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Extending the Search for Symmetries in the Genetic Code

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Abstract

We report on the search for symmetries in the genetic code involving the medium rank simple Lie algebras $B_6 = \mathfrak{so}(13)$ and $D_7 = \mathfrak{so}(14)$, in the context of the algebraic approach originally proposed by one of the present authors, which aims at explaining the degeneracies encountered in the genetic code as the result of a sequence of symmetry breakings that have occurred during its evolution.

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1 Introduction

The genetic code constitutes a dictionary for protein synthesis that to each codon (triplet of nucleic bases in DNA or RNA) associates one of the twenty fundamental aminoacids or the termination signal. It summarizes, in condensed form, the rules governing the complex process of translation that occurs in the ribosome, involving all three forms of RNA (mRNA, tRNA and rRNA) together with various proteins and other auxiliary molecules. The degeneracy of this code, resulting simply from the fact that the number of aminoacids (20) is much smaller than that of codons (64), immediately suggests an analysis based on invariance principles, that is, on the concept of symmetry. The main idea is to construct a codon space, i.e., a 64-dimensional vector space carrying an irreducible representation of a group G. The irreducibility of this representation is lost, step by step, by imposing that in each step the symmetry is lowered to a maximal subgroup of G. The result is a chain of subgroups $G \supset G_1 \supset \ldots \supset G_n$ which must generate precisely twenty-one subspaces to allocate the aminoacids and the termination signal. This process of spontaneous symmetry breaking has been used before, in various fields of science. The crucial point in such a search for symmetries is to find the ancestor group G and the possible chains of subgroups.

The first search for symmetries in the genetic code along this line of reasoning, as outlined in Ref. [1] (see also Ref. [2] and Refs [3, 4] for comments), has been carried out within the class of ordinary compact Lie groups G or, what amounts to the same thing, of semisimple Lie algebras $\mathfrak{g}$, based on Cartan's classification of semisimple Lie algebras, Dynkin's classification of their maximal semisimple subalgebras [5, 6] and the tables of branching rules of McKay and Patera [7]. This search has shown that, on the one hand, there is no simple Lie algebra that can generate the standard genetic code directly in its present form. On the other hand, it turned out that the standard genetic code can be obtained from the sympletic algebra $\mathfrak{sp}(6)$ if, in the last step, the process of symmetry breaking is allowed to be gently interrupted, or frozen. This means that a few of the multiplets (subspaces) that have resulted from previous steps of the process do not participate in the symmetry reduction implied by the last step.

Two restrictions have been imposed on the search reported in Ref. [1]. The first refers to the last phase of the process, after the ancestor algebra has already been broken to a direct sum of $\mathfrak{su}(2)$-subalgebras. In this last phase, further steps consist in breaking one or several of these $\mathfrak{su}(2)$-subalgebras completely, which in the language of atomic or nuclear physics can be implemented by introducing an appropriate generator L_z into the Hamiltonian. Another possibility that was also considered is to introduce instead its square L_z^2, which leads to a softer form of symmetry breaking. However, the possibility of performing both of these breakings sequentially, with freezing applied to the second step only, was not contemplated. The second restriction was the exclusion of "diagonal breaking", which in the case of $\mathfrak{su}(2)$-subalgebras amounts to a Clebsch-Gordan summation law for quantum numbers.

In Ref. [8], this search has been enlarged as a consequence of a new geometric interpretation of the operator L_z^2 (rather than L_z): it implements symmetry breaking of the compact Lie group $SU(2)$ to its maximal non-connected subgroup $O(2)$ (rather than its maximal connected subgroup $SO(2)$). This point of view makes it perfectly acceptable to allow a symmetry breaking from $SU(2)$ to $O(2)$ followed by one from $O(2)$ to $SO(2)$. As a result, there appeared a new chain based on the exceptional group G_2. Soon after, a second chain based on G_2 was discovered. The main problem with both of these G_2 models, which will appear again in the analysis of the medium rank algebras $\mathfrak{so}(13)$ and $\mathfrak{so}(14)$ to be reported in this paper, is the large number of multiplets (subspaces) which must be frozen, in contrast to biological information that seems to speak in favor of a small amount of freezing.

A complete review of the full process has been given in Ref. [9], eliminating the restrictions pointed out above for the low rank algebras. Here, we extend this analysis to the medium rank algebras $\mathfrak{so}(13)$ and $\mathfrak{so}(14)$. In this sense, the present paper is a companion of the extensive review [9]. Of course, we were aware from the very beginning that the analysis of hundreds of chains, required to handle this case, is a formidable task. However, when attempting to fully settle the question of the existence of symmetries behind a fundamental biological structure such as that underlying the organization of the genetic code, we cannot afford to discard any reasonable possibilities "a priori".

Another aspect that deserves to be commented is that, as already proposed in Ref. [1]. a similar analysis for Lie superalgebras has already been completed [10, 11], whereas the same investigation in the context of finite groups is under way and has so far given rise to partial results [12].

This article, which has grown out of previous work by Y.M.M. Hornos (unpublished) and the authors [13, 14], is organized as follows. Sect. 2 is dedicated to exposing the general strategy adopted in the search that is reported in the following three sections. Given the fact that branching from the medium rank simple Lie algebras $B_6 = \mathfrak{so}(13)$ and $D_7 = \mathfrak{so}(14)$ leads to thousands of chains that are impossible to analyze case by case, our strategy is to first formulate a set of rules that allows to discard the great majority of these chains without detailed analysis, thus reducing the number of "surviving" chains to manageable proportions. Sections 3-5 report the details of this search, organized in three phases and in a tree-like structure, where each branch is followed up to the point where it can either be discarded or else be shown to produce the pattern of degeneracies observed in the genetic code. In Sect. 6, we summarize the results, and in Sect. 7, we comment on conclusions that can be drawn from our work.

Given the highly technical nature of Sections 3-5, in which the material has been organized into enormous lists, the reader is strongly advised to concentrate first on Sect. 2 and then on Sect. 6, consulting Sections 3-5 according to necessity or to his/her personal preferences and interests, in an encyclopaedic manner.

2 General Strategy

The mathematical prcedure for analyzing the degeneracies of the genetic code is based on repeated symmetry reduction to maximal subalgebras, leading to descending chains of subalgebras, each of which is maximal in the previous one. For this purpose, it is sufficient to consider only semisimple subalgebras, because a possible non-trivial center does not contribute to dimensions or branching rules. Therefore, the explicit construction of all possible chains requires, as a pre-requisite, a complete classification of the maximal semisimple subalgebras of semisimple Lie algebras – a problem solved long ago by Dynkin [5, 6].

An important first step in Dynkin's classification is to realize that the general problem can be reduced to that of classifying the maximal semisimple subalgebras of simple Lie algebras, namely by means of a theorem ([5], Theorem 15.1) stating that the maximal semisimple subalgebras $\mathfrak{g}'$ of a semisimple Lie algebra $\mathfrak{g}$ are of two types:

a) the *simple type*: up to an isomorphism, including an appropriate permutation of the simple ideals that constitute $\mathfrak{g}$, we have $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and $\mathfrak{g}' = \mathfrak{g}_0' \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0$ is one of the simple ideals of $\mathfrak{g}$, $\mathfrak{g}_1$ is the direct sum of the other simple ideals of $\mathfrak{g}$ and $\mathfrak{g}_0'$ is a maximal semisimple subalgebra of $\mathfrak{g}_0$,

b) the *diagonal type*: up to an isomorphism, including an appropriate permutation of the simple ideals that constitute $\mathfrak{g}$, we have $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and $\mathfrak{g}' = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0$ is one of the simple ideals of $\mathfrak{g}$ that occurs (at least) twice in $\mathfrak{g}$, $\mathfrak{g}_1$ is the direct sum of the other simple ideals of $\mathfrak{g}$ and the embedding of $\mathfrak{g}_0$ into $\mathfrak{g}_0 \oplus \mathfrak{g}_0$ is the diagonal one, mapping $X_0 \in \mathfrak{g}_0$ to $(X_0, X_0) \in \mathfrak{g}_0 \oplus \mathfrak{g}_0$.

Correspondingly, the chains can be classified into *simple chains*, i.e., chains that do not involve breaking to a subalgebra of diagonal type at any point, and *diagonal chains*, i.e., chains that do so at some point.

The task of classifying the maximal semisimple subalgebras of simple Lie algebras is much more difficult and forms the core of Dynkin's work. The results for the simple Lie algebras of interest here are assembled in Table 1; they can also be read off from the tables of McKay and Patera [7] that are themselves, in this respect, derived from Dynkin's work and have been verified independently by one of the present authors [14].

Cartan Label of Simple Lie Algebra	Cartan Label of Maximal Semisimple Subalgebra
D_7	A_6, B_6, C_3, C_2, G_2, $A_3 \oplus D_4$, $A_1 \oplus B_5$, $C_2 \oplus B_4$, $B_3 \oplus B_3$, $A_1 \oplus A_1 \oplus D_5$
A_6	A_5, B_3, $A_1 \oplus A_4$, $A_2 \oplus A_3$
B_6	D_6, A_1, $A_1 \oplus D_5$, $C_2 \oplus D_4$, $A_3 \oplus B_3$, $A_1 \oplus A_1 \oplus B_4$
D_6	A_5, B_5, $A_3 \oplus A_3$, $A_1 \oplus B_4$, $C_2 \oplus B_3$, $A_1 \oplus C_3$, $A_1 \oplus A_1 \oplus A_1$, $A_1 \oplus A_1 \oplus D_4$
A_5	A_4, A_3, C_3, A_2, $A_1 \oplus A_3$, $A_2 \oplus A_2$, $A_1 \oplus A_2$
B_5	D_5, A_1, $A_1 \oplus D_4$, $A_3 \oplus C_2$, $A_1 \oplus A_1 \oplus B_3$
D_5	A_4, B_4, C_2, $A_1 \oplus B_3$, $C_2 \oplus C_2$, $A_1 \oplus A_1 \oplus A_3$
A_4	A_3, C_2, $A_1 \oplus A_2$
B_4	D_4, A_1, $A_1 \oplus A_3$, $A_1 \oplus A_1$, $A_1 \oplus A_1 \oplus C_2$
D_4	B_3, A_2, $A_1 \oplus C_2$, $A_1 \oplus A_1 \oplus A_1 \oplus A_1$
A_3	A_2, C_2, $A_1 \oplus A_1$
B_3	G_2, A_3, $A_1 \oplus A_1 \oplus A_1$
C_3	A_2, A_1, $A_1 \oplus C_2$, $A_1 \oplus A_1$
A_2	$A_1^{(1)}$, $A_1^{(2)}$
C_2	A_1, $A_1 \oplus A_1$
G_2	A_2, A_1, $A_1 \oplus A_1$

Table 1: Maximal semisimple subalgebras of some simple Lie algebras ($A_2 \supset A_1^{(1)}$ means $\mathfrak{su}(3) \supset \mathfrak{su}(2)$, $A_2 \supset A_1^{(2)}$ means $\mathfrak{su}(3) \supset \mathfrak{so}(3)$)

Here and in what follows, the subalgebras are ordered according to the following criteria: first by increasing number of simple summands, then by decreasing total rank and finally by decreasing rank for the simple summand of highest rank.

The general strategy for analyzing the chains that result from repeated symmetry reduction to maximal semisimple subalgebras consists in proceeding along each chain one step at a time and analyzing, after each step, whether the resulting pattern of degeneracies is still compatible with that of the genetic code, as shown in Table 2. If not, the chain is *non-surviving* and may be discarded without further analysis. Otherwise, the chain is *surviving* (up to the stage considered), which means that we must proceed to analyze the possible next steps of the symmetry breaking.

Dimension of Multiplet	Number of Multiplets	Amino Acids
6	3	Arg, Leu, Ser
4	5	Ala, Gly, Pro, Thr, Val
3	2	Ile, TERM
2	9	Asn, Asp, Cys, Gln, Glu, His, Lys, Phe, Tyr
1	2	Met, Trp

Table 2: Dimensions and multiplicities in the standard genetic code

To organize the multitude of chains that have to be analyzed, we have found it convenient to first of all divide the whole process into three distinct phases (rather than two, as stated in Ref. [9]):

Phase 1: Breaking of the primordial symmetry to $\mathfrak{su}(2)$*-symmetries.*

During the first phase, symmetry breaking proceeds through chains of maximal semisimple subalgebras, either by simple breaking or by diagonal breaking of pairs of simple subalgebras different from $\mathfrak{su}(2)$. Every such chain will necessarily terminate in a direct sum of p copies of $\mathfrak{su}(2)$, where p may range from 1 up to the rank of the original simple Lie algebra.

Phase 2: Diagonal breaking of the $\mathfrak{su}(2)$*-symmetries.*

The second phase consists in subjecting the chains that have survived up to the end of phase 1 to diagonal breaking of the $\mathfrak{su}(2)$-summands. When there are p such summands, there will *a priori* be $\binom{p}{2}$ possibilities of diagonal breaking, but this number can be substantially reduced when the previously obtained distribution of multiplets is invariant under certain permutations between the $\mathfrak{su}(2)$-summands.

Phase 3: Breaking of the $\mathfrak{su}(2)$*-symmetries.*
The third phase consists in subjecting the chains that have survived up to the end of phase 2 to breaking of (some of) the $\mathfrak{su}(2)$-subalgebras, either to $\mathfrak{o}(2)$ or to $\mathfrak{so}(2)$, including the possibility of freezing in the last step, as explained in Ref. [9].

Next, it is useful to assemble the chains into families characterized by the same final algebra and the same distribution of multiplets under it. This reflects the fact that if

$$\mathfrak{g} \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \ldots \supset \mathfrak{g}_r \supset \mathfrak{h}$$

and

$$\mathfrak{g} \supset \mathfrak{g}'_1 \supset \mathfrak{g}'_2 \supset \ldots \supset \mathfrak{g}'_{r'} \supset \mathfrak{h}'$$

are two descending chains of subalgebras starting out from the same initial algebra $\mathfrak{g}$ and ending in final subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$ of $\mathfrak{g}$ that are conjugate in $\mathfrak{g}$, then the distributions of multiplets with respect to $\mathfrak{h}$ and to $\mathfrak{h}'$ obtained from any given representation of $\mathfrak{g}$ along the two chains are identical – except when the last step of the symmetry reduction (from $\mathfrak{g}_r$ to $\mathfrak{h}$ and/or from $\mathfrak{g}'_{r'}$ to $\mathfrak{h}'$) involves freezing [9]. Of course, it is then sufficient to analyze the further fate of only one chain in each such family.

Our next task is to specify the criteria used to identify non-surviving chains. To this end, we observe first of all that there are a few simple rules that allow to decide whether the symmetry breaking must be stopped at the stage considered or whether it must proceed at least one stage further. As explained in Ref. [9], the process stops whenever we encounter

- *more than* 21 *multiplets,*
- *more than* 2 *singlets,*
- *more than* 4 *odd-dimensional multiplets,*
- *not enough multiplets of dimension* $\geqslant 6$ *or* $\geqslant 4$,

whereas it must be continued whenever we encounter

- *less than* 21 *multiplets,*
- *multiplets of dimension* $\geqslant 7$,
- *more than* 3 *multiplets of dimension* 6,
- *multiplets of dimension* 5.

In the limiting case of exactly 21 multiplets, the chain must either reproduce precisely the degeneracies of the genetic code or else must be discarded.

The simplest of these conditions is certainly the one referring to the total number of multiplets. When it exceeds 21, this can be taken as indication that we should discard the chain in question, return to the previous stage and look for other possibilities to proceed from there. In the following, we shall therefore apply the classification of chains as surviving or non-surviving only when the total number of multiplets does not exceed 21: all others could be considered as trivially non-surviving. There are then various properties that allow to identify non-surviving chains.

- *Total pairing: all multiplets occur in (identical or complex conjugate) pairs,*
- *Singlet excess: more than 2 singlets,*
- *Odd multiplet excess: more than 4 odd-dimensional multiplets,*
- *Large multiplet defect: not enough multiplets of dimension $\geqslant 6$ or $\geqslant 4$.*

These criteria for exclusion of chains are "hereditary" in the sense of being based on features, not shared by the distribution of multiplets found in the genetic code, that cannot be removed by any further symmetry breaking and therefore allow to discard the chain together with all of its possible descendants. Apart from these, there exist further criteria for exclusion of chains which are only "partially hereditary" in the sense of allowing to discard a certain chain and some but not all of its descendants; this is the case for a set of rules regarding the possibility of producing the 3 sextets and 2 triplets of the genetic code and collectively referred to as the *sextet/triplet generating rules.*

To explain the content of these rules, which apply to the third phase of the symmetry breaking process, assume that we are given a surviving chain (where "surviving" now means that it has survived up to the end of the first and/or second phase): it will necessarily end in the direct sum of, say, p copies of $\mathfrak{su}(2)$ where, for chains starting out from the codon representations of $B_6 = \mathfrak{so}(13)$ or $D_7 = \mathfrak{so}(14)$, p may range from 1 up to 6. Each of the multiplets appearing in the final distribution of multiplets for any such chain is characterized by its highest weight, which can be written in the form $(k_1, \ldots, k_p)$ with non-negative integers $k_1, \ldots, k_p$; its dimension is $(k_1+1) \cdot \ldots \cdot (k_p+1)$. Suppose now that we perform a symmetry reduction by breaking the j-th $\mathfrak{su}(2)$, that is, from $\mathfrak{su}_1(2) \oplus \ldots \oplus \mathfrak{su}_p(2)$ either to $\mathfrak{su}_1(2) \oplus \ldots \oplus \mathfrak{so}_j(2) \oplus \ldots \oplus \mathfrak{su}_p(2)$ or to $\mathfrak{su}_1(2) \oplus \ldots \oplus \mathfrak{o}_j(2) \oplus \ldots \oplus \mathfrak{su}_p(2)$. In the first case, the multiplet $(k_1, \ldots, k_p)$ will split into $k_j + 1$ multiplets of dimension $(k_1 + 1) \cdot \ldots \cdot (k_{j-1} + 1) \cdot (k_{j+1} + 1) \cdot \ldots \cdot (k_p + 1)$, whereas in the second case, it will split into

$$\tfrac{1}{2}\,(k_j + 1) \text{ multiplets of dimension } 2\,(k_1 + 1) \cdot \ldots \cdot (k_{j-1} + 1) \cdot (k_{j+1} + 1) \cdot \ldots \cdot (k_p + 1)$$

if k_j is odd and into

$$\begin{gathered} \tfrac{1}{2}\,k_j \text{ multiplets of dimension } 2\,(k_1 + 1) \cdot \ldots \cdot (k_{j-1} + 1) \cdot (k_{j+1} + 1) \cdot \ldots \cdot (k_p + 1) \\ \text{plus} \\ 1 \text{ multiplet of dimension } (k_1 + 1) \cdot \ldots \cdot (k_{j-1} + 1) \cdot (k_{j+1} + 1) \cdot \ldots \cdot (k_p + 1) \end{gathered}$$

if k_j is even. Repeating this process, we see that any breaking whatsoever of the multiplet $(k_1, \ldots, k_p)$ will lead to a distribution of sub-multiplets whose dimensions are products of certain of the numbers $k_j + 1$, corresponding to those $\mathfrak{su}(2)$'s that have remained unbroken, and when breaking to $\mathfrak{o}(2)$ is involved, some power of 2. In particular, sub-multiplets whose dimension is a multiple of 3 can only appear if at least one of the numbers $k_j + 1$ is a multiple of 3 (and the corresponding $\mathfrak{su}_j(2)$ is unbroken); moreover, sextets/triplets can only appear if at least one of them is actually equal to 6 or 3 / 3. Finally, observe that if any one of the integers k_j is greater than 5, then the corresponding $\mathfrak{su}_j(2)$ must be broken, since otherwise there would remain multiplets of dimension $\geqslant 7$. Thus we arrive at

Rule 1: During the last phase of the symmetry breaking process, sub-multiplets whose dimension is a multiple of 3 can only appear as the result of breaking multiplets $(k_1, \ldots, k_p)$ whose dimension is a multiple of 3, that is, for which at least one of the integers $k_1, \ldots, k_p$ equals 2 mod 3. More specifically, the triplets come from multiplets $(k_1, \ldots, k_p)$ where at least one of the integers $k_1, \ldots, k_p$ equals 2, whereas the sextets come from multiplets $(k_1, \ldots, k_p)$ where at least one of the integers $k_1, \ldots, k_p$ equals 2 or 5.

Rule 2: During the last phase of the symmetry breaking process, breaking a multiplet whose dimension is a multiple of 3 results in a collection of sub-multiplets whose dimnsion is a multiple of 3 either for *all* or for *none* of the sub-multiplets within the collection.

Based on these rules, we can make a list of all multiplets $(k_1, \ldots, k_p)$ appearing in the final distribution of multiplets for one of the surviving chains for $B_6 = \mathfrak{so}(13)$ or $D_7 = \mathfrak{so}(14)$ whose dimension is a multiple of 3 and that under further breaking can produce sextets and/or triplets. This information is collected in Table 3, except that we omit breakings that do not produce any sub-multiplets whose dimension is a multiple of 3 (1 sextet into 6 singlets or 3 doublets, rather than 2 triplets, for example) or that do not succeed in eliminating all sub-multiplets of dimension > 6 (1 multiplet of dimension 24 into 2 multiplets of dimension 12, rather than 4 sextets or 8 triplets, for example). Without loss of generality, the integer that, according to rule 1, must be equal to 2 or 5 is assumed to be k_1, and when we write $(k_1, 0, \ldots)$, $(k_1, k_2, 0, \ldots)$, $(k_1, k_2, k_3, 0, \ldots)$, etc., the dots indicate that the remaining k's are meant to be zero.

Multiplet	Dimension	Possible breakings into sextets/triplets
$(5,0,\ldots)$	6	1 sextet (unbroken)
$(2,0,\ldots)$	3	1 triplet (unbroken)
$(5,1,0,\ldots)$	12	2 sextets
$(2,6,0,\ldots)$	21	3 sextets and 1 triplet
		7 triplets
$(2,5,0,\ldots)$	18	3 sextets
		6 triplets
$(2,4,0,\ldots)$	15	2 sextets and 1 triplet
		5 triplets
$(2,3,0,\ldots)$	12	2 sextets
		4 triplets
$(2,2,0,\ldots)$	9	1 sextet and 1 triplet
		3 triplets
$(2,1,0,\ldots)$	6	1 sextet (unbroken)
		2 triplets
$(2,1,3,0,\ldots)$	24	4 sextets
		8 triplets
$(2,1,2,0,\ldots)$	18	3 sextets
		2 sextets and 2 triplets
		6 triplets
$(2,1,1,0,\ldots)$	12	2 sextets
		4 triplets

Table 3: Sextet/triplet generating rules

The simplest consequence of these sextet/triplet generating rules is of course the observation, already formalized in Ref. [9], that a chain will be non-surviving when there are not enough multiplets whose dimension is a multiple of 3: adding up the dimensions of all these multiplets gives a number called d_3 in Ref. [9] that must be at least 24 in order for the chain to be surviving.

3 First Phase of Symmetry Breaking

In what follows, we list the chains that result from carrying out the symmetry breaking up to the end of the first phase. Except when explicitly stated otherwise, the total number of multiplets obtained after the last step is always less than 21, so that according to the criteria stated above, the procedure must be continued; in particular no freezing is allowed in this phase. We also present the relevant information on the distribution of multiplets obtained after the last step that allows to discard the non-surviving chains.

$B_6 = \mathfrak{so}(13)$

1. $B_6 \supset D_6$:

   1.1. $B_6 \supset D_6 \supset A_5$: continuing the symmetry breaking process, we obtain the following chains, all of which can be excluded:

   1.1.1. $B_6 \supset D_6 \supset A_5 \supset A_4$: 4 quintets and 4 singlets, as well as total pairing.

   1.1.2. $B_6 \supset D_6 \supset A_5 \supset A_3$: total pairing.

   1.1.3. $B_6 \supset D_6 \supset A_5 \supset C_3$: 4 singlets.

   1.1.4. $B_6 \supset D_6 \supset A_5 \supset A_2$: total pairing.

   1.1.5. $B_6 \supset D_6 \supset A_5 \supset A_1 \oplus A_3$: 4 singlets.

   1.1.6. $B_6 \supset D_6 \supset A_5 \supset A_2 \oplus A_2$: 4 nonets, 8 triplets and 4 singlets.

   1.1.7. $B_6 \supset D_6 \supset A_5 \supset A_1 \oplus A_2$: continuing the symmetry breaking process, we obtain the following chains:

   1.1.7.1. $B_6 \supset D_6 \supset A_5 \supset A_1 \oplus A_2 \supset A_1 \oplus A_1^{(1)}$: this chain exhibits precisely 21 multiplets: 3 sextets, 5 quartets, 4 triplets, 5 doublets and 4 singlets.

   1.1.7.2. $B_6 \supset D_6 \supset A_5 \supset A_1 \oplus A_2 \supset A_1 \oplus A_1^{(2)}$:
   2 nonets, 2 quintets and 4 singlets.

   1.2. $B_6 \supset D_6 \supset B_5$: total pairing.

   1.3. $B_6 \supset D_6 \supset A_3 \oplus A_3$: total pairing.

   1.4. $B_6 \supset D_6 \supset A_1 \oplus B_4$: total pairing.

   1.5. $B_6 \supset D_6 \supset C_2 \oplus B_3$: total pairing.

