A GENERALIZED CR EQUATION WITH ISOLATED SINGULARITIES.

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Abstract. The generalized CR equation $u_{\overline{z}} = au + b\overline{u} + f$ is studied when the coefficients a and b have a finite number of singular points inside the domain. Solutions are constructed via the study of an associated integral operator and the existence of nontrivial solutions of the associated homogeneous equation is established.

1. Introduction

The study of generalized CR equations

$$\frac{\partial u}{\partial \overline{z}} = a(z)u + b(z)\overline{u} + f(z)$$

in a domain $\Omega \subset \mathbb{C}$ was initiated by L. Bers and I.N. Vekua in [3] and [14]. This equation is of fundamental importance and has applications in many areas (see for example [5] and [11] and the references therein). The initiators of the theory considered the elliptic case when the coefficients are in $L^p(\Omega)$ with $p > 2$ and this situation is now well understood (see [1] for a comprehensive presentation). The case of degenerate coefficients (either on the boundary of the domain or inside the domain) is of current interest. Of particular interest to us, and in view of application to the study of deformation of surfaces [4], we consider equations involving a finite number of isolated singular points. Such type of equations were considered in [2],[6], [7], [9], [10], [12], [13].

In this paper we consider the equation

$$\frac{\partial u}{\partial \overline{z}} = \frac{A(z)}{L(z)}u + \frac{B(z)}{L(z)}\overline{u} + F(z),$$

where $L(z) = \displaystyle\prod_{j=1}^{N}(z - z_j)$ and $z_1, \cdots, z_N$ are distinct points in the domain Ω. It should be noted that the case $N = 1$ is studied in [6], through the use of associated systems of ordinary differential equations when the coefficients

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A, B depend only on the argument θ of z and in [7] when the coefficients could also depend on $|z|$ but with small norm. The main results, Theorems 2.1 and 2.3 describe the solutions of such equations. To prove Theorem 2.1, we make use of and associated integral operator (4.1) and its adjoint with respect to a real bilinear form, which is inspired by the recent result in [10] by A.B. Rasulov and A.P. Soldatov when the case of a single singular point and small coefficients is considered. Theorem 2.3 shows the existence of nontrivial solutions for the homogeneous equation ($F = 0$).

2. Main Results

Let $\Omega \subset \mathbb{C}$ be a relatively compact domain, $S = \{z_1, \cdots, z_N\}$ be a collection of N distinct points in Ω and $A(z)$, $B(z) \in L^\infty(\Omega) \cap C^\infty(\overline{\Omega}\backslash S)$. Assume that for every $j \in \{1, \cdots, N\}$, there exist $0 < \tau_j < 1$, $\delta_j > 0$, 2π-periodic functions $p_j(\theta)$, $q_j(\theta)$ and functions $A_j(r\mathrm{e}^{i\theta})$, $B_j(r\mathrm{e}^{i\theta})$ such that A_j, B_j, $\in L^\infty(D(0,\delta_j)) \cap C^\infty(D(0,\delta_j)\backslash\{0\})$, where $D(0,\delta_j)$ denotes the open disc centered at 0 and with radius δ_j, and

$$\begin{aligned} A(z_j + r\mathrm{e}^{i\theta}) &= p_j(\theta) + r^{\tau_j} A_j(r\mathrm{e}^{i\theta}) \\ B(z_j + r\mathrm{e}^{i\theta}) &= q_j(\theta) + r^{\tau_j} B_j(r\mathrm{e}^{i\theta})\,. \end{aligned} \tag{2.1}$$

Consider

$$\begin{aligned} \gamma_j &= \frac{1}{\pi}\int_0^{2\pi} \mathrm{e}^{-2i\theta} p_j(\theta)\, d\theta, \quad j = 1, \cdots N, \\ L(z) &= \prod_{j=1}^{N} (z - z_j) \quad \text{and} \quad M(z) = \prod_{j=1}^{N} |z - z_j|^{\gamma_j}. \end{aligned} \tag{2.2}$$

Our goal is to understand the solutions of the equation

$$\frac{\partial u}{\partial \overline{z}} = \frac{A(z)}{L(z)} u + \frac{B(z)}{L(z)} \overline{u} + F(z), \tag{2.3}$$

when the nonhomogeneous term $F(z)$ vanishes at the set of singular points $\{z_1, \ldots, z_N\}$.

For positive numbers m and p, with $m \in \mathbb{Z}^+$ and $p > 1$, consider the Banach space

$$E_{m,p} = \left\{ f : \Omega \longrightarrow \mathbb{C} \;:\; \frac{f(z)}{L(z)^m} \in L^p(\Omega) \right\}$$

equipped with the norm

$$\|f\|_{m,p} = \left\| \frac{f(z)}{L(z)^m} \right\|_{L^p(\Omega)}.$$

The main results of this paper are the following theorems.

