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Abstract. W−operators are discrete set operators that are both translation invariant and locally defined within a finite window W. A particularly interesting property of W−operators is that they have a sup-decomposition in terms of a family sup-generating operators, that are parameterized by the operator. The sup-decomposition has a parallel structure that usually is not efficient for computation in conventional sequential machines. In this paper, we formalize the problem of transforming the sup-decompositions into purely sequential decompositions (when they exist). The techniques were developed for general W−operators, specialized for increasing W−operators and applied on operators built by alternating compositions of dilations and erosions.

Key words: W−operators, sup-decomposition, sequential decomposition, basis.

1. Introduction

W−operators are discrete set operators that are both translation invariant and locally defined within a finite window W. Due to their great utility in binary image analysis, they have been intensively studied. One of the most successful approaches to W−operators is Mathematical Morphology [7, 5, 3].

Mathematical Morphology on subsets can be understood as a formal language, that is built from two families of simple operators (dilation and erosion) and four operations on set operators (union, intersection, complement and composition). The phrases of the language are called morphological operators and their semantics are set operators.

A particularly interesting property of W−operators is that they have a sup-decompositions, that is, they can be decomposed in terms of a family of sup-generating operators (i.e., the uniquely intersection of erosions and complemented dilations), that are parameterized by the operator basis [1] (i.e., a family of maximal intervals). Moreover, the operator basis represents uniquely the operator.

A central problem in Binary Image Analysis is the design of W−operators. A successful technique for designing W−operators consists in the estimation of the optimal operator basis, according to some statistical error measure, from collections of input-output image pairs [2].

The sup-decomposition parameterized by the basis has some nice properties as unique characterization of W−operators and simple structure to formalize prior knowledge about families of operators considered in the optimal design. However, it has a serious drawback: its parallel structure usually is not efficient for computation in conventional sequential machines.

A general approach to conciliate the good and the bad properties of the sup-decomposition is to estimate the optimal operator basis and transform it into an equivalent morphological operator with a more sequential structure.

In this paper, we formalize the problem of transforming the sup-decompositions into purely sequential decompositions (when they exist). The theory proposed consists in the formulation and solution of discrete equations in lattice spaces, whose space of solutions have a strong combinatory nature. The techniques were developed for general W−operators, specialized for increasing W−operators and applied on operators built by alternating compositions of dilations and erosions.

The results presented here extend to general W-operators some results given by Jones [6] for increasing translation invariant operator and introduce some new bounds for the particular case of increasing W−operators.

Following this introduction, Section 2 recalls some basic properties of collections of maximal intervals. Section 3 gives some new properties for collections of W−operators. Section 4 introduces and gives bounds for the dilation factorization equation. Section 5 specializes the results of Section 4 for the family of increasing W−operators and introduces new bounds for the structuring element of the dilation. Section 6 shows how to imply the results of Section 5 to compute the structuring elements of morphological operators built by alternating compositions of dilations and erosions from their basis. Finally, in Section 7, we discuss the results presented and give some future steps of this research.

2. Lattice of Collection of Maximal Intervals

Let E be a non empty set and let W be a finite subset of E. Let $\mathcal{P}(W)$ denote the powerset of W. Elements of $\mathcal{P}(W)$ will be denoted by capital letters $A, B, C, ...$ Let $\subseteq$ be the usual inclusion relation on sets. The pair $(\mathcal{P}(W), \subseteq)$ is a Complete Boolean Lattice [4]. The intersection and union of X and Y in $\mathcal{P}(E)$ are, respectively, $X \cap Y$ and $X \cup Y$. The complementary set of $X \in \mathcal{P}(W)$ with respect to W, denoted X^c_W or, simply, X^c, when no confusion is possible, is $X^c_W = \{x \in W : x \notin X\}$.

The following proposition is an immediate consequence of the usual inclusion relation on sets.

Proposition 1 *Let $A, B \in \mathcal{P}(E)$. If $A \subseteq B$ and $A \neq B$, then $|A| < |B|$.*

Let $\mathcal{P}(\mathcal{P}(W))$ be the collection of all subcollections of $\mathcal{P}(W)$. Elements of $\mathcal{P}(\mathcal{P}(W))$ will be denoted by capital script letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, ...$ If $\subseteq$ is the usual inclusion relation on sets, then the pair $(\mathcal{P}(\mathcal{P}(W)), \subseteq)$ is a Complete Boolean Lattice.

Let $A, B \in \mathcal{P}(W)$ such that $A \subseteq B$. An *interval* of extremities A and B is the subset $[A, B]$ of $\mathcal{P}(W)$ given by $[A, B] = \{X \in \mathcal{P}(W) : A \subseteq X \subseteq B\}$. The sets A and B are called, respectively, the *left* and *right extremities* of the interval $[A, B]$. Collections of intervals contained in $\mathcal{P}(W)$ will be denoted by capital bold face letters as $\mathbf{A}_W, \mathbf{B}_W, \mathbf{C}_W, \ldots$, or, simply, $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$, when no confusion is possible.

An *interval* $[A, B]$ in a collection of intervals $\mathbf{X}$ is called *maximal* if and only if (iff) there does not exist an interval $[A', B']$ in $\mathbf{X}$, distinct of $[A, B]$, such that $[A, B] \subseteq [A', B']$.

The collection of all maximal intervals of $\mathbf{X}$ is denoted $Max(\mathbf{X})$. Of course, if all the intervals in $\mathbf{X}$ are maximal, then $\mathbf{X} = Max(\mathbf{X})$.

Let $\mathcal{X}$ be a subcollection of $\mathcal{P}(W)$. The collection of all maximal intervals contained in $\mathcal{X}$ is denoted $\mathbf{M}(\mathcal{X})$, that is,

$$\mathbf{M}(\mathcal{X}) = Max(\{[A, B] \subseteq \mathcal{P}(W) : [A, B] \subseteq \mathcal{X}\}).$$

We denote by $\cup\mathbf{X}$ the collection of all elements of $\mathcal{P}(W)$ that are elements of intervals in $\mathbf{X}$, that is,

$$\cup\mathbf{X} = \{X \in \mathcal{P}(W) : X \in [A, B],\ [A, B] \in \mathbf{X}\}.$$

Let Π_W denote the set $\{\mathbf{M}(\mathcal{X}) : \mathcal{X} \subseteq \mathcal{P}(W)\}$. We will define the partial order $\leq$ on the elements of Π_W by setting, for all $\mathbf{X}, \mathbf{Y} \in \Pi_W$,

$$\mathbf{X} \leq \mathbf{Y} \quad \Leftrightarrow \quad \forall [A, B] \in \mathbf{X},\ \exists [A', B'] \in \mathbf{Y} : [A, B] \subseteq [A', B'].$$

The poset $(\Pi_W, \leq)$ constitutes a Complete Boolean Lattice [3]. The supremum and infimum operations in the lattice $(\Pi_W, \leq)$ are given, respectively, by for any $\mathbf{X}, \mathbf{Y} \in \Pi_W$,

$$\mathbf{X} \sqcup \mathbf{Y} = \mathbf{M}((\cup\mathbf{X}) \cup (\cup\mathbf{Y})) \text{ and}$$
$$\mathbf{X} \sqcap \mathbf{Y} = \mathbf{M}((\cup\mathbf{X}) \cap (\cup\mathbf{Y})).$$

Let $W', W \in \mathcal{P}(E)$ such that $W' \supseteq W$. We define the set $\Pi_{W'/W} \subseteq \Pi_{W'}$ as $\Pi_{W'/W} = \{\mathbf{X} \in \Pi_{W'} : x \in W^c \Rightarrow x \in X, \forall X \in [A, B], \forall [A, B] \in \mathbf{X}$ or $x \notin X, \forall X \in [A, B], \forall [A, B] \in \mathbf{X}\}$.