   1.6. $B_6 \supset D_6 \supset A_1 \oplus C_3$: continuing the symmetry breaking process, we obtain the following chains:

   1.6.1. $B_6 \supset D_6 \supset A_1 \oplus C_3 \supset A_1 \oplus A_2$:
   distribution of multiplets identical to that from the chain 1.1.7:
   $B_6 \supset D_6 \supset A_5 \supset A_1 \oplus A_2$.

   1.6.2. $B_6 \supset D_6 \supset A_1 \oplus C_3 \supset A_1 \oplus A_1$: surviving.

1.6.3. $B_6 \supset D_6 \supset A_1 \oplus C_3 \supset A_1 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1$: surviving.

1.6.4. $B_6 \supset D_6 \supset A_1 \oplus C_3 \supset A_1 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$: surviving.

1.6.5. $B_6 \supset D_6 \supset A_1 \oplus C_3 \supset A_1 \oplus A_1 \oplus A_1$: surviving.

1.7. $B_6 \supset D_6 \supset A_1 \oplus A_1 \oplus D_4$: continuing the symmetry breaking process, we obtain the following chains:

1.7.1. $B_6 \supset D_6 \supset A_1 \oplus A_1 \oplus D_4 \supset A_1 \oplus A_1 \oplus B_3$: total pairing.

1.7.2. $B_6 \supset D_6 \supset A_1 \oplus A_1 \oplus D_4 \supset A_1 \oplus A_1 \oplus A_2$: total pairing.

1.7.3. $B_6 \supset D_6 \supset A_1 \oplus A_1 \oplus D_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$: surviving.

1.7.4. $B_6 \supset D_6 \supset A_1 \oplus A_1 \oplus D_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: surviving.

1.7.5. $B_6 \supset D_6 \supset A_1 \oplus A_1 \oplus D_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: surviving.

1.8. $B_6 \supset D_6 \supset A_1 \oplus A_1 \oplus A_1$:
distribution of multiplets identical to that from the chain 1.6.5 :
$B_6 \supset D_6 \supset A_1 \oplus C_3 \supset A_1 \oplus A_1 \oplus A_1$.

2. $B_6 \supset A_1$: does not reproduce the genetic code.

3. $B_6 \supset A_1 \oplus D_5$: total pairing.

4. $B_6 \supset C_2 \oplus D_4$: in the next step of the symmetry breaking process, we break the D_4-subalgebra, observing that this must be done at some stage since otherwise the dimension of all multiplets would remain a multiple of 8; this leads to the following chains:

4.1. $B_6 \supset C_2 \oplus D_4 \supset C_2 \oplus B_3$: total pairing.

4.2. $B_6 \supset C_2 \oplus D_4 \supset C_2 \oplus A_2$: total pairing.

4.3. $B_6 \supset C_2 \oplus D_4 \supset C_2 \oplus A_1 \oplus C_2$: in the next step of the symmetry breaking process, we either perform a diagonal breaking of the C_2-subalgebras or else we break the last C_2-subalgebra, observing that this must be done at some stage since otherwise there would remain multiplets whose dimensions are multiples of 5; then breaking the remaining C_2-subalgebra in the last step leads to the following chains:

4.3.1. $B_6 \supset C_2 \oplus D_4 \supset C_2 \oplus A_1 \oplus C_2 \supset (C_2)_{13} \oplus A_1$:
distribution of multiplets identical with that obtained from the $\mathfrak{sp}(6)$ model after the first step.

4.3.2. $B_6 \supset C_2 \oplus D_4 \supset C_2 \oplus A_1 \oplus C_2 \supset C_2 \oplus A_1 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1$: surviving.

4.3.3. $B_6 \supset C_2 \oplus D_4 \supset C_2 \oplus A_1 \oplus C_2 \supset C_2 \oplus A_1 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$:
distribution of multiplets identical to that from the chain 1.7.3 :
$B_6 \supset D_6 \supset A_1 \oplus A_1 \oplus D_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$.

4.3.4. $B_6 \supset C_2 \oplus D_4 \supset C_2 \oplus A_1 \oplus C_2 \supset C_2 \oplus A_1 \oplus A_1 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$: surviving.

4.3.5. $B_6 \supset C_2 \oplus D_4 \supset C_2 \oplus A_1 \oplus C_2 \supset C_2 \oplus A_1 \oplus A_1 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 1.7.4:
$B_6 \supset D_6 \supset A_1 \oplus A_1 \oplus D_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$.

4.4. $B_6 \supset C_2 \oplus D_4 \supset C_2 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: continuing the symmetry breaking process, we obtain the following chains:

4.4.1. $B_6 \supset C_2 \oplus D_4 \supset C_2 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: surviving.

4.4.2. $B_6 \supset C_2 \oplus D_4 \supset C_2 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 1.7.5:
$B_6 \supset D_6 \supset A_1 \oplus A_1 \oplus D_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$.

5. $B_6 \supset A_3 \oplus B_3$: total pairing.

6. $B_6 \supset A_1 \oplus A_1 \oplus B_4$: continuing the symmetry breaking process, we obtain the following chains:

6.1. $B_6 \supset A_1 \oplus A_1 \oplus B_4 \supset A_1 \oplus A_1 \oplus D_4$:
distribution of multiplets identical to that from the chain 1.7:
$B_6 \supset D_6 \supset A_1 \oplus A_1 \oplus D_4$.

6.2. $B_6 \supset A_1 \oplus A_1 \oplus B_4 \supset A_1 \oplus A_1 \oplus A_1$: surviving.

6.3. $B_6 \supset A_1 \oplus A_1 \oplus B_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_3$: total pairing.

6.4. $B_6 \supset A_1 \oplus A_1 \oplus B_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$: surviving.

6.5. $B_6 \supset A_1 \oplus A_1 \oplus B_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus C_2$:
distribution of multiplets identical to that from the chain 4.4:
$B_6 \supset C_2 \oplus D_4 \supset C_2 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$.

$D_7 = so(14)$

As a preliminary remark, we observe that the maximal subalgebras B_6, C_3, C_2 and G_2 of D_7 can be disregarded because the codon representations of D_7 remain irreducible under restriction to these subalgebras, so that these cases are covered by our previous analysis here (B_6) and in ref. [9] (C_3, C_2 and G_2).

1. $D_7 \supset A_6$:

1.1. $D_7 \supset A_6 \supset A_5$:
distribution of multiplets identical to that from the B_6-chain 1.1:
$B_6 \supset D_6 \supset A_5$.

1.2. $D_7 \supset A_6 \supset B_3$: continuing the symmetry breaking process, we obtain the following chains, all of which can be excluded:

1.2.1. $D_7 \supset A_6 \supset B_3 \supset A_3$: total pairing.

1.2.2. $D_7 \supset A_6 \supset B_3 \supset G_2$:
1 multiplet of dimension 27, 3 septets and 2 singlets.

1.2.3. $D_7 \supset A_6 \supset B_3 \supset A_1 \oplus A_1 \oplus A_1$: 2 nonets, 4 triplets and 2 singlets.

1.3. $D_7 \supset A_6 \supset A_2 \oplus A_3$: continuing the symmetry breaking process, we obtain the following chains, all of which can be excluded:

1.3.1. $D_7 \supset A_6 \supset A_2 \oplus A_3 \supset A_1^{(1)} \oplus A_3$: 4 singlets.

1.3.2. $D_7 \supset A_6 \supset A_2 \oplus A_3 \supset A_1^{(2)} \oplus A_3$: any posterior breaking of the A_3-algebra leads to a distribution of multiplets identical with that created by posterior breaking of the A_2-algebra in one of the following three cases and thus contains far too many odd-dimensional multiplets.

1.3.3. $D_7 \supset A_6 \supset A_2 \oplus A_3 \supset A_2 \oplus A_2$:
4 nonets, 8 triplets and 4 singlets.

1.3.4. $D_7 \supset A_6 \supset A_2 \oplus A_3 \supset A_2 \oplus C_2$:
1 multiplet of dimension 15, 1 quintet, 3 triplets and 3 singlets.

1.3.5. $D_7 \supset A_6 \supset A_2 \oplus A_3 \supset A_2 \oplus A_1 \oplus A_1$: 2 nonets, 4 triplets and 2 singlets.

1.4. $D_7 \supset A_6 \supset A_1 \oplus A_4$: continuing the symmetry breaking process, we obtain the following chains, all of which can be excluded:

1.4.1. $D_7 \supset A_6 \supset A_1 \oplus A_4 \supset A_1 \oplus A_3$: 4 singlets.

1.4.2. $D_7 \supset A_6 \supset A_1 \oplus A_4 \supset A_1 \oplus C_2 \supset A_1 \oplus A_1$:
2 septets, 2 quintets, 2 triplets and 2 singlets.

1.4.3. $D_7 \supset A_6 \supset A_1 \oplus A_4 \supset A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1$: 4 triplets and 4 singlets.

1.4.4. $D_7 \supset A_6 \supset A_1 \oplus A_4 \supset A_1 \oplus A_1 \oplus A_2$: 4 triplets and 4 singlets.

2. $D_7 \supset A_3 \oplus D_4$: in the next step of the symmetry breaking process, we break the D_4-subalgebra, observing that this must be done at some stage since otherwise the dimension of all multiplets would remain a multiple of 8; this leads to the following chains:

2.1. $D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus B_3$: total pairing.

2.2. $D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_2$: total pairing.

2.3. $D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2$: in the next step of the symmetry breaking process, we break the A_3-subalgebra, observing that this must be done at some stage since otherwise the dimension of all multiplets would remain a multiple of 4; this leads to the following chains:

2.3.1. $D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset A_2 \oplus A_1 \oplus C_2$: in the next step of the symmetry breaking process, we break the C_2-subalgebra, observing that this must be done at some stage since otherwise there would remain multiplets whose dimensions are multiples of 5; then breaking the A_2-subalgebra in the last step leads to the following chains:

2.3.1.1. $D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset A_2 \oplus A_1 \oplus C_2 \supset A_1^{(1)} \oplus A_1 \oplus A_1$: surviving.

2.3.1.2. $D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset A_2 \oplus A_1 \oplus C_2 \supset A_1^{(2)} \oplus A_1 \oplus A_1$: surviving.

2.3.1.3. $D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset A_2 \oplus A_1 \oplus C_2 \supset A_1^{(1)} \oplus A_1 \oplus A_1 \oplus A_1$: surviving.

2.3.1.4. $D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset A_2 \oplus A_1 \oplus C_2 \supset A_1^{(2)} \oplus A_1 \oplus A_1 \oplus A_1$: surviving.

2.3.2. $D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset C_2 \oplus A_1 \oplus C_2$: distribution of multiplets identical to that from the B_6-chain 4.3: $B_6 \supset C_2 \oplus D_4 \supset C_2 \oplus A_1 \oplus C_2$.

2.3.3. $D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus C_2$: continuing the symmetry breaking process, we obtain the following chains:

2.3.3.1. $D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus C_2$
$\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$: surviving.

2.3.3.2. $D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus C_2$
$\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: surviving.

2.4. $D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: continuing the symmetry breaking process, we obtain the following chains:

2.4.1. $D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset A_2 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: continuing the symmetry breaking process, we obtain the following chains:

2.4.1.1. $D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset A_2 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$
$\supset A_1^{(1)} \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: surviving.

2.4.1.2. $D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset A_2 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$
$\supset A_1^{(2)} \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: surviving.

2.4.2. $D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset C_2 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the B_6-chain 4.4: $B_6 \supset C_2 \oplus D_4 \supset C_2 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$.

2.4.3. $D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: surviving.

3. $D_7 \supset A_1 \oplus B_5$: continuing the symmetry breaking process, we obtain the following chains:

3.1. $D_7 \supset A_1 \oplus B_5 \supset A_1 \oplus D_5$: total pairing.

3.2. $D_7 \supset A_1 \oplus B_5 \supset A_1 \oplus A_1$: surviving.

3.3. $D_7 \supset A_1 \oplus B_5 \supset A_1 \oplus A_1 \oplus D_4$: continuing the symmetry breaking process, we obtain the following chains:

3.3.1. $D_7 \supset A_1 \oplus B_5 \supset A_1 \oplus A_1 \oplus D_4 \supset A_1 \oplus A_1 \oplus B_3$: total pairing.

3.3.2. $D_7 \supset A_1 \oplus B_5 \supset A_1 \oplus A_1 \oplus D_4 \supset A_1 \oplus A_1 \oplus A_2$: total pairing.

3.3.3. $D_7 \supset A_1 \oplus B_5 \supset A_1 \oplus A_1 \oplus D_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus C_2$:
distribution of multiplets identical to that from the chain 2.3.3 :
$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus C_2$.

3.3.4. $D_7 \supset A_1 \oplus B_5 \supset A_1 \oplus A_1 \oplus D_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$:
distribution of multiplets identical to that from the chain 2.4.3 :
$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$.

3.4. $D_7 \supset A_1 \oplus B_5 \supset A_1 \oplus A_3 \oplus C_2$: total pairing.

3.5. $D_7 \supset A_1 \oplus B_5 \supset A_1 \oplus A_1 \oplus A_1 \oplus B_3$: continuing the symmetry breaking process, we obtain the following chains:

3.5.1. $D_7 \supset A_1 \oplus B_5 \supset A_1 \oplus A_1 \oplus A_1 \oplus B_3 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_3$: total pairing.

3.5.2. $D_7 \supset A_1 \oplus B_5 \supset A_1 \oplus A_1 \oplus A_1 \oplus B_3 \supset A_1 \oplus A_1 \oplus A_1 \oplus G_2$: continuing the symmetry breaking process, we obtain the following chains:

3.5.2.1. $D_7 \supset A_1 \oplus B_5 \supset A_1 \oplus A_1 \oplus A_1 \oplus B_3 \supset A_1 \oplus A_1 \oplus A_1 \oplus G_2$
$\supset A_1 \oplus A_1 \oplus A_1 \oplus A_2$: total pairing.

3.5.2.2. $D_7 \supset A_1 \oplus B_5 \supset A_1 \oplus A_1 \oplus A_1 \oplus B_3 \supset A_1 \oplus A_1 \oplus A_1 \oplus G_2$
$\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$: surviving.

3.5.2.3. $D_7 \supset A_1 \oplus B_5 \supset A_1 \oplus A_1 \oplus A_1 \oplus B_3 \supset A_1 \oplus A_1 \oplus A_1 \oplus G_2$
$\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: surviving.

3.5.3. $D_7 \supset A_1 \oplus B_5 \supset A_1 \oplus A_1 \oplus A_1 \oplus B_3 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$:
distribution of multiplets identical to that from the chain 2.4.3 :
$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$.

4. $D_7 \supset C_2 \oplus B_4$: in the next step of the symmetry breaking process, we break the B_4-subalgebra, observing that this must be done at some stage since otherwise the dimensions of all multiplets would remain multiples of 16; this leads to the following chains:

4.1. $D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus D_4$: in the next step of the symmetry breaking process, we break the D_4-subalgebra, observing that this must be done at some stage since otherwise the dimensions of all multiplets would remain multiples of 8; this leads to the following chains:

4.1.1. $D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus D_4 \supset C_2 \oplus B_3$: total pairing.

4.1.2. $D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus D_4 \supset C_2 \oplus A_2$: total pairing.

4.1.3. $D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus D_4 \supset C_2 \oplus A_1 \oplus C_2$:
distribution of multiplets identical to that from the B_6-chain 4.3 :
$B_6 \supset C_2 \oplus D_4 \supset C_2 \oplus A_1 \oplus C_2$.

4.1.4. $D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus D_4 \supset C_2 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$:
distribution of multiplets identical to that from the B_6-chain 4.4 :
$B_6 \supset C_2 \oplus D_4 \supset C_2 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$.

4.2. $D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1$: continuing the symmetry breaking process, we obtain the following chains:

4.2.1. $D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1 \supset A_1 \oplus A_1$: surviving.

4.2.2. $D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1$:
distribution of multiplets identical to that from the B_6-chain 6.2 :
$B_6 \supset A_1 \oplus A_1 \oplus B_4 \supset A_1 \oplus A_1 \oplus A_1$.

4.3. $D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1 \oplus A_3$: total pairing.

4.4. $D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1 \oplus A_1$: continuing the symmetry breaking process, we obtain the following chains:

4.4.1. $D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1$: surviving.

4.4.2. $D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$:
distribution of multiplets identical to that from the B_6-chain 6.4 :
$B_6 \supset A_1 \oplus A_1 \oplus B_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$.

4.5. $D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1 \oplus A_1 \oplus C_2$: continuing the symmetry breaking process, we either perform a diagonal breaking of the C_2-subalgebras and then break the resulting diagonal C_2-subalgebra, or else we break first one of the two C_2-subalgebras and then the other one; this leads to the following chains:

4.5.1. $D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1 \oplus A_1 \oplus C_2 \supset (C_2)_{14} \oplus A_1 \oplus A_1$
$\supset A_1 \oplus A_1 \oplus A_1$: surviving.

4.5.2. $D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1 \oplus A_1 \oplus C_2 \supset (C_2)_{14} \oplus A_1 \oplus A_1$
$\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$: surviving.

4.5.3. $D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1 \oplus A_1 \oplus C_2 \supset C_2 \oplus A_1 \oplus A_1 \oplus A_1$
$\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$: surviving.

4.5.4. $D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1 \oplus A_1 \oplus C_2 \supset C_2 \oplus A_1 \oplus A_1 \oplus A_1$
$\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$:
distribution of multiplets identical to that from the chain 2.4.2.1 :
$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset C_2 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$
$\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$.

4.5.5. $D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1 \oplus A_1 \oplus C_2 \supset C_2 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$
$\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$:
distribution of multiplets identical to that from the chain 2.4.2.1 :

$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset C_2 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$
$\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$.

4.5.6. $D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1 \oplus A_1 \oplus C_2 \supset C_2 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$
$\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$:
distribution of multiplets identical to that from the chain 2.4.2.2:
$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset C_2 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$
$\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$.

5. $D_7 \supset B_3 \oplus B_3$: continuing the symmetry breaking process, we obtain the following chains:

5.1. $D_7 \supset B_3 \oplus B_3 \supset B_3 \oplus A_3$: total pairing.

5.2. $D_7 \supset B_3 \oplus B_3 \supset B_3 \oplus G_2$: in the next step of the symmetry breaking process, we break the B_3-subalgebra, observing that this must be done at some stage since otherwise the dimensions of all multiplets would remain multiples of 8; this leads to the following chains:

5.2.1. $D_7 \supset B_3 \oplus B_3 \supset B_3 \oplus G_2 \supset A_3 \oplus G_2$: total pairing.

5.2.2. $D_7 \supset B_3 \oplus B_3 \supset B_3 \oplus G_2 \supset G_2 \oplus G_2$: continuing the symmetry breaking process, we either perform a diagonal breaking of the G_2-subalgebras or else we break first one of the two G_2-subalgebras and then the other one; this leads to the following chains:

5.2.2.1. $D_7 \supset B_3 \oplus B_3 \supset B_3 \oplus G_2 \supset G_2 \oplus G_2 \supset (G_2)_{12}$:
1 multiplet of dimension 27, 3 septets and 2 singlets.

5.2.2.2. $D_7 \supset B_3 \oplus B_3 \supset B_3 \oplus G_2 \supset G_2 \oplus G_2 \supset G_2 \oplus A_1 \supset A_1 \oplus A_1$:
surviving.

5.2.2.3. $D_7 \supset B_3 \oplus B_3 \supset B_3 \oplus G_2 \supset G_2 \oplus G_2 \supset G_2 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1$:
surviving.

5.2.2.4. $D_7 \supset B_3 \oplus B_3 \supset B_3 \oplus G_2 \supset G_2 \oplus G_2 \supset G_2 \oplus A_1 \oplus A_1$
$\supset A_1 \oplus A_1 \oplus A_1$: surviving.

5.2.2.5. $D_7 \supset B_3 \oplus B_3 \supset B_3 \oplus G_2 \supset G_2 \oplus G_2 \supset G_2 \oplus A_1 \oplus A_1$
$\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$: surviving.

5.2.3. $D_7 \supset B_3 \oplus B_3 \supset B_3 \oplus G_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus G_2$:
distribution of multiplets identical to that from the chain 3.5.2:
$D_7 \supset A_1 \oplus B_5 \supset A_1 \oplus A_1 \oplus A_1 \oplus B_3 \supset A_1 \oplus A_1 \oplus A_1 \oplus G_2$.

5.3. $D_7 \supset B_3 \oplus B_3 \supset B_3 \oplus A_1 \oplus A_1 \oplus A_1$:
distribution of multiplets identical to that from the chain 3.5:
$D_7 \supset A_1 \oplus B_5 \supset A_1 \oplus A_1 \oplus A_1 \oplus B_3$.

5.4. $D_7 \supset B_3 \oplus B_3 \supset (B_3)_{12}$:
distribution of multiplets identical to that from the chain 1.2:
$D_7 \supset A_6 \supset B_3$.

6. $D_7 \supset A_1 \oplus A_1 \oplus D_5$: continuing the symmetry breaking process, we obtain the following chains:

6.1. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus A_4$: continuing the symmetry breaking process, we obtain the following chains:

6.1.1. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus A_4 \supset A_1 \oplus A_1 \oplus A_3$: continuing the symmetry breaking process, we obtain the following chains:

6.1.1.1. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus A_4 \supset A_1 \oplus A_1 \oplus A_3 \supset A_1 \oplus A_1 \oplus A_2$: total pairing.

6.1.1.2. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus A_4 \supset A_1 \oplus A_1 \oplus A_3 \supset A_1 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1$: surviving.

6.1.1.3. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus A_4 \supset A_1 \oplus A_1 \oplus A_3 \supset A_1 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$: non-surviving
(only 2 octets that by themselves cannot produce 3 sextets, and no other multiplets of dimension > 4).

6.1.1.4. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus A_4 \supset A_1 \oplus A_1 \oplus A_3 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$:
distribution of multiplets identical to that from the chain 4.5.2:
$D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1 \oplus A_1 \oplus C_2 \supset (C_2)_{14} \oplus A_1 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$.

6.1.2. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus A_4 \supset A_1 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1$:
distribution of multiplets identical to that from the chain 4.5.1:
$D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1 \oplus A_1 \oplus C_2 \supset (C_2)_{14} \oplus A_1 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1$.

6.1.3. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus A_4 \supset A_1 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$:
distribution of multiplets identical to that from the chain 4.5.2:
$D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1 \oplus A_1 \oplus C_2 \supset (C_2)_{14} \oplus A_1 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$.

6.1.4. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus A_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_2$: continuing the symmetry breaking process, we obtain the following chains:

6.1.4.1. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus A_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1^{(1)}$: non-surviving
(only 2 octets that by themselves cannot produce 3 sextets, and no other multiplets of dimension > 4).

6.1.4.2. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus A_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1^{(2)}$: surviving.

6.2. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus B_4$: continuing the symmetry breaking process, we obtain the following chains:

6.2.1. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus B_4 \supset A_1 \oplus A_1 \oplus D_4$: continuing the symmetry breaking process, we obtain the following chains:

6.2.1.1. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus B_4 \supset A_1 \oplus A_1 \oplus D_4 \supset A_1 \oplus A_1 \oplus B_3$: total pairing.

6.2.1.2. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus B_4 \supset A_1 \oplus A_1 \oplus D_4 \supset A_1 \oplus A_1 \oplus A_2$: total pairing.

6.2.1.3. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus B_4 \supset A_1 \oplus A_1 \oplus D_4$
$\supset A_1 \oplus A_1 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$:
distribution of multiplets identical to that from the B_6-chain 4.3.3:
$B_6 \supset C_2 \oplus D_4 \supset C_2 \oplus A_1 \oplus C_2 \supset (A_1 \oplus A_1) \oplus A_1 \oplus A_1$.

6.2.1.4. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus B_4 \supset A_1 \oplus A_1 \oplus D_4$
$\supset A_1 \oplus A_1 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$:
distribution of multiplets identical to that from the B_6-chain 4.3.5:
$B_6 \supset C_2 \oplus D_4 \supset C_2 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$.

6.2.1.5. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus B_4 \supset A_1 \oplus A_1 \oplus D_4$
$\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$:
distribution of multiplets identical to that from the chain 2.4.3:
$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$
$\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$.

6.2.2. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus B_4 \supset A_1 \oplus A_1 \oplus A_1$:
distribution of multiplets identical to that from the chain 4.2.2:
$D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1$.

6.2.3. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus B_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_3$: total pairing.

6.2.4. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus B_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$:
distribution of multiplets identical to that from the chain 4.4.2:
$D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$.

6.2.5. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus B_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus C_2$:
distribution of multiplets identical to that from the chain 2.4.2:
$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset C_2 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$.

6.3. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus C_2$: continuing the symmetry breaking process, we obtain the following chains:

6.3.1. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1$: surviving.

6.3.2. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$: surviving.

6.4. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus A_1 \oplus B_3$:
distribution of multiplets identical to that from the chain 3.5:
$D_7 \supset A_1 \oplus B_5 \supset A_1 \oplus A_1 \oplus A_1 \oplus B_3$.

6.5. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus C_2 \oplus C_2$:
distribution of multiplets identical to that from the chain 4.5:
$D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1 \oplus A_1 \oplus C_2$.

6.6. $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_3$:
distribution of multiplets identical to that from the chain 2.4:
$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$.

4 Second Phase of Symmetry Breaking

In this section we list the chains that result from the surviving chains of phase one by applying diagonal breaking in all possible ways, except that chains obtained from each other by permutation of the A_1-summands are written only once. We also recall that if a chain is identified as non-surviving due to the sextet/triplet generating rules, it may still give rise to surviving descendants, obtained by further diagonal breaking.

