$ , say  $(\hat{\theta}, \hat{\lambda}, \hat{\eta})$ , are the simultaneous solutions of the equations  $\partial \log L / \partial \theta = 0$ ,  $\partial \log L / \partial \lambda = 0$ , and  $\partial \log L / \partial \eta = 0$ . The Fisher information matrix for the estimators of  $(\theta, \lambda, \eta)$  corresponding to (28) is too complicated to be presented here.

### 13 A simulation study

Here, we assess the performance of the maximum likelihood estimates given by (25)-(27) with respect to sample size  $n$ . The assessment is based on a simulation study:

1. generate 10000 samples of size  $n$  from (7). The inversion method is used to generate samples, i.e variates of the GEP distribution are generated using

$$X = -\frac{1}{\lambda} \log \left[ -\frac{1}{\theta} \log \left\{ \frac{[1 - \eta - \exp(-\theta)]U + \exp(-\theta)}{1 - \eta U} \right\} \right],$$

where  $U \sim U(0, 1)$  is a uniform variate on the unit interval.

2. compute the maximum likelihood estimates for the 10000 samples, say  $(\hat{\theta}_i, \hat{\lambda}_i, \hat{\eta}_i)$  for  $i = 1, 2, \dots, 10000$ .
3. compute the biases and mean squared errors given by

$$bias_1(n) = \frac{1}{10000} \sum_{i=1}^{10000} (\hat{\theta}_i - \theta),$$

$$bias_2(n) = \frac{1}{10000} \sum_{i=1}^{10000} (\hat{\lambda}_i - \lambda),$$

$$bias_3(n) = \frac{1}{10000} \sum_{i=1}^{10000} (\hat{\eta}_i - \eta),$$

and

$$MSE_1(n) = \frac{1}{10000} \sum_{i=1}^{10000} (\hat{\theta}_i - \theta)^2,$$

$$MSE_2(n) = \frac{1}{10000} \sum_{i=1}^{10000} (\hat{\lambda}_i - \lambda)^2,$$

$$MSE_3(n) = \frac{1}{10000} \sum_{i=1}^{10000} (\hat{\eta}_i - \eta)^2.$$

We repeat these steps for  $n = 10, 20, \dots, 1000$  with  $\theta = 1$ ,  $\lambda = 1$  and  $\eta = 0.5$ , so computing  $bias_1(n)$ ,  $bias_2(n)$ ,  $bias_3(n)$  and  $MSE_1(n)$ ,  $MSE_2(n)$ ,  $MSE_3(n)$  for  $n = 10, 20, \dots, 1000$ .

[Figures 4 and 5 about here.]

Figures 4 and 5 show how the five biases and the five mean squared errors vary with respect to  $n$ . The broken line in Figure 4 corresponds to the biases being zero. The following observations can be made:

1. the biases for the parameters,  $\lambda$  and  $\eta$ , are generally negative,
2. the magnitude of bias for each parameter decreases to zero as  $n \rightarrow \infty$ ,
3. the mean squared errors for each parameter decrease to zero as  $n \rightarrow \infty$ ,
4. the mean squared errors appear largest for the parameter  $\theta$ ,
5. the mean squared errors appear smallest for the parameter  $\eta$ .

We have presented results for only one choice for  $(\theta, \lambda, \eta)$ , namely that  $(\theta, \lambda, \eta) = (1, 1, 0.5)$ . But the results are similar for other choices.

## 14 The log-GEP regression model

Let  $X$  be a random variable having the cumulative distribution function (3). We then say that the random variable  $Y = \sigma \log X$  has a log-GEP distribution. Writing  $\lambda = \exp(-\mu/\sigma)$ , the cumulative distribution function and probability density function of  $Y$  can be expressed as

$$G(y) = \frac{\exp \left\{ -\theta \exp \left[ -\exp \left( \frac{y-\mu}{\sigma} \right) \right] \right\} - \exp(-\theta)}{1 - \exp(-\theta) - \eta \left[ 1 - \exp \left\{ -\theta \exp \left[ -\exp \left( \frac{y-\mu}{\sigma} \right) \right] \right\} \right]} \quad (32)$$

and

$$g(y) = \frac{\theta \lambda (1 - \eta) [1 - \exp(-\theta)] \exp \left\{ -\exp \left( \frac{y-\mu}{\sigma} \right) - \theta \exp \left[ -\exp \left( \frac{y-\mu}{\sigma} \right) \right] \right\}}{\left\{ 1 - \exp(-\theta) - \eta \left[ 1 - \exp \left\{ -\theta \exp \left[ -\exp \left( \frac{y-\mu}{\sigma} \right) \right] \right\} \right] \right\}^2}, \quad (33)$$

respectively, for  $-\infty < x < \infty$ ,  $\theta > 0$ ,  $0 < \eta < 1$ ,  $-\infty < \mu < \infty$  and  $\sigma > 0$ . The standardized forms of (32) and (33) for  $\mu = 0$  and  $\sigma = 1$  are

$$G(y) = \frac{\exp \{ -\theta \exp [ -\exp (y) ] \} - \exp(-\theta)}{1 - \exp(-\theta) - \eta [1 - \exp \{ -\theta \exp [ -\exp (y) ] \}]} \quad (34)$$

and

$$g(y) = \frac{\theta \lambda (1 - \eta) [1 - \exp(-\theta)] \exp \{ -\exp (y) - \theta \exp [ -\exp (y) ] \}}{\{1 - \exp(-\theta) - \eta [1 - \exp \{ -\theta \exp [ -\exp (y) ] \}]\}^2}, \quad (35)$$

respectively, for  $-\infty < x < \infty$ ,  $\theta > 0$  and  $0 < \eta < 1$ .

Suppose  $x_1, x_2, \dots, x_n$  is a random sample of lifetimes from (3). In many practical applications, the lifetimes  $x_i$  are affected by explanatory variables such as the cholesterol level, blood pressure and many others. Let  $\mathbf{v}_i = (v_{i1}, \dots, v_{ip})^T$  be the explanatory variable vector associated with the  $i$ th response variable  $x_i$  for  $i = 1, \dots, n$ . Consider a sample  $(y_1, \mathbf{v}_1), \dots, (y_n, \mathbf{v}_n)$  of  $n$  independent observations, where each random response is defined by  $y_i = \min\{\log(x_i), \log(c_i)\}$ , where  $\log(x_i)$  and  $\log(c_i)$  are the log-lifetime and log-censoring, respectively. We assume non-informative censoring and that the observed lifetimes and censoring times are independent.

A linear regression model for the response variable  $y_i$  based on the log-GEP distribution is

$$y_i = \mathbf{v}_i^T \boldsymbol{\beta} + \sigma z_i, \quad i = 1, \dots, n, \quad (36)$$

where the random error  $z_i$  follows the distribution given by (34)-(35),  $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)^T$ ,  $\sigma > 0$ ,  $\theta > 0$  and  $0 < \eta < 1$  are unknown scalar parameters and  $\mathbf{v}_i$  is the explanatory variable vector modeling the location parameter  $\mu_i = \mathbf{v}_i^T \boldsymbol{\beta}$ . So, the location parameter vector  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)^T$  of the log-GEP model has a linear structure  $\boldsymbol{\mu} = \mathbf{V}\boldsymbol{\beta}$ , where  $\mathbf{V} = (\mathbf{v}_1, \dots, \mathbf{v}_n)^T$  is a known model matrix. The log-EP regression model is defined by (36) with  $\eta \rightarrow 0$ .

Let  $F$  and  $C$  be the sets of individuals for which  $y_i$  is the log-lifetime or log-censoring, respectively. The log-likelihood function for the model parameters  $\boldsymbol{\Phi} = (\theta, \eta, \sigma, \boldsymbol{\beta}^T)^T$  can be written from (34)-(35) and (36) as

$$\begin{aligned} \log L(\boldsymbol{\Phi}) = & q \log \{ \theta \lambda (1 - \eta) [1 - \exp(-\theta)] \} - \sum_{i \in F} \exp \left( \frac{y_i - \mathbf{v}_i^T \boldsymbol{\beta}}{\sigma} \right) \\ & - \sum_{i \in F} \theta \exp \left[ - \exp \left( \frac{y_i - \mathbf{v}_i^T \boldsymbol{\beta}}{\sigma} \right) \right] \\ & - 2 \sum_{i \in F} \log \left\{ 1 - \exp(-\theta) - \eta \left[ 1 - \exp \left\{ - \theta \exp \left[ - \exp \left( \frac{y_i - \mathbf{v}_i^T \boldsymbol{\beta}}{\sigma} \right) \right] \right\} \right] \right\} \\ & + \sum_{i \in C} \log \frac{(1 - \eta) \left[ 1 - \exp \left\{ - \theta \exp \left[ - \exp \left( \frac{y_i - \mathbf{v}_i^T \boldsymbol{\beta}}{\sigma} \right) \right] \right\} \right]}{1 - \exp(-\theta) - \eta \left[ 1 - \exp \left\{ - \theta \exp \left[ - \exp \left( \frac{y_i - \mathbf{v}_i^T \boldsymbol{\beta}}{\sigma} \right) \right] \right\} \right]}, \quad (37) \end{aligned}$$

where  $q$  is the observed number of failures. The maximum likelihood estimates  $\hat{\boldsymbol{\Phi}}$  of  $\boldsymbol{\Phi}$  can be obtained by maximizing the log-likelihood function (37). From the fitted model (36), the survival function for  $y_i$  can be estimated by

$$1 - \hat{G}(y_i) = \frac{(1 - \hat{\eta}) \left[ 1 - \exp \left\{ - \hat{\theta} \exp \left[ - \exp \left( \frac{y_i - \mathbf{v}_i^T \hat{\boldsymbol{\beta}}}{\hat{\sigma}} \right) \right] \right\} \right]}{1 - \exp(-\hat{\theta}) - \hat{\eta} \left[ 1 - \exp \left\{ - \hat{\theta} \exp \left[ - \exp \left( \frac{y_i - \mathbf{v}_i^T \hat{\boldsymbol{\beta}}}{\hat{\sigma}} \right) \right] \right\} \right]}.$$

Under general regularity conditions, the asymptotic distribution of  $\sqrt{n}(\hat{\boldsymbol{\Phi}} - \boldsymbol{\Phi})$  is multivariate normal  $N_{p+3}(\mathbf{0}, \mathbf{K}(\boldsymbol{\Phi})^{-1})$ , where  $\mathbf{K}(\boldsymbol{\Phi})$  is the expected information matrix. The asymptotic covariance matrix  $\mathbf{K}(\boldsymbol{\Phi})^{-1}$  of  $\hat{\boldsymbol{\Phi}}$  can be approximated by the inverse of the  $(p+3) \times (p+3)$  observed



information matrix  $\mathbf{J}(\Phi)$  and then the asymptotic inference for the parameter vector  $\Phi$  can be based on the normal approximation  $N_{p+3}(\mathbf{0}, \mathbf{J}(\Phi)^{-1})$  for  $\hat{\Phi}$ .

The multivariate normal  $N_{p+3}(\mathbf{0}, \mathbf{J}(\Phi)^{-1})$  distribution can be used to construct approximate confidence regions for some parameters in  $\Phi$  and for the hazard and survival functions. In fact, a  $100(1 - \alpha)\%$  asymptotic confidence interval for each parameter  $\theta_r$  is given by

$$ACI_r = \left( \hat{\phi}_r - z_{\alpha/2} \sqrt{-\hat{J}^{r,r}}, \hat{\phi}_r + z_{\alpha/2} \sqrt{-\hat{J}^{r,r}} \right),$$

where  $-\hat{J}^{r,r}$  represents the  $r$ th diagonal element of the inverse of the estimated observed information matrix  $\mathbf{J}(\hat{\Phi})^{-1}$  and  $z_{\alpha/2}$  is the  $1 - \alpha/2$  quantile of the standard normal distribution. The asymptotic normality is also useful for testing goodness of fit of some sub-models and for comparing some special sub-models using the likelihood ratio statistic.

We can investigate if the log-GEP regression model is a good model to fit the data under investigation. Clearly, the likelihood ratio statistic can be used to discriminate between the log-EP and log-GEP regression models since they are nested models. In this case, the hypotheses to be tested are  $H_0 : \eta = 0$  versus  $H_1 : H_0$  is not true, and the likelihood ratio statistic reduces to  $w = 2\{\log L(\hat{\Phi}) - \log L(\tilde{\Phi})\}$ , where  $\hat{\Phi}$  is the maximum likelihood estimate of  $\Phi$  under  $H_0$ . The null hypothesis is rejected if  $w > \chi^2_{1-\alpha}(1)$ , where  $\chi^2_{1-\alpha}(1)$  is the  $1 - \alpha/2$  quantile of the chi-square distribution with one degree of freedom.

## 15 Applications

### 15.1 First application: the GEP model

Here, we consider a possible application of the GEP distribution. We consider a real data set of adult numbers of *Tribolium confusum* (Eugene *et al.*, 2002). These data were also used by Cordeiro and de Castro (2011), where they fitted some members of the family of Kw generalized distributions. In that study they found that the best model is a Kw-normal model.

We compare the fit of the GEP distribution with the following distributions: Weibull (W), exponential Poisson (EP) (Kuş, 2007), exponential-Poisson generalized (EPG) (Barreto-Souza and Cribari-Neto, 2009), Kw-normal (Kw-N) and beta normal (BN) (Eugene *et al.*, 2002). Table 1 presents the maximum likelihood estimates of the parameters together with their standard errors.

For comparison of nested models, which is the case when comparing the GEP model with the exponential Poisson (EP) model, we can compute the maximum values of the unrestricted and restricted log-likelihoods to obtain the likelihood ratio statistic (LRS). For testing  $H_0 : \eta = 0$  versus  $H_1 : \eta > 0$ , we consider the LRS,  $w_n = 2(l_{GEP} - l_{EP})$ , where  $l_{EP}$  and  $l_{GEP}$  are the log-likelihoods for the model under the restricted hypothesis  $H_0$  and under the unrestricted hypothesis  $H_1$ . Taking into account that the test is performed in the boundary of the parameter space, following Maller and Zhou (1995), the LRS,  $w_n$ , is assumed to be asymptotically distributed as a symmetric mixture of a chi-squared distribution with one degree of freedom and a point-mass at zero. Then,  $\lim_{n \rightarrow \infty} P(w_n \leq c) = 1/2 + 1/2 P(\chi^2_1 \leq c)$ , where  $\chi^2_1$  denotes a chi-square random variable with one degree of freedom. Large positive values of  $w_n$  give favorable evidence to the full model. We have  $w_n$  equal to 1187.628 with a  $p$ -value  $\approx 0$ . This is evidence in favor of the GEP model.

[Tables 1-2 and Figure 6 about here.]

Also, we compare the GEP model with other models by inspection of the Akaike's information criterion (AIC,  $-2\ell(\hat{\theta})+2p$ ) and Schwarz's Bayesian information criterion (BIC,  $-2\ell(\hat{\theta})+p\log(n)$ ), where  $p$  is the number of parameters in the model and  $n$  is sample size. The last two columns of Table 1 show the estimated statistics AIC and BIC. The GEP model is judged to give the best fit with respect to each criterion.