Proposition 2 *Let $W', W \in \mathcal{P}(E)$ such that $W' \supseteq W$. The mapping $\mathcal{W}(\cdot)$, from $(\Pi_W, \leq)$ to $(\Pi_{W'/W}, \leq)$, defined by*

$$\mathcal{W}(\mathbf{X}_W) = \{[A', B'] \subseteq \mathcal{P}(W') : A' = A \text{ and } B' = B \cup W^c,\ [A, B] \in \mathbf{X}_W\}$$

constitutes a lattice isomorphism between the lattices $(\Pi_W, \leq)$ and $(\Pi_{W'/W}, \leq)$. The inverse of the mapping $\mathcal{W}(\cdot)$ is the mapping $\mathcal{W}^{-1}(\cdot)$, from $\Pi_{W'/W}$ to Π_W, defined by

$$\mathcal{W}^{-1}(\mathbf{X}_{W'}) = \{[A, B] \subseteq \mathcal{P}(W) : A = A' \text{ and } B = B' \cap W,\ [A', B'] \in \mathbf{X}_{W'}\}.$$

As a consequence of Proposition 2, if $W \subseteq W'$ we can change the representation of a collection of maximal intervals $\mathbf{X}_W \in \Pi_W$ to $\mathbf{X}_{W'} \in \Pi_{W'/W} \subseteq \Pi_{W'}$, and vice-versa.

Let E be a non empty set, that is an Abelian group with respect to a binary operation denoted by $+$. The zero element of $(E, +)$ is denoted by o.

The *transpose* of a subset $X \in \mathcal{P}(E)$ is the subset X^t, given by $X^t = \{x \in E : -x \in X\}$.

For any $X \in \mathcal{P}(E)$ and $y \in E$, X_y denotes the translation of X by y, that is, $X_y = \{x \in E : x - y \in X\}$.

For any $\mathbf{X} \in \Pi_W$ and $y \in E$, $\mathbf{X}_y \in \Pi_{W_y}$ denotes the translation of all intervals of $\mathbf{X}$ by y, that is, $\mathbf{X}_y = \{[A_y, B_y] \in \Pi_{W_y} : [A, B] \in \mathbf{X}\}$.

Let $X, Y \in \mathcal{P}(E)$. The ***Minkowski addition*** and ***subtraction*** of X and Y are, respectively, the subsets $X \oplus Y$ and $X \ominus Y$ given by $X \oplus Y = \cup\{X_y : y \in Y\}$ and $X \ominus Y = \cap\{X_{-y} : y \in Y\}$.

Two important properties of Minkowski addition are commutativity (i.e., $X \oplus Y = Y \oplus X$) [5, p. 81] and associativity (i.e., $(X \oplus Y) \oplus Z = Y \oplus (X \oplus Z)$) [5, p. 82, Eq. 4.29].

The next result is another property of the Minkowski addition and it is a immediate consequence of its definition and the fact that, for any $A, B \in \mathcal{P}(E)$ and $h \in E$, $(A \cup B)^t = A^t \cup B^t$ and $(A_h)^t = (A^t)_{-h}$.

Proposition 3 *If $X, Y \in \mathcal{P}(E)$, then $(X \oplus Y)^t = X^t \oplus Y^t$.*

Let $X, Y \in \mathcal{P}(E)$. We say Y is an *invariant* of X iff $X = (X \ominus Y) \oplus Y$. Zhuang and Haralick [9, Proposition 5] stated the following result.

Proposition 4 *Let $X, Y, Z \in \mathcal{P}(E)$. If $Z = X \oplus Y$, then X and Y are both invariants of Z.*

Proposition 5 *Let $X, Y \in \mathcal{P}(E)$. If Y is an invariant of X, then, for any $t \in E$, Y_t is also an invariant of X.*

Let $\mathbf{X}_W \in \Pi_W$ and $C \in \mathcal{P}(E)$. The ***Minkowski addition*** and ***subtraction*** of $\mathbf{X}_W$ by C are, respectively, the collection of maximal intervals $\mathbf{X}_W \oplus C \in \Pi_{W \oplus C}$ and $\mathbf{X}_W \ominus C \in \Pi_{W \oplus C^t}$ given by $\mathbf{X}_W \oplus C = \sqcup\{(\mathbf{X}_h)_{W \oplus C} : h \in C\}$ and $\mathbf{X}_W \ominus C = \sqcap\{(\mathbf{X}_{-h})_{W \oplus C^t} : h \in C\}$.

Technically, the collection $\mathbf{X}_W \oplus C \in \Pi_{W \oplus C}$ can be built in the following way. For each $h \in C$, translate the intervals in $\mathbf{X}$ by h, in order to get the collection $(\mathbf{X}_h)_{W_h} \in \Pi_{W_h}$. Since $W_h \subseteq W \oplus C$, for any $h \in C$, change the representation of $(\mathbf{X}_h)_{W_h} \in \Pi_{W_h}$ to $(\mathbf{X}_h)_{W \oplus C} \in \Pi_{W \oplus C}$. Finally, take the supremum of the collection $(\mathbf{X}_h)_{W \oplus C}$, for all $h \in C$. The collection $\mathbf{X} \ominus C$ is built in a similar way.

3. Lattice of W-operators

A mapping from $\mathcal{P}(E)$ to $\mathcal{P}(E)$ is called an *operator*. The operators will be denoted by lower case Greek letters $\alpha, \beta, \gamma, \ldots$ The set of all operators will be denoted by Ψ. The set Ψ inherits the Complete Boolean Lattice structure of $(\mathcal{P}(E), \subseteq)$ by setting, for any $\psi_1, \psi_2 \in \Psi$,

$$\psi_1 \leq \psi_2 \quad \Leftrightarrow \quad \psi_1(X) \subseteq \psi_2(X) \quad (X \in \mathcal{P}(E)).$$

The supremum and infimum of two operators ψ_1 and ψ_2 of Ψ verify, respectively, $(\psi_1 \vee \psi_2)(X) = \psi_1(X) \cup \psi_2(X)$ and $(\psi_1 \wedge \psi_2)(X) = \psi_1(X) \cap \psi_2(X)$, for any $X \in \mathcal{P}(E)$. The operator $\nu \in \Psi$ defined by $\nu(X) = X^c$, for any $X \in \mathcal{P}(E)$, is called *negation operator*. The complementary operator of an operator $\psi \in \Psi$, denoted $\nu\psi$, verifies $\nu(\psi(X)) = (\psi(X))^c$, for any $X \in \mathcal{P}(E)$.

The *dual operator* of the operator ψ, denoted by ψ^*, is $\psi^* = \nu\psi\nu$.

Let $C \in \mathcal{P}(E)$. The *dilation* and *erosion* by C are the operators δ_C and ε_C given by, for any $X \in \mathcal{P}(E)$, $\delta_C(X) = X \oplus C$ and $\varepsilon_C(X) = X \ominus C$.

The following proposition is a property of dilations [5, p. 82, Eq. 4.25].

Proposition 6 *If $X, C \in \mathcal{P}(E)$, then, for any $h \in E$, $\delta_C(X_h) = \delta_{C_h}(X) = (\delta_C(X))_h$.*

As a consequence of Proposition 6, we have the following result.

Corollary 7 *Let $X, Y, C \in \mathcal{P}(E)$. If $\delta_C(X) = Y$, then, for any $h \in E$, $\delta_{C_h}(X_{-h}) = Y$.*

Proof:

$$\begin{aligned} \delta_C(X) = Y \quad &\Leftrightarrow \quad (\delta_C(X))_h = Y_h \\ &\Leftrightarrow \quad \delta_{C_h}(C) = Y_h \\ &\qquad\qquad \text{(by Proposition 6)} \\ &\Leftrightarrow \quad (\delta_{C_h}(X))_{-h} = Y \\ &\Leftrightarrow \quad \delta_{C_h}(X_{-h}) = Y \\ &\qquad\qquad \text{(by Proposition 6).} \end{aligned}$$

■

An operator ψ is called *translation invariant* (t.i.) iff, for any $x \in E$ and $X \in \mathcal{P}(E)$, $\psi(X_x) = \psi(X)_x$.

Let W be a finite subset of E. An operator ψ is called *locally defined* within W iff, for any $x \in E$ and $X \in \mathcal{P}(E)$, $x \in \psi(X) \Leftrightarrow x \in \psi(X \cap W_x)$.

An operator ψ is called *W–operator* iff it is both t.i. and locally defined within W. The set of all W–operators will be denoted by Ψ_W. The pair $(\Psi_W, \leq)$ constitutes a sublattice of the lattice $(\Psi, \leq)$ [3].