$B_6 = so(13)$

- Chain 1.6.2:

  $B_6 \supset D_6 \supset A_1 \oplus C_3 \supset A_1 \oplus A_1$: non-surviving.

  This chain may be discarded due to the sextet/triplet generating rules in view of the fact that the two multiplets of dimension 18 that occur and that must of course be broken are incapable of generating the three sextets and two triplets in the genetic code: one of them has highest weight (1-8) and thus cannot produce any sextets or triplets at all while the other one has highest weight (2-5) and thus can only produce three sextets or six triplets or no sextets and triplets at all.

  1.6.2.1 $\ldots \supset (A_1)_{12}$: does not reproduce the genetic code.

- Chain 1.6.3:

  $B_6 \supset D_6 \supset A_1 \oplus C_3 \supset A_1 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1$: non-surviving.

  This chain may be discarded due to the sextet/triplet generating rules since it exhibits only one multiplet of dimension 12 and one sextet ($d_3 = 18$).

  1.6.3.1 $\ldots \supset (A_1)_{12} \oplus A_1$: total pairing.

  1.6.3.2 $\ldots \supset (A_1)_{13} \oplus A_1$: total pairing.

  1.6.3.3 $\ldots \supset (A_1)_{23} \oplus A_1$: **surviving.**

  1.6.3.3.1 $\ldots \supset (A_1)_{12}$: does not reproduce the genetic code.

- Chain 1.6.4:

  $B_6 \supset D_6 \supset A_1 \oplus C_3 \supset A_1 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$: non-surviving.

  This chain may be discarded due to the sextet/triplet generating rules since it exhibits only three sextets ($d_3 = 18$).

  1.6.4.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1$: total pairing.

  1.6.4.2 $\ldots \supset (A_1)_{23} \oplus A_1 \oplus A_1$: **surviving.**

  1.6.4.2.1 $\ldots \supset (A_1)_{12} \oplus A_1$: total pairing.

1.6.4.2.2 $\ldots \supset (A_1)_{13} \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules in view of the fact that the six sextets that occur and that may or may not be broken are incapable of generating the three sextets and two triplets in the genetic code: they are organized into three identical sextets with highest weight (2-1) plus another three identical sextets with highest weight (1-2) and thus cannot be broken so as to produce precisely two triplets.

1.6.4.2.2.1 $\ldots \supset (A_1)_{12}$: does not reproduce the genetic code.

1.6.4.2.3 $\ldots \supset (A_1)_{23} \oplus A_1$: total pairing.

- Chain 1.6.5:

$B_6 \supset D_6 \supset A_1 \oplus C_3 \supset A_1 \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules in view of the fact that the two multiplets of dimension 18 that occur and that must of course be broken are incapable of generating the three sextets and two triplets in the genetic code: they have highest weights (2-1-2) and (1-2-2) and so each of them can only produce three sextets or two sextets and two triplets or six triplets or no sextets and triplets at all, which cannot be combined to yield three sextets and two triplets.

1.6.5.1 $\ldots \supset (A_1)_{12} \oplus A_1$: total pairing.

1.6.5.2 $\ldots \supset (A_1)_{13} \oplus A_1$: distribution of multiplets identical to that from the chain 1.6.3.3.

- Chain 1.7.3:

$B_6 \supset D_6 \supset A_1 \oplus A_1 \oplus D_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules since it exhibits only two sextets ($d_3 = 12$).

1.7.3.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1$: total pairing.

1.7.3.2 $\ldots \supset (A_1)_{13} \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules since it exhibits only one multiplet of dimension 12 and one sextet ($d_3 = 18$).

1.7.3.2.1 $\ldots \supset (A_1)_{12} \oplus A_1$: total pairing.

1.7.3.2.2 $\ldots \supset (A_1)_{13} \oplus A_1$: total pairing.

1.7.3.2.3 $\ldots \supset (A_1)_{23} \oplus A_1$: distribution of multiplets identical to that from the chain 1.6.3.3.

1.7.3.3 $\ldots \supset (A_1)_{14} \oplus A_1 \oplus A_1$: **surviving.**

1.7.3.3.1 $\ldots \supset (A_1)_{12} \oplus A_1$: total pairing.

1.7.3.3.2 $\ldots \supset (A_1)_{13} \oplus A_1$: total pairing.

1.7.3.3.3 $\ldots \supset (A_1)_{23} \oplus A_1$: distribution of multiplets identical to that from the chain 1.6.3.3.

1.7.3.4 $\ldots \supset (A_1)_{34} \oplus A_1 \oplus A_1$: total pairing.

- Chain 1.7.4:

$B_6 \supset D_6 \supset A_1 \oplus A_1 \oplus D_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus C_2$
$\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules since it exhibits only two sextets ($d_3 = 12$).

1.7.4.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1 \oplus A_1$: total pairing.

1.7.4.2 $\ldots \supset (A_1)_{13} \oplus A_1 \oplus A_1 \oplus A_1$: non-surviving.
This chain may be discarded due to the sextet/triplet generating rules since it exhibits only three sextets ($d_3 = 18$).

1.7.4.2.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1$: total pairing.

1.7.4.2.2 $\ldots \supset (A_1)_{23} \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 1.6.4.2.

1.7.4.3 $\ldots \supset (A_1)_{14} \oplus A_1 \oplus A_1 \oplus A_1$: **surviving.**

1.7.4.3.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1$: total pairing.

1.7.4.3.2 $\ldots \supset (A_1)_{13} \oplus A_1 \oplus A_1$: total pairing.

1.7.4.3.3 $\ldots \supset (A_1)_{14} \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 1.6.4.2.

1.7.4.3.4 $\ldots \supset (A_1)_{23} \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 1.6.4.2.

1.7.4.3.5 $\ldots \supset (A_1)_{24} \oplus A_1 \oplus A_1$: distribution of multiplets identical with that obtained from the $\mathfrak{sp}(6)$-model after the second step.

1.7.4.3.6 $\ldots \supset (A_1)_{34} \oplus A_1 \oplus A_1$: total pairing.

1.7.4.4 $\ldots \supset (A_1)_{34} \oplus A_1 \oplus A_1 \oplus A_1$: total pairing.

1.7.4.5 $\ldots \supset (A_1)_{45} \oplus A_1 \oplus A_1 \oplus A_1$: **surviving.**

1.7.4.5.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 1.6.4.2.

1.7.4.5.2 $\ldots \supset (A_1)_{14} \oplus A_1 \oplus A_1$: total pairing.

1.7.4.5.3 $\ldots \supset (A_1)_{23} \oplus A_1 \oplus A_1$: total pairing.

- Chain 1.7.5:

$B_6 \supset D_6 \supset A_1 \oplus A_1 \oplus D_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules since it exhibits no triplets, sextets etc. ($d_3 = 0$).

1.7.5.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: total pairing.

1.7.5.2 $\ldots \supset (A_1)_{13} \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 1.7.4.

- Chain 4.3.2:

$B_6 \supset C_2 \oplus D_4 \supset C_2 \oplus A_1 \oplus C_2 \supset C_2 \oplus A_1 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules since it exhibits no triplets, sextets etc. ($d_3 = 0$).

4.3.2.1 $\ldots \supset (A_1)_{12} \oplus A_1$: non-surviving.
This chain may be discarded due to the sextet/triplet generating rules since it exhibits only one multiplet of dimension 12 and one sextet ($d_3 = 18$).

4.3.2.1.1 $\ldots \supset (A_1)_{12}$: does not reproduce the genetic code.

4.3.2.2 $\ldots \supset (A_1)_{13} \oplus A_1$: **surviving.**

4.3.2.2.1 $\ldots \supset (A_1)_{12}$: does not reproduce the genetic code.

4.3.2.3 $\ldots \supset (A_1)_{23} \oplus A_1$: total pairing.

- Chain 4.3.4:

$B_6 \supset C_2 \oplus D_4 \supset C_2 \oplus A_1 \oplus C_2 \supset C_2 \oplus A_1 \oplus A_1 \oplus A_1$
$\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules since it exhibits only one multiplet of dimension 12 ($d_3 = 12$).

4.3.4.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1$: non-surviving.
This chain may be discarded due to the sextet/triplet generating rules since it exhibits only three sextets ($d_3 = 18$).

4.3.4.1.1 $\ldots \supset (A_1)_{12} \oplus A_1$: total pairing.

4.3.4.1.2 $\ldots \supset (A_1)_{23} \oplus A_1$: **surviving.**

4.3.4.1.2.1 $\ldots \supset (A_1)_{12}$: does not reproduce the genetic code.

4.3.4.2 $\ldots \supset (A_1)_{13} \oplus A_1 \oplus A_1$: **surviving.**

4.3.4.2.1 $\ldots \supset (A_1)_{12} \oplus A_1$: total pairing.

4.3.4.2.2 $\ldots \supset (A_1)_{13} \oplus A_1$: distribution of multiplets identical to that from the chain 4.3.4.1.2.

4.3.4.2.3 $\ldots \supset (A_1)_{23} \oplus A_1$: total pairing.

4.3.4.3 $\ldots \supset (A_1)_{23} \oplus A_1 \oplus A_1$: total pairing.

4.3.4.4 $\ldots \supset (A_1)_{34} \oplus A_1 \oplus A_1$: non-surviving.
This chain may be discarded due to the sextet/triplet generating rules in view of the fact that the two multiplets of dimension 12 that occur and that must of course be broken are incapable of generating the three sextets

and two triplets in the genetic code: they have highest weights (2-3-0) and (0-3-2) and so each of them can only produce two sextets or four triplets or no sextets and triplets at all, which cannot be combined to yield an odd number of sextets.

4.3.4.4.1 $\ldots \supset (A_1)_{12} \oplus A_1$: distribution of multiplets identical to that from the chain 4.3.4.1.2.

4.3.4.4.2 $\ldots \supset (A_1)_{13} \oplus A_1$: total pairing.

- Chain 4.4.1:

$B_6 \supset C_2 \oplus D_4 \supset C_2 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: non-surviving.
This chain may be discarded due to the sextet/triplet generating rules since it exhibits no triplets, sextets etc. ($d_3 = 0$).

4.4.1.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1 \oplus A_1$: non-surviving.
This chain may be discarded due to the sextet/triplet generating rules since it exhibits only two sextets ($d_3 = 12$).

4.4.1.1.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1$: total pairing.

4.4.1.1.2 $\ldots \supset (A_1)_{13} \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 4.3.4.1.

4.4.1.1.3 $\ldots \supset (A_1)_{23} \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 4.3.4.2.

4.4.1.1.4 $\ldots \supset (A_1)_{34} \oplus A_1 \oplus A_1$: total pairing.

4.4.1.2 $\ldots \supset (A_1)_{23} \oplus A_1 \oplus A_1 \oplus A_1$: total pairing.

4.4.1.3 $\ldots \supset (A_1)_{34} \oplus A_1 \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 4.3.4.

- Chain 6.2:

$B_6 \supset A_1 \oplus A_1 \oplus B_4 \supset A_1 \oplus A_1 \oplus A_1$: non-surviving.
This chain may be discarded due to the sextet/triplet generating rules since it exhibits no triplets, sextets etc. ($d_3 = 0$).

6.2.1 $\ldots \supset (A_1)_{12} \oplus A_1$: total pairing.

6.2.2 $\ldots \supset (A_1)_{13} \oplus A_1$: non-surviving.
This chain may be discarded due to the sextet/triplet generating rules since it exhibits only one multiplet of dimension 12 and one sextet ($d_3 = 18$).

6.2.2.1 $\ldots \supset (A_1)_{12}$: does not reproduce the genetic code.

- Chain 6.4:

$B_6 \supset A_1 \oplus A_1 \oplus B_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$: non-surviving.
This chain may be discarded due to the sextet/triplet generating rules since it exhibits no triplets, sextets etc. ($d_3 = 0$).

6.4.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1$: total pairing.

6.4.2 $\ldots \supset (A_1)_{13} \oplus A_1 \oplus A_1$: non-surviving.
This chain may be discarded due to the sextet/triplet generating rules since it exhibits only one multiplet of dimension 12 and one sextet ($d_3 = 18$).

6.4.2.1 $\ldots \supset (A_1)_{12} \oplus A_1$: total pairing.

6.4.2.2 $\ldots \supset (A_1)_{13} \oplus A_1$: total pairing.

6.4.2.3 $\ldots \supset (A_1)_{23} \oplus A_1$: **surviving.**

6.4.2.1 $\ldots \supset (A_1)_{12}$: does not reproduce the genetic code.

Summarizing, we have found a total of 9 chains for B_6 that survive up to the end of the second phase and must be subjected to the third phase (breaking of the A_1-subalgebras).

$D_7 = \mathfrak{so}(14)$

- Chain 2.3.1.1:
$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset A_2 \oplus A_1 \oplus C_2 \supset A_1^{(1)} \oplus A_1 \oplus A_1$: non-surviving.
This chain may be discarded due to the sextet/triplet generating rules since it exhibits only one sextet and two triplets ($d_3 = 12$).

2.3.1.1.1 $\ldots \supset (A_1)_{12} \oplus A_1$: non-surviving.
This chain may be discarded due to the sextet/triplet generating rules since it exhibits only one multiplet of dimension 12 and two triplets ($d_3 = 18$).

2.3.1.1.1.1 $\ldots \supset (A_1)_{12}$: does not reproduce the genetic code.

2.3.1.1.2 $\ldots \supset (A_1)_{13} \oplus A_1$: **surviving.**

2.3.1.1.2.1 $\ldots \supset (A_1)_{12}$: does not reproduce the genetic code.

2.3.1.1.3 $\ldots \supset (A_1)_{23} \oplus A_1$: total pairing, as well as 4 quintets and 4 triplets.

- Chain 2.3.1.2:
$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset A_2 \oplus A_1 \oplus C_2 \supset A_1^{(2)} \oplus A_1 \oplus A_1$: **surviving.**

2.3.1.2.1 $\ldots \supset (A_1)_{12} \oplus A_1$:
1 multiplet of dimension 15, 2 quintets, 2 triplets and 1 singlet.

2.3.1.2.2 $\ldots \supset (A_1)_{13} \oplus A_1$: 1 nonet, 1 septet, 2 quintets and 2 triplets.

2.3.1.2.3 $\ldots \supset (A_1)_{23} \oplus A_1$: total pairing, as well as
2 multiplets of dimension 15, 2 nonets, 2 quintets and 2 triplets.

- Chain 2.3.1.3:
$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset A_2 \oplus A_1 \oplus C_2 \supset A_1^{(1)} \oplus A_1 \oplus A_1 \oplus A_1$: non-surviving.
This chain may be discarded due to the sextet/triplet generating rules since it exhibits only one sextet and two triplets ($d_3 = 12$).

2.3.1.3.1 ... $\supset (A_1)_{12} \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules since it exhibits only two sextets and two triplets ($d_3 = 18$).

2.3.1.3.1.1 ... $\supset (A_1)_{12} \oplus A_1$: total pairing, as well as 4 triplets and 4 singlets.

2.3.1.3.1.2 ... $\supset (A_1)_{23} \oplus A_1$: 4 triplets and 4 singlets.

2.3.1.3.2 ... $\supset (A_1)_{13} \oplus A_1 \oplus A_1$: **surviving.**

2.3.1.3.2.1 ... $\supset (A_1)_{12} \oplus A_1$: total pairing, as well as 4 triplets and 4 singlets.

2.3.1.3.2.2 ... $\supset (A_1)_{13} \oplus A_1$: 4 triplets and 4 singlets.

2.3.1.3.2.3 ... $\supset (A_1)_{23} \oplus A_1$: total pairing, as well as 4 triplets and 4 singlets.

2.3.1.3.3 ... $\supset (A_1)_{23} \oplus A_1 \oplus A_1$: total pairing, as well as 4 triplets and 4 singlets.

2.3.1.3.4 ... $\supset (A_1)_{34} \oplus A_1 \oplus A_1$: 4 triplets and 4 singlets.

- Chain 2.3.1.4:

$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset A_2 \oplus A_1 \oplus C_2 \supset A_1^{(2)} \oplus A_1 \oplus A_1 \oplus A_1$: **surviving.**

2.3.1.4.1 ... $\supset (A_1)_{12} \oplus A_1 \oplus A_1$: 1 quintet, 3 triplets and 2 singlets.

2.3.1.4.2 ... $\supset (A_1)_{13} \oplus A_1 \oplus A_1$: **surviving.**

2.3.1.4.2.1 ... $\supset (A_1)_{12} \oplus A_1$: total pairing, as well as 2 quintets, 6 triplets and 4 singlets.

2.3.1.4.2.2 ... $\supset (A_1)_{13} \oplus A_1$: 1 nonet, 1 quintet, 5 triplets and 3 singlets.

2.3.1.4.2.3 ... $\supset (A_1)_{23} \oplus A_1$: total pairing, as well as 2 nonets, 4 triplets and 2 singlets.

2.3.1.4.3 ... $\supset (A_1)_{23} \oplus A_1 \oplus A_1$: total pairing, as well as 2 nonets, 4 triplets and 2 singlets.

2.3.1.4.4 ... $\supset (A_1)_{34} \oplus A_1 \oplus A_1$: 2 nonets, 4 triplets and 2 singlets.

- Chain 2.3.3.1:

$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus C_2$
$\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules since it exhibits no triplets, sextets etc. ($d_3 = 0$).

2.3.3.1.1 ... $\supset (A_1)_{12} \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 2.3.1.2.

2.3.3.1.2 ... $\supset (A_1)_{13} \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules in view of the fact that the multiplet of dimension 24 that occurs and that must of course be broken is incapable of generating the three sextets and two triplets in the genetic code: it has highest weight (2-1-3) and so can only produce four sextets or eight triplets or no sextets and triplets at all.

2.3.3.1.2.1 $\ldots \supset (A_1)_{12} \oplus A_1$:
1 multiplet of dimension 15, 2 quintets, 2 triplets and 1 singlet.

2.3.3.1.2.2 $\ldots \supset (A_1)_{13} \oplus A_1$: total pairing.

2.3.3.1.2.3 $\ldots \supset (A_1)_{23} \oplus A_1$: distribution of multiplets identical with that obtained from the $\mathfrak{g}_2$ model after the first step.

2.3.3.1.2.3.1 $\ldots \supset (A_1)_{12}$: does not reproduce the genetic code.

2.3.3.1.3 $\ldots \supset (A_1)_{14} \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules in view of the fact that the three multiplets of dimension 12 that occur and that must of course be broken are incapable of generating the three sextets and two triplets in the genetic code: they have highest weights (5-1-0), (2-1-1) and (1-1-2) and so the first can only produce two sextets or no sextets and triplets at all whereas each of the other two can only produce two sextets or four triplets or no sextets and triplets at all, which cannot be combined to yield an odd number of sextets.

2.3.3.1.3.1 $\ldots \supset (A_1)_{12} \oplus A_1$: 1 nonet, 1 septet, 2 quintets and 2 triplets.

2.3.3.1.3.2 $\ldots \supset (A_1)_{13} \oplus A_1$: total pairing.

2.3.3.1.3.3 $\ldots \supset (A_1)_{23} \oplus A_1$: distribution of multiplets identical to that from the chain 2.3.3.1.2.3.

2.3.3.1.4 $\ldots \supset (A_1)_{34} \oplus A_1 \oplus A_1$: total pairing.

- Chain 2.3.3.2:

$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus C_2$
$\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules since it exhibits only one multiplet of dimension 12 ($d_3 = 12$).

2.3.3.2.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 2.3.1.4.

2.3.3.2.2 $\ldots \supset (A_1)_{13} \oplus A_1 \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules in view of the fact that the two multiplets of dimension 12 that occur and that must of course be broken are incapable of generating the three sextets and two triplets in the genetic code: they have highest weights (2-1-1-0) and (2-1-0-1) and so each of them can only produce two sextets or four triplets or no sextets and triplets at all, which cannot be combined to yield an odd number of sextets.

2.3.3.2.2.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1$: 1 quintet, 3 triplets and 2 singlets.

2.3.3.2.2.2 $\ldots \supset (A_1)_{13} \oplus A_1 \oplus A_1$: total pairing.

2.3.3.2.2.3 $\ldots \supset (A_1)_{23} \oplus A_1 \oplus A_1$: **surviving.**

2.3.3.2.2.3.1 $\ldots \supset (A_1)_{12} \oplus A_1$: total pairing, as well as 2 quintets, 6 triplets and 4 singlets.

2.3.3.2.2.3.2 $\ldots \supset (A_1)_{13} \oplus A_1$: total pairing, as well as 2 nonets, 4 triplets and 2 singlets.

2.3.3.2.2.3.3 $\ldots \supset (A_1)_{23} \oplus A_1$: 2 nonets, 4 triplets and 2 singlets.

2.3.3.2.2.4 $\ldots \supset (A_1)_{34} \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules in view of the fact that the three multiplets of dimension 12 that occur and that must of course be broken are incapable of generating the three sextets and two triplets in the genetic code: they have highest weights (2-1-1) and (1-2-1) (twice) and so each of them can only produce two sextets or four triplets or no sextets and triplets at all, which cannot be combined to yield an odd number of sextets.

2.3.3.2.2.4.1 $\ldots \supset (A_1)_{12} \oplus A_1$: total pairing.

2.3.3.2.2.4.2 $\ldots \supset (A_1)_{13} \oplus A_1$: 2 nonets, 4 triplets and 2 singlets.

2.3.3.2.2.4.3 $\ldots \supset (A_1)_{23} \oplus A_1$: 1 nonet, 1 quintet, 5 triplets and 3 singlets.

2.3.3.2.3 $\ldots \supset (A_1)_{14} \oplus A_1 \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules in view of the fact that the three multiplets of dimension 12 that occur and that must of course be broken are incapable of generating the three sextets and two triplets in the genetic code: they have highest weights (2-1-1-0), (2-1-0-1) and (1-1-2-0) and so each of them can only produce two sextets or four triplets or no sextets and triplets at all, which cannot be combined to yield an odd number of sextets.

2.3.3.2.3.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 2.3.1.4.2.

2.3.3.2.3.2 $\ldots \supset (A_1)_{13} \oplus A_1 \oplus A_1$: total pairing.

2.3.3.2.3.3 $\ldots \supset (A_1)_{14} \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 2.3.3.2.2.4.

2.3.3.2.3.4 $\ldots \supset (A_1)_{23} \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 2.3.3.2.2.3.

2.3.3.2.3.5 $\ldots \supset (A_1)_{24} \oplus A_1 \oplus A_1$: **surviving**.

2.3.3.2.3.5.1 $\ldots \supset (A_1)_{12} \oplus A_1$: 1 nonet, 1 quintet, 5 triplets and 3 singlets.

2.3.3.2.3.5.2 $\ldots \supset (A_1)_{13} \oplus A_1$: total pairing, as well as 2 nonets, 4 triplets and 2 singlets.

2.3.3.2.3.6 $\ldots \supset (A_1)_{34} \oplus A_1 \oplus A_1$: total pairing.

2.3.3.2.4 $\ldots \supset (A_1)_{34} \oplus A_1 \oplus A_1 \oplus A_1$: total pairing.

2.3.3.2.5 $\ldots \supset (A_1)_{45} \oplus A_1 \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules in view of the fact that the two multiplets of dimension 12 that occur and that must of course be broken are incapable of generating the three sextets and two triplets in the genetic code: they have highest weights (2-1-1-0) and (0-1-1-2) and so each of them can only produce two sextets or four triplets or no sextets and triplets at all, which cannot be combined to yield an odd number of sextets.

2.3.3.2.5.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 2.3.3.2.2.4.

2.3.3.2.5.2 $\ldots \supset (A_1)_{14} \oplus A_1 \oplus A_1$: total pairing.

2.3.3.2.5.3 $\ldots \supset (A_1)_{23} \oplus A_1 \oplus A_1$: 2 nonets, 4 triplets and 2 singlets.

- Chain 2.4.1.1:

$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset A_2 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$
$\supset A_1^{(1)} \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules since it exhibits no triplets, sextets etc. ($d_3 = 0$).

2.4.1.1.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules since it exhibits only two sextets ($d_3 = 12$).

2.4.1.1.1.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1$: total pairing.

2.4.1.1.1.2 $\ldots \supset (A_1)_{13} \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 2.3.1.3.1.

2.4.1.1.1.3 $\ldots \supset (A_1)_{23} \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 2.3.1.3.2.

2.4.1.1.1.4 $\ldots \supset (A_1)_{34} \oplus A_1 \oplus A_1$: total pairing.

2.4.1.1.2 $\ldots \supset (A_1)_{23} \oplus A_1 \oplus A_1 \oplus A_1$: total pairing.

2.4.1.1.3 $\ldots \supset (A_1)_{34} \oplus A_1 \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 2.3.1.3.

- Chain 2.4.1.2:

$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset A_2 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$
$\supset A_1^{(2)} \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules in view of the fact that the four multiplets of dimension 12 that occur and that must of course be broken are incapable of generating the three sextets and two triplets in the genetic code: they have highest weights (2-1-0-1-0), (2-1-0-0-1), (2-0-1-1-0) and (2-0-1-0-1) and so each of them can only produce two sextets or four triplets or no sextets and triplets at all, which cannot be combined to yield an odd number of sextets.