The fitted probability density functions superimposed to the histogram of the data in Figure 6 reinforce the results in Table 1. The beta normal and Kw-normal distributions appear almost indistinguishable. This claim is further strengthened by the comparison between observed and expected frequencies in Table 2. The mean absolute deviation between expected and observed frequencies (given in the last row) reaches the minimum value for the GEP model.

## 15.2 Second application: the log-GEP regression model

In this section, we illustrate the usefulness of the log-GEP regression model. Lawless (2003) reports an experiment in which specimens of solid epoxy electrical-insulation are studied in an accelerated voltage life test. The sample size is  $n = 60$ . The percentage of censored observations is 10%. Three levels of voltage are considered: 52.5, 55.0 and 57.5. The variables involved in the study are:  $t_i$  - failure times for epoxy insulation specimens in minutes;  $cens_i$  - censoring indicator (0=censoring, 1=lifetime observed);  $v_{i1}$  - voltage in kV.

Now, we consider the model

$$y_i = \beta_0 + \beta_1 v_{i1} + z_i,$$

where  $Y_i$  follows the log-GEP distribution (33) for  $i = 1, 2, \dots, 60$ . The maximum likelihood estimates of the model parameters are calculated using the procedure NLMixed in SAS. The convergence was achieved using the reparametrization  $\eta = \exp(\eta_0)/[1 + \exp(\eta_0)]$ . This guarantees that the estimate of  $\eta$  is in  $(0, 1)$ . Iterative maximization of the logarithm of the likelihood function (37) starts with initial values for  $\beta_0$  and  $\beta_1$  taken from the fit of the log-exponential regression model with  $a = b = 1$ .

The log-beta Weibull (LBW) and log-exponentiated Weibull (LEW) distributions are very popular models in survival analysis, see, for example, Ortega *et al.* (2011). The probability density function of the former is

$$f(y; a, b, \sigma, \mu) = \frac{1}{\sigma B(a, b)} \exp \left\{ \left( \frac{y - \mu}{\sigma} \right) - b \exp \left( \frac{y - \mu}{\sigma} \right) \right\} \left\{ 1 - \exp \left[ - \exp \left( \frac{y - \mu}{\sigma} \right) \right] \right\}^{a-1}$$

for  $-\infty < y < \infty$ ,  $\sigma > 0$  and  $-\infty < \mu < \infty$ . Here,  $\mu$  is the location parameter,  $\sigma$  is the dispersion parameter and  $a$  and  $b$  are shape parameters. For  $b = 1$ , we obtain the LEW model. For  $a = b = 1$ , we obtain the log-Weibull model. For  $\mu = \beta_0 + \beta_1 v_{i1}$ , we obtain the LBW regression model.

Table 3 gives the maximum likelihood estimates (and the corresponding standard errors in parentheses) of the model parameters and the values of the following statistics: AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion) and CAIC (Consistent Akaike Information Criterion). The  $p$  values are given within square brackets. The computations were done using the subroutine NLMixed in SAS. These results indicate that the log-GEP model has the lowest AIC, BIC and CAIC values among the fitted models. Therefore, it could be chosen as the best model.

[Table 3 about here.]

We note from the fitted log-GEP regression model that  $x_1$  is significant at 1% and that there is a significant difference between the voltages 52.5, 55.0 and 57.5 for the survival times.

## 16 Univariate generalizations

While constructing the GEP distribution, we took failure times to follow the exponential distribution. The exponential distribution was chosen because it is the first and the most widely used model for failure times. In practice, other distributions can be chosen to model failure times.

In this section, we provide a general treatment by taking the probability density function of the cumulative distribution function of failure times to be given by  $f(\cdot)$  and  $F(\cdot)$ , respectively. In this case, (3) generalizes to

$$G(x) = \frac{\exp[-\theta + \theta F(x)] - \exp(-\theta)}{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x)]} \quad (38)$$

for  $x > 0$ ,  $\theta > 0$  and  $0 < \eta < 1$ . The corresponding probability density function and hazard rate function are

$$g(x) = \frac{\theta(1 - \eta)[1 - \exp(-\theta)]f(x)\exp[-\theta + \theta F(x)]}{\{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x)]\}^2} \quad (39)$$

and

$$h(x) = \frac{\theta[1 - \exp(-\theta)]f(x)\exp[-\theta + \theta F(x)]}{\{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x)]\}}, \quad (40)$$

respectively, for  $x > 0$ ,  $\theta > 0$  and  $0 < \eta < 1$ .

The shapes of (39) can be studied by taking their derivatives. Note that

$$\frac{\partial \log g(x)}{\partial x} = \frac{f'(x)}{f(x)} + \theta f'(y) - \frac{2\theta\eta f(x)\exp[-\theta + \theta F(x)]}{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x)]}$$

and

$$\begin{aligned} \frac{\partial^2 \log g(x)}{\partial x^2} &= \frac{f''(x)}{f(x)} - \left(\frac{f'(x)}{f(x)}\right)^2 + \theta f''(x) - \frac{2\theta\eta [\theta f^2(x) + f'(x)] \exp[-\theta + \theta F(x)]}{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x)]} \\ &\quad + \frac{2\theta^2\eta^2 f^2(x) \exp[-2\theta + 2\theta F(x)]}{\{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x)]\}^2}. \end{aligned}$$

The modes of (39) are the roots of  $d \log g(x)/dx = 0$ , say  $x = x_0$ . The mode will correspond to a maximum if  $d \log g(x)/dx > 0$  for all  $x < x_0$  and  $d \log g(x)/dx < 0$  for all  $x > x_0$ . The mode will correspond to a minimum if  $d \log g(x)/dx < 0$  for all  $x < x_0$  and  $d \log g(x)/dx > 0$  for all  $x > x_0$ . The mode will correspond to a point of inflexion if either  $d \log g(x)/dx > 0$  for all  $x \neq x_0$  or  $d \log g(x)/dx < 0$  for all  $x \neq x_0$ .

The shapes of (40) can be studied similarly by taking their derivatives. Note that

$$\frac{\partial \log h(x)}{\partial x} = \frac{\partial \log g(x)}{\partial x} + \frac{\theta f(x) \exp[-\theta + \theta F(x)]}{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x)]} + \frac{\theta\eta f(x) \exp[-\theta + \theta F(x)]}{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x)]}$$

and

$$\begin{aligned} \frac{\partial^2 \log h(x)}{\partial x^2} = & \frac{\partial^2 \log g(x)}{\partial x^2} + \frac{\theta^2 f^2(x) \exp[-\theta + \theta F(x)]}{1 - \exp[-\theta + \theta F(x)]} + \frac{\theta f'(x) \exp[-\theta + \theta F(x)]}{1 - \exp[-\theta + \theta F(x)]} \\ & - \frac{\theta^2 \eta^2 f^2(x) \exp[-2\theta + 2\theta F(x)]}{\{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x)]\}^2} \\ & + \frac{\theta^2 \eta f^2(x) \exp[-\theta + \theta F(x)]}{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x)]} \\ & + \frac{\theta \eta f'(x) \exp[-\theta + \theta F(x)]}{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x)]}. \end{aligned}$$

The modes of (40) are the roots of  $d \log h(x)/dx = 0$ , say  $x = x_0$ . The mode will correspond to a maximum if  $d \log h(x)/dx > 0$  for all  $x < x_0$  and  $d \log h(x)/dx < 0$  for all  $x > x_0$ . The mode will correspond to a minimum if  $d \log h(x)/dx < 0$  for all  $x < x_0$  and  $d \log h(x)/dx > 0$  for all  $x > x_0$ . The mode will correspond to a point of inflexion if either  $d \log h(x)/dx > 0$  for all  $x \neq x_0$  or  $d \log h(x)/dx < 0$  for all  $x \neq x_0$ .

The asymptotes of (39) and (40) can be studied by taking limits as  $x \rightarrow 0, \infty$ . The asymptotes of (39) are given by

$$g(x) \sim \frac{\theta(1-\eta)}{1 - \exp(-\theta)} f(x)$$

as  $x \rightarrow \infty$  and

$$g(x) \sim \theta \exp(-\theta) f(x)$$

as  $x \rightarrow 0$ . The asymptotes of (40) are given by

$$h(x) \sim \frac{f(x)}{1 - F(x)}$$

as  $x \rightarrow \infty$  and

$$h(x) \sim \theta \exp(-\theta) f(x)$$

as  $x \rightarrow 0$ .

Using the series expansion, (4), we can express (38) and (39) as mixtures:

$$G(x) = \frac{\exp[-\theta + \theta F(x)] - \exp(-\theta)}{1 - \exp(-\theta) - \eta} \sum_{k=0}^{\infty} \binom{-1}{k} \left[ \frac{\eta}{1 - \exp(-\theta) - \eta} \right]^k \exp[-k\theta + k\theta F(x)]$$

and

$$g(x) = \frac{\theta(1-\eta)[1 - \exp(-\theta)]}{[1 - \exp(-\theta) - \eta]^2} \sum_{k=0}^{\infty} \binom{-2}{k} \left[ \frac{\eta}{1 - \exp(-\theta) - \eta} \right]^k f(x) \exp[-(k+1)\theta + (k+1)\theta F(x)].$$

These mixture representations can be used derive expressions for moment generating function, characteristic function, cumulant generating function, moments, conditional moments, and others corresponding to (38).

The quantile function corresponding to (38) is

$$G^{-1}(u) = F^{-1} \left[ \frac{1}{\theta} \log \left\{ \frac{[1 - \exp(-\theta) - \eta] u + \exp(-\theta)}{(1 - \eta u) \exp(-\theta)} \right\} \right]$$

for  $0 < u < 1$ , where  $F^{-1}(\cdot)$  denotes the inverse function of  $F(\cdot)$ .

Let  $X_1, X_2, \dots, X_n$  be a random sample from (38). Let  $X_{i:n}$  denote the  $i$ th order statistic. Following arguments similar to those in Section 8, the probability density function and the cumulative distribution function of  $X_{i:n}$  can be represented as

$$g_{i:n}(x) = \frac{n! \theta (1 - \eta)^{n-i+1} \exp(-i\theta)}{(i-1)!(n-i)! [1 - \exp(-\theta) - \eta]^{n+1}} \sum_{j=0}^{i-1} \sum_{k=0}^{n-i} \sum_{l=0}^{\infty} \binom{i-1}{j} \binom{n-i}{k} \binom{-n-1}{l} K(j, k, l), \quad (41)$$

and

$$G_{i:n}(x) = \frac{1}{[1 - \exp(-\theta) - \eta]^n} \sum_{m=0}^{\infty} \sum_{j=i}^n \sum_{k=0}^j \sum_{l=0}^{n-j} \binom{-n}{m} \binom{n}{j} \binom{j}{k} \binom{n-j}{l} L(m, j, k, l), \quad (42)$$

respectively, where

$$K(j, k, l) = (-1)^{i-1-j+k} \frac{\eta^l \exp[-(l+k)\theta]}{[1 - \exp(-\theta) - \eta]^l} f(x) \exp[(j+k+l+1)\theta F(x)]$$

and

$$L(m, j, k, l) = \frac{\eta^m (1 - \eta)^{n-j} (-1)^{j-k+l} \exp[-(j+m+l)\theta]}{[1 - \exp(-\theta) - \eta]^m} \exp[(m+k+l)\theta F(x)].$$

The representations, (41) and (42), can be used to derive similar expressions for moment generating function, characteristic function, cumulant generating function, moments, conditional moments, and others of  $X_{i:n}$ .

We now consider estimation of the unknown parameters of (38) by the method of maximum likelihood. Suppose  $F(\cdot)$  and  $f(\cdot)$  are parameterized by  $\Phi$ , a vector of length  $q$ . Let  $x_1, x_2, \dots, x_n$  be a random sample from (38). Then the log-likelihood function is

$$\begin{aligned} \log L(\theta, \eta, \Phi) &= n \log \{ \theta (1 - \eta) [1 - \exp(-\theta)] \} - n\theta + \sum_{i=1}^n \log f(x_i; \Phi) + \theta \sum_{i=1}^n \log F(x_i; \Phi) \\ &\quad - 2 \sum_{i=1}^n \log \{ 1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x_i; \Phi)] \}. \end{aligned} \quad (43)$$

The derivatives of (43) with respect to  $\theta$ ,  $\eta$  and  $\Phi$  are:

$$\begin{aligned} \frac{\partial \log L}{\partial \theta} &= \frac{n}{\theta} + \frac{n}{\exp(\theta) - 1} - n + \sum_{i=1}^n F(x_i; \Phi) \\ &\quad - 2 \exp(-\theta) \sum_{i=1}^n \frac{1 + \eta [F(x_i; \Phi) - 1] \exp[\theta F(x_i; \Phi)]}{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x_i; \Phi)]}, \end{aligned} \quad (44)$$

$$\frac{\partial \log L}{\partial \eta} = \frac{n}{\eta - 1} - 2 \sum_{i=1}^n \frac{\{\exp[-\theta + \theta F(x_i; \Phi)] - 1\}}{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x_i; \Phi)]}, \quad (45)$$

$$\begin{aligned} \frac{\partial \log L}{\partial \Phi} = & \sum_{i=1}^n \frac{\partial f(x_i; \Phi) / \partial \Phi}{f(x_i; \Phi)} + \theta \sum_{i=1}^n \frac{\partial F(x_i; \Phi)}{\partial \Phi} \\ & - 2\theta \eta \exp(-\theta) \sum_{i=1}^n \frac{\partial F(x_i; \Phi) / \partial \Phi \exp[\theta F(x_i; \Phi)]}{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x_i; \Phi)]}. \end{aligned} \quad (46)$$

The maximum likelihood estimates of  $(\theta, \eta, \Phi)$ , say  $(\hat{\theta}, \hat{\eta}, \hat{\Phi})$ , are the simultaneous solutions of the equations  $\partial \log L / \partial \theta = 0$ ,  $\partial \log L / \partial \eta = 0$ , and  $\partial \log L / \partial \Phi = 0$ .