The *kernel* of an operator $\psi \in \Psi_W$ is the set $\mathcal{K}_W(\psi)$ given by $\mathcal{K}_W(\psi) = \{X \in \mathcal{P}(W) : o \in \psi(X)\}$.

Barrera and Salas [3] stated the following lattice isomorphism between the complete lattices $(\Pi_W, \leq)$ and $(\Psi_W, \leq)$.

Theorem 8 *The mapping $\mathbf{M}(\mathcal{K}_W(\cdot))$ from $(\Psi_W, \leq)$ to $(\Pi_W, \leq)$ constitutes a lattice isomorphism between the lattices $(\Psi_W, \leq)$ and $(\Pi_W, \leq)$. The inverse of the mapping $\mathbf{M}(\mathcal{K}_W(\cdot))$ is the mapping $\mathcal{K}_W^{-1}(\cup(\cdot))$, where $\mathcal{K}_W^{-1}(\cdot)$ is defined by*

$$\mathcal{K}_W^{-1}(\mathcal{X})(X) = \{x \in E : (X - x) \cap W \in \mathcal{X}\}.$$

For any operator $\psi \in \Psi_W$, the *basis* of ψ is the collection $\mathbf{B}_W(\psi)$ of all maximal intervals contained in $\mathcal{K}_W(\psi)$, that is, $\mathbf{B}_W(\psi) = \mathbf{M}(\mathcal{K}_W(\psi))$.

An important consequence of Theorem 8 is that the basis of a W–operator ψ characterizes it uniquely.

Given $\psi \in \Psi_W$ and $h \in E$, the operator ψ_h is locally defined within W_{-h}. The following proposition [3] shows how to build the basis of ψ_h from the basis of ψ.

Proposition 9 *If $\psi \in \Pi_W$ and $h \in E$, then* $\mathbf{B}_{W_{-h}}(\psi_h) = (\mathbf{B}_W(\psi))_{-h}$.

Given $\psi \in \Psi_W$ and $C \in \mathcal{P}(E)$, the operators $\delta_C\psi$ and $\varepsilon_C\psi$ are locally defined, respectively, within $W \oplus C^t$ and $W \oplus C$. The following proposition [3] shows how to build the basis of $\delta_C\psi$ and $\varepsilon_C\psi$ from the basis of ψ.

Proposition 10 *If $\psi \in \Pi_W$ and $C \in \mathcal{P}(E)$, then*

$$\mathbf{B}_{W\oplus C^t}(\delta_C\psi) = \mathbf{B}_W(\psi) \oplus C^t \text{ and } \mathbf{B}_{W\oplus C}(\varepsilon_C\psi) = \mathbf{B}_W(\psi) \ominus C^t.$$

The next result is a consequence of Theorem 8 and Proposition 10.

Proposition 11 *If $\mathbf{X}, \mathbf{Y} \in \Pi_W$ and $C \in \mathcal{P}(E)$, then*

$$(\mathbf{X} \sqcup \mathbf{Y}) \oplus C = (\mathbf{X} \oplus C) \sqcup (\mathbf{Y} \oplus C).$$

Proof: By the lattice isomorphism between $(\Pi_W, \leq)$ and $(\Psi_W, \leq)$, there exist $\psi_1, \psi_2 \in \Psi_W$ such that $\mathbf{B}_W(\psi_1) = \mathbf{X}$, $\mathbf{B}_W(\psi_2) = \mathbf{Y}$ and $\mathbf{X} \sqcup \mathbf{Y} = \mathbf{B}_W(\psi_1 \vee \psi_2)$. Thus, by Proposition 10, we have that $(\mathbf{X} \sqcup \mathbf{Y}) \oplus C = \mathbf{B}_{W\oplus C}(\delta_{C^t}(\psi_1 \vee \psi_2))$. Since $\delta_{C^t}(\psi_1 \vee \psi_2) = (\delta_{C^t}\psi_1) \vee (\delta_{C^t}\psi_2)$ [5, p. 82, Eq. 4.27], then, by Theorem 8, $\mathbf{B}_{W\oplus C}(\delta_{C^t}(\psi_1 \vee \psi_2)) = \mathbf{B}_{W\oplus C}(\delta_{C^t}\psi_1) \sqcup \mathbf{B}_{W\oplus C}(\delta_{C^t}\psi_2)$. Since, by Proposition 10, $\mathbf{B}_{W\oplus C}(\delta_{C^t}\psi_1) = \mathbf{B}_W(\psi_1) \oplus C$ and $\mathbf{B}_{W\oplus C}(\delta_{C^t}\psi_2) = \mathbf{B}_W(\psi_2) \oplus C$, then $\mathbf{B}_{W\oplus C}(\delta_{C^t}\psi_1) \sqcup \mathbf{B}_{W\oplus C}(\delta_{C^t}\psi_2) = (\mathbf{B}_W(\psi_1) \oplus C) \sqcup (\mathbf{B}_W(\psi_2) \oplus C)$. Therefore, $(\mathbf{X} \sqcup \mathbf{Y}) \oplus C = (\mathbf{X} \oplus C) \sqcup (\mathbf{Y} \oplus C)$. ■

An operator ψ is called *increasing* iff $\forall X, Y \in \mathcal{P}(E)$, if $X \subseteq Y$, then $\psi(X) \subseteq \psi(Y)$.

Since dilation and erosion are increasing operators, the next result holds.

Proposition 12 *Let $\mathbf{X}, \mathbf{Y} \in \Pi_W$ and $C \in \mathcal{P}(E)$. If $\mathbf{X} \leq \mathbf{Y}$, then*

$$\mathbf{X} \oplus C \leq \mathbf{Y} \oplus C \text{ and } \mathbf{X} \ominus C \leq \mathbf{Y} \ominus C.$$

Proof: By the lattice isomorphism between $(\Pi_W, \leq)$ and $(\Psi_W, \leq)$, there exist $\psi_1, \psi_2 \in \Psi_W$ such that $\mathbf{B}_W(\psi_1) = \mathbf{X}$, $\mathbf{B}_W(\psi_2) = \mathbf{Y}$ and $\psi_1 \leq \psi_2$. Then, $\delta_{C^t}\psi_1 \leq \delta_{C^t}\psi_2$, since δ_{C^t} is increasing. Thus, by Theorem 8, $\mathbf{B}_{W\oplus C}(\delta_{C^t}\psi_1) \leq \mathbf{B}_{W\oplus C}(\delta_{C^t}\psi_2)$ and, by Proposition 10, $\mathbf{B}_W(\psi_1) \oplus C \leq \mathbf{B}_W(\psi_2) \oplus C$. Therefore, $\mathbf{X} \oplus C \leq \mathbf{Y} \oplus C$. Similarly, one can prove that $\mathbf{X} \ominus C \leq \mathbf{Y} \ominus C$. ■

The next result is a consequence of Corollary 7, Theorem 8 and Propositions 9 and 10.

Proposition 13 *Let $\mathbf{X}_W \in \Pi_W$, $C \in \mathcal{P}(E)$ and $\mathbf{Y}_{W'} \in \Pi_{W'}$. If $\mathbf{X}_W \oplus C^t = \mathbf{Y}_{W'}$, then, for any $h \in \mathcal{P}(E)$, $(\mathbf{X}_h)_{W_h} \oplus (C_h)^t = \mathbf{Y}_{W'}$.*

Proof: By Theorem 8, there exist $\psi \in \Psi_W$ and $\psi' \in \Psi_{W'}$ such that $\mathbf{B}_W(\psi) = \mathbf{X}_W$ and $\mathbf{B}_{W'}(\psi') = \mathbf{Y}_{W'}$. Since $\mathbf{X}_W \oplus C^t = \mathbf{Y}_{W'}$, by Proposition 10, $\delta_C\psi = \psi'$. Hence, for any $h \in E$, by Corollary 7, $\delta_{C_h}\psi_{-h} = \psi'$. By Proposition 10, $\mathbf{B}_{W'}(\psi') = \mathbf{B}_{W_h}(\psi_{-h}) \oplus (C_h)^t$. Thus, by Proposition 9, $\mathbf{B}_{W'}(\psi') = (\mathbf{B}_W(\psi))_h \oplus (C_h)^t$. Therefore, $(\mathbf{X}_h)_{W_h} \oplus (C_h)^t = \mathbf{Y}_{W'}$. ■