2.4.1.2.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules in view of the fact that the two multiplets of dimension 12 that occur and that must of course be broken are incapable of generating the three sextets and two triplets in the genetic code: they have highest weights (2-1-1-0) and (2-1-0-1) and so each of them can only produce two sextets or four triplets or no sextets and triplets at all, which cannot be combined to yield an odd number of sextets.

2.4.1.2.1.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1$: total pairing.

2.4.1.2.1.2 $\ldots \supset (A_1)_{13} \oplus A_1 \oplus A_1$: 1 quintet, 3 triplets and 2 singlets.

2.4.1.2.1.3 $\ldots \supset (A_1)_{23} \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 2.3.1.4.2.

2.4.1.2.1.4 $\ldots \supset (A_1)_{34} \oplus A_1 \oplus A_1$: total pairing.

2.4.1.2.2 $\ldots \supset (A_1)_{23} \oplus A_1 \oplus A_1 \oplus A_1$: total pairing.

2.4.1.2.3 $\ldots \supset (A_1)_{34} \oplus A_1 \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 2.3.1.4.

- Chain 2.4.3 :

$$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$$
$$\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 : \text{non-surviving.}$$

This chain may be discarded due to the sextet/triplet generating rules since it exhibits no triplets, sextets etc. ($d_3 = 0$).

2.4.3.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 2.4.1.2.

2.4.3.2 $\ldots \supset (A_1)_{13} \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules in view of the fact that the two multiplets of dimension 12 that occur and that must of course be broken are incapable of generating the three sextets and two triplets in the genetic code: they have highest weights (2-1-0-1-0) and (2-1-0-0-1) and so each of them can only produce two sextets or four triplets or no sextets and triplets at all, which cannot be combined to yield an odd number of sextets.

2.4.3.2.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 2.4.1.2.1.

2.4.3.2.2 $\ldots \supset (A_1)_{13} \oplus A_1 \oplus A_1 \oplus A_1$: total pairing.

2.4.3.2.3 $\ldots \supset (A_1)_{14} \oplus A_1 \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 2.3.3.2.2.

2.4.3.2.4 $\ldots \supset (A_1)_{23} \oplus A_1 \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules in view of the fact that the four multiplets of dimension 12 that occur and that must of course be broken are incapable of generating the three sextets and two triplets in the genetic code: they have highest weights (2-1-1-0), (2-1-0-1), (1-2-1-0) and (1-2-0-1) and so each of them can only produce two sextets or four triplets or no sextets and triplets at all, which cannot be combined to yield an odd number of sextets.

2.4.3.2.4.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1$: total pairing.

2.4.3.2.4.2 $\ldots \supset (A_1)_{13} \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 2.3.3.2.2.3.

2.4.3.2.5 $\ldots \supset (A_1)_{24} \oplus A_1 \oplus A_1 \oplus A_1$: **surviving.**

2.4.3.2.5.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1$: 1 quintet, 3 triplets and 2 singlets.

2.4.3.2.5.2 $\ldots \supset (A_1)_{13} \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 2.3.3.2.2.3.

2.4.3.2.5.3 $\ldots \supset (A_1)_{14} \oplus A_1 \oplus A_1$: total pairing.

2.4.3.2.5.4 $\ldots \supset (A_1)_{34} \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 2.3.3.2.3.5.

2.4.3.2.6 $\ldots \supset (A_1)_{34} \oplus A_1 \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 2.3.3.2.3.

2.4.3.2.7 $\ldots \supset (A_1)_{45} \oplus A_1 \oplus A_1 \oplus A_1$: total pairing.

2.4.3.3 $\ldots \supset (A_1)_{34} \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: total pairing.

2.4.3.4 $\ldots \supset (A_1)_{35} \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$: distribution of multiplets identical to that from the chain 2.3.3.2.

- Chain 3.2:

  $D_7 \supset A_1 \oplus B_5 \supset A_1 \oplus A_1$: non-surviving.

  This chain may be discarded due to the sextet/triplet generating rules since it exhibits only one multiplet of dimension 12 ($d_3 = 12$).

  3.2.1 $\ldots \supset (A_1)_{12}$: does not reproduce the genetic code.

- Chain 3.5.2.2:

  $D_7 \supset A_1 \oplus B_5 \supset A_1 \oplus A_1 \oplus A_1 \oplus B_3 \supset A_1 \oplus A_1 \oplus A_1 \oplus G_2$
  $\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$: non-surviving

  This chain may be discarded due to the sextet/triplet generating rules since it exhibits no triplets, sextets etc. ($d_3 = 0$).

  3.5.2.2.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1$: **surviving.**

3.5.2.2.2 ... $\supset (A_1)_{14} \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules in view of the fact that the two multiplets of dimension 12 that occur and that must of course be broken are incapable of generating the three sextets and two triplets in the genetic code: they have highest weights (5-1-0) and (5-0-1) and so each of them can only produce two sextets or no sextets and triplets at all, that is, no triplets.

3.5.2.2.2.1 ... $\supset (A_1)_{12} \oplus A_1$:

1 nonet, 2 septets, 1 quintet, 1 triplet and 1 singlet.

3.5.2.2.2.2 ... $\supset (A_1)_{23} \oplus A_1$: total pairing.

3.5.2.2.3 ... $\supset (A_1)_{23} \oplus A_1 \oplus A_1$: total pairing.

3.5.2.2.4 ... $\supset (A_1)_{24} \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules since it exhibits only one multiplet of dimension 12 ($d_3 = 12$).

3.5.2.2.4.1 ... $\supset (A_1)_{12} \oplus A_1$: 1 nonet, 2 septets, 1 quintet, 1 triplet and 1 singlet.

3.5.2.2.4.2 ... $\supset (A_1)_{13} \oplus A_1$: total pairing.

3.5.2.2.4.3 ... $\supset (A_1)_{23} \oplus A_1$: **surviving.**

3.5.2.2.4.3.1 ... $\supset (A_1)_{12}$: does not reproduce the genetic code.

- Chain 3.5.2.3:

$D_7 \supset A_1 \oplus B_5 \supset A_1 \oplus A_1 \oplus A_1 \oplus B_3 \supset A_1 \oplus A_1 \oplus A_1 \oplus G_2$
$\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$:

distribution of multiplets identical to that from the chain 2.4.3.2.

- Chain 4.2.1:

$D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1 \supset A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules since it exhibits no triplets, sextets etc. ($d_3 = 0$).

4.2.1.1 ... $\supset (A_1)_{12}$: does not reproduce the genetic code.

- Chain 4.4.1:

$D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules since it exhibits no triplets, sextets etc. ($d_3 = 0$).

4.4.1.1 ... $\supset (A_1)_{12} \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules since it exhibits only one multiplet of dimension 12 and one sextet ($d_3 = 18$).

4.4.1.1.1 ... $\supset (A_1)_{12}$: does not reproduce the genetic code.

4.4.1.2 ... $\supset (A_1)_{23} \oplus A_1$: total pairing.

- Chain 4.5.1:

  $D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1 \oplus A_1 \oplus C_2 \supset (C_2)_{14} \oplus A_1 \oplus A_1$
  $\supset A_1 \oplus A_1 \oplus A_1$: non-surviving.

  This chain may be discarded due to the sextet/triplet generating rules since it exhibits only two sextets ($d_3 = 12$).

  4.5.1.1 $\ldots \supset (A_1)_{12} \oplus A_1$: non-surviving.
  This chain may be discarded due to the sextet/triplet generating rules since it exhibits only three sextets ($d_3 = 18$).

  4.5.1.1.1 $\ldots \supset (A_1)_{12}$: does not reproduce the genetic code.

  4.5.1.2 $\ldots \supset (A_1)_{23} \oplus A_1$: total pairing.

- Chain 4.5.2:

  $D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1 \oplus A_1 \oplus C_2 \supset (C_2)_{14} \oplus A_1 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$:
  distribution of multiplets identical to that from the B_6-chain 1.7.4.5.

- Chain 4.5.3:

  $D_7 \supset C_2 \oplus B_4 \supset C_2 \oplus A_1 \oplus A_1 \oplus C_2 \supset C_2 \oplus A_1 \oplus A_1 \oplus A_1$
  $\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$: non-surviving.

  This chain may be discarded due to the sextet/triplet generating rules since it exhibits no triplets, sextets etc. ($d_3 = 0$).

  4.5.3.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1$: non-surviving.
  This chain may be discarded due to the sextet/triplet generating rules since it exhibits only one multiplet of dimension 12 ($d_3 = 12$).

  4.5.3.1.1 $\ldots \supset (A_1)_{12} \oplus A_1$: total pairing.

  4.5.3.1.2 $\ldots \supset (A_1)_{13} \oplus A_1$: distribution of
  multiplets identical to that from the chain 4.5.1.1.

  4.5.3.1.3 $\ldots \supset (A_1)_{23} \oplus A_1$: non-surviving.
  This chain may be discarded due to the sextet/triplet generating rules in view of the fact that the two multiplets of dimension 12 that occur and that must of course be broken are incapable of generating the three sextets and two triplets in the genetic code: they have highest weights (3-2) and (2-3) and so each of them can only produce two sextets or four triplets or no sextets and triplets at all, which cannot be combined to yield an odd number of sextets.

  4.5.3.1.3.1 $\ldots \supset (A_1)_{12}$: does not reproduce the genetic code.

  4.5.3.2 $\ldots \supset (A_1)_{14} \oplus A_1 \oplus A_1$: distribution of
  multiplets identical to that from the chain 4.5.1.

  4.5.3.3 $\ldots \supset (A_1)_{23} \oplus A_1 \oplus A_1$: total pairing.

- Chain 5.2.2.2:
  $D_7 \supset B_3 \oplus B_3 \supset B_3 \oplus G_2 \supset G_2 \oplus G_2 \supset G_2 \oplus A_1 \supset A_1 \oplus A_1$: non-surviving.
  This chain may be discarded due to the sextet/triplet generating rules since it exhibits no triplets, sextets etc. ($d_3 = 0$).

  5.2.2.2.1 $\ldots \supset (A_1)_{12}$: does not reproduce the genetic code.

- Chain 5.2.2.3:
  $D_7 \supset B_3 \oplus B_3 \supset B_3 \oplus G_2 \supset G_2 \oplus G_2 \supset G_2 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1$:
  distribution of multiplets identical to that from the chain 3.5.2.2.1.

- Chain 5.2.2.4:
  $D_7 \supset B_3 \oplus B_3 \supset B_3 \oplus G_2 \supset G_2 \oplus G_2 \supset G_2 \oplus A_1 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1$:
  distribution of multiplets identical to that from the chain 3.5.2.2.1.

- Chain 5.2.2.5:
  $D_7 \supset B_3 \oplus B_3 \supset B_3 \oplus G_2 \supset G_2 \oplus G_2 \supset G_2 \oplus A_1 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$:
  distribution of multiplets identical to that from the chain 2.4.3.2.5.

- Chain 6.1.1.2:
  $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus A_4 \supset A_1 \oplus A_1 \oplus A_3 \supset A_1 \oplus A_1 \oplus C_2$
  $\supset A_1 \oplus A_1 \oplus A_1$: non-surviving.
  This chain may be discarded due to the sextet/triplet generating rules since it exhibits no triplets, sextets etc. ($d_3 = 0$).

  6.1.1.2.1 $\ldots \supset (A_1)_{12} \oplus A_1$: total pairing.

  6.1.1.2.2 $\ldots \supset (A_1)_{13} \oplus A_1$: non-surviving.
  This chain may be discarded due to the sextet/triplet generating rules since it exhibits only one sextet and two triplets ($d_3 = 12$).

  6.1.1.2.2.1 $\ldots \supset (A_1)_{12}$: does not reproduce the genetic code.

- Chain 6.1.4.2:
  $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus A_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_2$
  $\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1^{(2)}$: non-surviving.
  This chain may be discarded due to the sextet/triplet generating rules in view of the fact that the two multiplets of dimension 12 that occur and that must of course be broken, together with the four sextets that occur and that may or may not be broken, are incapable of generating the three sextets and two triplets in the genetic code: the former have highest weights (1-0-1-2) and (0-1-1-2) and so each of them can only produce two sextets or four triplets or no sextets and triplets at all, whereas the latter are organized into two identical sextets with highest weight (1-0-0-2) plus another two identical sextets with highest weight (0-1-0-2); all this cannot be combined to yield an odd number of sextets.

6.1.4.2.1 $\ldots \supset (A_1)_{12} \oplus A_1 \oplus A_1$: total pairing.

6.1.4.2.2 $\ldots \supset (A_1)_{13} \oplus A_1 \oplus A_1$: **surviving.**

6.1.4.2.2.1 $\ldots \supset (A_1)_{12} \oplus A_1$: total pairing.

6.1.4.2.2.2 $\ldots \supset (A_1)_{13} \oplus A_1$: 1 quintet, 3 triplets and 2 singlets.

6.1.4.2.2.3 $\ldots \supset (A_1)_{23} \oplus A_1$: **surviving.**

6.1.4.2.2.3.1 $\ldots \supset (A_1)_{12}$: does not reproduce the genetic code.

6.1.4.2.3 $\ldots \supset (A_1)_{14} \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules in view of the fact that the multiplet of dimension 12 that occurs and that must of course be broken, together with the two sextets that occur and that may or may not be broken, are incapable of generating the three sextets and two triplets in the genetic code: the former has highest weight (2-1-1) and so can only produce two sextets or four triplets or no sextets and triplets at all, whereas the latter are organized into two identical sextets with highest weight (2-1-0); all this cannot be combined to yield an odd number of sextets.

6.1.4.2.3.1 $\ldots \supset (A_1)_{12} \oplus A_1$: total pairing.

6.1.4.2.3.2 $\ldots \supset (A_1)_{13} \oplus A_1$: 1 quintet, 3 triplets and 2 singlets.

6.1.4.2.3.3 $\ldots \supset (A_1)_{23} \oplus A_1$: distribution of multiplets identical to that from the chain 6.1.4.2.2.3.

6.1.4.2.4 $\ldots \supset (A_1)_{34} \oplus A_1 \oplus A_1$: non-surviving.

This chain may be discarded due to the sextet/triplet generating rules in view of the fact that the four sextets that occur and that may or may not be broken, are incapable of generating the three sextets and two triplets in the genetic code: they are organized into two identical sextets with highest weight (2-1-0) plus another two identical sextets with highest weight (2-0-1), which cannot be combined to yield an odd number of sextets.

6.1.4.2.4.1 $\ldots \supset (A_1)_{12} \oplus A_1$: 1 quintet, 3 triplets and 2 singlets.

6.1.4.2.4.2 $\ldots \supset (A_1)_{23} \oplus A_1$: total pairing.

- Chain 6.3.1 :
  $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1$:
  distribution of multiplets identical to that from the chain 3.5.2.2.2.

- Chain 6.3.2 :
  $D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$:
  distribution of multiplets identical to that from the chain 2.4.3.2.4.

Summarizing, we have found a total of 12 chains for D_7 that survive up to the end of the second phase and must be subjected to the third phase (breaking of the A_1-subalgebras).

5 Third Phase of Symmetry Breaking

In the last phase of the symmetry breaking process, we subject the surviving chains from the first two phases to breaking of one or several of the A_1-subalgebras, which requires a rather long and tedious case-by-case analysis of each of the corresponding tables (in which the abbreviation "HW" stands for "highest weight").

$B_6 = \mathfrak{so}(13)$

1. Chain 1.6.3.3:

$B_6 \supset D_6 \supset A_1 \oplus C_3 \supset A_1 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \supset (A_1)_{23} \oplus A_1$

B_6		D_6		$A_1 \oplus C_3$		$(A_1)^2 \oplus C_2$		$(A_1)^3$		$(A_1)^2$	
HW	d	HW	d	HW	d	HW	d	HW	d	HW	d
(000001)	64	(000010)	32	(2-100)	18	(2-0-10)	12	(2-0-3)	12	(3-2)	12
						(2-1-00)	6	(2-1-0)	6	(1-2)	6
				(0-001)	14	(0-1-01)	10	(0-1-4)	10	(5-0)	6
										(3-0)	4
						(0-0-10)	4	(0-0-3)	4	(3-0)	4
		(000001)	32	(1-010)	28	(1-1-10)	16	(1-1-3)	16	(4-1)	10
										(2-1)	6
						(1-0-01)	10	(1-0-4)	10	(4-1)	10
						(1-0-00)	2	(1-0-0)	2	(0-1)	2
				(3-000)	4	(3-0-00)	4	(3-0-0)	4	(0-3)	4
1 subspace		2 subspaces		4 subspaces		8 subspaces		8 subspaces		10 subspaces	

In this chain, the first $\mathfrak{su}(2)$ must be broken, in order to eliminate the multiplets of dimension 10. Now observe that

1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 20 multiplets with $d_3 = 18$,
2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 35 multiplets with $d_3 = 12$,
3. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 13 multiplets with $d_3 = 12$,
4. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 21 multiplets with $d_3 = 12$.

Note that option 4 leads to a scheme which does not allow for freezing and is remote from the genetic code, with 1 sextet, 4 quintets, 5 quartets, 2 triplets, 3 doublets and 6 singlets. In the case of option 2, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplets of dimension > 6 must not be frozen and will therefore provide 4 triplets and 10 doublets, which is certainly too many.

2. Chain 1.6.4.2:

$B_6 \supset D_6 \supset A_1 \oplus C_3 \supset A_1 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset (A_1)_{23} \oplus A_1 \oplus A_1$

B_6		D_6		$A_1 \oplus C_3$		$(A_1)^2 \oplus C_2$		$(A_1)^4$		$(A_1)^3$	
HW	d	HW	d	HW	d	HW	d	HW	d	HW	d
(000001)	64	(000010)	32	(2-100)	18	(2-0-10)	12	(2-0-1-0)	6	(1-2-0)	6
								(2-0-0-1)	6	(0-2-1)	6
						(2-1-00)	6	(2-1-0-0)	6	(1-2-0)	6
				(0-001)	14	(0-1-01)	10	(0-1-1-1)	8	(2-0-1)	6
										(0-0-1)	2
								(0-1-0-0)	2	(1-0-0)	2
						(0-0-10)	4	(0-0-1-0)	2	(1-0-0)	2
								(0-0-0-1)	2	(0-0-1)	2
		(000001)	32	(1-010)	28	(1-1-10)	16	(1-1-1-0)	8	(2-1-0)	6
										(0-1-0)	2
								(1-1-0-1)	8	(1-1-1)	8
						(1-0-01)	10	(1-0-1-1)	8	(1-1-1)	8
								(1-0-0-0)	2	(0-1-0)	2
						(1-0-00)	2	(1-0-0-0)	2	(0-1-0)	2
				(3-000)	4	(3-0-00)	4	(3-0-0-0)	4	(0-3-0)	4
1 subspace		2 subspaces		4 subspaces		8 subspaces		13 subspaces		15 subspaces	

In this chain, breaking the third $\mathfrak{su}(2)$ to $\mathfrak{o}(2)$ has no effect and will be disregarded; moreover, at least one of the three $\mathfrak{su}(2)$'s must be broken down to $\mathfrak{so}(2)$ because otherwise the two octets would remain unbroken. Now observe that

1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 17 multiplets with $d_3 = 18$ (only two of the five sextets are broken into a quartet and a doublet each),
2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 25 multiplets with $d_3 = 18$,
3. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 19 multiplets with $d_3 = 12$ (only three of the five sextets are broken into a quartet and a doublet each and the only quartet is broken into two doublets),
4. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 30 multiplets with $d_3 = 12$,
5. breaking the third $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates precisely 21 multiplets with $d_3 = 30$.

Note that option 5 leads to an interesting scheme which, without any freezing, comes close to the genetic code but is slightly different, with 3 sextets, 5 quartets, 4 triplets, 5 doublets and 4 singlets. In the cases of options 2 and 4, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, it turns out that there is no possibility of generating, for example, the correct number of singlets (2): we can only get 0 or 4 singlets in the first case and 0, 4, 6 or 10 singlets in the second case, apart from other deviations.

3. Chain 1.7.3.3:

$$B_6 \supset D_6 \supset A_1 \oplus A_1 \oplus D_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset (A_1)_{14} \oplus A_1 \oplus A_1$$

B_6		D_6		$(A_1)^2 \oplus D_4$		$(A_1)^3 \oplus C_2$		$(A_1)^4$		$(A_1)^3$	
HW	d	HW	d	HW	d	HW	d	HW	d	HW	d
(000001)	64	(000010)	32	(0-1-0010)	16	(0-1-0-01)	10	(0-1-0-4)	10	(4-1-0)	10
						(0-1-2-00)	6	(0-1-2-0)	6	(0-1-2)	6
				(1-0-0001)	16	(1-0-1-10)	16	(1-0-1-3)	16	(4-0-1)	10
										(2-0-1)	6
		(000001)	32	(0-1-0001)	16	(0-1-1-10)	16	(0-1-1-3)	16	(3-1-1)	16
				(1-0-0010)	16	(1-0-0-01)	10	(1-0-0-4)	10	(5-0-0)	6
										(3-0-0)	4
						(1-0-2-00)	6	(1-0-2-0)	6	(1-0-2)	6
1 subspace		2 subspaces		4 subspaces		6 subspaces		6 subspaces		8 subspaces	

In this chain, the first $su(2)$ must be broken, in order to eliminate the multiplets of dimension 10. Moreover, breaking the second $su(2)$ to $o(2)$ has no effect and will be disregarded. Now observe that, in a first step,

1. breaking the first $su(2)$ down to $o(2)$ generates 17 multiplets with $d_3 = 12$,
2. breaking the first $su(2)$ down to $so(2)$ generates 30 multiplets with $d_3 = 12$,
3. breaking the second $su(2)$ down to $so(2)$ generates 11 multiplets with $d_3 = 24$,
4. breaking the third $su(2)$ down to $o(2)$ generates 10 multiplets with $d_3 = 12$,
5. breaking the third $su(2)$ down to $so(2)$ generates 15 multiplets with $d_3 = 12$.

In the case of option 2, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the two multiplets of dimension 10 must not be frozen and will therefore provide 10 doublets, which is already one too many. Therefore, the only surviving option for continuing the symmetry breaking process is 3. Hence in a second step,

3.1. breaking the first $su(2)$ down to $o(2)$ generates 20 multiplets with $d_3 = 12$,
3.2. breaking the first $su(2)$ down to $so(2)$ generates 40 multiplets with $d_3 = 12$,
3.3. breaking the third $su(2)$ down to $o(2)$ generates 14 multiplets with $d_3 = 12$,
3.4. breaking the third $su(2)$ down to $so(2)$ generates precisely 21 multiplets with $d_3 = 12$.

Note that option 3.4 leads to a scheme which does not allow for freezing and is remote from the genetic code, with 1 sextet, 4 quintets, 5 quartets, 2 triplets, 3 doublets and 6 singlets. In the case of option 3.2, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the two multiplets of dimension 5 must not be frozen and will therefore provide 10 singlets, which is certainly too many.

4. Chain 1.7.4.3:

$$B_6 \supset D_6 \supset A_1 \oplus A_1 \oplus D_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$$
$$\supset (A_1)_{14} \oplus A_1 \oplus A_1 \oplus A_1$$

B_6		D_6		$(A_1)^2 \oplus D_4$		$(A_1)^3 \oplus C_2$		$(A_1)^5$		$(A_1)^4$	
HW	d	HW	d	HW	d	HW	d	HW	d	HW	d
(000001)	64	(000010)	32	(0-1-0010)	16	(0-1-0-01)	10	(0-1-0-1-1)	8	(1-1-0-1)	8
								(0-1-0-0-0)	2	(0-1-0-0)	2
						(0-1-2-00)	6	(0-1-2-0-0)	6	(0-1-2-0)	6
				(1-0-0001)	16	(1-0-1-10)	16	(1-0-1-1-0)	8	(2-0-1-0)	6
										(0-0-1-0)	2
								(1-0-1-0-1)	8	(1-0-1-1)	8
		(000001)	32	(0-1-0001)	16	(0-1-1-10)	16	(0-1-1-1-0)	8	(1-1-1-0)	8
								(0-1-1-0-1)	8	(0-1-1-1)	8
				(1-0-0010)	16	(1-0-0-01)	10	(1-0-0-1-1)	8	(2-0-0-1)	6
										(0-0-0-1)	2
								(1-0-0-0-0)	2	(1-0-0-0)	2
						(1-0-2-00)	6	(1-0-2-0-0)	6	(1-0-2-0)	6
1 subspace		2 subspaces		4 subspaces		6 subspaces		10 subspaces		12 subspaces	

In this chain, breaking the second and the fourth $su(2)$ to $o(2)$ has no effect and will be disregarded; moreover, at least two of the four $su(2)$'s must be broken down to $so(2)$ because otherwise at least one of the four octets would remain unbroken. Note also the symmetry of the distribution of multiplets in the penultimate column under simultaneous exchange of the first with the third and the second with the fourth $su(2)$. Now observe that, in a first step,

1. breaking the first or third $su(2)$ down to $o(2)$ generates 14 multiplets with $d_3 = 12$ (only two of the four sextets are broken into a quartet and a doublet each),
2. breaking the first or third $su(2)$ down to $so(2)$ generates precisely 21 multiplets with $d_3 = 12$,
3. breaking the second or fourth $su(2)$ down to $so(2)$ generates 17 multiplets with $d_3 = 24$.