For interval estimation of  $(\theta, \eta, \Phi)$  and tests of hypothesis, one requires the Fisher information matrix. We can express the observed Fisher information matrix of  $(\hat{\theta}, \hat{\eta}, \hat{\Phi})$  as

$$\mathbf{J} = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{12} & J_{22} & J_{23} \\ J_{13} & J_{23} & J_{33} \end{pmatrix},$$

where

$$\begin{aligned} J_{11} = & -\frac{\partial^2 \log L}{\partial \hat{\theta}^2} = \frac{n}{\hat{\theta}^2} + \frac{n \exp(\hat{\theta})}{[\exp(\hat{\theta}) - 1]^2} \\ & + \sum_{i=1}^n \frac{\exp(-\hat{\theta}) - \hat{\eta} [F(x_i; \hat{\Phi}) - 1]^2 \exp[-\hat{\theta} + \hat{\theta} F(x_i; \hat{\Phi})]}{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i; \hat{\Phi})]} \\ & + \sum_{i=1}^n \left\{ \frac{\exp(-\hat{\theta}) + \hat{\eta} [F(x_i; \hat{\Phi}) - 1] \exp[-\hat{\theta} + \hat{\theta} F(x_i; \hat{\Phi})]}{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i; \hat{\Phi})]} \right\}^2, \\ J_{12} = & -\frac{\partial^2 \log L}{\partial \hat{\theta} \partial \hat{\eta}} = 2 \sum_{i=1}^n \frac{[F(x_i; \hat{\Phi}) - 1] \exp[-\hat{\theta} + \hat{\theta} F(x_i; \hat{\Phi})]}{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i; \hat{\Phi})]} \\ & + \sum_{i=1}^n \frac{\exp(-2\hat{\theta}) \{1 + \hat{\eta} [F(x_i; \hat{\Phi}) - 1] \exp[2\hat{\theta} F(x_i; \hat{\Phi})]\}}{\{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i; \hat{\Phi})]\}^2} \\ & - \sum_{i=1}^n \frac{\exp(-\hat{\theta}) \{1 + \hat{\eta} [F(x_i; \hat{\Phi}) - 1] \exp[\hat{\theta} F(x_i; \hat{\Phi})]\}}{\{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i; \hat{\Phi})]\}^2}, \end{aligned}$$

$$\begin{aligned}
J_{13} = & -\frac{\partial^2 \log L}{\partial \hat{\theta} \partial \hat{\Phi}} = -\sum_{i=1}^n \frac{\partial F(x_i; \hat{\Phi})}{\partial \hat{\Phi}} \\
& + 2\hat{\theta}\hat{\eta} \sum_{i=1}^n \frac{\partial F(x_i; \hat{\Phi}) / \partial \hat{\Phi} [F(x_i; \hat{\Phi}) - 1] \exp[-\hat{\theta} + \hat{\theta}F(x_i; \hat{\Phi})]}{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta}F(x_i; \hat{\Phi})]} \\
& + 2\hat{\eta} \sum_{i=1}^n \frac{\partial F(x_i; \hat{\Phi}) / \partial \hat{\Phi} \exp[-\hat{\theta} + \hat{\theta}F(x_i; \hat{\Phi})]}{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta}F(x_i; \hat{\Phi})]} \\
& + \hat{\theta} \sum_{i=1}^n \frac{\partial F(x_i; \hat{\Phi}) / \partial \hat{\Phi} \exp[-2\hat{\theta} + \hat{\theta}F(x_i; \hat{\Phi})]}{\{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta}F(x_i; \hat{\Phi})]\}^2} \\
& + \hat{\theta}\hat{\eta} \sum_{i=1}^n \frac{\partial F(x_i; \hat{\Phi}) / \partial \hat{\Phi} [F(x_i; \hat{\Phi}) - 1] \exp[-2\hat{\theta} + 2\hat{\theta}F(x_i; \hat{\Phi})]}{\{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta}F(x_i; \hat{\Phi})]\}^2},
\end{aligned}$$

$$\begin{aligned}
J_{22} = & -\frac{\partial^2 \log L}{\partial \hat{\eta}^2} = \frac{n}{(1 - \hat{\eta})^2} - \\
& 2 \sum_{i=1}^n \left\{ \frac{\exp[-\hat{\theta} + \hat{\theta}F(x_i; \hat{\Phi})] - 1}{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta}F(x_i; \hat{\Phi})]} \right\}^2,
\end{aligned}$$

$$\begin{aligned}
J_{23} = & -\frac{\partial^2 \log L}{\partial \hat{\eta} \partial \hat{\Phi}} = 2\hat{\theta} \sum_{i=1}^n \frac{\partial F(x_i; \hat{\Phi}) / \partial \hat{\Phi} \exp[-\hat{\theta} + \hat{\theta}F(x_i; \hat{\Phi})]}{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta}F(x_i; \hat{\Phi})]} \\
& + 2\hat{\theta}\hat{\eta} \sum_{i=1}^n \frac{\partial F(x_i; \hat{\Phi}) / \partial \hat{\Phi} \exp[-\hat{\theta} + \hat{\theta}F(x_i; \hat{\Phi})]}{\{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta}F(x_i; \hat{\Phi})]\}^2} \\
& - 2\hat{\theta}\hat{\eta} \sum_{i=1}^n \frac{\partial F(x_i; \hat{\Phi}) / \partial \hat{\Phi} \exp[-2\hat{\theta} + 2\hat{\theta}F(x_i; \hat{\Phi})]}{\{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta}F(x_i; \hat{\Phi})]\}^2},
\end{aligned}$$



$$\begin{aligned}
J_{33} = & -\frac{\partial^2 \log L}{\partial \hat{\theta} \partial \hat{\Phi}^2} = -\sum_{i=1}^n \frac{\partial^2 f(x_i; \hat{\Phi}) / \partial \hat{\Phi}^2}{f(x_i; \hat{\Phi})} + \sum_{i=1}^n \left[ \frac{\partial f(x_i; \hat{\Phi}) / \partial \hat{\Phi}}{f(x_i; \hat{\Phi})} \right]^2 + \hat{\theta} \sum_{i=1}^n \frac{\partial^2 F(x_i; \hat{\Phi})}{\partial \hat{\Phi}^2} \\
& - 2\hat{\theta}^2 \hat{\eta}^2 \sum_{i=1}^n \frac{[\partial F(x_i; \hat{\Phi}) / \partial \hat{\Phi}]^2 \exp[-2\hat{\theta} + 2\hat{\theta} F(x_i; \hat{\Phi})]}{\{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i; \hat{\Phi})]\}^2} \\
& + 2\hat{\theta} \hat{\eta} \sum_{i=1}^n \frac{[\partial F(x_i; \hat{\Phi}) / \partial \hat{\Phi}]^2 \exp[-\hat{\theta} + \hat{\theta} F(x_i; \hat{\Phi})]}{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i; \hat{\Phi})]} \\
& + 2\hat{\theta} \hat{\eta} \sum_{i=1}^n \frac{\partial^2 F(x_i; \hat{\Phi}) / \partial \hat{\Phi}^2 \exp[-\hat{\theta} + \hat{\theta} F(x_i; \hat{\Phi})]}{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i; \hat{\Phi})]}.
\end{aligned}$$

For large  $n$ , the distribution of  $\sqrt{n}(\hat{\theta} - \theta, \hat{\eta} - \eta, \hat{\Phi} - \Phi)$  approximates to a  $(q+2)$  variate normal distribution with zero means and variance-covariance matrix  $J^{-1}$ . The properties of  $(\hat{\theta}, \hat{\eta}, \hat{\Phi})$  can be derived based on this normal approximation.

Finally, consider the case of multicensoring as described in Section 12. In the case of multicensoring, the log-likelihood function is:

$$\log L(\theta, \eta, \Phi) = \sum_{i=1}^{n_0} \log g(t_i; \Phi) + \sum_{i=1}^{n_1} \log [G(s_i; \Phi) - G(s_{i-1}; \Phi)] + \sum_{i=1}^{n_2} \log [1 - G(r_i; \Phi)], \quad (47)$$

where  $g(\cdot)$  and  $G(\cdot)$  are given by (39) and (38), respectively. The first derivatives of the log-likelihood function with respect to the parameters  $\theta, \eta$  and  $\Phi$  are:

$$\begin{aligned}
\frac{\partial \log L}{\partial \theta} = & \sum_{i=1}^{n_0} \frac{1}{g(t_i; \Phi)} \frac{\partial g(t_i; \Phi)}{\partial \theta} + \sum_{i=1}^{n_1} \frac{1}{G(s_i; \Phi) - G(s_{i-1}; \Phi)} \left[ \frac{\partial G(s_i; \Phi)}{\partial \theta} - \frac{\partial G(s_{i-1}; \Phi)}{\partial \theta} \right] \\
& - \sum_{i=1}^{n_2} \frac{1}{1 - G(r_i; \Phi)} \frac{\partial G(r_i; \Phi)}{\partial \theta}, \quad (48)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \log L}{\partial \eta} = & \sum_{i=1}^{n_0} \frac{1}{g(t_i; \Phi)} \frac{\partial g(t_i; \Phi)}{\partial \eta} + \sum_{i=1}^{n_1} \frac{1}{G(s_i; \Phi) - G(s_{i-1}; \Phi)} \left[ \frac{\partial G(s_i; \Phi)}{\partial \eta} - \frac{\partial G(s_{i-1}; \Phi)}{\partial \eta} \right] \\
& - \sum_{i=1}^{n_2} \frac{1}{1 - G(r_i; \Phi)} \frac{\partial G(r_i; \Phi)}{\partial \eta}, \quad (49)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \log L}{\partial \Phi} = & \sum_{i=1}^{n_0} \frac{1}{g(t_i; \Phi)} \frac{\partial g(t_i; \Phi)}{\partial \Phi} + \sum_{i=1}^{n_1} \frac{1}{G(s_i; \Phi) - G(s_{i-1}; \Phi)} \left[ \frac{\partial G(s_i; \Phi)}{\partial \Phi} - \frac{\partial G(s_{i-1}; \Phi)}{\partial \Phi} \right] \\
& - \sum_{i=1}^{n_2} \frac{1}{1 - G(r_i; \Phi)} \frac{\partial G(r_i; \Phi)}{\partial \Phi}. \quad (50)
\end{aligned}$$

The first term in (47) is the same as (43) with  $(x_i, n)$  replaced by  $(t_i, n_0)$ . Also the first terms in (48)-(50) are the same as (44)-(46) with  $(x_i, n)$  replaced by  $(t_i, n_0)$ . So, it is sufficient to find

explicit expressions for the partial derivatives in (48)-(50). They are

$$\frac{\partial G(x; \Phi)}{\partial \theta} = \frac{[F(x; \Phi) - 1] \exp[-\theta + \theta F(x; \Phi)] + \exp(-\theta)}{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x; \Phi)]} - \frac{\exp(-2\theta) \{\exp[\theta F(x; \Phi)] - 1\}}{\{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x; \Phi)]\}^2} - \frac{\eta \{\exp[\theta F(x; \Phi)] - 1\} [F(x; \Phi) - 1] \exp[-2\theta + \theta F(x; \Phi)]}{\{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x; \Phi)]\}^2},$$

$$\frac{\partial G(x; \Phi)}{\partial \eta} = \frac{\exp(-\theta) \{1 - \exp[-\theta + \theta F(x; \Phi)]\} \{\exp[\theta F(x; \Phi)] - 1\}}{\{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x; \Phi)]\}^2},$$

$$\frac{\partial G(x; \Phi)}{\partial \Phi} = \frac{\theta \partial F(x; \Phi) / \partial \Phi \exp[-\theta + \theta F(x; \Phi)]}{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x; \Phi)]} - \frac{\theta \eta \partial F(x; \Phi) / \partial \Phi \exp[-2\theta + \theta F(x; \Phi)] \{\exp[\theta F(x; \Phi)] - 1\}}{\{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x; \Phi)]\}^2}.$$

The maximum likelihood estimates of  $(\theta, \eta, \Phi)$ , say  $(\hat{\theta}, \hat{\eta}, \hat{\Phi})$ , are the simultaneous solutions of the equations  $\partial \log L / \partial \theta = 0$ ,  $\partial \log L / \partial \eta = 0$ , and  $\partial \log L / \partial \Phi = 0$ . The Fisher information matrix for the estimators of  $(\theta, \eta, \Phi)$  corresponding to (43) is too complicated to be presented here.

## 17 Bivariate generalizations

Suppose now that each unit of the system suffers from two types of failures. Let  $F(x, y)$  and  $f(x, y)$  denote the joint cumulative distribution function and the joint probability density function of the failure times for each unit. Then the joint cumulative distribution function of the system failure times are:

$$G(x, y) = \frac{\exp[-\theta + \theta F(x, y)] - \exp(-\theta)}{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)]} \quad (51)$$

for  $x > 0, y > 0, \theta > 0$  and  $0 < \eta < 1$ . The joint probability density function is:

$$g(x, y) = \frac{\theta(1 - \eta) [1 - \exp(-\theta)] \exp[-\theta + \theta F(x, y)] A(x, y)}{\{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)]\}^3} \quad (52)$$

for  $x > 0, y > 0, \theta > 0$  and  $0 < \eta < 1$ , where

$$A(x, y) = \theta \{1 - \exp(-\theta) - \eta - \eta \exp[-\theta + \theta F(x, y)]\} \frac{\partial F(x, y)}{\partial x} \frac{\partial F(x, y)}{\partial y} + \{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)]\} \frac{\partial^2 F(x, y)}{\partial x \partial y}.$$

The marginal cumulative distribution functions are:

$$G(x) = \frac{\exp[-\theta + \theta F(x)] - \exp(-\theta)}{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x)]}$$

and

$$G(y) = \frac{\exp[-\theta + \theta F(y)] - \exp(-\theta)}{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(y)]}$$

for  $x > 0, y > 0, \theta > 0$  and  $0 < \eta < 1$ . The marginal probability density functions are:

$$g(x) = \frac{\theta(1 - \eta)[1 - \exp(-\theta)] f(x) \exp[-\theta + \theta F(x)]}{\{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x)]\}^2}$$

and

$$g(y) = \frac{\theta(1 - \eta)[1 - \exp(-\theta)] f(y) \exp[-\theta + \theta F(y)]}{\{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(y)]\}^2}$$

for  $x > 0, y > 0, \theta > 0$  and  $0 < \eta < 1$ . The conditional cumulative distribution functions are:

$$G(x|y) = \frac{\exp[\theta F(x, y)] - 1}{\exp[\theta F(y)] - 1} \frac{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(y)]}{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)]}$$

and

$$G(y|x) = \frac{\exp[\theta F(x, y)] - 1}{\exp[\theta F(x)] - 1} \frac{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x)]}{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)]}$$

for  $x > 0, y > 0, \theta > 0$  and  $0 < \eta < 1$ . The conditional probability density functions are:

$$g(x|y) = \frac{\{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(y)]\}^2 \exp[\theta F(x, y) - \theta F(y)] A(x, y)}{f(y) \{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)]\}^3}$$

and

$$g(y|x) = \frac{\{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x)]\}^2 \exp[\theta F(x, y) - \theta F(x)] A(x, y)}{f(x) \{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)]\}^3}$$

for  $x > 0, y > 0, \theta > 0$  and  $0 < \eta < 1$ . Also

$$\frac{\partial G(x, y)}{\partial x} = \frac{\theta \partial F(x, y) / \partial x \exp[-\theta + \theta F(x, y)] \{(1 - \eta)[1 - \exp(-\theta)] + \eta \exp[-\theta + \theta F(x, y)]\}}{\{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)]\}^2}$$

and

$$\frac{\partial G(x, y)}{\partial y} = \frac{\theta \partial F(x, y) / \partial y \exp[-\theta + \theta F(x, y)] \{(1 - \eta)[1 - \exp(-\theta)] + \eta \exp[-\theta + \theta F(x, y)]\}}{\{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)]\}^2}$$

for  $x > 0, y > 0, \theta > 0$  and  $0 < \eta < 1$ .