4. The Dilation Factorization Equation

Problem: *Given a collection of maximal intervals* $\mathbf{Y}_{W'} \in \Pi_{W'}$ *and a set* $C \in \mathcal{P}(E)$, *find all collections of maximal intervals* $\mathbf{X}_W \in \Pi_W$ *such that*

$$\mathbf{X}_W \oplus C^t = \mathbf{Y}_{W'}. \tag{1}$$

By Theorem 8, there exist $\psi \in \Psi_W$ and $\psi' \in \Psi_{W'}$ such that $\mathbf{B}_W(\psi) = \mathbf{X}_W$ and $\mathbf{B}_{W'}(\psi') = \mathbf{Y}_{W'}$. So, by Proposition 10, we have that $\mathbf{X}_W \oplus C^t = \mathbf{B}_{W \oplus C^t}(\delta_C \psi)$. Thus, given a set $C \in \mathcal{P}(E)$, the above problem can be equivalently viewed as the problem of finding all $W-$operators $\psi \in \Psi_W$ such that $\delta_C \psi = \psi'$. Moreover, since the operators $\delta_C \psi$ and ψ' are locally defined within, respectively, $W \oplus C^t$ and W', the windows W and W' satisfy $W \oplus C^t = W'$.

4.1. An Upper Bound for W

The next result states an upper bound for the window W in the Equation (1) and is a direct consequence of the adjunction relation given in [5, p. 84, Eq. 4.41].

Proposition 14 *Let* $W, C, W' \in \mathcal{P}(E)$. *If* $W \oplus C^t = W'$, *then* $W \subseteq W' \ominus C^t$.

Given a set $C \in \mathcal{P}(E)$ and a collection of maximal intervals $\mathbf{Y}_{W'}$, by Propositions 14 and 2, the collections $\mathbf{X}_W \in \Pi_W$ that satisfy the Equation (1) can be changed its representation to $\mathbf{X}_{W' \ominus C^t} \in \Pi_{W' \ominus C^t}$. So, we can consider that $W = W' \ominus C^t$.

4.2. An Upper Bound for $\mathbf{X}_W$

In this section, we state an upper bound for $\mathbf{X}_W$. For that, we need first some preliminary results.

Proposition 15 *If* $C \in \mathcal{P}(E)$, *then* $o \in C^t \oplus C$.

The next result is an immediate consequence of Proposition 15 and the definition and associative property of the Minkowski addition.

Proposition 16 *If* $W, C \in \mathcal{P}(E)$, *then* $W \subseteq (W \oplus C^t) \oplus C$.

As a consequence of Proposition 2 and Proposition 16, we can change the representation of any collection of maximal intervals $\mathbf{X}_W \in \Pi_W$ to $\mathbf{X}_{W''} \in \Pi_{W''}$, where $W'' = (W \oplus C^t) \oplus C$.

The following theorem states an upper bound for all $\mathbf{X}_W \in \Pi_W$ that satisfy Equation (1).

Theorem 17 *Let* $\mathbf{Y}_{W'} \in \Pi_{W'}$ *and* $C \in \mathcal{P}(E)$. *For any* $\mathbf{X}_W \in \Pi_W$ *such that* $\mathbf{X}_W \oplus C^t = \mathbf{Y}_{W'}$, *then* $\mathbf{X}_{W''} \leq \mathbf{Y}_{W'} \ominus C^t$, *where* $W'' = (W \oplus C^t) \oplus C$.

Proof: Since $\mathbf{X}_W \oplus C^t = \mathbf{Y}_{W'}$, then $(\mathbf{X}_W \oplus C^t) \ominus C^t = \mathbf{Y}_{W'} \ominus C^t$. Let ψ be the W-operator such that $\mathbf{X}_W = \mathbf{B}_W(\psi)$. Note that, by Proposition 2 and Proposition 16, the collection $\mathbf{X}_W$ can change its representation to $\mathbf{X}_{W''} = \mathbf{B}_{W''}(\psi)$. By Proposition 10, $(\mathbf{X}_W \oplus C^t) \ominus C^t = \mathbf{B}_{W''}(\varepsilon_C\delta_C\psi)$. Since $\varepsilon_C\delta_C$ is a closing and closing is extensive [5, p. 91, Eq. 4.66], then $\psi \leq (\varepsilon_C\delta_C)\psi$. Hence, by lattice isomorphism between $(\Pi_{W''}, \leq)$ and $(\Psi_{W''}, \leq)$, $\mathbf{B}_{W''}(\psi) \leq \mathbf{B}_{W''}(\varepsilon_C\delta_C\psi)$ and, therefore, $\mathbf{X}_{W''} \leq (\mathbf{X}_W \oplus C^t) \ominus C^t = \mathbf{Y}_{W'} \ominus C^t$. ■

4.3. Lower Bounds for $\mathbf{X}_W$

Now, we will state the lower bounds for $\mathbf{X}_W$. For that, we need the following result.

Proposition 18 *Let $\mathbf{Y}_{W'} \in \Pi_{W'}$, $\mathbf{X}_W \in \Pi_W$ and $C \in \mathcal{P}(E)$. If $\mathbf{X}_W \oplus C^t = \mathbf{Y}_{W'}$, then, for each interval $[A', B'] \in \mathbf{Y}_{W'}$, there exist $[P,Q], [X,Y] \in \mathbf{X}_W$ and $a, b \in C^t$ such that $P_a = A' \subseteq L_b$ and $(W_a)^c_{W'} \cup R_a \subseteq B' = (W_b)^c_{W'} \cup Y_b$, where $R \subseteq Q$ and $L \supseteq X$.*

Proof: Since $\mathbf{Y}_{W'} = \mathbf{X}_W \oplus C^t = \sqcup\{(\mathbf{X}_h)_{W \oplus C^t} : h \in C^t\}$, then $\mathbf{Y}_{W'}$ is the supremum of the collections $(\mathbf{X}_h)_{W_h}$, for each $h \in C^t$, where the representation of the intervals in $(\mathbf{X}_h)_{W_h}$ is changed to $(\mathbf{X}_h)_{W \oplus C^t}$. So, each interval $[A', B'] \in \mathbf{Y}_{W'}$ is built by the supremum of the translations of the intervals in $\mathbf{X}_W$. Thus, there exists an interval in $\mathbf{X}_W$ such that its translation contributes to generate the left extremity of the interval $[A', B']$. In the same way, there exists another interval in $\mathbf{X}_W$ such that its translation contributes to generate the right extremity of the interval $[A', B']$.

Hence, for any interval $[A', B'] \in \mathbf{Y}_{W'}$, there exist $[P, Q], [X, Y] \in \mathbf{X}_W$ and points $a, b \in C^t$ such that $[P, R]_a \subseteq [A', B']$, $R \subseteq Q$ and $P_a = A'$; $[L, Y]_b \subseteq [A', B']$, $L \supseteq X$ and $(W_b)^c_{W'} \cup Y_b = B'$. Since $[P, R]_a \subseteq [A', B']$, then $(W_a)^c_{W'} \cup R_a \subseteq B'$. Since $[L, Y]_b \subseteq [A', B']$, then $A' \subseteq L_b$. ■

Given an interval $[A', B'] \subseteq \mathcal{P}(W')$ and a set $C \in \mathcal{P}(E)$, we define the collections of intervals $\mathcal{L}_W^{[A',B'],C}$ and $\mathcal{R}_W^{[A',B'],C}$, contained in $\mathcal{P}(\mathcal{P}(W))$, where $W = W' \ominus C^t$, as $\mathcal{L}_W^{[A',B'],C} = \{[A'_{-x}, B \cap W] : B \subseteq B'_{-x},\ x \in C^t\}$ and $\mathcal{R}_W^{[A',B'],C} = \{[A, B'_{-x} \cap W] : A'_{-x} \subseteq A,\ x \in C^t\}$. We define the set $\mathcal{H}_W^{[A',B'],C}$ as $\mathcal{H}_W^{[A',B'],C} = \{\{[P,Q],[R,S]\} : [P,Q] \in \mathcal{L}_W^{[A',B'],C}, [R,S] \in \mathcal{R}_W^{[A',B'],C}\}$.