Note that option 2 leads to an interesting scheme which, without any freezing, comes close to the genetic code but is slightly different, with 1 octet, 1 sextet, 6 quartets, 2 triplets, 9 doublets and 2 singlets. Therefore, the only surviving option for continuing the symmetry breaking process is 3; for the sake of definiteness, we choose to break the fourth $su(2)$. Hence in a second step,

3.1. breaking the first $su(2)$ down to $o(2)$ generates 20 multiplets with $d_3 = 12$,
3.2. breaking the first $su(2)$ down to $so(2)$ generates 30 multiplets with $d_3 = 12$,
3.3. breaking the second $su(2)$ down to $so(2)$ generates 24 multiplets with $d_3 = 24$,
3.4. breaking the third $su(2)$ down to $o(2)$ generates 19 multiplets with $d_3 = 12$,
3.5. breaking the third $su(2)$ down to $so(2)$ generates 28 multiplets with $d_3 = 12$.

In the cases of options 3.2, 3.3 and 3.5, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. A closer analysis reveals that in all three cases, we are able to produce the correct number of sextets (3), triplets (2) and singlets (2), but there is no possibility of generating the correct number of quartets (5) and of doublets (9): we can only get 2, 4, 6 or 8 quartets and, correspondingly, 15, 11, 7 or 3 doublets.

5. Chain 1.7.4.5:

$B_6 \supset D_6 \supset A_1 \oplus A_1 \oplus D_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$
$\quad \supset (A_1)_{45} \oplus A_1 \oplus A_1 \oplus A_1$

B_6		D_6		$(A_1)^2 \oplus D_4$		$(A_1)^3 \oplus C_2$		$(A_1)^5$		$(A_1)^4$	
HW	d	HW	d	HW	d	HW	d	HW	d	HW	d
(000001)	64	(000010)	32	(0-1-0010)	16	(0-1-0-01)	10	(0-1-0-1-1)	8	(2-0-1-0)	6
										(0-0-1-0)	2
								(0-1-0-0-0)	2	(0-0-1-0)	2
						(0-1-2-00)	6	(0-1-2-0-0)	6	(0-0-1-2)	6
				(1-0-0001)	16	(1-0-1-10)	16	(1-0-1-1-0)	8	(1-1-0-1)	8
								(1-0-1-0-1)	8	(1-1-0-1)	8
		(000001)	32	(0-1-0001)	16	(0-1-1-10)	16	(0-1-1-1-0)	8	(1-0-1-1)	8
								(0-1-1-0-1)	8	(1-0-1-1)	8
				(1-0-0010)	16	(1-0-0-01)	10	(1-0-0-1-1)	8	(2-1-0-0)	6
										(0-1-0-0)	2
								(1-0-0-0-0)	2	(0-1-0-0)	2
						(1-0-2-00)	6	(1-0-2-0-0)	6	(0-1-0-2)	6
1 subspace		2 subspaces		4 subspaces		6 subspaces		10 subspaces		12 subspaces	

In this chain, breaking the second and the third $su(2)$ to $o(2)$ has no effect and will be disregarded. Note also the symmetry of the distribution of multiplets in the penultimate column under exchange of the second with the third and the first with the fourth $su(2)$. Now observe that

1. breaking the second or third $su(2)$ down to $so(2)$ generates 18 multiplets with $d_3 = 24$, among which there are already 4 triplets and 4 singlets,

2. breaking the first or fourth $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 14 multiplets with $d_3 = 12$ (only two of the four sextets are broken into a quartet and a doublet each),
3. breaking the first or fourth $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 20 multiplets with $d_3 = 12$.

Therefore, there is no surviving option for continuing the symmetry breaking process.

6. Chain 4.3.2.2 (with one intermediate step omitted):

$B_6 \supset C_2 \oplus D_4 \supset C_2 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \supset (A_1)_{13} \oplus A_1$

B_6		$C_2 \oplus D_4$		$C_2 \oplus A_1 \oplus C_2$		$(A_1)^3$		$(A_1)^2$	
HW	d	HW	d	HW	d	HW	d	HW	d
(000001)	64	(10-0010)	32	(10-0-01)	20	(3-0-4)	20	(7-0)	8
								(5-0)	6
								(3-0)	4
								(1-0)	2
				(10-2-00)	12	(3-2-0)	12	(3-2)	12
		(10-0001)	32	(10-1-10)	32	(3-1-3)	32	(6-1)	14
								(4-1)	10
								(2-1)	6
								(0-1)	2
1 subspace		2 subspaces		3 subspaces		3 subspaces		9 subspaces	

In this chain, the first $\mathfrak{su}(2)$ must be broken, in order to eliminate the multiplets of dimension 10 and 14. Now observe that

1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 22 multiplets with $d_3 = 12$,
2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 40 multiplets with $d_3 = 12$,
3. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 10 multiplets with $d_3 = 12$,
4. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 15 multiplets with $d_3 = 12$.

In the cases of options 1 and 2, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplets of dimension > 6 must not be frozen. In the case of option 1, we see that all resulting multiplets still have even dimension. In the case of option 2, we get at least 8 singlets and 4 triplets, which is certainly too many.

7. Chain 4.3.4.1.2 (with one intermediate step omitted):

$$B_6 \supset C_2 \oplus D_4 \supset C_2 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset (A_1)_{12} \oplus A_1 \oplus A_1 \supset (A_1)_{23} \oplus A_1$$

B_6		$C_2 \oplus D_4$		$C_2 \oplus A_1 \oplus C_2$		$(A_1)^4$		$(A_1)^3$		$(A_1)^2$	
HW	d	HW	d	HW	d	HW	d	HW	d	HW	d
(000001)	64	(10-0010)	32	(10-0-01)	20	(3-0-1-1)	16	(3-1-1)	16	(2-3)	12
										(0-3)	4
						(3-0-0-0)	4	(3-0-0)	4	(0-3)	4
				(10-2-00)	12	(3-2-0-0)	12	(5-0-0)	6	(0-5)	6
								(3-0-0)	4	(0-3)	4
								(1-0-0)	2	(0-1)	2
		(10-0001)	32	(10-1-10)	32	(3-1-1-0)	16	(4-1-0)	10	(1-4)	10
								(2-1-0)	6	(1-2)	6
						(3-1-0-1)	16	(4-0-1)	10	(1-4)	10
								(2-0-1)	6	(1-2)	6
1 subspace		2 subspaces		3 subspaces		5 subspaces		9 subspaces		10 subspaces	

In this chain, the second $\mathfrak{su}(2)$ must be broken, in order to eliminate the multiplets of dimension 10. Now observe that

1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 11 multiplets with $d_3 = 18$,
2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 16 multiplets with $d_3 = 18$,
3. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 22 multiplets with $d_3 = 12$,
4. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 40 multiplets with $d_3 = 12$.

In the cases of options 3 and 4, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplets of dimension > 6 must not be frozen. In the case of option 3, we see that all resulting multiplets still have even dimension. In the case of option 4, there is no possibility of generating the correct number of triplets (2): we can only get 0 or 4 triplets.

8. Chain 4.3.4.2 (with one intermediate step omitted):

$$B_6 \supset C_2 \oplus D_4 \supset C_2 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset (A_1)_{13} \oplus A_1 \oplus A_1$$

B_6		$C_2 \oplus D_4$		$C_2 \oplus A_1 \oplus C_2$		$(A_1)^4$		$(A_1)^3$	
HW	d	HW	d	HW	d	HW	d	HW	d
(000001)	64	(10-0010)	32	(10-0-01)	20	(3-0-1-1)	16	(4-0-1)	10
								(2-0-1)	6
						(3-0-0-0)	4	(3-0-0)	4
				(10-2-00)	12	(3-2-0-0)	12	(3-2-0)	12
		(10-0001)	32	(10-1-10)	32	(3-1-1-0)	16	(4-1-0)	10
								(2-1-0)	6
						(3-1-0-1)	16	(3-1-1)	16
1 subspace		2 subspaces		3 subspaces		5 subspaces		7 subspaces	

In this chain, the first $\mathfrak{su}(2)$ must be broken, in order to eliminate the multiplets of dimension 10. Moreover, breaking the third $\mathfrak{su}(2)$ to $\mathfrak{o}(2)$ has no effect and will be disregarded. Now observe that, in a first step,

1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 16 multiplets with $d_3 = 12$,
2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 28 multiplets with $d_3 = 12$,
3. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 8 multiplets with $d_3 = 12$,
4. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 12 multiplets with $d_3 = 12$,
5. breaking the third $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 10 multiplets with $d_3 = 24$.

In the case of option 2, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplet of dimension 12 and the two multiplets of dimension 10 must not be frozen and will therefore provide 4 triplets and 10 doublets, which is certainly too many. Therefore, the only surviving option for continuing the symmetry breaking process is 5. Hence in a second step,

5.1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 23 multiplets with $d_3 = 12$,
5.2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 40 multiplets with $d_3 = 12$,
5.3. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 11 multiplets with $d_3 = 12$,
5.4. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 16 multiplets with $d_3 = 12$.

In the cases of options 5.1 and 5.2, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplets of dimension > 6 and of dimension 5 must not be frozen. In the case of option 5.1, we are able to produce the correct number of sextets (3), triplets (2) and singlets (2), but there is no possibility of generating the correct number of quartets (5) and of doublets (9): we can only get 6 or 7 quartets and, correspondingly, 7 or 5 doublets. In the case of option 5.2, we get at least 10 singlets, which is certainly too many.

9. Chain 6.4.2.3:

$B_6 \supset A_1 \oplus A_1 \oplus B_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset (A_1)_{13} \oplus A_1 \oplus A_1 \supset (A_1)_{23} \oplus A_1$

B_6		$(A_1)^2 \oplus B_4$		$(A_1)^4$		$(A_1)^3$		$(A_1)^2$	
HW	d	HW	d	HW	d	HW	d	HW	d
(000001)	64	(1-0-0001)	32	(1-0-3-1)	16	(4-0-1)	10	(1-4)	10
						(2-0-1)	6	(1-2)	6
				(1-0-1-3)	16	(2-0-3)	12	(3-2)	12
						(0-0-3)	4	(3-0)	4
		(0-1-0001)	32	(0-1-3-1)	16	(3-1-1)	16	(2-3)	12
								(0-3)	4
				(0-1-1-3)	16	(1-1-3)	16	(4-1)	10
								(2-1)	6
1 subspace		2 subspaces		4 subspaces		6 subspaces		8 subspaces	

In this chain, both the first and the second $su(2)$ must be broken, in order to eliminate all multiplets whose dimension is a multiple of 5. Note also the symmetry of the distribution of multiplets in the penultimate column under exchange of the first with the second $su(2)$. On the other hand, observe that

1. breaking the first or second $su(2)$ down to $o(2)$ generates 14 multiplets with $d_3 = 18$,
2. breaking the first or the second $su(2)$ down to $so(2)$ generates 24 multiplets with $d_3 = 18$.

Therefore, there is no possibility to continue the symmetry breaking process to perform the necessary breaking of the other $su(2)$.

$D_7 = \mathfrak{so}(14)$

1. Chain 2.3.1.1.2:

$$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset A_2 \oplus A_1 \oplus C_2 \supset A_1^{(1)} \oplus A_1 \oplus A_1 \supset (A_1)_{13} \oplus A_1$$

D_7		$A_3 \oplus D_4$		$A_3 \oplus A_1 \oplus C_2$		$A_2 \oplus A_1 \oplus C_2$		$(A_1)^3$		$(A_1)^2$	
HW	d	HW	d	HW	d	HW	d	HW	d	HW	d
(0000010)	64	(001-0010)	32	(001-0-01)	20	(01-0-01)	15	(1-0-4)	10	(5-0)	6
										(3-0)	4
								(0-0-4)	5	(4-0)	5
						(00-0-01)	5	(0-0-4)	5	(4-0)	5
				(001-2-00)	12	(01-2-00)	9	(1-2-0)	6	(1-2)	6
								(0-2-0)	3	(0-2)	3
						(00-2-00)	3	(0-2-0)	3	(0-2)	3
		(100-0001)	32	(100-1-10)	32	(10-1-10)	24	(1-1-3)	16	(4-1)	10
										(2-1)	6
								(0-1-3)	8	(3-1)	8
						(00-1-10)	8	(0-1-3)	8	(3-1)	8
1 subspace		2 subspaces		3 subspaces		6 subspaces		9 subspaces		11 subspaces	

In this chain, the first $\mathfrak{su}(2)$ must be broken, in order to eliminate the multiplets of dimension 5 and 10. Now observe that

1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 23 multiplets with $d_3 = 12$,
2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 40 multiplets with $d_3 = 12$,
3. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 15 multiplets with $d_3 = 12$,
4. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates precisely 21 multiplets with $d_3 = 12$.

Note that option 4 leads to a scheme which does not allow for freezing and is remote from the genetic code, with 1 sextet, 4 quintets, 5 quartets, 2 triplets, 3 doublets and 6 singlets. In the cases of options 1 and 2, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplets of dimension > 6 and of dimension 5 must not be frozen. In the case of option 1, we are able to produce the correct number of sextets (3), triplets (2) and singlets (2), but there is no possibility of generating the correct number of quartets (5) and doublets (9): we get at least 6 quartets and at most 7 doublets. In the case of option 2, the multiplets of dimension 5 will provide at least 10 singlets, which is certainly too many.

2. Chain 2.3.1.2:

$$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset A_2 \oplus A_1 \oplus C_2 \supset A_1^{(2)} \oplus A_1 \oplus A_1$$

D_7		$A_3 \oplus D_4$		$A_3 \oplus A_1 \oplus C_2$		$A_2 \oplus A_1 \oplus C_2$		$(A_1)^3$	
HW	d	HW	d	HW	d	HW	d	HW	d
(0000010)	64	(001-0010)	32	(001-0-01)	20	(01-0-01)	15	(2-0-4)	15
						(00-0-01)	5	(0-0-4)	5
				(001-2-00)	12	(01-2-00)	9	(2-2-0)	9
						(00-2-00)	3	(0-2-0)	3
		(100-0001)	32	(100-1-10)	32	(10-1-10)	24	(2-1-3)	24
						(00-1-10)	8	(0-1-3)	8
1 subspace		2 subspaces		3 subspaces		6 subspaces		6 subspaces	

In this chain, the third $\mathfrak{su}(2)$ must be broken, in order to eliminate the multiplets of dimension 5 and 15. Now observe that, in a first step,

1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 9 multiplets with $d_3 = 12$,
2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 12 multiplets with $d_3 = 12$,
3. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 8 multiplets with $d_3 = 48$,
4. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 12 multiplets with $d_3 = 48$, among which there are already 8 odd-dimensional multiplets, including 3 triplets and 3 singlets,
5. breaking the third $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 12 multiplets with $d_3 = 51$,
6. breaking the third $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 20 multiplets with $d_3 = 51$, among which there are already 12 odd-dimensional multiplets, including 6 triplets and 5 singlets.

Therefore, the only surviving options for continuing the symmetry breaking process are 3 and 5. Hence in a second step,

3.1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 12 multiplets with $d_3 = 0$,
3.2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 16 multiplets with $d_3 = 0$,
3.3. breaking the second $\mathfrak{o}(2)$ down to $\mathfrak{so}(2)$ generates 10 multiplets with $d_3 = 48$, among which there are already 6 odd-dimensional multiplets, including 3 singlets,
3.4. breaking the third $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 14 multiplets with $d_3 = 48$,
3.5. breaking the third $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 22 multiplets with $d_3 = 48$,

5.1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 18 multiplets with $d_3 = 12$,

5.2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 24 multiplets with $d_3 = 12$,

5.3. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 14 multiplets with $d_3 = 48$: this gives the same distribution of multiplets as option 3.4,

5.4. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 20 multiplets with $d_3 = 48$, among which there are already 6 odd-dimensional multiplets, namely 3 triplets and 3 singlets,

5.5. breaking the third $\mathfrak{o}(2)$ down to $\mathfrak{so}(2)$ generates 20 multiplets with $d_3 = 51$, among which there are already 12 odd-dimensional multiplets, including 6 triplets and 5 singlets.

In the cases of option 3.5 and 5.2, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplets of dimension > 6 and of dimension 5 must not be frozen. In the case of option 3.5, the multiplets of dimension 15 and of dimension 5 will break into 5 triplets and 5 singlets, respectively, so we get at least 6 triplets and 6 singlets. In the case of option 5.2, the multiplet of dimension 9 will break into 3 triplets, so we get at least 3 triplets and 6 odd-dimensional multiplets. We are thus left with a single surviving option for continuing the symmetry breaking process, namely $3.4 = 5.3$, which consists in breaking the second and the third $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$, generating 14 multiplets with $d_3 = 48$. Hence in a third step,

3.4.1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates precisely 21 multiplets with $d_3 = 0$,

3.4.2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 28 multiplets with $d_3 = 0$,

3.4.3. breaking the second $\mathfrak{o}(2)$ down to $\mathfrak{so}(2)$ generates 12 multiplets with $d_3 = 48$, among which there are already 8 odd-dimensional multiplets, including 3 triplets and 3 singlets,

3.4.4. breaking the third $\mathfrak{o}(2)$ down to $\mathfrak{so}(2)$ generates 22 multiplets with $d_3 = 48$.

Note that option 3.4.1 leads to a scheme which does not allow for freezing and is remote from the genetic code, with 2 octets, 7 quartets, 8 doublets and 4 singlets. In the cases of options 3.4.2 and 3.4.4, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplets of dimension > 6 must not be frozen. In the case of option 3.4.2, we are able to produce the correct number of sextets (3), triplets (2) and singlets (2), but there is no possibility of generating the correct number of quartets (5) and doublets (9): we can only get 8 quartets and, correspondingly, 3 doublets. In the case of option 3.4.4, the multiplets of dimension 12 will break into 2 sextets, so we get at least 5 sextets.

3. Chain 2.3.1.3.2:

$$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset A_2 \oplus A_1 \oplus C_2 \supset A_1^{(1)} \oplus A_1 \oplus A_1 \oplus A_1 \supset (A_1)_{13} \oplus A_1 \oplus A_1$$

D_7		$A_3 \oplus D_4$		$A_3 \oplus A_1 \oplus C_2$		$A_2 \oplus A_1 \oplus C_2$		$(A_1)^4$		$(A_1)^3$	
HW	*d*	HW	*d*	HW	*d*	HW	*d*	HW	*d*	HW	*d*
(0000010)	64	(001-0010)	32	(001-0-01)	20	(01-0-01)	15	(1-0-1-1)	8	(2-0-1)	6
										(0-0-1)	2
								(1-0-0-0)	2	(1-0-0)	2
								(0-0-1-1)	4	(1-0-1)	4
								(0-0-0-0)	1	(0-0-0)	1
						(00-0-01)	5	(0-0-1-1)	4	(1-0-1)	4
								(0-0-0-0)	1	(0-0-0)	1
				(001-2-00)	12	(01-2-00)	9	(1-2-0-0)	6	(1-2-0)	6
								(0-2-0-0)	3	(0-2-0)	3
						(00-2-00)	3	(0-2-0-0)	3	(0-2-0)	3
		(100-0001)	32	(100-1-10)	32	(10-1-10)	24	(1-1-1-0)	8	(2-1-0)	6
										(0-1-0)	2
								(1-1-0-1)	8	(1-1-1)	8
								(0-1-1-0)	4	(1-1-0)	4
								(0-1-0-1)	4	(0-1-1)	4
						(00-1-10)	8	(0-1-1-0)	4	(1-1-0)	4
								(0-1-0-1)	4	(0-1-1)	4
1 subspace		2 subspaces		3 subspaces		6 subspaces		15 subspaces		17 subspaces	

In this chain, breaking the third $\mathfrak{su}(2)$ to $\mathfrak{o}(2)$ has no effect and will be disregarded; moreover, at least one of the three $\mathfrak{su}(2)$'s must be broken down to $\mathfrak{so}(2)$ because otherwise the octet would remain unbroken. Now observe that

1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 19 multiplets with $d_3 = 12$ (only two of the three sextets are broken into a quartet and a doublet each),
2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 28 multiplets with $d_3 = 12$,
3. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 20 multiplets with $d_3 = 12$ (only one of the three sextets is broken into a quartet and a doublet and the two triplets are broken into a doublet and a singlet each),
4. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 30 multiplets with $d_3 = 12$,
5. breaking the third $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 24 multiplets with $d_3 = 24$.

In the cases of options 2, 4 and 5, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. A closer analysis reveals that in all three cases, we are able to produce the correct number of sextets (3), triplets (2) and singlets (2), but there is no possibility of generating the correct number of quartets (5) and doublets (9): we can only get 4, 6 or 8 quartets and, correspondingly, 11, 7 or 3 doublets.

4. Chain 2.3.1.4:

$$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset A_2 \oplus A_1 \oplus C_2 \supset A_1^{(2)} \oplus A_1 \oplus A_1 \oplus A_1$$

D_7		$A_3 \oplus D_4$		$A_3 \oplus A_1 \oplus C_2$		$A_2 \oplus A_1 \oplus C_2$		$(A_1)^4$	
HW	d	HW	d	HW	d	HW	d	HW	d
(0000010)	64	(001-0010)	32	(001-0-01)	20	(01-0-01)	15	(2-0-1-1)	12
								(2-0-0-0)	3
						(00-0-01)	5	(0-0-1-1)	4
								(0-0-0-0)	1
				(001-2-00)	12	(01-2-00)	9	(2-2-0-0)	9
						(00-2-00)	3	(0-2-0-0)	3
		(100-0001)	32	(100-1-10)	32	(10-1-10)	24	(2-1-1-0)	12
								(2-1-0-1)	12
						(00-1-10)	8	(0-1-1-0)	4
								(0-1-0-1)	4
1 subspace		2 subspaces		3 subspaces		6 subspaces		10 subspaces	

In this chain, breaking the third or fourth $\mathfrak{su}(2)$ to $\mathfrak{o}(2)$ has no effect and will be disregarded; moreover, at least one of the first two $\mathfrak{su}(2)$'s must be broken because otherwise the nonet would remain unbroken. Note also the symmetry of the distribution of multiplets in the penultimate column under exchange of the third with the fourth $\mathfrak{su}(2)$. Now observe that, in a first step,

1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 15 multiplets with $d_3 = 12$,
2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 20 multiplets with $d_3 = 12$,
3. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 12 multiplets with $d_3 = 48$,
4. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 18 multiplets with $d_3 = 48$, among which there are already 4 triplets and 4 singlets,
5. breaking the third $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 14 multiplets with $d_3 = 51$.

Therefore, the only surviving options for continuing the symmetry breaking process are 3 and 5. Hence in a second step,

3.1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 18 multiplets with $d_3 = 0$,

3.2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 24 multiplets with $d_3 = 0$,

3.3. breaking the second $\mathfrak{o}(2)$ down to $\mathfrak{so}(2)$ generates 18 multiplets with $d_3 = 48$: this gives the same distribution of multiplets as option 4 above,

3.4. breaking the third $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 16 multiplets with $d_3 = 48$,

5.1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates precisely 21 multiplets with $d_3 = 12$,

5.2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 28 multiplets with $d_3 = 12$,

5.3. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 16 multiplets with $d_3 = 48$: this gives the same distribution of multiplets as option 3.4 above,

5.4. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 24 multiplets with $d_3 = 48$,

5.5. breaking the fourth $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 20 multiplets with $d_3 = 51$, among which there are already 6 triplets and 5 singlets.

Note that option 5.1 leads to an interesting scheme which, without any freezing, comes close to the genetic code but is slightly different, with 1 octet, 1 sextet, 6 quartets, 2 triplets, 9 doublets and 2 singlets. In the cases of options 3.2, 5.2 and 5.4, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplets of dimension > 6 must not be frozen. In the case of option 3.2, the two multiplets of dimension 12 will break into 3 quartets each, so we get at least 9 quartets and at most 2 sextets. In the cases of options 5.2 and 5.4, the multiplet of dimension 9 will break into 3 triplets, so we get at least 4 triplets and 6 odd-dimensional multiplets. We are thus left with a single surviving option for continuing the symmetry breaking process, namely 3.4 = 5.3, which consists in breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ and the third $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$, generating 16 multiplets with $d_3 = 48$. Hence in a third step,

3.4.1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 24 multiplets with $d_3 = 0$,

3.4.2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 32 multiplets with $d_3 = 0$,

3.4.3. breaking the second $\mathfrak{o}(2)$ down to $\mathfrak{so}(2)$ generates 24 multiplets with $d_3 = 48$,

3.4.4. breaking the fourth $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 22 multiplets with $d_3 = 48$.

In all cases, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplet of dimension 12 must not be frozen. In the case of option 3.4.1, the multiplet of dimension 12 will break into an octet and a quartet. In the cases of options 3.4.3 and 3.4.4, there remain too many multiplets whose dimension is a multiple of 3 (sextets and triplets), since we still have $d_3 = 48$. Finally, in the case of option 3.4.2, we are able to produce the correct number of sextets (3), triplets (2) and singlets (2), but there is no possibility of generating the correct number of quartets (5) and doublets (9): we can only get 4 quartets and, correspondingly, 11 doublets.