Note that (51) and (52) can be expressed as mixtures:

$$G(x, y) = \frac{\exp[-\theta + \theta F(x, y)] - \exp(-\theta)}{1 - \exp(-\theta) - \eta} \sum_{k=0}^{\infty} \binom{-1}{k} \left[ \frac{\eta}{1 - \exp(-\theta) - \eta} \right]^k \exp[-k\theta + k\theta F(x, y)]$$

and

$$g(x, y) = \frac{\theta(1 - \eta)[1 - \exp(-\theta)]}{[1 - \exp(-\theta) - \eta]^3} \sum_{k=0}^{\infty} \binom{-3}{k} \frac{\eta^k A(x, y) \exp[-(k+1)\theta + (k+1)\theta F(x, y)]}{[1 - \exp(-\theta) - \eta]^k}$$

These representations can be used to derive expressions for joint moment generating function, joint characteristic function, joint cumulant generating function, product moments and others corresponding to (51).

We now consider estimation of the unknown parameters of (51) by the method of maximum likelihood. Suppose  $F(\cdot, \cdot)$  and  $f(\cdot, \cdot)$  are parameterized by  $\Phi$ . Let  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$  be a random sample from (51). Then the log-likelihood function is

$$\begin{aligned} \log L(\theta, \eta, \Phi) = & n \log \{ \theta(1 - \eta) [1 - \exp(-\theta)] \} - n\theta + \theta \sum_{i=1}^n \log F(x_i, y_i; \Phi) + \sum_{i=1}^n \log A(x_i, y_i) \\ & - 3 \sum_{i=1}^n \log \{ 1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x_i, y_i; \Phi)] \}. \end{aligned} \quad (53)$$

The derivatives of (53) with respect to  $\theta$ ,  $\eta$  and  $\Phi$  are:

$$\begin{aligned} \frac{\partial \log L}{\partial \theta} = & \frac{n}{\theta} + \frac{n}{\exp(\theta) - 1} - n + \sum_{i=1}^n F(x_i, y_i; \Phi) + \sum_{i=1}^n \frac{\partial A(x_i, y_i) / \partial \theta}{A(x_i, y_i)} \\ & - 3 \exp(-\theta) \sum_{i=1}^n \frac{1 + \eta [F(x_i, y_i; \Phi) - 1] \exp[\theta F(x_i, y_i; \Phi)]}{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x_i, y_i; \Phi)]}, \end{aligned} \quad (54)$$

$$\begin{aligned} \frac{\partial \log L}{\partial \eta} = & \frac{n}{\eta - 1} + \sum_{i=1}^n \frac{\partial A(x_i, y_i) / \partial \eta}{A(x_i, y_i)} \\ & - 3 \sum_{i=1}^n \frac{\exp[-\theta + \theta F(x_i, y_i; \Phi)] - 1}{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x_i, y_i; \Phi)]}, \end{aligned} \quad (55)$$

$$\begin{aligned} \frac{\partial \log L}{\partial \Phi} = & \sum_{i=1}^n \frac{\theta \partial F(x_i, y_i; \Phi) / \partial \Phi}{F(x_i, y_i; \Phi)} + \sum_{i=1}^n \frac{\partial A(x_i, y_i) / \partial \Phi}{A(x_i, y_i)} \\ & - 3\theta \exp(-\theta) \sum_{i=1}^n \frac{\partial F(x_i, y_i; \Phi) / \partial \Phi \exp[\theta F(x_i, y_i; \Phi)]}{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x_i, y_i; \Phi)]}, \end{aligned} \quad (56)$$

where

$$\begin{aligned} \frac{\partial A(x, y)}{\partial \theta} = & \{1 - \exp(-\theta) - \eta - \eta \exp[-\theta + \theta F(x, y)]\} \frac{\partial F(x, y)}{\partial x} \frac{\partial F(x, y)}{\partial y} \\ & + \theta \{ \exp(-\theta) - \eta [F(x, y) - 1] \exp[-\theta + \theta F(x, y)] \} \frac{\partial F(x, y)}{\partial x} \frac{\partial F(x, y)}{\partial y} \\ & + \{ \exp(-\theta) - \eta [F(x, y) - 1] \exp[-\theta + \theta F(x, y)] \} \frac{\partial^2 F(x, y)}{\partial x \partial y}, \\ \frac{\partial A(x, y)}{\partial \eta} = & -\theta \{1 + \exp[-\theta + \theta F(x, y)]\} \frac{\partial F(x, y)}{\partial x} \frac{\partial F(x, y)}{\partial y} \\ & + \{ \exp[-\theta + \theta F(x, y)] - 1 \} \frac{\partial^2 F(x, y)}{\partial x \partial y}, \end{aligned}$$

$$\begin{aligned}\frac{\partial A(x, y)}{\partial \Phi} &= -\theta^2 \eta \exp[-\theta + \theta F(x, y)] \frac{\partial F(x, y)}{\partial \Phi} \frac{\partial F(x, y)}{\partial x} \frac{\partial F(x, y)}{\partial y} \\ &\quad - \theta \eta \exp[-\theta + \theta F(x, y)] \frac{\partial^2 F(x, y)}{\partial x \partial \Phi} \frac{\partial F(x, y)}{\partial y} \\ &\quad - \theta \eta \exp[-\theta + \theta F(x, y)] \frac{\partial^2 F(x, y)}{\partial y \partial \Phi} \frac{\partial F(x, y)}{\partial x}.\end{aligned}$$

The maximum likelihood estimates of  $(\theta, \eta, \Phi)$ , say  $(\hat{\theta}, \hat{\eta}, \hat{\Phi})$ , are the simultaneous solutions of the equations  $\partial \log L / \partial \theta = 0$ ,  $\partial \log L / \partial \eta = 0$ , and  $\partial \log L / \partial \Phi = 0$ .

For interval estimation of  $(\theta, \eta, \Phi)$  and tests of hypothesis, one requires the Fisher information matrix. We can express the observed Fisher information matrix of  $(\hat{\theta}, \hat{\eta}, \hat{\Phi})$  as

$$\mathbf{J} = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{12} & J_{22} & J_{23} \\ J_{13} & J_{23} & J_{33} \end{pmatrix},$$

where

$$\begin{aligned}J_{11} &= -\frac{\partial^2 \log L}{\partial \theta^2} = \frac{n}{\hat{\theta}^2} + \frac{n \exp(\hat{\theta})}{[\exp(\hat{\theta}) - 1]^2} \\ &\quad - 3 \exp(-\hat{\theta}) \sum_{i=1}^n \frac{1}{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]} \\ &\quad + 3 \hat{\eta} \sum_{i=1}^n \frac{[F(x_i, y_i; \hat{\Phi}) - 1]^2 \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]}{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]} \\ &\quad - 3 \exp(-2\hat{\theta}) \sum_{i=1}^n \frac{1}{\{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]\}^2} \\ &\quad - 6 \hat{\eta} \sum_{i=1}^n \frac{[F(x_i, y_i; \hat{\Phi}) - 1] \exp[-2\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]}{\{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]\}^2} \\ &\quad - 3 \hat{\eta}^2 \sum_{i=1}^n \frac{[F(x_i, y_i; \hat{\Phi}) - 1]^2 \exp[-2\hat{\theta} + 2\hat{\theta} F(x_i, y_i; \hat{\Phi})]}{\{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]\}^2} \\ &\quad - \sum_{i=1}^n \frac{\partial^2 A(x_i, y_i) / \partial \hat{\theta}^2}{A(x_i, y_i)} + \sum_{i=1}^n \left[ \frac{\partial A(x_i, y_i) / \partial \hat{\theta}}{A(x_i, y_i)} \right]^2,\end{aligned}$$

$$\begin{aligned}
J_{12} = -\frac{\partial^2 \log L}{\partial \hat{\theta} \hat{\eta}} &= 3 \sum_{i=1}^n \frac{[F(x_i, y_i; \hat{\Phi}) - 1] \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]}{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]} \\
&\quad - 3 \sum_{i=1}^n \frac{\exp[-2\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]}{\{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]\}^2} \\
&\quad + 3 \sum_{i=1}^n \frac{\exp(-\hat{\theta})}{\{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]\}^2} \\
&\quad - 3\hat{\eta} \sum_{i=1}^n \frac{[F(x_i, y_i; \hat{\Phi}) - 1] \exp[-2\hat{\theta} + 2\hat{\theta} F(x_i, y_i; \hat{\Phi})]}{\{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]\}^2} \\
&\quad + 3\hat{\eta} \sum_{i=1}^n \frac{[F(x_i, y_i; \hat{\Phi}) - 1] \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]}{\{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]\}^2} \\
&\quad - \sum_{i=1}^n \frac{\partial^2 A(x_i, y_i) / \partial \hat{\theta} \partial \hat{\eta}}{A(x_i, y_i)} + \sum_{i=1}^n \frac{\partial A(x_i, y_i) / \partial \hat{\theta} \partial A(x_i, y_i) / \partial \hat{\eta}}{A^2(x_i, y_i)},
\end{aligned}$$

$$\begin{aligned}
J_{13} = -\frac{\partial^2 \log L}{\partial \hat{\theta} \hat{\Phi}} &= 3\hat{\theta} \hat{\eta} \sum_{i=1}^n \frac{\partial F(x_i, y_i; \hat{\Phi}) / \partial \hat{\Phi} [F(x_i, y_i; \hat{\Phi}) - 1] \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]}{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]} \\
&\quad + 3\hat{\eta} \sum_{i=1}^n \frac{\partial F(x_i, y_i; \hat{\Phi}) / \partial \hat{\Phi} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]}{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]} \\
&\quad - 3\hat{\theta} \hat{\eta} \sum_{i=1}^n \frac{\partial F(x_i, y_i; \hat{\Phi}) / \partial \hat{\Phi} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]}{\{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]\}^2} \\
&\quad - 3\hat{\theta} \hat{\eta}^2 \sum_{i=1}^n \frac{\partial F(x_i, y_i; \hat{\Phi}) / \partial \hat{\Phi} [F(x_i, y_i; \hat{\Phi}) - 1] \exp[-2\hat{\theta} + 2\hat{\theta} F(x_i, y_i; \hat{\Phi})]}{\{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]\}^2} \\
&\quad - \sum_{i=1}^n \frac{\partial^2 A(x_i, y_i) / \partial \hat{\theta} \partial \hat{\Phi}}{A(x_i, y_i)} + \sum_{i=1}^n \frac{\partial A(x_i, y_i) / \partial \hat{\theta} \partial A(x_i, y_i) / \partial \hat{\Phi}}{A^2(x_i, y_i)},
\end{aligned}$$

$$\begin{aligned}
J_{22} = -\frac{\partial^2 \log L}{\partial \hat{\eta}^2} &= \frac{n}{(1 - \hat{\eta})^2} - 3 \sum_{i=1}^n \left\{ \frac{\exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})] - 1}{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]} \right\}^2 \\
&\quad - \sum_{i=1}^n \frac{\partial^2 A(x_i, y_i) / \partial \hat{\eta}^2}{A(x_i, y_i)} + \sum_{i=1}^n \left[ \frac{\partial A(x_i, y_i) / \partial \hat{\eta}}{A(x_i, y_i)} \right]^2,
\end{aligned}$$

$$\begin{aligned}
J_{23} &= -\frac{\partial^2 \log L}{\partial \hat{\eta} \partial \hat{\Phi}} = 3\hat{\theta} \sum_{i=1}^n \frac{\partial F(x_i, y_i; \hat{\Phi}) / \partial \hat{\Phi} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]}{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]} \\
&\quad - 3\hat{\theta} \hat{\eta} \sum_{i=1}^n \frac{\partial F(x_i, y_i; \hat{\Phi}) / \partial \hat{\Phi} \exp[-2\hat{\theta} + 2\hat{\theta} F(x_i, y_i; \hat{\Phi})]}{\{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]\}^2} \\
&\quad + 3\hat{\theta} \hat{\eta} \sum_{i=1}^n \frac{\partial F(x_i, y_i; \hat{\Phi}) / \partial \hat{\Phi} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]}{\{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]\}^2} \\
&\quad - \sum_{i=1}^n \frac{\partial^2 A(x_i, y_i) / \partial \hat{\eta} \partial \hat{\Phi}}{A(x_i, y_i)} + \sum_{i=1}^n \frac{\partial A(x_i, y_i) / \partial \hat{\eta} \partial A(x_i, y_i) / \partial \hat{\Phi}}{A^2(x_i, y_i)}, \\
J_{33} &= -\frac{\partial^2 \log L}{\partial \hat{\Phi}^2} = -3\hat{\theta}^2 \hat{\eta}^2 \sum_{i=1}^n \left\{ \frac{\partial F(x_i, y_i; \hat{\Phi}) / \partial \hat{\Phi} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]}{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]} \right\}^2 \\
&\quad - 3\hat{\theta}^2 \hat{\eta} \sum_{i=1}^n \frac{[\partial F(x_i, y_i; \hat{\Phi}) / \partial \hat{\Phi}]^2 \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]}{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]} \\
&\quad - 3\hat{\theta} \hat{\eta} \sum_{i=1}^n \frac{\partial^2 F(x_i, y_i; \hat{\Phi}) / \partial \hat{\Phi}^2 \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]}{1 - \exp(-\hat{\theta}) - \hat{\eta} + \hat{\eta} \exp[-\hat{\theta} + \hat{\theta} F(x_i, y_i; \hat{\Phi})]} \\
&\quad - \hat{\theta} \sum_{i=1}^n \frac{\partial^2 F(x_i, y_i) / \partial \hat{\Phi}^2}{F(x_i, y_i)} + \sum_{i=1}^n \left[ \frac{\partial F(x_i, y_i) / \partial \hat{\Phi}}{F(x_i, y_i)} \right]^2 \\
&\quad - \sum_{i=1}^n \frac{\partial^2 A(x_i, y_i) / \partial \hat{\Phi}^2}{A(x_i, y_i)} + \sum_{i=1}^n \left[ \frac{\partial A(x_i, y_i) / \partial \hat{\Phi}}{A(x_i, y_i)} \right]^2,
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial^2 A(x, y)}{\partial \theta^2} &= \exp(-\theta) \{1 + [F(x, y) - 1] \exp[\theta F(x, y)]\} \frac{\partial F(x, y)}{\partial x} \frac{\partial F(x, y)}{\partial y} \\
&\quad + \{\exp(-\theta) - \eta [F(x, y) - 1] \exp[-\theta + \theta F(x, y)]\} \frac{\partial F(x, y)}{\partial x} \frac{\partial F(x, y)}{\partial y} \\
&\quad + \theta \left\{ -\exp(-\theta) - \eta [F(x, y) - 1]^2 \exp[-\theta + \theta F(x, y)] \right\} \frac{\partial F(x, y)}{\partial x} \frac{\partial F(x, y)}{\partial y} \\
&\quad + \left\{ -\exp(-\theta) + \eta [F(x, y) - 1]^2 \exp[-\theta + \theta F(x, y)] \right\} \frac{\partial^2 F(x, y)}{\partial x \partial y},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 A(x, y)}{\partial \theta \partial \eta} &= \exp(-\theta) \{\exp[-\theta + \theta F(x, y)] - 1\} \frac{\partial F(x, y)}{\partial x} \frac{\partial F(x, y)}{\partial y} \\
&\quad - \theta [F(x, y) - 1] \exp[-\theta + \theta F(x, y)] \frac{\partial F(x, y)}{\partial x} \frac{\partial F(x, y)}{\partial y} \\
&\quad + [F(x, y) - 1] \exp[-\theta + \theta F(x, y)] \frac{\partial^2 F(x, y)}{\partial x \partial y},
\end{aligned}$$