Let $I = \{1, 2, 3, \cdots, n\}$ be a set of indices. Let $\mathbf{Y}_{W'} = \{[A'_i, B'_i] : i \in I\}$ be a collection of maximal intervals in $\Pi_{W'}$ and $C \in \mathcal{P}(E)$. We define the set $\mathcal{S}_W^{\mathbf{Y}_{W'},C} = \mathcal{H}_W^{[A'_1,B'_1],C} \times \mathcal{H}_W^{[A'_2,B'_2],C} \times \cdots \times \mathcal{H}_W^{[A'_n,B'_n],C}$

Given a collection of maximal intervals $\mathbf{Y}_{W'} \in \Pi_{W'}$ and a subset $C \in \mathcal{P}(E)$, let us define the set of collection of intervals $\Theta_W^{\mathbf{Y}_{W'},C}$, where $W = W' \ominus C^t$, by $\Theta_W^{\mathbf{Y}_{W'},C} = \{\mathbf{Z}_W \in \Pi_W : \mathbf{Z}_W = \sqcup\{\mathbf{S}^i_W,\ i \in I\},\ (\mathbf{S}^1_W, \mathbf{S}^2_W, \cdots, \mathbf{S}^n_W) \in \mathcal{S}_W^{\mathbf{Y}_{W'},C}\}$

The next result states the lower bounds for $\mathbf{X}_W$ in Equation (1).

Theorem 19 *Let $C \in \mathcal{P}(E)$ and $\mathbf{Y}_{W'} \in \Pi_{W'}$. For all $\mathbf{X}_W \in \Pi_W$ such that $\mathbf{X}_W \oplus C^t = \mathbf{Y}_{W'}$, there exists $\mathbf{Z}_W \in \Theta_W^{\mathbf{Y}_{W'},C}$ such that $\mathbf{Z}_W \leq \mathbf{X}_W$.*

Proof: Let $\mathbf{Y}_{W'} = \{[A'_i, B'_i] : i \in I\}$. By Proposition 18, for each interval $[A'_i, B'_i] \in \mathbf{Y}_{W'}$, there exist the points $a_i, b_i \in C^t$ and the intervals $[P_i, Q_i], [X_i, Y_i] \in \mathbf{X}_W$ such that $(P_i)_{a_i} = A'_i \subseteq (L_i)_{b_i}$ and $(W_{a_i})^c_{W'} \cup (R_i)_{a_i} \subseteq B'_i = (W_{b_i})^c_{W'} \cup (Y_i)_{b_i}$, with $R_i \subseteq Q_i$ and $L_i \supseteq X_i$. Let $\mathbf{S}^i_W = \{[P_i, R_i], [L_i, Y_i]\}$. Let $\mathbf{Z}_W$ be the collection of intervals in Π_W such that $\mathbf{Z}_W = \sqcup\{\mathbf{S}^i_W : i \in I\}$. Since $[P_i, R_i] \subseteq [P_i, Q_i]$ and $[L_i, Y_i] \subseteq [X_i, Y_i]$, then $\mathbf{S}^i_W \leq \{[P_i, Q_i], [X_i, Y_i]\} \leq \mathbf{X}_W$. So, $\mathbf{Z}_W \leq \mathbf{S}^i_W \leq \sqcup\{\mathbf{S}^i_W : i \in I\} \leq \mathbf{X}_W$.

In order to prove that $\mathbf{Z}_W \in \Theta_W^{\mathbf{Y}_{W'}, C}$, we have to show that $\mathbf{S}^i_W \in \mathcal{H}_W^{[A'_i, B'_i], C}$. For that, we must show that $[P_i, R_i] \in \mathcal{L}_W^{[A'_i, B'_i], C}$ and $[L_i, Y_i] \in \mathcal{R}_W^{[A'_i, B'_i], C}$. Since $(P_i)_{a_i} = A'_i$, $(W_{a_i})^c_{W'} \cup (R_i)_{a_i} \subseteq B'_i$ and $a_i \in C^t$, then, $P_i = (A'_i)_{-a_i}$ and $W^c_{W'_{-a_i}} \cup R_i \subseteq (B'_i)_{-a_i}$. Hence, $[P_i, R_i] \in \mathcal{L}_W^{[A'_i, B'_i], C}$. In a similar way, one can prove that $[L_i, Y_i] \in \mathcal{R}_W^{[A'_i, B'_i], C}$. ■

As a consequence of Theorem 19, all lower bounds for $\mathbf{X}_W$ in Equation (1) are in $\Theta_W^{\mathbf{Y}_{W'}, C}$.

4.4. Finding Solutions of Equation (1)

This section presents the algorithm for solving the Equation (1).

Algorithm Search $(C, \mathbf{Y}_{W'})$:

Input: A set $C \in \mathcal{P}(E)$ and a collection of intervals $\mathbf{Y}_{W'} \in \Pi_{W'}$.
Output: The collections $\mathbf{X}_W \in \Pi_W$, where $W = W' \ominus C^t$, such that $\mathbf{X}_W \oplus C^t = \mathbf{Y}_{W'}$.

```
begin
  W'' ← (W ⊕ C^t) ⊕ C;
  for each Z_W ∈ Θ_W^{Y_W',C} do
    for each X_W such that Z_W'' ≤ X_W'' ≤ Y_W' ⊖ C^t do
      if X_W ⊕ C^t = Y_W' then
        output X_W;
end.
```

---

5. Increasing Operators Simplification

In this section, we recall some known properties of increasing operators. In addition, we show how the search space of the solutions of Equation (1) can be reduced when we restrict the problem to the increasing W−operators.

We denote by Ω_W the set of all increasing W−operators.

Let us define the set $\mathcal{I}_W \subseteq \Pi_W$ as the set of all collections of maximal intervals that are the basis of increasing W−operators, that is, $\mathcal{I}_W = \{\mathbf{B}_W(\psi) \in \Pi_W : \psi \in \Omega_W\}$.

A very interesting property of basis of increasing W−operators is given in the following proposition.

Proposition 20 *Let ψ be a W-operator. Then, ψ is an increasing operator iff for any interval $[A,B] \in \mathbf{B}_W(\psi)$, $B = W$.*

Thus, by Proposition 20, the right extremity of any interval in every collection of maximal intervals in $\mathcal{I}_W$ is the window W. For simplicity, where there is no risk of confusion, we denote the intervals $[A,W]$ of $\mathbf{X} \in \mathcal{I}_W$ by $[A]$. Furthermore, the partial order $\leq$ on the elements of $\mathcal{I}_W$ can be simplified in the following way.

For all $\mathbf{X}, \mathbf{Y} \in \mathcal{I}_W$,

$$\begin{aligned} \mathbf{X} \leq \mathbf{Y} \quad &\Leftrightarrow \quad \forall [X] \in \mathbf{X},\ \exists [Y] \in \mathbf{Y} : [X] \subseteq [Y]. \\ &\Leftrightarrow \quad \forall [X] \in \mathbf{X},\ \exists [Y] \in \mathbf{Y} : Y \supseteq X. \end{aligned}$$

Now, consider the problem, presented in Section 4, restricted to the increasing W-operators, that is, given a collection $\mathbf{Y}_{W'} \in \mathcal{I}_{W'}$ and a set $C \in \mathcal{P}(E)$, find all collections of maximal intervals $\mathbf{X}_W \in \mathcal{I}_W$ such that $\mathbf{X}_W \oplus C^t = \mathbf{Y}_{W'}$.

5.1. Lower Bound Simplification

We can get a new lower bound for $\mathbf{X}_W \in \mathcal{I}_W$ that is solution of the problem. For that, we need the following result, that is a particular case of the Proposition 18, when the right extremity of the intervals in $\mathbf{Y}_{W'}$ and $\mathbf{X}_W$ are, respectively, the windows W' and W.

Proposition 21 *Let $\mathbf{Y}_{W'} \in \mathcal{I}_{W'}$, $\mathbf{X}_W \in \mathcal{I}_W$ and $C \in \mathcal{P}(E)$. If $\mathbf{Y}_{W'} = \mathbf{X}_W \oplus C^t$, then, for each interval $[A'] \in \mathbf{Y}_{W'}$, there exist an interval $[P] \in \mathbf{X}_W$ and a point $h \in C^t$ such that $P_h = A'$.*

Given an interval $[A'] \subseteq \mathcal{P}(W')$ and a subset $C \in \mathcal{P}(E)$, we define the sets $\mathcal{L}_W^{[A'],C}$ and $\mathcal{H}_W^{[A'],C}$ as $\mathcal{L}_W^{[A'],C} = \{[A'_{-x}, W] : x \in C^t\}$ and $\mathcal{H}_W^{[A'],C} = \{\{[P,W]\} : [P,W] \in \mathcal{L}_W^{[A'],C}\}$.