5. Chain 2.3.1.4.2:

$$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset A_2 \oplus A_1 \oplus C_2 \supset A_1^{(2)} \oplus A_1 \oplus A_1 \oplus A_1 \supset (A_1)_{13} \oplus A_1 \oplus A_1$$

D_7		$A_3 \oplus D_4$		$A_3 \oplus A_1 \oplus C_2$		$A_2 \oplus A_1 \oplus C_2$		$(A_1)^4$		$(A_1)^3$	
HW	d	HW	d	HW	d	HW	d	HW	d	HW	d
(0000010)	64	(001-0010)	32	(001-0-01)	20	(01-0-01)	15	(2-0-1-1)	12	(3-0-1)	8
										(1-0-1)	4
								(2-0-0-0)	3	(2-0-0)	3
						(00-0-01)	5	(0-0-1-1)	4	(1-0-1)	4
								(0-0-0-0)	1	(0-0-0)	1
				(001-2-00)	12	(01-2-00)	9	(2-2-0-0)	9	(2-2-0)	9
						(00-2-00)	3	(0-2-0-0)	3	(0-2-0)	3
		(100-0001)	32	(100-1-10)	32	(10-1-10)	24	(2-1-1-0)	12	(3-1-0)	8
										(1-1-0)	4
								(2-1-0-1)	12	(2-1-1)	12
						(00-1-10)	8	(0-1-1-0)	4	(1-1-0)	4
								(0-1-0-1)	4	(0-1-1)	4
1 subspace		2 subspaces		3 subspaces		6 subspaces		10 subspaces		12 subspaces	

In this chain, breaking the third $\mathfrak{su}(2)$ to $\mathfrak{o}(2)$ has no effect and will be disregarded; moreover, at least one of the first two $\mathfrak{su}(2)$'s must be broken because otherwise the nonet would remain unbroken. Now observe that, in a first step,

1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 17 multiplets with $d_3 = 12$,
2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 28 multiplets with $d_3 = 12$,
3. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 14 multiplets with $d_3 = 24$ (only the nonet is broken into a sextet and a triplet and one of the two triplets is broken into a doublet and a singlet),
4. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates precisely 21 multiplets with $d_3 = 24$,
5. breaking the third $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 17 multiplets with $d_3 = 27$.

Note that option 4 leads to an interesting scheme which, without any freezing, comes close to the genetic code but is slightly different, with 1 octet, 2 sextets, 4 quartets, 4 triplets, 6 doublets and 4 singlets. In the case of option 2, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplet of dimension 9 must not be frozen and will therefore provide 3 triplets, so we get at least 4 triplets and 6 odd-dimensional multiplets and at most 2 sextets. Therefore, the only surviving options for continuing the symmetry breaking process are 3 and 5. Hence in a second step, 3.1. breaking the first $su(2)$ down to $o(2)$ generates 20 multiplets with $d_3 = 0$,

3.2. breaking the first $su(2)$ down to $so(2)$ generates 32 multiplets with $d_3 = 0$,

3.3. breaking the second $o(2)$ down to $so(2)$ generates precisely 21 multiplets with $d_3 = 24$: this gives the same distribution of multiplets as option 4 above,

3.4. breaking the third $su(2)$ down to $so(2)$ generates 19 multiplets with $d_3 = 27$,

5.1. breaking the first $su(2)$ down to $o(2)$ generates 24 multiplets with $d_3 = 12$,

5.2. breaking the first $su(2)$ down to $so(2)$ generates 40 multiplets with $d_3 = 12$,

5.3. breaking the second $su(2)$ down to $o(2)$ generates 19 multiplets with $d_3 = 27$: this gives the same distribution of multiplets as option 3.4 above,

5.4. breaking the second $su(2)$ down to $so(2)$ generates 28 multiplets with $d_3 = 24$.

In the cases of options 3.2, 5.1, 5.2 and 5.4, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplets of dimension > 6 must not be frozen. In the case of option 3.2, the multiplet of dimension 12 will break into 3 quartets, so we get at most 1 sextet. In the cases of options 5.2 and 5.4, the multiplet of dimension 9 will break into 3 triplets, so we get at least 4 triplets and 6 odd-dimensional multiplets and at most 2 sextets. In the case of option 5.1, we are able to produce the correct number of sextets (3), triplets (2) and singlets (2), but there is no possibility of generating the correct number of quartets (5) and doublets (9): we can only get 4 or 6 quartets and, correspondingly, 11 or 7 doublets. We are thus left with a single surviving option for continuing the symmetry breaking process, namely 3.4 = 5.3, which consists in breaking the second $su(2)$ down to $o(2)$ and the third $su(2)$ down to $so(2)$, generating 19 multiplets with $d_3 = 27$. From this, the distribution of multiplets found in the genetic code could be obtained by breaking the remaining octet into two quartets and one of the quartets into two doublets, but this is impossible since the available quartets come in identical pairs. In other words, in any possible further breaking, the number of quartets will be even and the number of doublets will be 3 mod 4.

6. Chain 2.3.3.2.2.3 (with one intermediate step omitted):

$$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$$
$$\supset (A_1)_{13} \oplus A_1 \oplus A_1 \oplus A_1 \supset (A_1)_{23} \oplus A_1 \oplus A_1$$

D_7		$A_3 \oplus D_4$		$A_3 \oplus A_1 \oplus C_2$		$(A_1)^5$		$(A_1)^4$		$(A_1)^3$	
HW	d	HW	d	HW	d	HW	d	HW	d	HW	d
(0000010)	64	(001-0010)	32	(001-0-01)	20	(1-1-0-1-1)	16	(1-1-1-1)	16	(2-1-1)	12
										(0-1-1)	4
						(1-1-0-0-0)	4	(1-1-0-0)	4	(1-1-0)	4
				(001-2-00)	12	(1-1-2-0-0)	12	(3-1-0-0)	8	(1-3-0)	8
								(1-1-0-0)	4	(1-1-0)	4
		(100-0001)	32	(100-1-10)	32	(1-1-1-1-0)	16	(2-1-1-0)	12	(2-2-0)	9
										(0-2-0)	3
								(0-1-1-0)	4	(2-0-0)	3
										(0-0-0)	1
						(1-1-1-0-1)	16	(2-1-0-1)	12	(1-2-1)	12
								(0-1-0-1)	4	(1-0-1)	4
1 subspace		2 subspaces		3 subspaces		5 subspaces		8 subspaces		11 subspaces	

In this chain, breaking the third $\mathfrak{su}(2)$ to $\mathfrak{o}(2)$ has no effect and will be disregarded. Now observe that, in a first step,

1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 14 multiplets with $d_3 = 24$,
2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 22 multiplets with $d_3 = 24$,
3. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 15 multiplets with $d_3 = 24$,
4. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 24 multiplets with $d_3 = 24$,
5. breaking the third $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 15 multiplets with $d_3 = 39$.

In the cases of options 2 and 4, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplets of dimension > 6 must not be frozen. In both cases, the multiplet of dimension 9 will break into 3 triplets, so we get at least 3 triplets and 6 odd-dimensional multiplets. Therefore, the only surviving options for continuing the symmetry breaking process are 1, 3 and 5. Hence in a second step,

1.1. breaking the first $\mathfrak{o}(2)$ down to $\mathfrak{so}(2)$ generates 22 multiplets with $d_3 = 24$,

1.2. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 19 multiplets with $d_3 = 0$,

1.3. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 30 multiplets with $d_3 = 0$,

1.4. breaking the third $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 19 multiplets with $d_3 = 24$,

3.1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 19 multiplets with $d_3 = 0$,
3.2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 30 multiplets with $d_3 = 0$,
3.3. breaking the second $\mathfrak{o}(2)$ down to $\mathfrak{so}(2)$ generates 24 multiplets with $d_3 = 24$,
3.4. breaking the third $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 20 multiplets with $d_3 = 24$.

5.1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 19 multiplets with $d_3 = 24$: this gives the same distribution of multiplets as option 1.4 above,
5.2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 30 multiplets with $d_3 = 24$,
5.3. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 20 multiplets with $d_3 = 24$: this gives the same distribution of multiplets as option 3.4 above,
5.4. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 32 multiplets with $d_3 = 24$.

In the cases of options 1.1, 1.3, 3.2, 3.3, 5.2 and 5.4, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplets of dimension > 6 must not be frozen. In the case of option 1.1, we are able to produce the correct number of sextets (3), triplets (2) and singlets (2), but there is no possibility of generating the correct number of quartets (5) and doublets (9): we get at least 6 quartets and at most 7 doublets. In the cases of options 1.3 and 3.2, the multiplet of dimension 12 will break into 3 quartets, so we get at least 6 quartets and at most 1 sextet. In the case of option 3.3, it suffices to freeze the multiplets coming from the nonet, one of the quartets and one of the triplets: **this will reproduce the genetic code**, as shown in Table 6 of the next section. In the cases of options 5.2 and 5.4, the multiplet of dimension 9 will break into 3 triplets, so we get at least 3 triplets and 6 odd-dimensional multiplets. We are thus left with two surviving options for continuing the symmetry breaking process, namely 1.4 = 5.1 and 3.4 = 5.3. From either of these, the distribution of multiplets found in the genetic code could be obtained by breaking one of the quartets into two doublets (and, in the first case, the remaining octet into two quartets), but this is impossible since the available quartets come in identical pairs (and since, in the first case, the remaining octet can only be broken into two quartets or four doublets, but not into one quartet and two doublets). In other words, in any possible further breaking, the number of quartets will be even and the number of doublets will be 3 mod 4.

7. Chain 2.3.3.2.3.5 (with one intermediate step omitted):

$$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$$
$$\supset (A_1)_{14} \oplus A_1 \oplus A_1 \oplus A_1 \supset (A_1)_{24} \oplus A_1 \oplus A_1$$

D_7		$A_3 \oplus D_4$		$A_3 \oplus A_1 \oplus C_2$		$(A_1)^5$		$(A_1)^4$		$(A_1)^3$	
HW	d	HW	d	HW	d	HW	d	HW	d	HW	d
(0000010)	64	(001-0010)	32	(001-0-01)	20	(1-1-0-1-1)	16	(2-1-0-1)	12	(2-2-0)	9
										(0-2-0)	3
								(0-1-0-1)	4	(2-0-0)	3
										(0-0-0)	1
						(1-1-0-0-0)	4	(1-1-0-0)	4	(1-1-0)	4
				(001-2-00)	12	(1-1-2-0-0)	12	(1-1-2-0)	12	(1-1-2)	12
		(100-0001)	32	(100-1-10)	32	(1-1-1-1-0)	16	(2-1-1-0)	12	(1-2-1)	12
								(0-1-1-0)	4	(1-0-1)	4
						(1-1-1-0-1)	16	(1-1-1-1)	16	(2-1-1)	12
										(0-1-1)	4
1 subspace		2 subspaces		3 subspaces		5 subspaces		7 subspaces		10 subspaces	

In this chain, at least one of the first two $\mathfrak{su}(2)$'s must be broken because otherwise the nonet would remain unbroken. Note also the symmetry of the distribution of multiplets in the penultimate column under exchange of the first with the second $\mathfrak{su}(2)$. Now observe that, in a first step,

1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 13 multiplets with $d_3 = 36$,
2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 20 multiplets with $d_3 = 36$, among which there are already 4 triplets and 4 singlets.
3. breaking the third $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 11 multiplets with $d_3 = 39$ (only one of the three multiplets of dimension 12 is broken into an octet and a quartet),
4. breaking the third $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 16 multiplets with $d_3 = 39$.

Therefore, the only surviving options for continuing the symmetry breaking process are 1, 3 and 4. Hence in a second step,

1.1. breaking the first $\mathfrak{o}(2)$ down to $\mathfrak{so}(2)$ generates 20 multiplets with $d_3 = 36$: this gives the same distribution of multiplets as option 2 above,

1.2. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 17 multiplets with $d_3 = 12$,

1.3. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 26 multiplets with $d_3 = 12$,

1.4. breaking the third $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 14 multiplets with $d_3 = 24$,

1.5. breaking the third $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 20 multiplets with $d_3 = 24$,

3.1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 14 multiplets with $d_3 = 24$: this gives the same distribution of multiplets as option 1.4 above,

3.2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 22 multiplets with $d_3 = 24$,

3.3. breaking the third $\mathfrak{o}(2)$ down to $\mathfrak{so}(2)$ generates 16 multiplets with $d_3 = 39$: this gives the same distribution of multiplets as option 4 above,

4.1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 20 multiplets with $d_3 = 24$: this gives the same distribution of multiplets as option 1.5 above,

4.2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 32 multiplets with $d_3 = 24$.

In the cases of options 1.3, 3.2 and 4.2, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplets of dimension > 6 must not be frozen. In the case of option 1.3, we are able to produce the correct number of sextets (3), triplets (2) and singlets (2), but there is no possibility of generating the correct number of quartets (5) and doublets (9): we can only get 6 or 7 or 8 or 9 quartets and, correspondingly, 7 or 5 or 3 or 1 doublets. In the cases of options 3.2 and 4.2, the multiplet of dimension 9 will break into 3 triplets, so we get at least 4 triplets and 6 odd-dimensional multiplets. We are thus left with two surviving options for continuing the symmetry breaking process, namely 1.4 and 1.5, which consist in breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ and the third $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ and $\mathfrak{so}(2)$, repsectively, generating 14 and 20 multiplets, respectively, with $d_3 = 48$. Hence in a third step,

1.4.1. breaking the first $\mathfrak{o}(2)$ down to $\mathfrak{so}(2)$ generates 22 multiplets with $d_3 = 24$: this gives the same distribution of multiplets as option 3.2 above,

1.4.2. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 18 multiplets with $d_3 = 0$,

1.4.3. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 28 multiplets with $d_3 = 0$,

1.4.4. breaking the third $\mathfrak{o}(2)$ down to $\mathfrak{so}(2)$ generates 20 multiplets with $d_3 = 24$: this gives the same distribution of multiplets as option 1.5 above,

1.5.1. breaking the first $\mathfrak{o}(2)$ down to $\mathfrak{so}(2)$ generates 32 multiplets with $d_3 = 24$: this gives the same distribution of multiplets as option 4.2 above,

1.5.2. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 25 multiplets with $d_3 = 0$,

1.5.3. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 40 multiplets with $d_3 = 0$.

In the cases of options 1.4.1, 1.4.3, 1.5.1 and 1.5.3, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplets of dimension > 6 must not be frozen. In the case of option 1.4.1 (which may differ from option 3.2 when freezing is taken into account), we are able to produce the correct number of sextets (3), triplets (2) and singlets (2), but there is no possibility of generating the correct number of quartets (5) and doublets (9): since the two multiplets of dimension 8 will break into 2 quartets each and since there are two other quartets that remain unbroken during the last step, we get at least 6 quartets and at most 7 doublets. In the case of option 1.4.3, the multiplet of dimension 12 will break into 3 quartets, so we get at most 1 sextet. In the cases of options 1.5.1 and 1.5.3, it suffices to break one quartet into two doublets and freeze all other multiplets: **this will reproduce the genetic code**, as shown in Table 7 and Table 8 of the next section.

8. Chain 2.4.3.2.5:

$$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$$
$$\supset (A_1)_{13} \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset (A_1)_{24} \oplus A_1 \oplus A_1 \oplus A_1$$

D_7		$A_3 \oplus D_4$		$A_3 \oplus (A_1)^4$		$(A_1)^6$		$(A_1)^5$		$(A_1)^4$	
HW	d	HW	d	HW	d	HW	d	HW	d	HW	d
(0000010)	64	(001-0010)	32	(001-0-1-1-0)	16	(1-1-0-1-1-0)	16	(1-1-1-1-0)	16	(2-1-1-0)	12
										(0-1-1-0)	4
				(001-1-0-0-1)	16	(1-1-1-0-0-1)	16	(2-1-0-0-1)	12	(1-2-0-1)	12
								(0-1-0-0-1)	4	(1-0-0-1)	4
		(100-0001)	32	(100-0-1-0-1)	16	(1-1-0-1-0-1)	16	(1-1-1-0-1)	16	(1-1-1-1)	16
				(100-1-0-1-0)	16	(1-1-1-0-1-0)	16	(2-1-0-1-0)	12	(2-2-0-0)	9
										(0-2-0-0)	3
								(0-1-0-1-0)	4	(2-0-0-0)	3
										(0-0-0-0)	1
1 subspace		2 subspaces		4 subspaces		4 subspaces		6 subspaces		9 subspaces	

In this chain, breaking the third and the fourth $su(2)$ to $o(2)$ has no effect and will be disregarded; moreover, at least two of the four $su(2)$'s must be broken down to $so(2)$ because otherwise there would be octets left over from the multiplet of dimension 16. Note also the symmetry of the distribution of multiplets in the penultimate column under simultaneous exchange of the first with the second and the third with the fourth $su(2)$. Now observe that, in a first step,

1. breaking the first or second $su(2)$ down to $o(2)$ generates 12 multiplets with $d_3 = 24$,
2. breaking the first or second $su(2)$ down to $so(2)$ generates 18 multiplets with $d_3 = 24$, among which there are already 4 triplets and 4 singlets,
3. breaking the third or fourth $su(2)$ down to $so(2)$ generates 12 multiplets with $d_3 = 39$.

Therefore, the only surviving options for continuing the symmetry breaking process are 1 and 3; for the sake of definiteness, we choose to break the second $su(2)$ in the first case and the fourth $su(2)$ in the second case. Hence in a second step,

1.1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 16 multiplets with $d_3 = 0$,
1.2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 24 multiplets with $d_3 = 0$,
1.3. breaking the second $\mathfrak{o}(2)$ down to $\mathfrak{so}(2)$ generates 18 multiplets with $d_3 = 24$: this gives the same distribution of multiplets as option 2 above,
1.4. breaking the third $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 15 multiplets with $d_3 = 24$,
1.5. breaking the fourth $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 16 multiplets with $d_3 = 24$,

3.1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 15 multiplets with $d_3 = 24$: up to the aforementioned symmetry operation of simultaneously exchanging the first with the second and the third with the fourth $\mathfrak{su}(2)$, this gives the same distribution of multiplets as option 1.4 above,
3.2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 24 multiplets with $d_3 = 24$,
3.3. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 16 multiplets with $d_3 = 24$: this gives the same distribution of multiplets as option 1.5 above,
3.4. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 24 multiplets with $d_3 = 24$,
3.5. breaking the third $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 16 multiplets with $d_3 = 39$.

In the cases of options 1.2, 3.2 and 3.4, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplets of dimension > 6 must not be frozen. In the case of option 1.2, the multiplet of dimension 16 will break into 2 octets. In the cases of options 3.2 and 3.4, the multiplet of dimension 9 will break into 3 triplets, so we get at least 4 triplets and 6 odd-dimensional multiplets. We are thus left with three surviving options for continuing the symmetry breaking process: 1.4, 1.5 and 3.5. Hence in a third step,

1.4.1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 20 multiplets with $d_3 = 0$,
1.4.2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 30 multiplets with $d_3 = 0$,
1.4.3. breaking the second $\mathfrak{o}(2)$ down to $\mathfrak{so}(2)$ generates 24 multiplets with $d_3 = 24$,
1.4.4. breaking the fourth $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 20 multiplets with $d_3 = 24$,

1.5.1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 20 multiplets with $d_3 = 0$,
1.5.2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 32 multiplets with $d_3 = 0$,
1.5.3. breaking the second $\mathfrak{o}(2)$ down to $\mathfrak{so}(2)$ generates 24 multiplets with $d_3 = 24$,
1.5.4. breaking the third $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 20 multiplets with $d_3 = 24$: this gives the same distribution of multiplets as option 1.4.4 above,

3.5.1. breaking the first or second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 20 multiplets with $d_3 = 24$: this gives the same distribution of multiplets as option 1.4.4 above,
3.5.2. breaking the first or second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 32 multiplets with $d_3 = 24$.

In the cases of options 1.4.2, 1.4.3, 1.5.2, 1.5.3 and 3.5.2, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplets of dimension > 6 must not be frozen. In the cases of options 1.4.2 and 1.4.3, the 3 multiplets of dimension 8 will break into 2 quartets each, so we get at least 6 quartets. In the case of option 1.5.2, the 2 multiplets of dimension 8 will break into 2 quartets each and the multiplet of dimension 12 will break into 3 quartets, so we get at least 7 quartets and at most 1 sextet. In the case of option 1.5.3, it suffices to freeze the multiplets coming from the nonet, one of the two quartets and one of the two triplets: **this will reproduce the genetic code**, as shown in Table 9 of the next section. In the case of option 3.5.2, the multiplet of dimension 9 will break into 3 triplets, so we get at least 4 triplets and 6 odd-dimensional multiplets. Finally, we are still left with a single surviving option for continuing the symmetry breaking process, namely 1.4.4 = 3.3.4 = 3.5.1, which consists in breaking the first or second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ and the third and fourth $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$, generating 20 multiplets with $d_3 = 24$. From this, the distribution of multiplets found in the genetic code could be obtained by breaking exactly one of the quartets into two doublets, but this is impossible since the freezing mechanism only allows us to get 0, 2, 4 or 6 quartets and, correspondingly, 19, 15, 11 or 7 doublets.

9. Chain 3.5.2.2.1:

$D_7 \supset A_1 \oplus B_5 \supset A_1 \oplus A_1 \oplus A_1 \oplus B_3 \supset A_1 \oplus A_1 \oplus A_1 \oplus G_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$
$\supset (A_1)_{12} \oplus A_1 \oplus A_1$

D_7		$A_1 \oplus B_5$		$(A_1)^3 \oplus B_3$		$(A_1)^3 \oplus G_2$		$(A_1)^4$		$(A_1)^3$	
HW	d	HW	d	HW	d	HW	d	HW	d	HW	d
(0000010)	64	(1-00001)	64	(1-1-0-001)	32	(1-1-0-01)	28	(1-1-0-6)	28	(2-0-6)	21
										(0-0-6)	7
						(1-1-0-00)	4	(1-1-0-0)	4	(2-0-0)	3
										(0-0-0)	1
				(1-0-1-001)	32	(1-0-1-01)	28	(1-0-1-6)	28	(1-1-6)	28
						(1-0-1-00)	4	(1-0-1-0)	4	(1-1-0)	4
1 subspace		1 subspace		2 subspaces		4 subspaces		4 subspaces		6 subspaces	

In this chain, the third $\mathfrak{su}(2)$ must be broken, in order to eliminate the multiplets of dimension 7, 21 and 28. Moreover, breaking the second $\mathfrak{su}(2)$ to $\mathfrak{o}(2)$ has no effect and will be disregarded. Now observe that, in a first step,

1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 8 multiplets with $d_3 = 0$,
2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 10 multiplets with $d_3 = 0$,
3. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 8 multiplets with $d_3 = 24$,

4. breaking the third $su(2)$ down to $o(2)$ generates 15 multiplets with $d_3 = 24$,
5. breaking the third $su(2)$ down to $so(2)$ generates 24 multiplets with $d_3 = 24$.

In the case of option 5, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplets of dimension > 6 must not be frozen, so that we get at least 8 triplets and 8 singlets. Therefore, the only surviving options for continuing the symmetry breaking process are 3 and 4. Hence in a second step,

3.1. breaking the first $su(2)$ down to $o(2)$ generates 10 multiplets with $d_3 = 0$,
3.2. breaking the first $su(2)$ down to $so(2)$ generates 16 multiplets with $d_3 = 0$,
3.3. breaking the third $su(2)$ down to $o(2)$ generates 20 multiplets with $d_3 = 24$,
3.4. breaking the third $su(2)$ down to $so(2)$ generates 32 multiplets with $d_3 = 24$,

4.1. breaking the first $su(2)$ down to $o(2)$ generates 20 multiplets with $d_3 = 0$,
4.2. breaking the first $su(2)$ down to $so(2)$ generates 32 multiplets with $d_3 = 0$,
4.3. breaking the second $su(2)$ down to $so(2)$ generates 20 multiplets with $d_3 = 24$: this gives the same distribution of multiplets as option 3.3 above,
4.4. breaking the third $o(2)$ down to $so(2)$ generates 24 multiplets with $d_3 = 24$.

In the cases of options 3.4, 4.2 and 4.4, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplets of dimension > 6 must not be frozen. In the case of option 3.4, we again get at least 8 triplets and 8 singlets. In the cases of options 4.2 and 4.4, the 3 multiplets of dimension 8 coming from the multiplet of dimension 28 will break into 2 quartets each, so we get at least 6 quartets. We are thus left with a single surviving option for continuing the symmetry breaking process, namely 3.3 = 4.3, which consists in breaking the second $su(2)$ down to $so(2)$ and the third $su(2)$ down to $o(2)$, generating 20 multiplets with $d_3 = 24$. From this, the distribution of multiplets found in the genetic code could be obtained by breaking exactly one of the quartets into two doublets, but this is impossible since the freezing mechanism only allows us to get 0 or 6 quartets and, correspondingly, 19 or 7 doublets.

10. Chain 3.5.2.2.4.3:

$$D_7 \supset A_1 \oplus B_5 \supset A_1 \oplus A_1 \oplus A_1 \oplus B_3 \supset A_1 \oplus A_1 \oplus A_1 \oplus G_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1$$
$$\supset (A_1)_{24} \oplus A_1 \oplus A_1 \supset (A_1)_{23} \oplus A_1$$

D_7		$A_1 \oplus B_5$		$(A_1)^3 \oplus B_3$		$(A_1)^3 \oplus G_2$		$(A_1)^4$		$(A_1)^3$		$(A_1)^2$	
HW	d	HW	d	HW	d	HW	d	HW	d	HW	d	HW	d
(0000010)	64	(1-00001)	64	(1-1-0-001)	32	(1-1-0-01)	28	(1-1-0-6)	28	(7-1-0)	16	(1-7)	16
										(5-1-0)	12	(1-5)	12
						(1-1-0-00)	4	(1-1-0-0)	4	(1-1-0)	4	(1-1)	4
				(1-0-1-001)	32	(1-0-1-01)	28	(1-0-1-6)	28	(6-1-1)	28	(2-6)	21
												(0-6)	7
						(1-0-1-00)	4	(1-0-1-0)	4	(0-1-1)	4	(2-0)	3
												(0-0)	1
1 subspace		1 subspace		2 subspaces		4 subspaces		4 subspaces		5 subspaces		7 subspaces	

In this chain, the second $\mathfrak{su}(2)$ must be broken, in order to eliminate the multiplets of dimension 7 and 21. Now observe that, in a first step,

1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 9 multiplets with $d_3 = 0$,
2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 14 multiplets with $d_3 = 0$,
3. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 18 multiplets with $d_3 = 24$,
4. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 32 multiplets with $d_3 = 24$.