$$\begin{aligned}
\frac{\partial^2 A(x, y)}{\partial \theta \partial \Phi} = & \theta \eta \exp[-\theta + \theta F(x, y)] \frac{\partial F(x, y)}{\partial \Phi} \frac{\partial F(x, y)}{\partial x} \frac{\partial F(x, y)}{\partial y} \\
& + \{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)]\} \frac{\partial^2 F(x, y)}{\partial x \partial \Phi} \frac{\partial F(x, y)}{\partial y} \\
& + \{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)]\} \frac{\partial^2 F(x, y)}{\partial y \partial \Phi} \frac{\partial F(x, y)}{\partial x} \\
& - \theta \eta [F(x, y) - 1]^2 \exp[-\theta + \theta F(x, y)] \frac{\partial F(x, y)}{\partial \Phi} \frac{\partial F(x, y)}{\partial x} \frac{\partial F(x, y)}{\partial y} \\
& - \theta \eta \exp[-\theta + \theta F(x, y)] \left( \frac{\partial F(x, y)}{\partial \Phi} \right)^2 \frac{\partial F(x, y)}{\partial x} \frac{\partial F(x, y)}{\partial y} \\
& + \theta \{ \exp(-\theta) - \eta [F(x, y) - 1] \exp[-\theta + \theta F(x, y)] \} \frac{\partial^2 F(x, y)}{\partial x \partial \Phi} \frac{\partial F(x, y)}{\partial y} \\
& + \theta \{ \exp(-\theta) - \eta [F(x, y) - 1] \exp[-\theta + \theta F(x, y)] \} \frac{\partial^2 F(x, y)}{\partial y \partial \Phi} \frac{\partial F(x, y)}{\partial x} \\
& + \{ \exp(-\theta) + \eta [F(x, y) - 1] \exp[-\theta + \theta F(x, y)] \} \frac{\partial^3 F(x, y)}{\partial x \partial y \partial \Phi} \\
& + \eta [1 + \theta F(x, y)] \exp[-\theta + \theta F(x, y)] \frac{\partial F(x, y)}{\partial \Phi} \frac{\partial^2 F(x, y)}{\partial x \partial y},
\end{aligned}$$

$$\frac{\partial^2 A(x, y)}{\partial \eta^2} = 0,$$

$$\begin{aligned}
\frac{\partial^2 A(x, y)}{\partial \eta \partial \Phi} = & -\theta^2 \exp[-\theta + \theta F(x, y)] \frac{\partial F(x, y)}{\partial \Phi} \frac{\partial F(x, y)}{\partial x} \frac{\partial F(x, y)}{\partial y} \\
& - \theta \{1 + \exp[-\theta + \theta F(x, y)]\} \frac{\partial^2 F(x, y)}{\partial x \partial \Phi} \frac{\partial F(x, y)}{\partial y} \\
& - \theta \{1 + \exp[-\theta + \theta F(x, y)]\} \frac{\partial^2 F(x, y)}{\partial y \partial \Phi} \frac{\partial F(x, y)}{\partial x} \\
& + \{ \exp[-\theta + \theta F(x, y)] - 1 \} \frac{\partial^3 F(x, y)}{\partial x \partial y \partial \Phi} \\
& + \theta \exp[-\theta + \theta F(x, y)] \frac{\partial F(x, y)}{\partial \Phi} \frac{\partial^2 F(x, y)}{\partial x \partial y},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 A(x, y)}{\partial \Phi^2} = & -\theta^3 \exp[-\theta + \theta F(x, y)] \left( \frac{\partial F(x, y)}{\partial \Phi} \right)^2 \frac{\partial F(x, y)}{\partial x} \frac{\partial F(x, y)}{\partial y} \\
& -\theta^2 \eta \exp[-\theta + \theta F(x, y)] \frac{\partial F(x, y)}{\partial \Phi} \frac{\partial^2 F(x, y)}{\partial x \partial \Phi} \frac{\partial F(x, y)}{\partial y} \\
& -\theta^2 \eta \exp[-\theta + \theta F(x, y)] \frac{\partial^2 F(x, y)}{\partial \Phi^2} \frac{\partial F(x, y)}{\partial x} \frac{\partial F(x, y)}{\partial y} \\
& -\theta^2 \eta \exp[-\theta + \theta F(x, y)] \frac{\partial F(x, y)}{\partial \Phi} \frac{\partial F(x, y)}{\partial x} \frac{\partial^2 F(x, y)}{\partial y \partial \Phi} \\
& -\theta^2 \eta \exp[-\theta + \theta F(x, y)] \frac{\partial F(x, y)}{\partial \Phi} \frac{\partial^2 F(x, y)}{\partial x \partial \Phi} \frac{\partial F(x, y)}{\partial y} \\
& -\theta \phi \exp[-\theta + \theta F(x, y)] \frac{\partial^3 F(x, y)}{\partial x \partial \Phi^2} \frac{\partial F(x, y)}{\partial y} \\
& -\theta \phi \exp[-\theta + \theta F(x, y)] \frac{\partial^2 F(x, y)}{\partial x \partial \Phi} \frac{\partial^2 F(x, y)}{\partial y \partial \Phi} \\
& -\theta^2 \eta \exp[-\theta + \theta F(x, y)] \frac{\partial F(x, y)}{\partial \Phi} \frac{\partial^2 F(x, y)}{\partial x^2} \frac{\partial^2 F(x, y)}{\partial y \partial \Phi} \\
& -\theta \eta \exp[-\theta + \theta F(x, y)] \frac{\partial^2 F(x, y)}{\partial x \partial \Phi} \frac{\partial^2 F(x, y)}{\partial y \partial \Phi} \\
& -\theta \eta \exp[-\theta + \theta F(x, y)] \frac{\partial^3 F(x, y)}{\partial y \partial \Phi^2} \frac{\partial F(x, y)}{\partial x}.
\end{aligned}$$

For large  $n$ , the distribution of  $\sqrt{n}(\hat{\theta} - \theta, \hat{\eta} - \eta, \hat{\Phi} - \Phi)$  approximates to a  $(q + 2)$  variate normal distribution with zero means and variance-covariance matrix  $\mathbf{J}^{-1}$ . The properties of  $(\hat{\theta}, \hat{\eta}, \hat{\Phi})$  can be derived based on this normal approximation.

Finally, we consider the case of multicensoring for bivariate data. We suppose that there are  $n$  bivariate failure times of which

- $n_0$  are known to occur at  $(x_1^{(0)}, y_1^{(0)}), \dots, (x_{n_0}^{(0)}, y_{n_0}^{(0)})$ .
- $n_1$  are known to have  $x$  components occurring at  $x_1^{(1)}, \dots, x_{n_1}^{(1)}$  and  $y$  components exceeding  $y_i^{(1)}, i = 1, \dots, n_1$  but not observed any longer.
- $n_2$  are known to have  $x$  components occurring at  $x_1^{(2)}, \dots, x_{n_2}^{(2)}$  and  $y$  components belonging to the interval  $[y_{i-1}^{(2)}, y_i^{(2)}], i = 1, \dots, n_2$ .
- $n_3$  are known to have  $y$  components occurring at  $y_1^{(3)}, \dots, y_{n_3}^{(3)}$  and  $x$  components exceeding  $x_i^{(3)}, i = 1, \dots, n_3$  but not observed any longer.
- $n_4$  are known to have  $y$  components occurring at  $y_1^{(4)}, \dots, y_{n_4}^{(4)}$  and  $x$  components belonging to the interval  $[x_{i-1}^{(4)}, x_i^{(4)}], i = 1, \dots, n_4$ .
- $n_5$  are known to have  $y$  components belonging to the interval  $[y_{i-1}^{(5)}, y_i^{(5)}], i = 1, \dots, n_5$  and  $x$  components belonging to the interval  $[x_{i-1}^{(5)}, x_i^{(5)}], i = 1, \dots, n_5$ .
- $n_6$  are known to have  $y$  components belonging to the interval  $[y_{i-1}^{(6)}, y_i^{(6)}], i = 1, \dots, n_6$  and  $x$  components exceeding  $x_i^{(6)}, i = 1, \dots, n_6$  but not observed any longer.

- $n_7$  are known to have  $x$  components belonging to the interval  $[x_{i-1}^{(7)}, x_i^{(7)}]$ ,  $i = 1, \dots, n_7$  and  $y$  components exceeding  $y_i^{(7)}$ ,  $i = 1, \dots, n_7$  but not observed any longer.
- $n_8$  are known to have  $x$  components exceeding  $x_i^{(8)}$ ,  $i = 1, \dots, n_8$  but not observed any longer and  $y$  components exceeding  $y_i^{(8)}$ ,  $i = 1, \dots, n_8$  but not observed any longer.

Note that  $n = n_0 + n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 + n_8$ .

In the multicensoring scheme described, the log-likelihood function, (53), becomes

$$\begin{aligned}
\log L(\theta, \eta, \Phi) = & \sum_{i=1}^{n_0} \log g(x_i^{(0)}, y_i^{(0)}) + \sum_{i=1}^{n_1} \log \left[ g(x_i^{(1)}) - \frac{\partial G(x_i^{(1)}, y_i^{(1)})}{\partial x_i^{(1)}} \right] \\
& + \sum_{i=1}^{n_2} \log \left[ \frac{\partial G(x_i^{(2)}, y_i^{(2)})}{\partial x_i^{(2)}} - \frac{\partial G(x_i^{(2)}, y_{i-1}^{(2)})}{\partial x_i^{(2)}} \right] \\
& + \sum_{i=1}^{n_3} \log \left[ g(y_i^{(3)}) - \frac{\partial G(x_i^{(3)}, y_i^{(3)})}{\partial y_i^{(3)}} \right] \\
& + \sum_{i=1}^{n_4} \log \left[ \frac{\partial G(x_i^{(4)}, y_i^{(4)})}{\partial y_i^{(4)}} - \frac{\partial G(x_{i-1}^{(4)}, y_i^{(4)})}{\partial x_i^{(4)}} \right] \\
& + \sum_{i=1}^{n_5} \log \left[ G(x_i^{(5)}, y_i^{(5)}) - G(x_{i-1}^{(5)}, y_i^{(5)}) - G(x_i^{(5)}, y_{i-1}^{(5)}) + G(x_{i-1}^{(5)}, y_{i-1}^{(5)}) \right] \\
& + \sum_{i=1}^{n_6} \log \left[ G(y_i^{(6)}) - G(x_i^{(6)}, y_i^{(6)}) - G(y_{i-1}^{(6)}) + G(x_i^{(6)}, y_{i-1}^{(6)}) \right] \\
& + \sum_{i=1}^{n_7} \log \left[ G(x_i^{(7)}) - G(x_{i-1}^{(7)}) - G(x_i^{(7)}, y_i^{(7)}) + G(x_{i-1}^{(7)}, y_i^{(7)}) \right] \\
& + \sum_{i=1}^{n_8} \log \left[ 1 - G(x_i^{(8)}) - G(y_i^{(8)}) + G(x_i^{(8)}, y_i^{(8)}) \right]. \tag{57}
\end{aligned}$$

The derivatives of (57) with respect to  $\theta$ ,  $\eta$  and  $\Phi$  are:

$$\begin{aligned}
\frac{\partial \log L(\theta, \eta, \Phi)}{\partial \theta} = & \sum_{i=1}^{n_0} \frac{\partial g(x_i^{(0)}, y_i^{(0)}) / \partial \theta}{\partial g(x_i^{(0)}, y_i^{(0)})} + \sum_{i=1}^{n_1} \frac{\frac{\partial g(x_i^{(1)})}{\partial \theta} - \frac{\partial^2 G(x_i^{(1)}, y_i^{(1)})}{\partial \theta \partial x_i^{(1)}}}{g(x_i^{(1)}) - \frac{\partial G(x_i^{(1)}, y_i^{(1)})}{\partial x_i^{(1)}}} \\
& + \sum_{i=1}^{n_2} \frac{\frac{\partial^2 G(x_i^{(2)}, y_i^{(2)})}{\partial \theta \partial x_i^{(2)}} - \frac{\partial^2 G(x_i^{(2)}, y_{i-1}^{(2)})}{\partial \theta \partial x_i^{(2)}}}{\frac{\partial G(x_i^{(2)}, y_i^{(2)})}{\partial x_i^{(2)}} - \frac{\partial G(x_i^{(2)}, y_{i-1}^{(2)})}{\partial x_i^{(2)}}} \\
& + \sum_{i=1}^{n_3} \frac{\frac{\partial g(y_i^{(3)})}{\partial \theta} - \frac{\partial^2 G(x_i^{(3)}, y_i^{(3)})}{\partial \theta \partial y_i^{(3)}}}{g(y_i^{(3)}) - \frac{\partial G(x_i^{(3)}, y_i^{(3)})}{\partial y_i^{(3)}}} \\
& + \sum_{i=1}^{n_4} \frac{\frac{\partial^2 G(x_i^{(4)}, y_i^{(4)})}{\partial \theta \partial y_i^{(4)}} - \frac{\partial^2 G(x_{i-1}^{(4)}, y_i^{(4)})}{\partial \theta \partial y_i^{(4)}}}{\frac{\partial G(x_i^{(4)}, y_i^{(4)})}{\partial y_i^{(4)}} - \frac{\partial G(x_{i-1}^{(4)}, y_i^{(4)})}{\partial y_i^{(4)}}} \\
& + \sum_{i=1}^{n_5} \frac{\frac{\partial G(x_i^{(5)}, y_i^{(5)})}{\partial \theta} - \frac{\partial G(x_{i-1}^{(5)}, y_i^{(5)})}{\partial \theta} - \frac{\partial G(x_i^{(5)}, y_{i-1}^{(5)})}{\partial \theta} + \frac{\partial G(x_{i-1}^{(5)}, y_{i-1}^{(5)})}{\partial \theta}}{G(x_i^{(5)}, y_i^{(5)}) - G(x_{i-1}^{(5)}, y_i^{(5)}) - G(x_i^{(5)}, y_{i-1}^{(5)}) + G(x_{i-1}^{(5)}, y_{i-1}^{(5)})} \\
& + \sum_{i=1}^{n_6} \frac{\frac{\partial G(y_i^{(6)})}{\partial \theta} - \frac{\partial G(x_i^{(6)}, y_i^{(6)})}{\partial \theta} - \frac{\partial G(y_{i-1}^{(6)})}{\partial \theta} + \frac{\partial G(x_i^{(6)}, y_{i-1}^{(6)})}{\partial \theta}}{G(y_i^{(6)}) - G(x_i^{(6)}, y_i^{(6)}) - G(y_{i-1}^{(6)}) + G(x_i^{(6)}, y_{i-1}^{(6)})} \\
& + \sum_{i=1}^{n_7} \frac{\frac{\partial G(x_i^{(7)})}{\partial \theta} - \frac{\partial G(x_{i-1}^{(7)})}{\partial \theta} - \frac{\partial G(x_i^{(7)}, y_i^{(7)})}{\partial \theta} + \frac{\partial G(x_{i-1}^{(7)}, y_i^{(7)})}{\partial \theta}}{G(x_i^{(7)}) - G(x_{i-1}^{(7)}) - G(x_i^{(7)}, y_i^{(7)}) + G(x_{i-1}^{(7)}, y_i^{(7)})} \\
& + \sum_{i=1}^{n_8} \frac{\frac{\partial G(x_i^{(8)})}{\partial \theta} - \frac{\partial G(y_i^{(8)})}{\partial \theta} + \frac{\partial G(x_i^{(8)}, y_i^{(8)})}{\partial \theta}}{1 - G(x_i^{(8)}) - G(y_i^{(8)}) + G(x_i^{(8)}, y_i^{(8)})}, \tag{58}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \log L(\theta, \eta, \Phi)}{\partial \eta} = & \sum_{i=1}^{n_0} \frac{\partial g(x_i^{(0)}, y_i^{(0)}) / \partial \eta}{\partial g(x_i^{(0)}, y_i^{(0)})} + \sum_{i=1}^{n_1} \frac{\frac{\partial g(x_i^{(1)})}{\partial \eta} - \frac{\partial^2 G(x_i^{(1)}, y_i^{(1)})}{\partial \eta \partial x_i^{(1)}}}{g(x_i^{(1)}) - \frac{\partial G(x_i^{(1)}, y_i^{(1)})}{\partial x_i^{(1)}}} \\
& + \sum_{i=1}^{n_2} \frac{\frac{\partial^2 G(x_i^{(2)}, y_i^{(2)})}{\partial \eta \partial x_i^{(2)}} - \frac{\partial^2 G(x_i^{(2)}, y_{i-1}^{(2)})}{\partial \eta \partial x_i^{(2)}}}{\frac{\partial G(x_i^{(2)}, y_i^{(2)})}{x_i^{(2)}} - \frac{\partial G(x_i^{(2)}, y_{i-1}^{(2)})}{x_i^{(2)}}} \\
& + \sum_{i=1}^{n_3} \frac{\frac{\partial g(y_i^{(3)})}{\partial \eta} - \frac{\partial^2 G(x_i^{(3)}, y_i^{(3)})}{\partial \eta \partial y_i^{(3)}}}{g(y_i^{(3)}) - \frac{\partial G(x_i^{(3)}, y_i^{(3)})}{\partial y_i^{(3)}}} \\
& + \sum_{i=1}^{n_4} \frac{\frac{\partial^2 G(x_i^{(4)}, y_i^{(4)})}{\partial \eta \partial y_i^{(4)}} - \frac{\partial^2 G(x_{i-1}^{(4)}, y_i^{(4)})}{\partial \eta \partial x_i^{(4)}}}{\frac{\partial G(x_i^{(4)}, y_i^{(4)})}{\partial y_i^{(4)}} - \frac{\partial G(x_{i-1}^{(4)}, y_i^{(4)})}{\partial x_i^{(4)}}} \\
& + \sum_{i=1}^{n_5} \frac{\frac{\partial G(x_i^{(5)}, y_i^{(5)})}{\partial \eta} - \frac{\partial G(x_{i-1}^{(5)}, y_i^{(5)})}{\partial \eta} - \frac{\partial G(x_i^{(5)}, y_{i-1}^{(5)})}{\partial \eta} + \frac{\partial G(x_{i-1}^{(5)}, y_{i-1}^{(5)})}{\partial \eta}}{G(x_i^{(5)}, y_i^{(5)}) - G(x_{i-1}^{(5)}, y_i^{(5)}) - G(x_i^{(5)}, y_{i-1}^{(5)}) + G(x_{i-1}^{(5)}, y_{i-1}^{(5)})} \\
& + \sum_{i=1}^{n_6} \frac{\frac{\partial G(y_i^{(6)})}{\partial \eta} - \frac{\partial G(x_i^{(6)}, y_i^{(6)})}{\partial \eta} - \frac{\partial G(y_{i-1}^{(6)})}{\partial \eta} + \frac{\partial G(x_i^{(6)}, y_{i-1}^{(6)})}{\partial \eta}}{G(y_i^{(6)}) - G(x_i^{(6)}, y_i^{(6)}) - G(y_{i-1}^{(6)}) + G(x_i^{(6)}, y_{i-1}^{(6)})} \\
& + \sum_{i=1}^{n_7} \frac{\frac{\partial G(x_i^{(7)})}{\partial \eta} - \frac{\partial G(x_{i-1}^{(7)})}{\partial \eta} - \frac{\partial G(x_i^{(7)}, y_i^{(7)})}{\partial \eta} + \frac{\partial G(x_{i-1}^{(7)}, y_i^{(7)})}{\partial \eta}}{G(x_i^{(7)}) - G(x_{i-1}^{(7)}) - G(x_i^{(7)}, y_i^{(7)}) + G(x_{i-1}^{(7)}, y_i^{(7)})} \\
& + \sum_{i=1}^{n_8} \frac{-\frac{\partial G(x_i^{(8)})}{\partial \eta} - \frac{\partial G(y_i^{(8)})}{\partial \eta} + \frac{\partial G(x_i^{(8)}, y_i^{(8)})}{\partial \eta}}{1 - G(x_i^{(8)}) - G(y_i^{(8)}) + G(x_i^{(8)}, y_i^{(8)})}, \tag{59}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \log L(\theta, \eta, \Phi)}{\partial \Phi} = & \sum_{i=1}^{n_0} \frac{\partial g(x_i^{(0)}, y_i^{(0)}) / \partial \Phi}{\partial g(x_i^{(0)}, y_i^{(0)})} + \sum_{i=1}^{n_1} \frac{\frac{\partial g(x_i^{(1)})}{\partial \Phi} - \frac{\partial^2 G(x_i^{(1)}, y_i^{(1)})}{\partial \Phi \partial x_i^{(1)}}}{g(x_i^{(1)}) - \frac{\partial G(x_i^{(1)}, y_i^{(1)})}{\partial x_i^{(1)}}} \\
& + \sum_{i=1}^{n_2} \frac{\frac{\partial^2 G(x_i^{(2)}, y_i^{(2)})}{\partial \Phi \partial x_i^{(2)}} - \frac{\partial^2 G(x_i^{(2)}, y_{i-1}^{(2)})}{\partial \Phi \partial x_i^{(2)}}}{\frac{\partial G(x_i^{(2)}, y_i^{(2)})}{\partial x_i^{(2)}} - \frac{\partial G(x_i^{(2)}, y_{i-1}^{(2)})}{\partial x_i^{(2)}}} \\
& + \sum_{i=1}^{n_3} \frac{\frac{\partial g(y_i^{(3)})}{\partial \Phi} - \frac{\partial^2 G(x_i^{(3)}, y_i^{(3)})}{\partial \Phi \partial y_i^{(3)}}}{g(y_i^{(3)}) - \frac{\partial G(x_i^{(3)}, y_i^{(3)})}{\partial y_i^{(3)}}} \\
& + \sum_{i=1}^{n_4} \frac{\frac{\partial^2 G(x_i^{(4)}, y_i^{(4)})}{\partial \Phi \partial y_i^{(4)}} - \frac{\partial^2 G(x_{i-1}^{(4)}, y_i^{(4)})}{\partial \Phi \partial y_i^{(4)}}}{\frac{\partial G(x_i^{(4)}, y_i^{(4)})}{\partial y_i^{(4)}} - \frac{\partial G(x_{i-1}^{(4)}, y_i^{(4)})}{\partial y_i^{(4)}}} \\
& + \sum_{i=1}^{n_5} \frac{\frac{\partial G(x_i^{(5)}, y_i^{(5)})}{\partial \Phi} - \frac{\partial G(x_{i-1}^{(5)}, y_i^{(5)})}{\partial \Phi} - \frac{\partial G(x_i^{(5)}, y_{i-1}^{(5)})}{\partial \Phi} + \frac{\partial G(x_{i-1}^{(5)}, y_{i-1}^{(5)})}{\partial \Phi}}{G(x_i^{(5)}, y_i^{(5)}) - G(x_{i-1}^{(5)}, y_i^{(5)}) - G(x_i^{(5)}, y_{i-1}^{(5)}) + G(x_{i-1}^{(5)}, y_{i-1}^{(5)})} \\
& + \sum_{i=1}^{n_6} \frac{\frac{\partial G(y_i^{(6)})}{\partial \Phi} - \frac{\partial G(x_i^{(6)}, y_i^{(6)})}{\partial \Phi} - \frac{\partial G(y_{i-1}^{(6)})}{\partial \Phi} + \frac{\partial G(x_{i-1}^{(6)}, y_{i-1}^{(6)})}{\partial \Phi}}{G(y_i^{(6)}) - G(x_i^{(6)}, y_i^{(6)}) - G(y_{i-1}^{(6)}) + G(x_{i-1}^{(6)}, y_{i-1}^{(6)})} \\
& + \sum_{i=1}^{n_7} \frac{\frac{\partial G(x_i^{(7)})}{\partial \Phi} - \frac{\partial G(x_{i-1}^{(7)})}{\partial \Phi} - \frac{\partial G(x_i^{(7)}, y_i^{(7)})}{\partial \Phi} + \frac{\partial G(x_{i-1}^{(7)}, y_i^{(7)})}{\partial \Phi}}{G(x_i^{(7)}) - G(x_{i-1}^{(7)}) - G(x_i^{(7)}, y_i^{(7)}) + G(x_{i-1}^{(7)}, y_i^{(7)})} \\
& + \sum_{i=1}^{n_8} \frac{\frac{\partial G(x_i^{(8)})}{\partial \Phi} - \frac{\partial G(y_i^{(8)})}{\partial \Phi} + \frac{\partial G(x_i^{(8)}, y_i^{(8)})}{\partial \Phi}}{1 - G(x_i^{(8)}) - G(y_i^{(8)}) + G(x_i^{(8)}, y_i^{(8)})}, \tag{60}
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial G(x, y)}{\partial \theta} = & \frac{\exp(-\theta) \{ [F(x, y) - 1] \exp[\theta F(x, y)] + 1 \}}{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)]} \\
& - \frac{\exp(-2\theta) \{ \exp[\theta F(x, y)] - 1 \} \{ 1 + \eta [F(x, y) - 1] \exp[\theta F(x, y)] \}}{\{ 1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)] \}^2},
\end{aligned}$$

$$\frac{\partial G(x, y)}{\partial \eta} = + \frac{\exp(-\theta) \{ \exp[\theta F(x, y)] - 1 \} \{ 1 - \exp[-\theta + \theta F(x, y)] \}}{\{ 1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)] \}^2},$$

$$\frac{\partial G(x, y)}{\partial \Phi} = \frac{\theta \partial F(x, y) / \partial \Phi \exp[-\theta + \theta F(x, y)]}{1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)]} - \frac{\theta \eta \exp(-2\theta) \partial F(x, y) / \partial \Phi \{ \exp[\theta F(x, y)] - 1 \} \exp[\theta F(x, y)]}{\{ 1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)] \}^2},$$

$$\begin{aligned} \frac{\partial^2 G(x, y)}{\partial x \partial \theta} = & \frac{(1 - \eta) [1 - \exp(-\theta)] \partial F(x, y) / \partial x \exp[-\theta + \theta F(x, y)]}{\{ 1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)] \}^2} \\ & + \frac{\theta(1 - \eta) \partial F(x, y) / \partial x \exp[-2\theta + \theta F(x, y)]}{\{ 1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)] \}^2} \\ & + \frac{\theta(1 - \eta) [1 - \exp(-\theta)] \partial F(x, y) / \partial x [F(x, y) - 1] \exp[-\theta + \theta F(x, y)]}{\{ 1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)] \}^2} \\ & - \frac{2\theta(1 - \eta) [1 - \exp(-\theta)] \partial F(x, y) / \partial x \exp[-2\theta + \theta F(x, y)]}{\{ 1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)] \}^3} \\ & - \frac{2\theta \eta (1 - \eta) [1 - \exp(-\theta)] \partial F(x, y) / \partial x [F(x, y) - 1] \exp[-2\theta + \theta F(x, y)]}{\{ 1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)] \}^3}, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 G(x, y)}{\partial x \partial \eta} = & - \frac{\theta [1 - \exp(-\theta)] \partial F(x, y) / \partial x \exp[-\theta + \theta F(x, y)]}{\{ 1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)] \}^2} \\ & - \frac{2\theta(1 - \eta) [1 - \exp(-\theta)] \partial F(x, y) / \partial x \exp[-2\theta + \theta F(x, y)]}{\{ 1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)] \}^3} \\ & + \frac{2\theta(1 - \eta) [1 - \exp(-\theta)] \partial F(x, y) / \partial x \exp[-\theta + \theta F(x, y)]}{\{ 1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)] \}^3}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 G(x, y)}{\partial x \partial \Phi} = & \frac{\theta^2(1 - \eta) [1 - \exp(-\theta)] \partial F(x, y) / \partial \Phi \partial F(x, y) / \partial x \exp[-\theta + \theta F(x, y)]}{\{ 1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)] \}^2} \\ & + \frac{\theta(1 - \eta) [1 - \exp(-\theta)] \partial^2 F(x, y) / (\partial x \partial \Phi) \exp[-\theta + \theta F(x, y)]}{\{ 1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)] \}^2} \\ & - \frac{2\eta \theta^2(1 - \eta) [1 - \exp(-\theta)] \partial F(x, y) / \partial x \partial F(x, y) / \partial \Phi \exp[-2\theta + \theta F(x, y)]}{\{ 1 - \exp(-\theta) - \eta + \eta \exp[-\theta + \theta F(x, y)] \}^3}. \end{aligned}$$

The remaining partial derivatives required for (58)-(60) follow from previous results:  $\partial g(x, y) / \partial \theta$ ,  $\partial g(x, y) / \partial \eta$ ,  $\partial g(x, y) / \partial \Phi$  follow from (54)-(56);  $\partial g(x) / \partial \theta$ ,  $\partial g(x) / \partial \eta$ ,  $\partial g(x) / \partial \Phi$  follow from (44)-(46); and,  $\partial G(x) / \partial \theta$ ,  $\partial G(x) / \partial \eta$ ,  $\partial G(x) / \partial \Phi$  follow from (48)-(50).

The maximum likelihood estimates of  $(\theta, \eta, \Phi)$ , say  $(\hat{\theta}, \hat{\eta}, \hat{\Phi})$ , are the simultaneous solutions of the equations  $\partial \log L / \partial \theta = 0$ ,  $\partial \log L / \partial \eta = 0$ , and  $\partial \log L / \partial \Phi = 0$ .

## Appendix

The calculations of the paper require the following lemmas.