Let $I = \{1, 2, 3, \cdots, n\}$ be a set of indices. Let $\mathbf{Y}_{W'} = \{[A'_i] : i \in I]\}$ be a collection of maximal intervals in $\mathcal{I}_{W'}$ and $C \in \mathcal{P}(E)$. We define the set $\mathcal{F}_W^{\mathbf{Y}_{W'},C} = \mathcal{H}_W^{[A'_1],C} \times \cdots \times \mathcal{H}_W^{[A'_n],C}$.

Given a collection of maximal intervals $\mathbf{Y}_{W'} \in \mathcal{I}_{W'}$ and a subset $C \in \mathcal{P}(E)$, let us define the set of collection of intervals $\Phi_W^{\mathbf{Y}_{W'},C}$, where $W = W' \ominus C^t$, by $\Phi_W^{\mathbf{Y}_{W'},C} = \{\mathbf{Z}_W \in \mathcal{I}_W : \mathbf{Z}_W = \sqcup\{\mathbf{F}_W^i,\ i \in I\},\ (\mathbf{F}_W^1, \mathbf{F}_W^2, \cdots, \mathbf{F}_W^n) \in \mathcal{F}_W^{\mathbf{Y}_{W'},C}\}$

Note that, by definition of the set $\Theta_W^{\mathbf{Y}_{W'},C}$ in Section 4.3, if the right extremity of the intervals in $\mathbf{Y}_{W'}$ and $\mathbf{Z}_W$ are, respectively, the windows W' and W, then $\Theta_W^{\mathbf{Y}_{W'},C}$ is reduced to the set $\Phi_W^{\mathbf{Y}_{W'},C}$. Thus, we can easily see that, $\Phi_W^{\mathbf{Y}_{W'},C} \subseteq \Theta_W^{\mathbf{Y}_{W'},C}$.

The following result states the lower bounds for $\mathbf{X}_W$ in Equation (1) for increasing operators and it is a particular case of Theorem 19.

Theorem 22 *Let $C \in \mathcal{P}(E)$ and $\mathbf{Y}_{W'} \in \mathcal{I}_{W'}$. For all $\mathbf{X}_W \in \mathcal{I}_W$ such that $\mathbf{X}_W \oplus C^t = \mathbf{Y}_{W'}$, there exists $\mathbf{Z}_W \in \Phi_W^{\mathbf{Y}_{W'},C}$ such that $\mathbf{Z}_W \leq \mathbf{X}_W$ and $\mathbf{Z}_W \oplus C^t = \mathbf{Y}_{W'}$.*

Proof: Let $\mathbf{Y}_{W'} = \{[A'_i] : i \in I\}$. By Proposition 21, for each interval $[A'_i] \in \mathbf{Y}_{W'}$, there exist $a_i \in C^t$ and $[P_i] \in \mathbf{X}_W$ such that $(P_i)_{a_i} = A'_i$. Let $\mathbf{F}^i_W = \{[P_i]\}$, for $i \in I$. Let $\mathbf{Z}_W$ be the collection of intervals in $\mathcal{I}_W$ such that $\mathbf{Z}_W = \sqcup\{\mathbf{F}^i_W : i \in I\}$.

The proof that $\mathbf{Z}_W \in \Phi_W^{\mathbf{Y}_{W'},C}$ and $\mathbf{Z}_W \leq \mathbf{X}_W$ can be done in a similar way that we did in Theorem 19.

Now, we prove that $\mathbf{Z}_W \oplus C^t = \mathbf{Y}_{W'}$. We divide this proof in two parts. In the first one, we prove that $\mathbf{Z}_W \oplus C^t \leq \mathbf{Y}_{W'}$ and, in the second one, we show that $\mathbf{Y}_{W'} \leq \mathbf{Z}_W \oplus C^t$.

Since $\mathbf{Z}_W \leq \mathbf{X}_W$, then, by Proposition 12, we can easily see that $\mathbf{Z}_W \oplus C^t \leq \mathbf{X}_W \oplus C^t = \mathbf{Y}_{W'}$.

Observe that $\mathbf{Y}_{W'} = \{[A'_i] : i \in I\} = \{[(A'_i)_{-a_i}, W'_{-a_i}]_{a_i} : i \in I\} \leq \sqcup\{\{[(A'_i)_{-a_i}, W]_c : c \in C^t\} : i \in I\} \leq \sqcup\{\{[(A'_i)_{-a_i}, W]\} \oplus C^t\} : i \in I\}$, since each $a_i \in C^t$. So, $\mathbf{Y}_{W'} \leq \sqcup\{\mathbf{F}^i_W \oplus C^t : i \in I\}$, since $\mathbf{F}^i_W = \{[(A'_i)_{-a_i}]\}$. Thus, by Proposition 11, $\mathbf{Y}_{W'} \leq \sqcup\{\mathbf{F}^i_W : i \in I\} \oplus C^t = \mathbf{Z}_W \oplus C^t$. ■

As a consequence of Theorem 22, all lower bounds for $\mathbf{X}_W$ in Equation (1) are in $\Phi_W^{\mathbf{Y}_{W'},C}$. In fact, each $\mathbf{Z}_W \in \Phi_W^{\mathbf{Y}_{W'},C}$ such that $\mathbf{Z}_W \oplus C^t = \mathbf{Y}_{W'}$ is a lower bound for $\mathbf{X}_W$.

Algorithm SEARCH_INCREASING $(C, \mathbf{Y}_{W'})$:

Input: A set $C \in \mathcal{P}(E)$ and a collection of intervals $\mathbf{Y}_{W'} \in \mathcal{I}_{W'}$.
Output: The collections $\mathbf{X}_W \in \mathcal{I}_W$, where $W = W' \ominus C^t$, such that $\mathbf{X}_W \oplus C^t = \mathbf{Y}_{W'}$.

```
begin
  W'' ← (W ⊕ C^t) ⊕ C;
  for each Z_W ∈ Φ_W^{Y_W',C} do
    if Z_W ⊕ C^t = Y_W' then
      for each X_W such that Z_W'' ≤ X_W'' ≤ Y_W' ⊖ C^t
        if X_W ⊕ C^t = Y_W' then
          output X_W;
end.
```

5.2. FEASIBLE SETS C FOR THE EQUATION (1)

In the Problem defined in Section 4, a subset $C \in \mathcal{P}(E)$ and $\mathbf{Y}_{W'} \in \Pi_{W'}$ are fixed. However, there exist subsets $C \in \mathcal{P}(E)$ for which Equation (1) has no solution.

Given $\mathbf{Y}_{W'} \in \mathcal{I}_{W'}$, the subsets C in $\mathcal{P}(E)$ such that Equation (1) has at least one solution are called *feasible sets*.

In this section, we study some properties of Equation (1) in order to give a necessary condition for the existence of feasible sets. Observe that, by Proposition 13, if a subset $C \in \mathcal{P}(E)$ is feasible, then, so is C_h, for any $h \in E$.

Let us state an equivalence relation on a generic collection of maximal intervals $\mathbf{X} \in \mathcal{I}_W$. Let $[A]$ and $[A']$ be two generic elements of $\mathbf{X}$. We will say that $[A]$ and $[A']$ are *equivalent under translation* iff the left extremity of one can be built by a translation of the other, that is, $[A] \equiv [A']$ iff there exists $h \in E$ such that $A = A'_{-h}$.

As the equivalence under translation is an equivalence relation (i.e., reflexive, symmetric and transitive), the set of their equivalence classes (i.e., the sets composed exactly of all the equivalent elements in $\mathbf{X}$) constitute a partition of $\mathbf{X}$.