In the case of option 4, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplets of dimension > 6 must not be frozen, so we get at least 7 triplets and 7 singlets. Therefore, the only surviving option for continuing the symmetry breaking process is 3. Hence in a second step,

3.1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 23 multiplets with $d_3 = 0$,
3.2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 36 multiplets with $d_3 = 0$,
3.3. breaking the second $\mathfrak{o}(2)$ down to $\mathfrak{so}(2)$ generates 32 multiplets with $d_3 = 24$.

In all cases, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. In the case of option 3.1, we are able to produce the correct number of sextets (3), triplets (2) and singlets (2), but there is no possibility of generating the correct number of quartets (5) and doublets (9): we can only get 8 quartets and, correspondingly, 3 doublets. In the cases of options 3.2 and 3.3, it suffices to freeze the multiplets coming from the multiplets of dimension 21, 16, 7, 4 and 3: **this will reproduce the genetic code**, as shown in Table 10 and Table 11 of the next section.

11. Chain 6.1.4.2.2:

$$D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus A_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1^{(2)}$$
$$\supset (A_1)_{13} \oplus A_1 \oplus A_1$$

D_7		$(A_1)^2 \oplus D_5$		$(A_1)^2 \oplus A_4$		$(A_1)^3 \oplus A_2$		$(A_1)^4$		$(A_1)^3$	
HW	d	HW	d	HW	d	HW	d	HW	d	HW	d
(0000010)	64	(0-1-00010)	32	(0-1-0010)	20	(0-1-1-01)	12	(0-1-1-2)	12	(1-1-2)	12
						(0-1-0-10)	6	(0-1-0-2)	6	(0-1-2)	6
						(0-1-0-00)	2	(0-1-0-0)	2	(0-1-0)	2
				(0-1-1000)	10	(0-1-0-10)	6	(0-1-0-2)	6	(0-1-2)	6
						(0-1-1-00)	4	(0-1-1-0)	4	(1-1-0)	4
				(0-1-0000)	2	(0-1-0-00)	2	(0-1-0-0)	2	(0-1-0)	2
		(1-0-00001)	32	(1-0-0100)	20	(1-0-1-10)	12	(1-0-1-2)	12	(2-0-2)	9
										(0-0-2)	3
						(1-0-0-01)	6	(1-0-0-2)	6	(1-0-2)	6
						(1-0-0-00)	2	(1-0-0-0)	2	(1-0-0)	2
				(1-0-0001)	10	(1-0-0-01)	6	(1-0-0-2)	6	(1-0-2)	6
						(1-0-1-00)	4	(1-0-1-0)	4	(2-0-0)	3
										(0-0-0)	1
				(1-0-0000)	2	(1-0-0-00)	2	(1-0-0-0)	2	(1-0-0)	2
1 subspace		2 subspaces		6 subspaces		12 subspaces		12 subspaces		14 subspaces	

In this chain, breaking the second $\mathfrak{su}(2)$ to $\mathfrak{o}(2)$ has no effect and will be disregarded. Now observe that, in a first step,

1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 16 multiplets with $d_3 = 48$,
2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 24 multiplets with $d_3 = 48$,
3. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 20 multiplets with $d_3 = 51$, among which there are already 4 triplets and 4 singlets,
4. breaking the third $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates precisely 21 multiplets with $d_3 = 12$,
5. breaking the third $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 28 multiplets with $d_3 = 12$.

Note that option 4 leads to an interesting scheme which, without any freezing, comes close to the genetic code but is slightly different, with 1 octet, 1 sextet, 6 quartets, 2 triplets, 9 doublets and 2 singlets. In the cases of options 2 and 5, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplets of dimension > 6 must not be frozen. In both cases, the multiplet of dimension 9 will break into 3 triplets, so we get at least 4 triplets and 6 odd-dimensional multiplets. Therefore, the only surviving option for continuing the symmetry breaking process is 1. Hence in a second step, 1.1. breaking the first $\mathfrak{o}(2)$ down to $\mathfrak{so}(2)$ generates 24 multiplets with $d_3 = 48$,

1.2. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 22 multiplets with $d_3 = 48$,

1.3. breaking the third $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 24 multiplets with $d_3 = 0$,

1.4. breaking the third $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 32 multiplets with $d_3 = 0$.

In all cases, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplet of dimension 12 must not be frozen. In the cases of options 1.1 and 1.2, we get at least 4 sextets. In the case of option 1.3, the multiplet of dimension 12 will break into an octet and a quartet. In the case of option 1.4, we get at most 4 quartets.

12. Chain 6.1.4.2.2.3:

$$D_7 \supset A_1 \oplus A_1 \oplus D_5 \supset A_1 \oplus A_1 \oplus A_4 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1^{(2)}$$
$$\supset (A_1)_{13} \oplus A_1 \oplus A_1 \supset (A_1)_{23} \oplus A_1$$

D_7		$(A_1)^2 \oplus D_5$		$(A_1)^2 \oplus A_4$		$(A_1)^3 \oplus A_2$		$(A_1)^4$		$(A_1)^3$		$(A_1)^2$	
HW	d	HW	d	HW	d	HW	d	HW	d	HW	d	HW	d
(0000010)	64	(0-1-00010)	32	(0-1-0010)	20	(0-1-1-01)	12	(0-1-1-2)	12	(1-1-2)	12	(3-1)	8
												(1-1)	4
						(0-1-0-10)	6	(0-1-0-2)	6	(0-1-2)	6	(3-0)	4
												(1-0)	2
						(0-1-0-00)	2	(0-1-0-0)	2	(0-1-0)	2	(1-0)	2
				(0-1-1000)	10	(0-1-0-10)	6	(0-1-0-2)	6	(0-1-2)	6	(3-0)	4
												(1-0)	2
						(0-1-1-00)	4	(0-1-1-0)	4	(1-1-0)	4	(1-1)	4
				(0-1-0000)	2	(0-1-0-00)	2	(0-1-0-0)	2	(0-1-0)	2	(1-0)	2
		(1-0-00001)	32	(1-0-0100)	20	(1-0-1-10)	12	(1-0-1-2)	12	(2-0-2)	9	(2-2)	9
										(0-0-2)	3	(2-0)	3
						(1-0-0-01)	6	(1-0-0-2)	6	(1-0-2)	6	(2-1)	6
						(1-0-0-00)	2	(1-0-0-0)	2	(1-0-0)	2	(0-1)	2
				(1-0-0001)	10	(1-0-0-01)	6	(1-0-0-2)	6	(1-0-2)	6	(2-1)	6
						(1-0-1-00)	4	(1-0-1-0)	4	(2-0-0)	3	(0-2)	3
										(0-0-0)	1	(0-0)	1
				(1-0-0000)	2	(1-0-0-00)	2	(1-0-0-0)	2	(1-0-0)	2	(0-1)	2
1 subspace		2 subspaces		6 subspaces		12 subspaces		12 subspaces		14 subspaces		17 subspaces	

In this chain, observe that, in a first step,

1. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 24 multiplets with $d_3 = 12$,
2. breaking the first $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 40 multiplets with $d_3 = 12$,
3. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{o}(2)$ generates 19 multiplets with $d_3 = 24$,
4. breaking the second $\mathfrak{su}(2)$ down to $\mathfrak{so}(2)$ generates 28 multiplets with $d_3 = 24$.

In the cases of options 1, 2 and 4, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplets of dimension > 6 must not be frozen. In the case of option 1, we are able to produce the correct number of sextets (3), triplets (2) and singlets (2), but there is no possibility of generating the correct number of quartets (5) and doublets (9): we can only get 6 or 4 quartets and, correspondingly, 7 or 11 doublets. In the cases of options 2 and 4, the multiplet of dimension 9 will break into 3 triplets, so we get at least 4 triplets and 6 odd-dimensional multiplets. Therefore, the only surviving option for continuing the symmetry breaking process is 3, which already presents the correct number of sextets (3), triplets (2) and singlets (2). However, since the remaining octet can only be broken into 2 quartets or 4 doublets and since the remaining 4 quartets come in two identical pairs, there is no possibility of generating the correct number of quartets (5) and doublets (9): in any possible further breaking, the number of quartets will be even and the number of doublets will be 3 mod 4.

6 Results

Having concluded the long and tedious case by case analysis reported in the previous three sections, we proceed to summarize the results.

- **There is no symmetry breaking scheme starting out from the simple Lie algebra $B_6 = \mathfrak{so}(13)$ capable of reproducing the degeneracies observed in the genetic code.**
- **There are altogether 6 symmetry breaking schemes starting out from the simple Lie algebra $D_7 = \mathfrak{so}(14)$ that do reproduce the degeneracies observed in the genetic code.**

In what follows, we shall summarize the symmetry breaking schemes based on simple Lie algebras of low or medium rank (i.e., except $\mathfrak{su}(64)$, $\mathfrak{so}(64)$ and $\mathfrak{sp}(64)$) that reproduce the distribution of multiplets observed in the standard genetic code. We begin with the single scheme based on $\mathfrak{sp}(6)$ and the two schemes based on $\mathfrak{g}_2$, which are included not only for the sake of completeness but also because, in the process of typesetting, the corresponding tables in Ref. [9] have been distorted in such a way that it has become impossible to infer which of the multiplets are subject to freezing in the last step.

Some comments about the notation used in the following tables seem in order. First, for the sake of brevity, we shall continue to characterize simple Lie algebras by their Dynkin labels, even though this may not always be the most intuitive manner to understand the embeddings as maximal subalgebras that are involved. For example, the chains

$$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset (A_1 \oplus A_1) \oplus A_1 \oplus (A_1 \oplus A_1) \tag{1}$$

and

$$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus (A_1 \oplus A_1) \oplus (A_1 \oplus A_1) \supset (A_1 \oplus A_1) \oplus (A_1 \oplus A_1) \oplus (A_1 \oplus A_1) \tag{2}$$

and

$$D_7 \supset A_1 \oplus B_5 \supset A_1 \oplus (A_1 \oplus A_1 \oplus B_3) \tag{3}$$

that appear in the following tables are readily understood by using the isomorphisms $A_1 \cong \mathfrak{so}(3)$, $A_1 \oplus A_1 \cong \mathfrak{so}(4)$, $C_2 \cong \mathfrak{so}(5)$ and $A_3 \cong \mathfrak{so}(6)$ to rewrite them as

$$\begin{aligned} \mathfrak{so}(14) &\supset \mathfrak{so}(6) \oplus \mathfrak{so}(8) \supset \mathfrak{so}(6) \oplus (\mathfrak{so}(3) \oplus \mathfrak{so}(5)) \\ &\supset (\mathfrak{so}(3) \oplus \mathfrak{so}(3)) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(4) \end{aligned} \tag{4}$$

and

$$\begin{aligned} \mathfrak{so}(14) &\supset \mathfrak{so}(6) \oplus \mathfrak{so}(8) \supset \mathfrak{so}(6) \oplus (\mathfrak{so}(4) \oplus \mathfrak{so}(4)) \\ &\supset (\mathfrak{so}(3) \oplus \mathfrak{so}(3)) \oplus \mathfrak{so}(4) \oplus \mathfrak{so}(4) \end{aligned} \tag{5}$$

and

$$\mathfrak{so}(14) \supset \mathfrak{so}(3) \oplus \mathfrak{so}(11) \supset \mathfrak{so}(3) \oplus (\mathfrak{so}(4) \oplus \mathfrak{so}(7)) \tag{6}$$

respectively. We also continue to abbreviate the direct sum of p copies of $\mathfrak{su}(2)$ by $(A_1)^p$ and the expression "highest weight" by "HW": in the case of $A_1 = \mathfrak{su}(2)$, this is $2s$ where s is the "spin" and in the case of $\mathfrak{o}(2)$ or $\mathfrak{so}(2)$, it is $2m$ where m is the "magnetic quantum number". In the third phase, breaking of the k^{th} $\mathfrak{su}(2)$ to $\mathfrak{o}(2)$ and to $\mathfrak{so}(2)$ will be indicated by the symbol $L^2_{k,z}$ and $L_{k,z}$, respectively, representing the operator that, as explained in Ref. [9], implements this breaking. Finally, freezing of multiplets will be indicated by shading.

1. The $\mathfrak{sp}(6)$-chain

$$\begin{aligned} \mathfrak{sp}(6) &\supset \mathfrak{sp}(4) \oplus \mathfrak{su}(2) \supset \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \\ &\supset \mathfrak{su}(2) \oplus \mathfrak{o}(2) \oplus \mathfrak{su}(2) \supset \mathfrak{su}(2) \oplus \mathfrak{o}(2) \oplus \mathfrak{so}(2) \end{aligned} \tag{7}$$

C_3		$C_2 \oplus A_1$		$(A_1)^3$		$L^2_{2,z}$		$(L^2_{2,z}, L_{3,z})$	
HW	d	HW	d	HW	d	HW	d	HW	d
(001)	64	(20,1)	20	(1,1,1)	8	(1,±1,1)	8	(1,±1,+1)	4
								(1,±1,−1)	4
				(2,0,1)	6	(2,0,1)	6	(2,0,+1)	3
								(2,0,−1)	3
				(0,2,1)	6	(0,±2,1)	4	(0,±2,+1)	2
								(0,±2,−1)	2
						(0,0,1)	2	(0,0,+1)	1
								(0,0,−1)	1
		(11,0)	16	(2,1,0)	6	(2,±1,0)	6	(2,±1,0)	6
				(1,2,0)	6	(1,±2,0)	4	(1,±2,0)	4
						(1,0,0)	2	(1,0,0)	2
				(1,0,0)	2	(1,0,0)	2	(1,0,0)	2
				(0,1,0)	2	(0,±1,0)	2	(0,±1,0)	2
		(10,2)	12	(1,0,2)	6	(1,0,2)	6	(1,0,+2)	2
								(1,0,−2)	2
								(1,0,0)	2
				(0,1,2)	6	(0,±1,2)	6	(0,±1,+2)	2
								(0,±1,−2)	2
								(0,±1,0)	2
		(01,1)	10	(1,1,1)	8	(1,±1,1)	8	(1,±1,+1)	4
								(1,±1,−1)	4
				(0,0,1)	2	(0,0,1)	2	(0,0,+1)	1
								(0,0,−1)	1
		(10,0)	4	(1,0,0)	2	(1,0,0)	2	(1,0,0)	2
				(0,1,0)	2	(0,±1,0)	2	(0,±1,0)	2
		(00,1)	2	(0,0,1)	2	(0,0,1)	2	(0,0,+1)	1
								(0,0,−1)	1
1 subspace		6 subspaces		14 subspaces		16 subspaces		27 subspaces	

Table 6.1: Branching of the codon representation of $\mathfrak{sp}(6)$ in the chain (7).

2. The first $\mathfrak{g}_2$-chain

$$\mathfrak{g}_2 \supset \mathfrak{su}(2) \oplus \mathfrak{su}(2) \supset \mathfrak{su}(2) \oplus \mathfrak{o}(2) \supset \mathfrak{so}(2) \oplus \mathfrak{o}(2) \tag{8}$$

G_2		$(A_1)^2$		$L^2_{2,z}$		$(L^2_{2,z}, L_{1,z})$	
HW	d	HW	d	HW	d	HW	d
(11)	64	(2,4)	15	(2,±4)	6	(+2,±4)	2
						(−2,±4)	2
						(0,±4)	2
				(2,±2)	6	(+2,±2)	2
						(−2,±2)	2
						(0,±2)	2
				(2,0)	3	(+2,0)	1
						(−2,0)	1
						(0,0)	1
		(1,5)	12	(1,±5)	4	(+1,±5)	2
						(−1,±5)	2
				(1,±3)	4	(+1,±3)	2
						(−1,±3)	2
				(1,±1)	4	(+1,±1)	2
						(−1,±1)	2
		(2,2)	9	(2,±2)	6	(+2,±2)	2
						(−2,±2)	2
						(0,±2)	2
				(2,0)	3	(+2,0)	1
						(−2,0)	1
						(0,0)	1
		(3,1)	8	(3,±1)	8	(+3,±1)	2
						(−3,±1)	2
						(+1,±1)	2
						(−1,±1)	2
		(1,3)	8	(1,±3)	4	(+1,±3)	2
						(−1,±3)	2
				(1,±1)	4	(+1,±1)	2
						(−1,±1)	2
		(0,4)	5	(0,±4)	2	(0,±4)	2
				(0,±2)	2	(0,±2)	2
				(0,0)	1	(0,0)	1
		(1,1)	4	(1,±1)	4	(+1,±1)	2
						(−1,±1)	2
		(0,2)	3	(0,±2)	2	(0,±2)	2
				(0,0)	1	(0,0)	1
1 subspace		8 subspaces		17 subspaces		36 subspaces	

Table 6.2: Branching of the codon representation of $\mathfrak{g}_2$ in the chain (8).

3. The second $\mathfrak{g}_2$-chain

$$\mathfrak{g}_2 \supset su(2) \oplus su(2) \supset su(2) \oplus o(2) \supset su(2) \oplus so(2) \tag{9}$$

G_2		$(A_1)^2$		$L^2_{2,z}$		$L_{2,z}$	
HW	d	HW	d	HW	d	HW	d
(11)	64	(2, 4)	15	(2, ±4)	6	(2, +4)	3
						(2, −4)	3
				(2, ±2)	6	(2, +2)	3
						(2, −2)	3
				(2, 0)	3	(2, 0)	3
		(1, 5)	12	(1, ±5)	4	(1, +5)	2
						(1, −5)	2
				(1, ±3)	4	(1, +3)	2
						(1, −3)	2
				(1, ±1)	4	(1, +1)	2
						(1, −1)	2
		(2, 2)	9	(2, ±2)	6	(2, +2)	3
						(2, −2)	3
				(2, 0)	3	(2, 0)	3
		(3, 1)	8	(3, ±1)	8	(3, +1)	4
						(3, −1)	4
		(1, 3)	8	(1, ±3)	4	(1, +3)	2
						(1, −3)	2
				(1, ±1)	4	(1, +1)	2
						(1, −1)	2
		(0, 4)	5	(0, ±4)	2	(0, +4)	1
						(0, −4)	1
				(0, ±2)	2	(0, +2)	1
						(0, −2)	1
				(0, 0)	1	(0, 0)	1
		(1, 1)	4	(1, ±1)	4	(1, +1)	2
						(1, −1)	2
		(0, 2)	3	(0, ±2)	2	(0, +2)	1
						(0, −3)	1
				(0, 0)	1	(0, 0)	1
1 subspace		8 subspaces		17 subspaces		30 subspaces	

Table 6.3: Branching of the codon representation of $\mathfrak{g}_2$ in the chain (9).

4. The $so(14)$-chain

$$
\begin{aligned}
D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \\
\supset (A_1)_{13} \oplus A_1 \oplus A_1 \oplus A_1 \supset (A_1)_{23} \oplus A_1 \oplus A_1 \supset \ldots
\end{aligned}
\tag{10}
$$

Tables 6.4 and 6.5 exhibit the branching of the codon representation of $so(14)$ along this chain (no. 6 of the previous section): Table 6.4 presents the first phase (first four columns) and second phase (last three columns) whereas Table 6.5 shows the third phase. Of course, it is evident from the analysis performed in Sects 3 and 4 that there are many other chains that, after completion of the second phase, lead to the same distribution of multiplets under $(A_1)^3$; they are represented in terms of the flow diagram shown in Fig. 6.1.

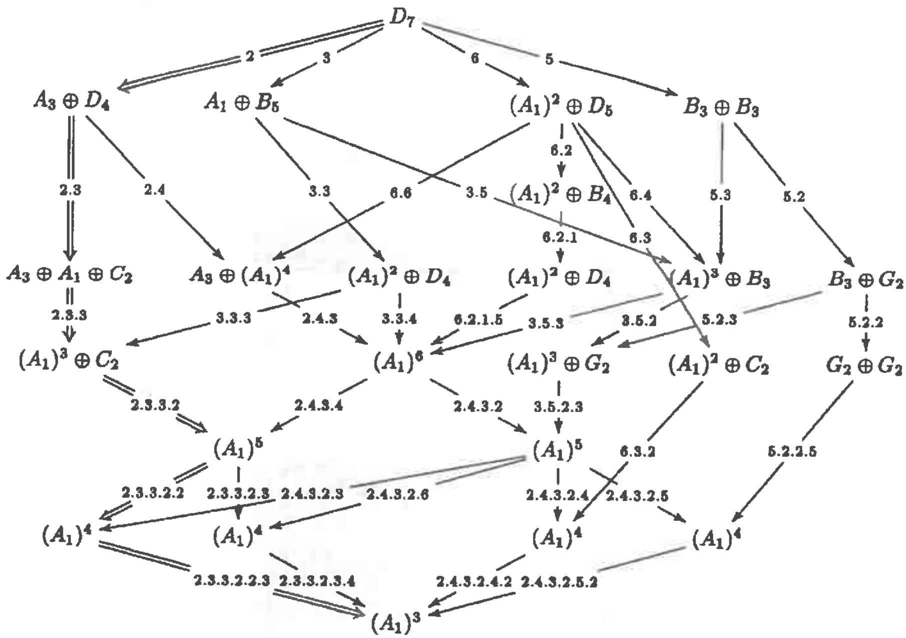

Fig. 6.1: Graphical representation of all $so(14)$ chains leading to the same distribution of multiplets under $(A_1)^3$ as the chain (10) (which is indicated by the double arrows).

D_7		$A_3 \oplus D_4$		$A_3 \oplus A_1 \oplus C_2$		$(A_1)^5$		$(A_1)^4$		$(A_1)^3$	
HW	d	**HW**	d	**HW**	d	**HW**	d	**HW**	d	**HW**	d
(0000010)	64	(001, 010)	32	(001, 0, 01)	20	(1, 1, 0, 1, 1)	16	(1, 1, 1, 1)	16	(2, 1, 1)	12
										(0, 1, 1)	4
						(1, 1, 0, 0, 0)	4	(1, 1, 0, 0)	4	(1, 1, 0)	4
				(001, 2, 00)	12	(1, 1, 2, 0, 0)	12	(3, 1, 0, 0)	8	(1, 3, 0)	8
								(1, 1, 0, 0)	4	(1, 1, 0)	4
		(100, 0001)	32	(100, 1, 10)	32	(1, 1, 1, 1, 0)	16	(2, 1, 1, 0)	12	(2, 2, 0)	9
										(0, 2, 0)	3
								(0, 1, 1, 0)	4	(2, 0, 0)	3
										(0, 0, 0)	1
						(1, 1, 1, 0, 1)	16	(2, 1, 0, 1)	12	(1, 2, 1)	12
								(0, 1, 0, 1)	4	(1, 0, 1)	4
1 subspace		**2 subspaces**		**3 subspaces**		**5 subspaces**		**8 subspaces**		**11 subspaces**	

Table 6.4: Branching of the codon representation of $\mathfrak{so}(14)$ in the chain (10): first two phases.

$(A_1)^3$		$L^2_{2,z}$		$L_{2,z}$	
HW	d	**HW**	d	**HW**	d
(2, 1, 1)	12	(2, ±1, 1)	12	(2, +1, 1)	6
				(2, −1, 1)	6
(0, 1, 1)	4	(0, ±1, 1)	4	(0, +1, 1)	2
				(0, −1, 1)	2
(1, 1, 0)	4	(1, ±1, 0)	4	(1, +1, 0)	2
				(1, −1, 0)	2
(1, 3, 0)	8	(1, ±3, 0)	4	(1, +3, 0)	2
				(1, −3, 0)	2
		(1, ±1, 0)	4	(1, +1, 0)	2
				(1, −1, 0)	2
(1, 1, 0)	4	(1, ±1, 0)	4	(1, +1, 0)	2
				(1, −1, 0)	2
(2, 2, 0)	9	(2, ±2, 0)	6	(2, +2, 0)	3
				(2, −2, 0)	3
		(2, 0, 0)	3	(2, 0, 0)	3
(0, 2, 0)	3	(0, ±2, 0)	2	(0, +2, 0)	1
				(0, −2, 0)	1
		(0, 0, 0)	1	(0, 0, 0)	1
(2, 0, 0)	3	(2, 0, 0)	3	(2, 0, 0)	3
(0, 0, 0)	1	(0, 0, 0)	1	(0, 0, 0)	1
(1, 2, 1)	12	(1, ±2, 1)	8	(1, +2, 1)	4
				(1, −2, 1)	4
		(1, 0, 1)	4	(1, 0, 1)	4
(1, 0, 1)	4	(1, 0, 1)	4	(1, 0, 1)	4
11 subspaces		**15 subspaces**		**24 subspaces**	

Table 6.5: Branching of the codon representation of $\mathfrak{so}(14)$ in the chain (10): third phase.