**Lemma 1** If a random variable  $X$  has the GEP distribution then

$$A(k) = E \{ \exp [-k\theta \exp(-\lambda X)] \} = \eta^{-k-1}(1-\eta) [1 - \exp(-\theta)] \sum_{l=0}^k \binom{k}{l} [\exp(-\theta) - 1 + \eta]^{k-l} b_l,$$

where  $b_l$  is given by

$$b_l = \begin{cases} \frac{[1 - \exp(-\theta)]^{l-1} [1 - (1-\eta)^{l-1}]}{l-1}, & \text{if } l \neq 1, \\ -\log(1-\eta), & \text{if } l = 1. \end{cases}$$

**Proof:** We can write

$$\begin{aligned} A(k) &= \theta \lambda (1-\eta) [1 - \exp(-\theta)] \int_0^\infty \frac{\exp [-\lambda x - (k+1)\theta \exp(-\lambda x)]}{[1 - \exp(-\theta) - \eta \{1 - \exp [-\theta \exp(-\lambda x)]\}]^2} dx \\ &= \eta^{-k-1}(1-\eta) [1 - \exp(-\theta)] \int_{[1 - \exp(-\theta)](1-\eta)}^{1 - \exp(-\theta)} z^{-2} [z - 1 + \exp(-\theta) + \eta]^k dz \\ &= \eta^{-k-1}(1-\eta) [1 - \exp(-\theta)] \sum_{l=0}^k \binom{k}{l} [\exp(-\theta) - 1 + \eta]^{k-l} \int_{[1 - \exp(-\theta)](1-\eta)}^{1 - \exp(-\theta)} z^{l-2} dz. \end{aligned}$$

The result follows by elementary integration.  $\square$

**Lemma 2** If a random variable  $X$  has the GEP distribution then

$$\begin{aligned} B(a, b, c) &= E \left\{ \frac{\exp [-a\lambda X - b\theta \exp(-\lambda X)]}{[1 - \exp(-\theta) - \eta \{1 - \exp [-\theta \exp(-\lambda X)]\}]^c} \right\} \\ &= \frac{\theta(1-\eta) [1 - \exp(-\theta)]}{[1 - \exp(-\theta) - \eta]^{2+c}} \sum_{k=0}^\infty \binom{-2-c}{k} \frac{\eta^k \gamma(a+2, (1+b+k)\theta)}{[1 - \exp(-\theta) - \eta]^k (1+b+k)^{a+2} \theta^{a+2}}. \end{aligned}$$

**Proof:** Using the series expansion, (4), we can write

$$\begin{aligned} B(a, b, c) &= \theta \lambda (1-\eta) [1 - \exp(-\theta)] \int_0^\infty \frac{\exp [-(a+1)\lambda x - (b+1)\theta \exp(-\lambda x)]}{[1 - \exp(-\theta) - \eta \{1 - \exp [-\theta \exp(-\lambda x)]\}]^{2+c}} dx \\ &= \theta(1-\eta) [1 - \exp(-\theta)] \int_0^1 \frac{y^{a+1} \exp [-(b+1)\theta y]}{[1 - \exp(-\theta) - \eta + \eta \exp(-\theta y)]^{2+c}} dy \\ &= \frac{\theta(1-\eta) [1 - \exp(-\theta)]}{[1 - \exp(-\theta) - \eta]^{2+c}} \sum_{k=0}^\infty \binom{-2-c}{k} \left[ \frac{\eta}{[1 - \exp(-\theta) - \eta]} \right]^k \\ &\quad \times \int_0^1 y^{a+1} \exp [-(b+k+1)\theta y] dy. \end{aligned}$$

The result follows by the definition of the incomplete gamma function.  $\square$

**Lemma 3** If a random variable  $X$  has the GEP distribution then

$$\begin{aligned} D(a, b, c, d) &= E \left\{ \frac{X^d \exp [-a\lambda X - b\theta \exp(-\lambda X)]}{[1 - \exp(-\theta) - \eta \{1 - \exp [-\theta \exp(-\lambda X)]\}]^c} \right\} \\ &= \frac{\theta(1-\eta) [1 - \exp(-\theta)]}{(-\lambda)^d [1 - \exp(-\theta) - \eta]^{2+c}} \sum_{k=0}^\infty \binom{-2-c}{k} \frac{\partial^d}{\partial s^d} \frac{\eta^k \gamma(s+a+2, (1+b+k)\theta)}{[1 - \exp(-\theta) - \eta]^k (1+b+k)^{a+2} \theta^{a+2}} \Big|_{s=0}. \end{aligned}$$



**Proof:** Using the series expansion, (4), we can write

$$\begin{aligned}
 D(a, b, c, d) &= \theta \lambda (1 - \eta) [1 - \exp(-\theta)] \int_0^\infty \frac{x^d \exp[-(a+1)\lambda x - (b+1)\theta \exp(-\lambda x)]}{[1 - \exp(-\theta) - \eta \{1 - \exp(-\theta \exp(-\lambda x))\}]^{2+c}} dx \\
 &= \theta (-\lambda)^{-d} (1 - \eta) [1 - \exp(-\theta)] \int_0^1 \frac{(\log y)^d y^{a+1} \exp[-(b+1)\theta y]}{[1 - \exp(-\theta) - \eta + \eta \exp(-\theta y)]^{2+c}} dy \\
 &= \frac{\theta (1 - \eta) [1 - \exp(-\theta)]}{(-\lambda)^d [1 - \exp(-\theta) - \eta]^{2+c}} \sum_{k=0}^{\infty} \binom{-2-c}{k} \left[ \frac{\eta}{1 - \exp(-\theta) - \eta} \right]^k \\
 &\quad \times \int_0^1 (\log y)^d y^{a+1} \exp[-(b+k+1)\theta y] dy \\
 &= \frac{\theta (1 - \eta) [1 - \exp(-\theta)]}{(-\lambda)^d [1 - \exp(-\theta) - \eta]^{2+c}} \sum_{k=0}^{\infty} \binom{-2-c}{k} \left[ \frac{\eta}{1 - \exp(-\theta) - \eta} \right]^k \\
 &\quad \times \frac{\partial^d}{\partial s^d} \int_0^1 y^{s+a+1} \exp[-(b+k+1)\theta y] dy \Big|_{s=0}.
 \end{aligned}$$

The result follows by the definition of the incomplete gamma function.  $\square$

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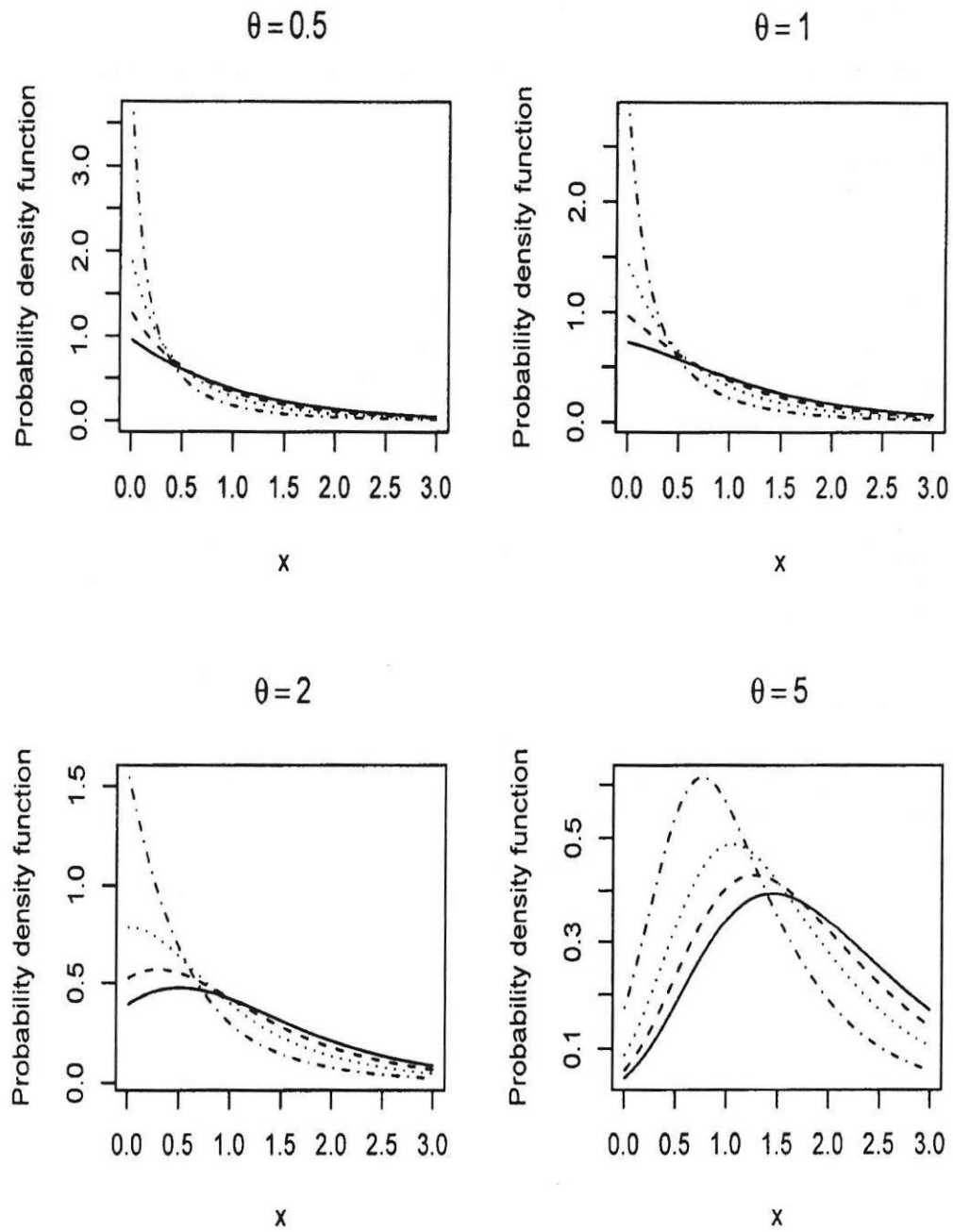
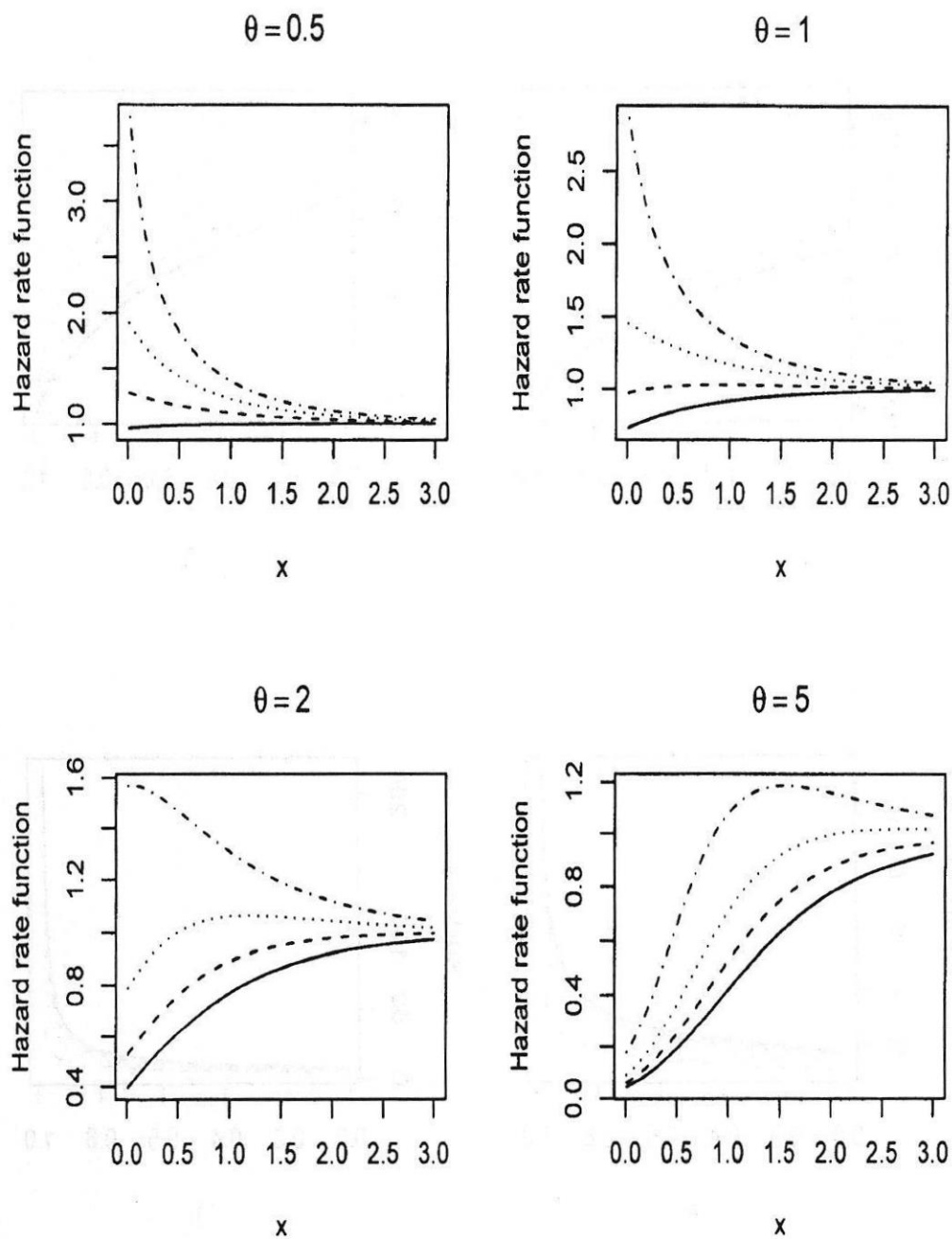


Figure 1 Plots of the probability density function, (7), for  $\lambda = 1$ ,  $\theta = 0.5, 1, 2, 5$ ,  $\eta = 0.2$  (solid curve),  $\eta = 0.4$  (curve of dashes),  $\eta = 0.6$  (curve of dots) and  $\eta = 0.8$  (curve of dots).



**Figure 2** Plots of the hazard rate function, (10), for  $\lambda = 1$ ,  $\theta = 0.5, 1, 2, 5$ ,  $\eta = 0.2$  (solid curve),  $\eta = 0.4$  (curve of dashes),  $\eta = 0.6$  (curve of dots) and  $\eta = 0.8$  (curve of dots).