Let $\mathbf{Z} \in \mathcal{I}_W$. We will denote by $\mathcal{C}(\mathbf{Z})$ the set of all equivalence classes (under translation) on $\mathbf{Z}$. We will denote by $\mathbf{E}(\mathbf{Z})$ a set composed by exactly one element of each equivalence class in $\mathcal{C}(\mathbf{Z})$, that is, $\mathbf{E}(\mathbf{Z})$ is a set such that $|\mathbf{E}(\mathbf{Z})| = |\mathcal{C}(\mathbf{Z})|$ and for each $\mathbf{X} \in \mathcal{C}(\mathbf{Z})$ there exists $[A] \in \mathbf{E}(\mathbf{Z})$ such that $[A] \in \mathbf{X}$.

Let $[A] \in \mathbf{X} \in \mathcal{I}_W$. We say that a left extremity A is *minimal* in $\mathbf{X}$ iff $|A| \leq |B|$, for any interval $[B] \in \mathbf{X}$. Clearly, if $|A| = |B|$, then the extremities of $[A]$ and $[B]$ are minimal.

Let $\mathbf{Z} \in \mathcal{I}_W$. Let us denote by $\mathbf{Min}(\mathbf{Z})$ the set of all intervals in $\mathbf{E}(\mathbf{Z})$ such that its left extremity is minimal in $\mathbf{E}(\mathbf{Z})$, that is, $\mathbf{Min}(\mathbf{Z}) = \{[A] \in \mathbf{E}(\mathbf{Z}) : A \text{ is minimal in } \mathbf{E}(\mathbf{Z})\}$.

Given a collection of maximal intervals $\mathbf{Z} \in \mathcal{I}_W$, for each set $A \in \mathcal{P}(E)$, let us define the set $S_A^{\mathbf{Z}} \in \mathcal{P}(E)$ as $S_A^{\mathbf{Z}} = \{h \in E : [A_{-h}] \in \mathbf{Z}\}$.

The next result gives a necessary condition for feasible sets.

Theorem 23 *Let $\mathbf{Y}_{W'} \in \mathcal{I}_{W'}$ and $C \in \mathcal{P}(E)$. If C is a feasible set, then, for any $[A'] \in \mathbf{Min}(\mathbf{Y}_{W'})$, there exists $a \in E$ such that $C_a \subseteq S_{A'}^{\mathbf{Y}_{W'}}$ and C is an invariant of $S_{A'}^{\mathbf{Y}_{W'}}$.*

Proof: Since C is feasible, there exists $\mathbf{X}_W \in \mathcal{I}_W$ such that $\mathbf{X}_W \oplus C^t = \mathbf{Y}_{W'}$. Given an interval $[A']$ in $\mathbf{Min}(\mathbf{Y}_{W'})$, let us denote $\mathbf{I}_{A'}^{\mathbf{X}_W}$ and $\mathbf{I}_{A'}^{\mathbf{Y}_{W'}}$ the intervals of $\mathbf{X}_W$ and $\mathbf{Y}_{W'}$, respectively, such that the translation of their left extremity is equal to A', that is, $\mathbf{I}_{A'}^{\mathbf{X}_W} = \{[X, W] \in \mathbf{X}_W : \exists h \in E, X_h = A'\} = \{[A'_{-x_1}], [A'_{-x_2}], \cdots, [A'_{-x_n}]\}$ and $\mathbf{I}_{A'}^{\mathbf{Y}_{W'}} = \{[X', W'] \in \mathbf{Y}_{W'} : \exists h \in E, X'_h = A'\} = \{[A'_{-y_1}], [A'_{-y_2}], \cdots, [A'_{-y_m}]\}$. Note that, $S_{A'}^{\mathbf{X}_W} = \{x_1, x_2, \cdots, x_n\}$ and $S_{A'}^{\mathbf{Y}_{W'}} = \{y_1, y_2, \cdots, y_m\}$. Thus, $\mathbf{I}_{A'}^{\mathbf{X}_W} = \{[A'_{-x}] : x \in S_{A'}^{\mathbf{X}_W}\}$ and $\mathbf{I}_{A'}^{\mathbf{Y}_{W'}} = \{[A'_{-y}] : y \in S_{A'}^{\mathbf{Y}_{W'}}\}$. Now, we will prove that $S_{A'}^{\mathbf{Y}_{W'}} = S_{A'}^{\mathbf{X}_W} \oplus C$.

Since A' is minimal in $\mathbf{E}(\mathbf{Y}_{W'})$, then A' is also minimal in $\mathbf{Y}_{W'}$.

On one hand, $S_{A'}^{\mathbf{X}_W} \oplus C \subseteq S_{A'}^{\mathbf{Y}_{W'}}$. In fact, let $[A'_{-x}] \in \mathbf{I}_{A'}^{\mathbf{X}_W}$ and $c \in C^t$. We will prove that $[A'_{-x+c}, W'] \in \mathbf{Y}_{W'}$. Suppose that $[A'_{-x+c}, W'] \notin \mathbf{Y}_{W'}$. Thus, there exists an interval $[X', W'] \in \mathbf{Y}_{W'}$ such that $X' \subseteq A'_{-x+c}$ and $X' \neq A'_{-x+c}$. So, by Proposition 1, $|X'| < |A'_{-x+c}|$. But, it contradicts the fact that A' is minimal in $\mathbf{Y}_{W'}$. Thus, $[A'_{-x+c}, W'] \in \mathbf{Y}_{W'}$. Hence, there exist $y \in S_{A'}^{\mathbf{Y}_{W'}}$ such that $[A'_{-x+c}] = [A'_{-y}]$, that is, $-x + c = -y$. Thus, $x + (-c) = y$ and , therefore $S_{A'}^{\mathbf{X}_W} \oplus C \subseteq S_{A'}^{\mathbf{Y}_{W'}}$, since $c \in C^t$.

On the other hand, $S_{A'}^{\mathbf{Y}_{W'}} \subseteq S_{A'}^{\mathbf{X}_W} \oplus C$. To prove this, we will show that, given an interval $[A'_{-y}] \in \mathbf{Y}_{W'}$, then there exist $x \in S_{A'}^{\mathbf{X}_W}$ and $c \in C^t$ such that $-y = -x + c$. By Proposition 21, there exist an interval $[P] \in \mathbf{X}_W$ and a point

$c \in C^t$ such that $P_c = A'_{-y}$. Thus, $[P] = [A'_{-c-y}]$ and, consequently, $[P] \in \mathrm{I}_{A'}^{\mathbf{X}_W}$. Thus, there exists $x \in S_{A'}^{\mathbf{X}_W}$ such that $[P] = [A'_{-x}]$. Thus, $-y = -x + c$. Therefore, $y = x + (-c)$, and, since $y \in S_{A'}^{\mathbf{Y}_{W'}}$, $x \in S_{A'}^{\mathbf{X}_W}$ and $c \in C^t$, then $S_{A'}^{\mathbf{Y}_{W'}} \subseteq S_{A'}^{\mathbf{X}_W} \oplus C$.

Since $S_{A'}^{\mathbf{Y}_{W'}} = S_{A'}^{\mathbf{X}_W} \oplus C$, then by Proposition 4, C is an invariant of $S_{A'}^{\mathbf{Y}_{W'}}$. It remains to show that there exists $a \in E$ such that $C_a \subseteq S_{A'}^{\mathbf{Y}_{W'}}$.

Let $a \in S_{A'}^{\mathbf{X}_W}$. By Corollary 7 and the definition of dilation, we have $S_{A'}^{\mathbf{Y}_{W'}} = (S_{A'}^{\mathbf{X}_W})_{-a} \oplus C_a$. Since $a \in S_{A'}^{\mathbf{X}_W}$, then $o \in (S_{A'}^{\mathbf{X}_W})_{-a}$, and, by definition and the commutativity property of the Minkowski addition, $C_a \subseteq C_a \oplus (S_{A'}^{\mathbf{X}_W})_{-a} = (S_{A'}^{\mathbf{X}_W})_{-a} \oplus C_a = S_{A'}^{\mathbf{Y}_{W'}}$. ■

Given a collection of maximal intervals $\mathbf{Y}_{W'} \in \mathcal{I}_{W'}$, as a consequence of Proposition 13, if C is feasible, then so is C_h, for any $h \in E$. By Theorem 23, if C is feasible, then, for any $[A'] \in \mathrm{Min}(\mathbf{Y}_{W'})$, a translation of C, say C_a, is a subset of $S_{A'}^{\mathbf{Y}_{W'}}$. Since C_a is also feasible, then the feasible sets can be found by searching $C \subseteq S_{A'}^{\mathbf{Y}_{W'}}$ such that C is an invariant of $S_{A'}^{\mathbf{Y}_{W'}}$.