5. The $\mathfrak{so}(14)$-chain

$$
\begin{aligned}
D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus C_2 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \\
\supset (A_1)_{14} \oplus A_1 \oplus A_1 \oplus A_1 \supset (A_1)_{24} \oplus A_1 \oplus A_1 \supset \ldots
\end{aligned}
\tag{11}
$$

Tables 6.6-6.8 exhibit the branching of the codon representation of $\mathfrak{so}(14)$ along this chain (no. 7 of the previous section): Table 6.6 presents the first phase (first four columns) and second phase (last three columns) whereas Table 6.7 shows the first option and Table 6.8 the second option for the third phase, which differ only in the last step. Of course, it is evident from the analysis performed in Sects 3 and 4 that there are many other chains that, after completion of the second phase, lead to the same distribution of multiplets under $(A_1)^3$; they are represented in terms of the flow diagram shown in Fig. 6.2.

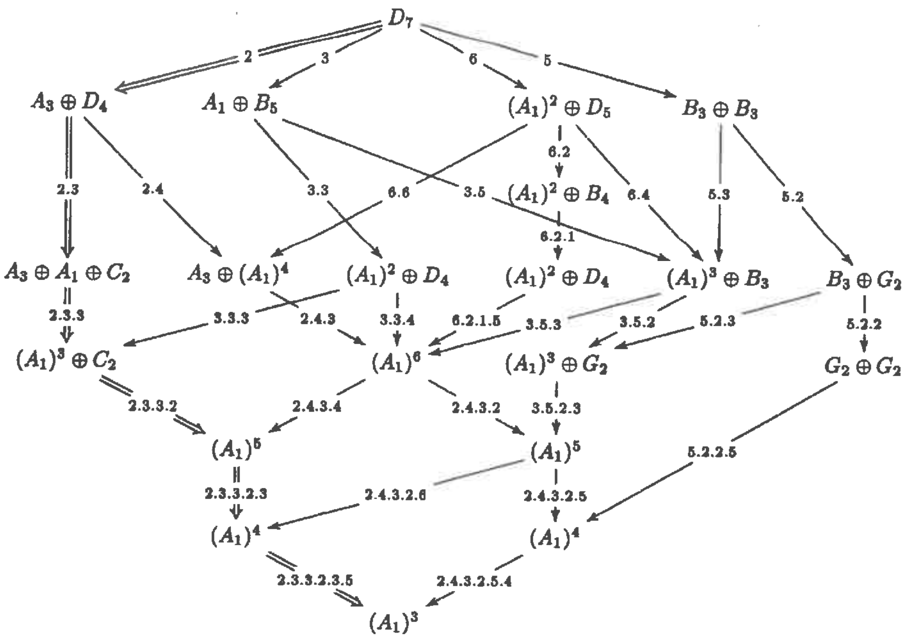

Fig. 6.2: Graphical representation of all $\mathfrak{so}(14)$ chains leading to the same distribution of multiplets under $(A_1)^3$ as the chain (11) (which is indicated by the double arrows).

D_7		$A_3 \oplus D_4$		$A_3 \oplus A_1 \oplus C_2$		$(A_1)^5$		$(A_1)^4$		$(A_1)^3$	
HW	d	HW	d	HW	d	HW	d	HW	d	HW	d
(0000010)	64	(001, 0010)	32	(001, 0, 01)	20	(1, 1, 0, 1, 1)	16	(2, 1, 0, 1)	12	(2, 2, 0)	9
										(0, 2, 0)	3
								(0, 1, 0, 1)	4	(2, 0, 0)	3
										(0, 0, 0)	1
						(1, 1, 0, 0, 0)	4	(1, 1, 0, 0)	4	(1, 1, 0)	4
				(001, 2, 00)	12	(1, 1, 2, 0, 0)	12	(1, 1, 2, 0)	12	(1, 1, 2)	12
		(100, 0001)	32	(100, 1, 10)	32	(1, 1, 1, 1, 0)	16	(2, 1, 1, 0)	12	(1, 2, 1)	12
								(0, 1, 1, 0)	4	(1, 0, 1)	4
						(1, 1, 1, 0, 1)	16	(1, 1, 1, 1)	16	(2, 1, 1)	12
										(0, 1, 1)	4
1 subspace		2 subspaces		3 subspaces		5 subspaces		7 subspaces		10 subspaces	

Table 6.6: Branching of the codon representation of $\mathfrak{so}(14)$ in the chain (11): first two phases.

$(A_1)^3$		$L^2_{1,z}$		$(L^2_{1,z}, L_{3,z})$		$(L_{1,z}, L_{3,z})$	
HW	d	HW	d	HW	d	HW	d
$(2,2,0)$	9	$(\pm 2,2,0)$	6	$(\pm 2,2,0)$	6	$(+2,2,0)$	3
						$(-2,2,0)$	3
		$(0,2,0)$	3	$(0,2,0)$	3	$(0,2,0)$	3
$(0,2,0)$	3	$(0,2,0)$	3	$(0,2,0)$	3	$(0,2,0)$	3
$(2,0,0)$	3	$(\pm 2,0,0)$	2	$(\pm 2,0,0)$	2	$(+2,0,0)$	1
						$(-2,0,0)$	1
		$(0,0,0)$	1	$(0,0,0)$	1	$(0,0,0)$	1
$(0,0,0)$	1	$(0,0,0)$	1	$(0,0,0)$	1	$(0,0,0)$	1
$(1,1,0)$	4	$(\pm 1,1,0)$	4	$(\pm 1,1,0)$	4	$(+1,1,0)$	2
						$(-1,1,0)$	2
$(1,1,2)$	12	$(\pm 1,1,2)$	12	$(\pm 1,1,+2)$	4	$(+1,1,+2)$	2
						$(-1,1,+2)$	2
				$(\pm 1,1,-2)$	4	$(+1,1,-2)$	2
						$(-1,1,-2)$	2
				$(\pm 1,1,0)$	4	$(+1,1,0)$	2
						$(-1,1,0)$	2
$(1,2,1)$	12	$(\pm 1,2,1)$	12	$(\pm 1,2,+1)$	6	$(+1,2,+1)$	3
						$(-1,2,+1)$	3
				$(\pm 1,2,-1)$	6	$(+1,2,-1)$	3
						$(-1,2,-1)$	3
$(1,0,1)$	4	$(\pm 1,0,1)$	4	$(\pm 1,0,+1)$	2	$(+1,0,+1)$	1
						$(-1,0,+1)$	1
				$(\pm 1,0,-1)$	2	$(+1,0,-1)$	1
						$(-1,0,-1)$	1
$(2,1,1)$	12	$(\pm 2,1,1)$	8	$(\pm 2,1,+1)$	4	$(+2,1,+1)$	2
						$(-2,1,+1)$	2
				$(\pm 2,1,-1)$	4	$(+2,1,-1)$	2
						$(-2,1,-1)$	2
		$(0,1,1)$	4	$(0,1,+1)$	2	$(0,1,+1)$	2
				$(0,1,-1)$	2	$(0,1,-1)$	2
$(0,1,1)$	4	$(0,1,1)$	4	$(0,1,+1)$	2	$(0,1,+1)$	2
				$(0,1,-1)$	2	$(0,1,-1)$	2
10 subspaces		13 subspaces		20 subspaces		32 subspaces	

Table 6.7: Branching of the codon representation of $\mathfrak{so}(14)$ in the chain (11): third phase, first option.

$(A_1)^3$		$L^2_{1,x}$		$(L^2_{1,x}, L_{3,x})$		$(L^2_{1,x}, L_{2,x}, L_{3,x})$	
HW	d	HW	d	HW	d	HW	d
$(2,2,0)$	9	$(\pm2,2,0)$	6	$(\pm2,2,0)$	6	$(\pm2,+2,0)$	2
						$(\pm2,-2,0)$	2
						$(\pm2,0,0)$	2
		$(0,2,0)$	3	$(0,2,0)$	3	$(0,+2,0)$	1
						$(0,-2,0)$	1
						$(0,0,0)$	1
$(0,2,0)$	3	$(0,2,0)$	3	$(0,2,0)$	3	$(0,+2,0)$	1
						$(0,-2,0)$	1
						$(0,0,0)$	1
$(2,0,0)$	3	$(\pm2,0,0)$	2	$(\pm2,0,0)$	2	$(\pm2,0,0)$	2
		$(0,0,0)$	1	$(0,0,0)$	1	$(0,0,0)$	1
$(0,0,0)$	1	$(0,0,0)$	1	$(0,0,0)$	1	$(0,0,0)$	1
$(1,1,0)$	4	$(\pm1,1,0)$	4	$(\pm1,1,0)$	4	$(\pm1,+1,0)$	2
						$(\pm1,-1,0)$	2
$(1,1,2)$	12	$(\pm1,1,2)$	12	$(\pm1,1,+2)$	4	$(\pm1,+1,+2)$	2
						$(\pm1,-1,+2)$	2
				$(\pm1,1,-2)$	4	$(\pm1,+1,-2)$	2
						$(\pm1,-1,-2)$	2
				$(\pm1,1,0)$	4	$(\pm1,+1,0)$	2
						$(\pm1,-1,0)$	2
$(1,2,1)$	12	$(\pm1,2,1)$	12	$(\pm1,2,+1)$	6	$(\pm1,+2,+1)$	2
						$(\pm1,-2,+1)$	2
						$(\pm1,0,+1)$	2
				$(\pm1,2,-1)$	6	$(\pm1,+2,-1)$	2
						$(\pm1,-2,-1)$	2
						$(\pm1,0,-1)$	2
$(1,0,1)$	4	$(\pm1,0,1)$	4	$(\pm1,0,+1)$	2	$(\pm1,0,+1)$	2
				$(\pm1,0,-1)$	2	$(\pm1,0,-1)$	2
$(2,1,1)$	12	$(\pm2,1,1)$	8	$(\pm2,1,+1)$	4	$(\pm2,+1,+1)$	2
						$(\pm2,-1,+1)$	2
				$(\pm2,1,-1)$	4	$(\pm2,+1,-1)$	2
						$(\pm2,-1,-1)$	2
		$(0,1,1)$	4	$(0,1,+1)$	2	$(0,+1,+1)$	1
						$(0,-1,+1)$	1
				$(0,1,-1)$	2	$(0,+1,-1)$	1
						$(0,-1,-1)$	1
$(0,1,1)$	4	$(0,1,1)$	4	$(0,1,+1)$	2	$(0,+1,+1)$	1
						$(0,-1,+1)$	1
				$(0,1,-1)$	2	$(0,+1,-1)$	1
						$(0,-1,-1)$	1
10 subspaces		13 subspaces		20 subspaces		40 subspaces	

Table 6.8: Branching of the codon representation of $\mathfrak{so}(14)$ in the chain (11): third phase, second option.

6. The $so(14)$-chain

$$D_7 \supset A_3 \oplus D_4 \supset A_3 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \\ \supset (A_1)_{13} \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \supset (A_1)_{24} \oplus A_1 \oplus A_1 \oplus A_1 \supset \ldots \tag{12}$$

Tables 6.9 and 6.10 exhibit the branching of the codon representation of $so(14)$ along this chain (no. 8 of the previous section): Table 6.9 presents the first phase (first four columns) and second phase (last three columns) whereas Table 6.10 shows the third phase. Of course, it is evident from the analysis performed in Sects 3 and 4 that there are many other chains that, after completion of the second phase, lead to the same distribution of multiplets under $(A_1)^4$; they are represented in terms of the flow diagram shown in Fig. 6.3.

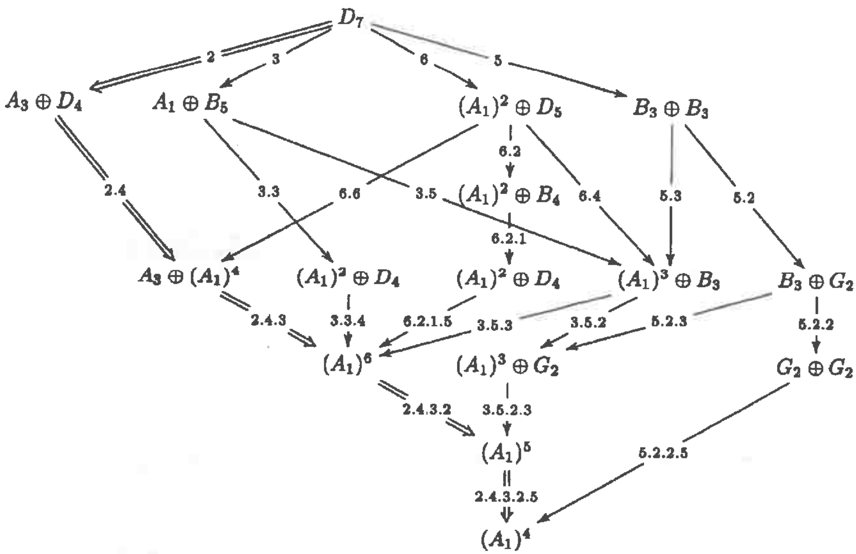

Fig. 6.3: Graphical representation of all $so(14)$ chains leading to the same distribution of multiplets under $(A_1)^4$ as the chain (12) (which is indicated by the double arrows).

Note that in this case, there is one chain that does not involve diagonal breaking: this is the $so(14)$-chain already reported in Ref. [9].

D_7		$A_3 \oplus D_4$		$A_3 \oplus (A_1)^4$		$(A_1)^6$		$(A_1)^5$		$(A_1)^4$	
HW	d	HW	d	HW	d	HW	d	HW	d	HW	d
(0000010)	64	(001-0010)	32	(001-0-1-1-0)	16	(1-1-0-1-1-0)	16	(1-1-1-1-0)	16	(2-1-1-0)	12
										(0-1-1-0)	4
				(001-1-0-0-1)	16	(1-1-1-0-0-1)	16	(2-1-0-0-1)	12	(1-2-0-1)	12
								(0-1-0-0-1)	4	(1-0-0-1)	4
		(100-0001)	32	(100-0-1-0-1)	16	(1-1-0-1-0-1)	16	(1-1-1-0-1)	16	(1-1-1-1)	16
				(100-1-0-1-0)	16	(1-1-1-0-1-0)	16	(2-1-0-1-0)	12	(2-2-0-0)	9
										(0-2-0-0)	3
								(0-1-0-1-0)	4	(2-0-0-0)	3
										(0-0-0-0)	1
1 subspace		2 subspaces		4 subspaces		4 subspaces		6 subspaces		9 subspaces	

Table 6.9: Branching of the codon representation of $so(14)$ in the chain (12): first two phases.

$(A_1)^4$		$L^2_{2,z}$		$(L^2_{2,z}, L_{4,z})$		$(L_{2,z}, L_{4,z})$	
HW	d	HW	d	HW	d	HW	d
$(2,1,1,0)$	12	$(2,\pm1,1,0)$	12	$(2,\pm1,1,0)$	12	$(2,+1,1,0)$	6
						$(2,-1,1,0)$	6
$(0,1,1,0)$	4	$(0,\pm1,1,0)$	4	$(0,\pm1,1,0)$	4	$(0,+1,1,0)$	2
						$(0,-1,1,0)$	2
$(1,2,0,1)$	12	$(1,\pm2,0,1)$	8	$(1,\pm2,0,+1)$	4	$(1,+2,0,+1)$	2
						$(1,-2,0,+1)$	2
				$(1,\pm2,0,-1)$	4	$(1,+2,0,-1)$	2
						$(1,-2,0,-1)$	2
		$(1,0,0,1)$	4	$(1,0,0,+1)$	2	$(1,0,0,+1)$	2
				$(1,0,0,-1)$	2	$(1,0,0,-1)$	2
$(1,0,0,1)$	4	$(1,0,0,1)$	4	$(1,0,0,+1)$	2	$(1,0,0,+1)$	2
				$(1,0,0,-1)$	2	$(1,0,0,-1)$	2
$(1,1,1,1)$	16	$(1,\pm1,1,1)$	16	$(1,\pm1,1,+1)$	8	$(1,+1,1,+1)$	4
						$(1,-1,1,+1)$	4
				$(1,\pm1,1,-1)$	8	$(1,+1,1,-1)$	4
						$(1,-1,1,-1)$	4
$(2,2,0,0)$	9	$(2,\pm2,0,0)$	6	$(2,\pm2,0,0)$	6	$(2,+2,0,0)$	3
						$(2,-2,0,0)$	3
		$(2,0,0,0)$	3	$(2,0,0,0)$	3	$(2,0,0,0)$	3
$(0,2,0,0)$	3	$(0,\pm2,0,0)$	2	$(0,\pm2,0,0)$	2	$(0,+2,0,0)$	1
						$(0,-2,0,0)$	1
		$(0,0,0,0)$	1	$(0,0,0,0)$	1	$(0,0,0,0)$	1
$(2,0,0,0)$	3	$(2,0,0,0)$	3	$(2,0,0,0)$	3	$(2,0,0,0)$	3
$(0,0,0,0)$	1	$(0,0,0,0)$	1	$(0,0,0,0)$	1	$(0,0,0,0)$	1
9 subspaces		12 subspaces		16 subspaces		24 subspaces	

Table 6.10: Branching of the codon representation of $so(14)$ in the chain (12): third phase.

7. The $\mathfrak{so}(14)$-chain

$$\begin{aligned} D_7 &\supset A_1 \oplus B_5 \supset A_1 \oplus A_1 \oplus A_1 \oplus B_3 \supset A_1 \oplus A_1 \oplus A_1 \oplus G_2 \\ &\supset A_1 \oplus A_1 \oplus A_1 \oplus A_1 \\ &\supset (A_1)_{24} \oplus A_1 \oplus A_1 \supset (A_1)_{23} \oplus A_1 \supset \ldots \end{aligned} \tag{13}$$

Tables 6.11-6.13 exhibit the branching of the codon representation of $\mathfrak{so}(14)$ along this chain (no. 10 of the previous section): Table 6.11 presents the first phase (first five columns) and second phase (last three columns) whereas Table 6.12 shows the first option and Table 6.13 the second option for the third phase, which differ only in the last step. Of course, it is evident from the analysis performed in Sects 3 and 4 that there are a couple of other chains that, after completion of the second phase, lead to the same distribution of multiplets under $(A_1)^2$; they are represented in terms of the flow diagram shown in Fig. 6.4.

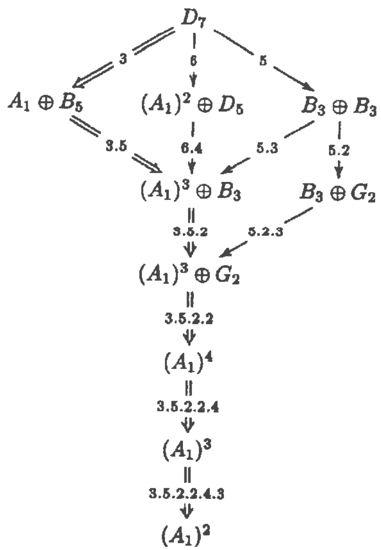

Fig. 6.4: Graphical representation of all $\mathfrak{so}(14)$ chains leading to the same distribution of multiplets under $(A_1)^2$ as the chain (13) (which is indicated by the double arrows).

D_7		$A_1 \oplus B_5$		$(A_1)^3 \oplus B_3$		$(A_1)^3 \oplus G_2$		$(A_1)^4$		$(A_1)^3$		$(A_1)^2$	
HW	d	HW	d	HW	d	HW	d	HW	d	HW	d	HW	d
(0000010)	64	(1-00001)	64	(1-1-0-001)	32	(1-1-0-01)	28	(1-1-0-6)	28	(7-1-0)	16	(1-7)	16
										(5-1-0)	12	(1-5)	12
						(1-1-0-00)	4	(1-1-0-0)	4	(1-1-0)	4	(1-1)	4
				(1-0-1-001)	32	(1-0-1-01)	28	(1-0-1-6)	28	(6-1-1)	28	(2-6)	21
												(0-6)	7
						(1-0-1-00)	4	(1-0-1-0)	4	(0-1-1)	4	(2-0)	3
												(0-0)	1
1 subspace		1 subspace		2 subspaces		4 subspaces		4 subspaces		5 subspaces		7 subspaces	

Table 6.11: Branching of the codon representation of $\mathfrak{so}(14)$ in the chain (13): first two phases.

$(A_1)^2$		$L^2_{2,x}$		$(L_{1,x}, L^2_{2,x})$	
HW	d	HW	d	HW	d
$(1,7)$	16	$(1,\pm7)$	4	$(+1,\pm7)$	2
				$(-1,\pm7)$	2
		$(1,\pm5)$	4	$(+1,\pm5)$	2
				$(-1,\pm5)$	2
		$(1,\pm3)$	4	$(+1,\pm3)$	2
				$(-1,\pm3)$	2
		$(1,\pm1)$	4	$(+1,\pm1)$	2
				$(-1,\pm1)$	2
$(1,5)$	12	$(1,\pm5)$	4	$(+1,\pm5)$	2
				$(-1,\pm5)$	2
		$(1,\pm3)$	4	$(+1,\pm3)$	2
				$(-1,\pm3)$	2
		$(1,\pm1)$	4	$(+1,\pm1)$	2
				$(-1,\pm1)$	2
$(1,1)$	4	$(1,\pm1)$	4	$(+1,\pm1)$	2
				$(-1,\pm1)$	2
$(2,6)$	21	$(2,\pm6)$	6	$(+2,\pm6)$	2
				$(-2,\pm6)$	2
				$(0,\pm6)$	2
		$(2,\pm4)$	6	$(+2,\pm4)$	2
				$(-2,\pm4)$	2
				$(0,\pm4)$	2
		$(2,\pm2)$	6	$(+2,\pm2)$	2
				$(-2,\pm2)$	2
				$(0,\pm2)$	2
		$(2,0)$	3	$(+2,0)$	1
				$(-2,0)$	1
				$(0,0)$	1
$(0,6)$	7	$(0,\pm6)$	2	$(0,\pm6)$	2
		$(0,\pm4)$	2	$(0,\pm4)$	2
		$(0,\pm2)$	2	$(0,\pm2)$	2
		$(0,0)$	1	$(0,0)$	1
$(2,0)$	3	$(2,0)$	3	$(+2,0)$	1
				$(-2,0)$	1
				$(0,0)$	1
$(0,0)$	1	$(0,0)$	1	$(0,0)$	1
7 subspaces		18 subspaces		36 subspaces	

Table 6.12: Branching of the codon representation of $so(14)$ in the chain (13): third phase, first option.

$(A_1)^2$		$L^2_{2,z}$		$L_{2,z}$	
HW	d	HW	d	HW	d
(1,7)	16	(1,±7)	4	(1,+7)	2
				(1,−7)	2
		(1,±5)	4	(1,+5)	2
				(1,−5)	2
		(1,±3)	4	(1,+3)	2
				(1,−3)	2
		(1,±1)	4	(1,+1)	2
				(1,−1)	2
(1,5)	12	(1,±5)	4	(1,+5)	2
				(1,−5)	2
		(1,±3)	4	(1,+3)	2
				(1,−3)	2
		(1,±1)	4	(1,+1)	2
				(1,−1)	2
(1,1)	4	(1,±1)	4	(1,+1)	2
				(1,−1)	2
(2,6)	21	(2,±6)	6	(2,+6)	3
				(2,−6)	3
		(2,±4)	6	(2,+4)	3
				(2,−4)	3
		(2,±2)	6	(2,+2)	3
				(2,−2)	3
		(2,0)	3	(2,0)	3
(0,6)	7	(0,±6)	2	(0,+6)	1
				(0,−6)	1
		(0,±4)	2	(0,+4)	1
				(0,−4)	1
		(0,±2)	2	(0,+2)	1
				(0,−2)	1
		(0,0)	1	(0,0)	1
(2,0)	3	(2,0)	3	(2,0)	3
(0,0)	1	(0,0)	1	(0,0)	1
7 subspaces		18 subspaces		32 subspaces	

Table 6.13: Branching of the codon representation of $so(14)$ in the chain (13): third phase, second option.

7 Concluding Remarks

The extensive analysis reported in this paper has shown that extending the search for symmetries in the genetic code from the simple Lie algebras of low rank ($\mathfrak{su}(2)$, $\mathfrak{su}(3)$, $\mathfrak{so}(5) = \mathfrak{sp}(4)$, $\mathfrak{g}_2$, $\mathfrak{su}(4)$ and $\mathfrak{sp}(6)$) to those of medium rank ($\mathfrak{so}(13)$ and $\mathfrak{so}(14)$) produces only a very limited number of new schemes, even when the possibilities of diagonal breaking and of indirect breaking, as explained in Ref. [9], are taken into account. On the other hand, it is obvious from Figs 6.1-6.4 that there is a wealth of $\mathfrak{so}(14)$-chains associated with these new schemes. In our view, this remarkable lack of uniqueness is a strong argument against the application of the whole method in this wider context, since it provides little information on the type of symmetry that has prevailed in the intermediate steps of the process and, therefore, on the manner in which the genetic code has evolved into its presently observed form. In an extreme form, the same argument has already been used in Ref. [9] to exclude the simple Lie algebras of high rank ($\mathfrak{su}(64)$, $\mathfrak{so}(64)$ and $\mathfrak{sp}(64)$) from the analysis, since these can produce any final distribution of multiplets whatsoever and can do so in millions of different ways. This situation is in sharp contrast to that encountered in the original model based on $\mathfrak{sp}(6)$ [1], where the whole chain is uniquely determined from the final distribution of multiplets: we take this as a strong indication in favor of the $\mathfrak{sp}(6)$-model.

In a forthcoming paper [15], we plan to present further arguments in favor of the $\mathfrak{sp}(6)$-model, based on an analysis of the possibility or impossibility to devise a mathematically compelling strategy for deriving an aminoacid and codon assignment, as has been done for the $\mathfrak{sp}(6)$-model in Ref. [8].

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