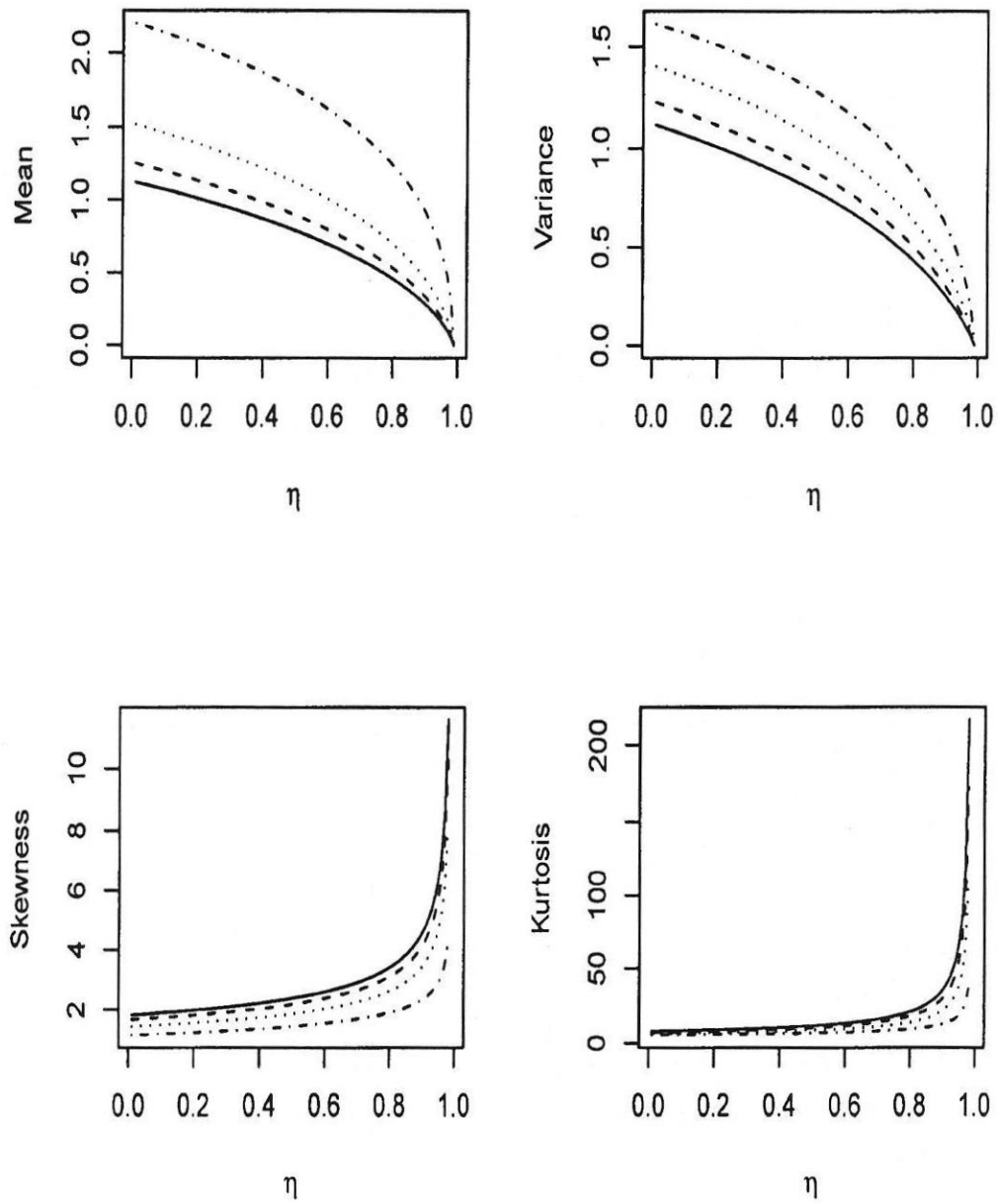


Figure 3 Mean, variance, skewness and kurtosis versus  $\eta$  for  $\lambda = 1$ ,  $\theta = 0.5$  (solid curve),  $\theta = 1$  (curve of dashes),  $\theta = 2$  (curve of dots) and  $\theta = 5$  (curve of dots and dashes).

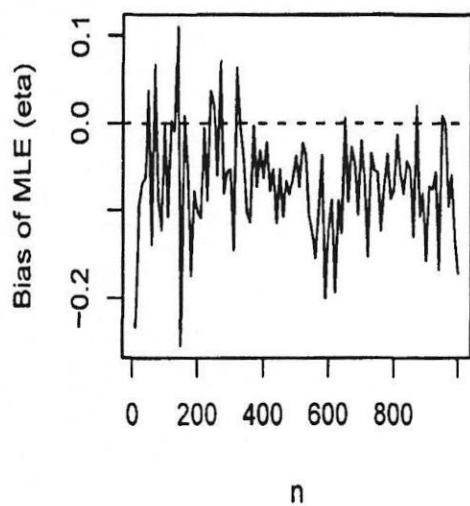
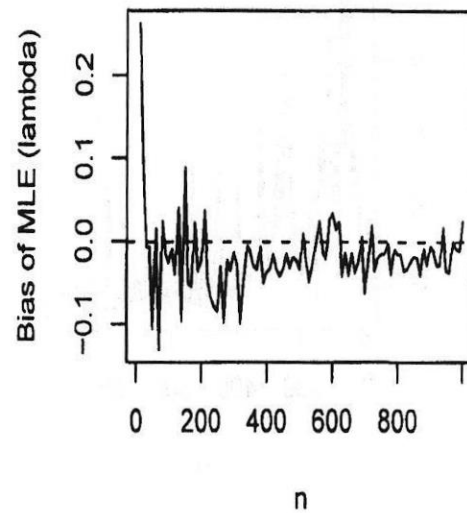
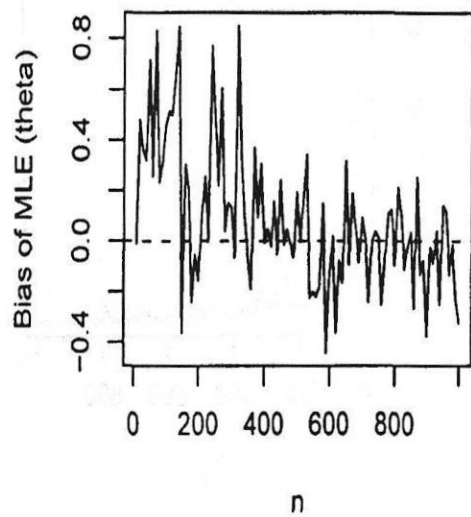


Figure 4  $bias_1(n)$  (top left),  $bias_2(n)$  (top right) and  $bias_3(n)$  (bottom left) versus  $n = 10, 20, \dots, 1000$ .

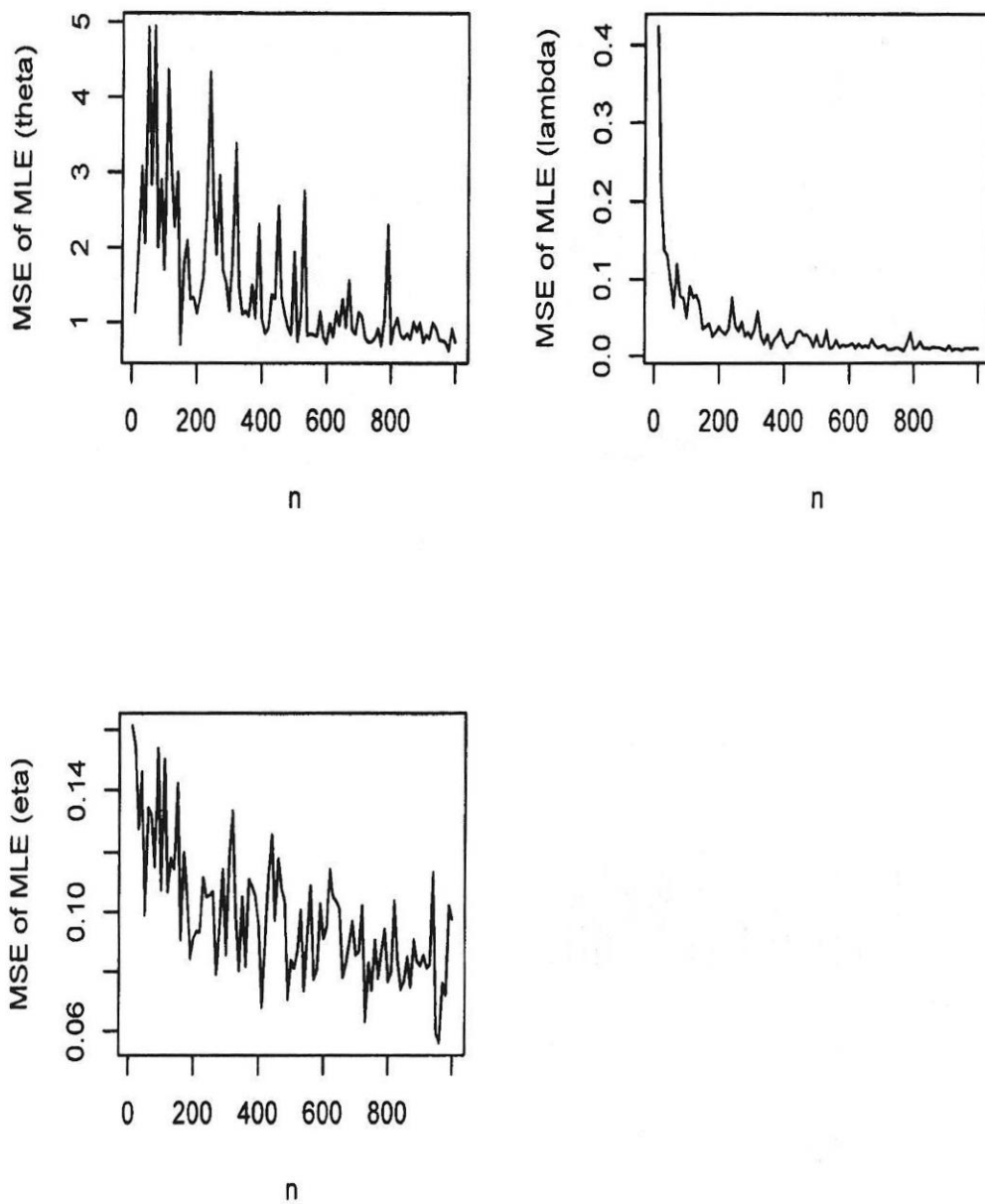


Figure 5  $MSE_1(n)$  (top left),  $MSE_2(n)$  (top right) and  $MSE_3(n)$  (bottom left) versus  $n = 10, 20, \dots, 1000$ .



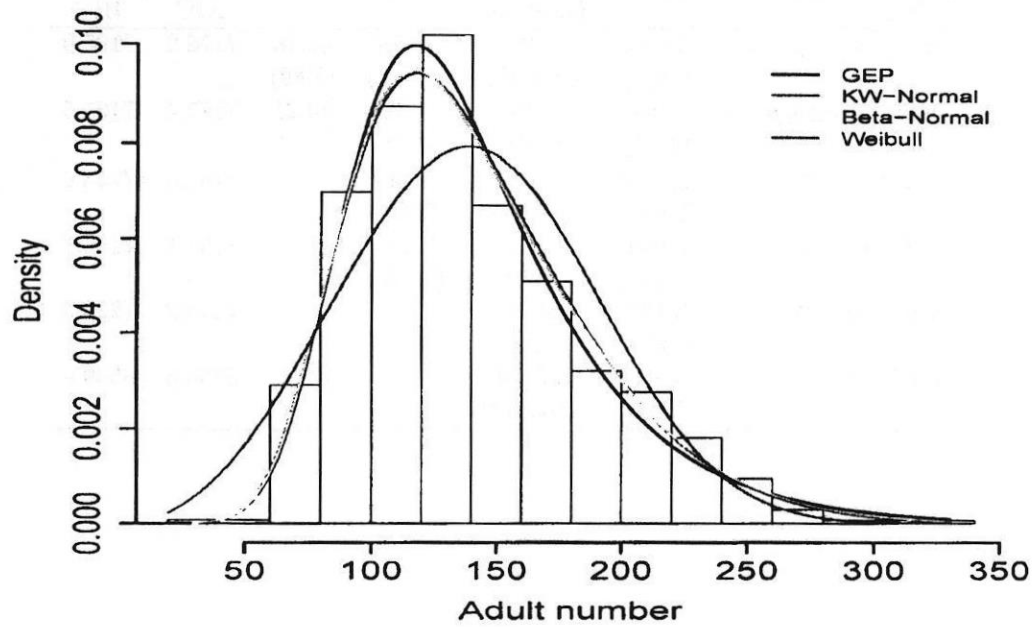


Figure 6 Histogram of adult numbers and the fitted probability density functions.

Table 1: Maximum likelihood estimates of the parameters and related statistics for the adult numbers data.

Model	Estimates				AIC	BIC
$B - N(a, b, \mu, \sigma)$	18.20 (12.7)	0.25 (0.0749)	12.69 (12.47)	42.76 (8.89)	7176.9	7195.0
$Kw - N(a, b, \mu, \sigma)$	1.85 (1.36)	0.67 (0.476)	7.37 (6.31)	14.21 (11.7)	7177.4	7195.5
$GEP(\lambda, \eta, \theta)$	0.0254 (0.001)	0.217 (0.167)	23.097 (2.044)		7145.0	7158.6
$EPG(\alpha, \lambda, \theta)$	0.889 (0.436)	0.025 (0.001)	30.288 (7.139)		7154.5	7168.1
Weibull( $\alpha, \beta$ )	3.187 (0.089)	155.914 (1.979)			7214.2	7223.3
$EP(\lambda, \theta)$	2.905 (0.282)	0.0024 (0.00014)			8330.5	8339.6

Table 2: Observed and expected frequencies of the adult numbers.

Adults number	Observed	Expected			
		<i>Kw</i> -normal	Beta normal	Weibull	GEP
30	1	0.21	0.22	7.64	0.21
50	1	5.77	5.67	22.86	6.23
70	40	37.43	37.39	45.33	38.88
90	96	91.85	93.70	71.37	93.42
110	122	125.53	127.54	94.73	128.48
130	140	123.95	123.73	108.30	125.48
150	92	102.41	100.77	107.09	100.69
170	70	75.95	74.30	91.25	71.71
190	44	52.00	50.97	66.40	47.57
210	38	33.18	32.80	40.79	30.24
230	25	19.80	19.86	20.85	18.74
250	13	11.06	11.32	8.74	12.44
270	4	5.79	6.08	2.95	6.92
290	1	2.84	3.08	0.79	4.16
310	1	1.31	1.47	0.16	2.50
330	2	0.56	0.66	0.03	1.49
Total	690	689.6	689.5	689.3	688.8
MAD		4.60	4.39	12.00	4.19

Table 3: Maximum likelihood estimates of the parameters and related statistics for the voltage data.

Model	$\eta_0$	$\theta$	$\beta_0$	$\beta_1$		AIC	CAIC	BIC
Log-GEP	2.0693 (4.0735)	5.4378 (2.1531)	19.9648 (2.7488) [<0.0001]	-0.2446 (0.0596) [0.0001]		135.0	135.8	143.4
Model	$a$	$b$	$\sigma$	$\beta_0$	$\beta_1$	AIC	CAIC	BIC
LBW	102.32 (1.6594)	0.6574 (0.3118)	3.4587 (0.6765)	10.3838 (3.3112) [0.0027]	-0.1803 (0.0607) [0.0043]	166.5	167.6	177.0
LEW	822.76 (13.2240)	1 (-)	5.7575 (5.7575)	4.8691 (4.8691) [0.3350]	-0.1764 (-0.1764) [0.0454]	164.4	165.1	172.8
Log-Weibull	1 (-)	1 (-)	0.8454 (0.090)	22.032 (3.046) [< 0.001]	-0.275 (0.055) [< 0.001]	173.4	173.8	179.7