Now, given a collection of maximal intervals $\mathbf{Y}_{W'} \in \mathcal{I}_{W'}$, we present an algorithm that outputs pairs $(C, \mathbf{X}_W) \in \mathcal{P}(E) \times \mathcal{I}_W$ such that $\mathbf{X}_W \oplus C^t = \mathbf{Y}_{W'}$.

Algorithm SEARCH_INCREASING_ALL ($\mathbf{Y}_{W'}$):

Input: A collection of intervals $\mathbf{Y}_{W'} \in \Pi_{W'}$.
Output: The pairs $(C, \mathbf{X}_W) \in \mathcal{P}(E) \times \mathcal{I}_W$, with $W = W' \ominus C^t$, such that $\mathbf{X}_W \oplus C^t = \mathbf{Y}_{W'}$.

```
begin
  let [A'] ∈ Min(Y_W') such that |S_A'^{Y_W'}| is minimum.
  for each C ⊆ S_A'^{Y_W'} such that C is an invariant of S_A'^{Y_W'}
    begin
      let {X_1, X_2, ..., X_n} be the output of
        SEARCH_INCREASING (C, Y_W');
      for i = 1, 2, ..., n do
        output the pair (C, X_i);
    end
end.
```

---

6. Compositions of Erosions and Dilations

In this section, given the basis of a W-operator ψ, that is an alternating composition of erosions and dilations, we describe how to find a representation of ψ using the algorithm presented in Section 5.2.

We denote by Υ_W the set of all W-operators that is an alternating composition of erosions and dilations. Note that the set of all alternating sequential filters [8], locally defined within a window W, is a subset of Υ_W.

Given $\psi \in \Psi_W$, the operators $\nu\psi$ and $\psi\nu$ are locally defined within W. Consequently, $\psi^* = \nu\psi\nu$ is also locally defined within W. In addition, if $\psi \in \Pi_W$ is increasing, then so is ψ^* [5, p. 46].

Let $I = \{1, 2, 3, \cdots, n\}$ be a set of indices. Given the basis of an increasing W–operator ψ, the next result shows how to build the basis of ψ^* from the basis of ψ. This proposition is a particular case of the result stated in [3].

Proposition 24 *If ψ is an increasing W–operator with basis $\mathbf{B}_W(\psi) = \{[A_i] : i \in I\}$, then the basis of its dual operator ψ^* is*

$$\mathbf{B}_W(\psi^*) = \sqcap\{[\{a\}] : a \in A_i^c, i \in I\}.$$

The following result show that the dual operator of an erosion is a dilation, and vice-versa [5, p. 84, Eq. 4.41].

Proposition 25 *If $C \in \mathcal{P}(E)$, then $\delta_C^* = \varepsilon_{C^t}$ and $\varepsilon_C^* = \delta_{C^t}$.*

The next result is an immediate consequence of the definition of dual operator and Proposition 25.

Corollary 26 *If $C \in \mathcal{P}(E)$ and $\psi \in \Pi_W$, then $(\varepsilon_{C^t}\psi)^* = \delta_C\psi^*$.*

The following result is a consequence of Proposition 10 and Corollary 26.

Corollary 27 *Let $\psi' \in \Pi_{W'}$, $C \in \mathcal{P}(E)$ and $\psi \in \Pi_W$. Then, $\psi' = \varepsilon_{C^t}\psi$ iff $\mathbf{B}_{W'}((\psi')^*) = \mathbf{B}_W(\psi^*) \oplus C^t$.*

Proof: Since $\psi' = \varepsilon_{C^t}\psi$, then, by Corollary 26, $(\psi')^* = \delta_C\psi^*$. Therefore, by Proposition 10, $\mathbf{B}_{W'}((\psi')^*) = \mathbf{B}_W(\psi^*) \oplus C^t$. ∎

If ψ is an operator in Υ_W, then a representation of ψ may start by a dilation or an erosion, that is, ψ may be rewritten by $\delta_{C_1}\psi_1$ or $\varepsilon_{C_2}\psi_2$. Recursively, ψ_1 may be rewritten by $\delta_{C_3}\psi_3$ or $\varepsilon_{C_4}\psi_4$, ψ_2 may be rewritten by $\delta_{C_5}\psi_5$ or $\varepsilon_{C_6}\psi_6$, and so on.

Given the basis of an operator $\psi \in \Upsilon_W$ that starts by a dilation, then, by Proposition 10, we can find a representation of ψ applying the procedure SEARCH_INCREASING_ALL for $\mathbf{B}_W(\psi)$. If $(C, \mathbf{X})$ is an output of the procedure SEARCH_INCREASING_ALL $(\mathbf{B}_W(\psi))$, then ψ can be rewritten by $\psi = \delta_C\psi_1$, where ψ_1 is the increasing W–operator such that the basis of ψ_1 is $\mathbf{X}$.

Given the basis of an operator $\psi \in \Upsilon_W$ that starts by an erosion, then, by Corollary 27, we can find a representation of ψ taking the dual of the basis of ψ and applying the procedure SEARCH_INCREASING_ALL for $\mathbf{B}_W(\psi^*)$. If $(C, \mathbf{X})$ is an output of SEARCH_INCREASING_ALL $(\mathbf{B}_W(\psi^*))$, then ψ can be rewritten by $\psi = \varepsilon_{C^t}\psi_1$, where ψ_1 is the increasing W–operator such that the basis of ψ_1^* is $\mathbf{X}^*$.

Thus, given the basis of an operator $\psi \in \Upsilon_W$, we will construct the tree that represents the space of all possible representations of ψ, using the algorithm SEARCH_INCREASING_ALL, presented in Section 5.2, and the properties given in Propositions 10 and 24 and Corollary 27

The *representation tree* is the tree such that the root is the basis of ψ. A node is a collection of maximal intervals $\mathbf{Y}_W \in \mathcal{I}_W$. If $\mathbf{Y}_W = \{[\{o\}]\}$, then $\mathbf{Y}_W$ has no descendants. If $\mathbf{Y}_W = \{[\{a\}]\}$, $a \neq o$, then the descendant of $\mathbf{Y}_W$ is $\{[\{o\}]\}$ and the edge that joins $\mathbf{Y}_W$ and its descendant is labeled $\delta_{\{a\}}$. In any other case, compute $\mathbf{Y}_W^*$ and apply the procedure SEARCH_INCREASING_ALL for $\mathbf{Y}_W$ and $\mathbf{Y}_W^*$. If $(C, \mathbf{X})$ is an output of SEARCH_INCREASING_ALL $(\mathbf{Y}_W)$, then $\mathbf{X}$ is a descendant of $\mathbf{Y}_W$ and the edge that joins $\mathbf{Y}_W$ to $\mathbf{X}$ is labeled δ_C. If $(C, \mathbf{X})$ is an output of the procedure SEARCH_INCREASING_ALL $(\mathbf{Y}_W^*)$, then $\mathbf{X}^*$ is a descendant of $\mathbf{Y}_W$ and the edge that joins $\mathbf{Y}_W$ to $\mathbf{X}^*$ is labeled ε_{C^t}.

Note that, given the basis of an operator $\psi \in \Upsilon_W$, the labels of the edges on the path from the root to a node $\mathbf{Y}_W = \{[\{o\}]\}$ forms a representation of ψ.

7. Conclusion

In this paper, we have studied the problem of transforming the sup-decomposition of $W-$operators, parameterized by their basis, into more efficient sequential decompositions (when they exist).

The solution of this problem depends essentially on the solution of the dilation factorization equation, that is a hard combinatorial problem. We have generalized this equation for the family of $W-$operators and given bounds for its space of solutions.

Moreover, we have gotten new bounds for the space of solutions of the dilation factorization equation constrained to the family of increasing $W-$operators and showed how to apply it to build sequential decompositions from the basis of alternating compositions of dilations and erosions.

The next steps of this research are the implementation of the technique proposed and the study of more restrict bounds for the family of alternating sequential filters.

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