Theorem 2.1. *Let A and B be functions satisfying (2.1), $m \in \mathbb{Z}^+$ and $p > 2$. Then for every function F in Ω such that $\dfrac{F(z)}{M(z)} \in E_{m+1,p}(\Omega)$, where $M(z)$ is given in (2.2), there exists a function $v \in E_{m,p}(\Omega) \cap C^\alpha(\Omega\backslash S)$, with*

$\alpha = (p-2)/p$ such that the function $u(z) = v(z)M(z)$ is a solution of the equation (2.3). Moreover, if in addition $F \in C^{k,\sigma}(\Omega\backslash S)$ with $k \in \mathbb{Z}^+$ and $0 < \sigma < 1$, then $u \in C^{k+1,\sigma}(\Omega\backslash S)$.

Remark 2.2. In the paper [10], equation (2.3) is studied in the presence of a single singular point p_0 (so $N = 1$) and when the coefficient B has small norm. In this case, the authors prove the existence of solutions of the form $v(z)/|z-p_0|^a$ with $a < 1$. In our case, we only require the number of singular points to be finite and there is no restriction on the size of the norms of the coefficients.

Theorem 2.3. *Let A and B be functions satisfying (2.1) and $k \in \mathbb{Z}^+$. The homogeneous equation*

$$\frac{\partial u}{\partial \overline{z}} = \frac{A(z)}{L(z)}u + \frac{B(z)}{L(z)}\overline{u} \tag{2.4}$$

has non trivial solutions in $C^k(\Omega)$. Moreover, for any $a > 0$, a nontrivial solution u can be chosen so that u vanishes to an order $\geq a$ at each singular point z_j.

The rest of the paper deals with the proof of these results.

3. Reduction to the case $A = 0$

In this section we show that the solvability of equation (2.3) can be reduced to an analogous equation where the coefficient $A = 0$. For this we start by proving the following lemma.

Lemma 3.1. *For $j = 1, \cdots, N$ let γ_j be as in (2.2). Then there exists a function $\mu \in L^\infty(\Omega) \cap C^\infty(\Omega\backslash S)$ such that*

$$w(z) = \sum_{j=1}^{N} \gamma_j \log|z - z_j| + \mu(z) \tag{3.1}$$

satisfies

$$\frac{\partial w(z)}{\partial \overline{z}} = \frac{A(z)}{L(z)}. \tag{3.2}$$

Proof. Let $\delta > 0$ be such that the discs $D(z_j, 2\delta)$, with $j = 1, \cdots, N$, are contained in Ω and are pairwise disjoint. Let $\phi_1, \cdots, \phi_N \in C^\infty(\mathbb{C})$ such that

$$\phi_j \equiv 1 \text{ in the disc } D(z_j, \delta), \quad \text{Supp}(\phi_j) \subset D(z_j, 2\delta)$$

and set $\phi_0 = 1 - \sum_{j=1}^{N} \phi_j$. Note that $\phi_0 \equiv 1$ in $\Omega\backslash(\bigcup_{j=1}^{N} D(z_j, 2\delta))$ and $\phi_0 \equiv 0$ in $\bigcup_{j=1}^{N} D(z_j, \delta)$. The solvability of the equation (3.2) can be reduced to those of the $N+1$ equations

$$\frac{\partial w_j}{\partial \bar{z}} = \frac{A\phi_j}{L}, \quad j = 0, \ldots, N \tag{3.3}$$

and taking $w = \sum_{j=0}^{N} w_j$. Note that since $\phi_0 \equiv 0$ in $\bigcup_{j=1}^{N} D(z_j, \delta)$, it follows that $\frac{A\phi_0}{L} \in C^\infty(\overline{\Omega})$ and so for $j = 0$, equation (3.3) has a solution $w_0 \in C^\infty(\overline{\Omega})$ (see [1]).

For $j = 1, \cdots, N$, we use polar coordinates around the point z_j, that is, set $z = z_j + r\mathrm{e}^{i\theta}$, and use property (2.1) of the function A to transform equation (3.3) into an equation of the form

$$\frac{\partial w_j}{\partial r} + \frac{i}{r}\frac{\partial w_j}{\partial \theta} = \frac{\gamma_j + \hat{p}_j(\theta)}{r} + r^{\tau_j - 1} c_j(r, \theta)\,, \tag{3.4}$$

where $\hat{p}_j(\theta)$ is a 2π-periodic, C^∞ function with zero average and $c_j(r, \theta)$ is a bounded function, C^∞ for $r > 0$. Since $r^{\tau_j - 1} c_j \in L^p(\overline{\Omega})$ with $2 < p < \frac{2}{1 - \tau_j}$, then equation

$$\frac{\partial v_j}{\partial r} + \frac{i}{r}\frac{\partial v_j}{\partial \theta} = r^{\tau_j - 1} c_j(r, \theta)\,,$$

has a solution $v_j \in C^\alpha(\Omega) \cap C^\infty(\overline{\Omega} \backslash \{z_j\})$ with $\alpha = (p-2)/p$ (see [1]).

The function

$$\zeta_j(r, \theta) = \gamma_j \log r - i \int_0^\theta \hat{p}_j(s)\, ds$$

satisfies

$$\frac{\partial \zeta_j}{\partial r} + \frac{i}{r}\frac{\partial \zeta_j}{\partial \theta} = \frac{\gamma_j + \hat{p}_j(\theta)}{r}\,.$$

It follows that the function

$$w_j(r, \theta) = \zeta_j(r, \theta) + v_j(r, \theta) = \gamma_j \log r + \underbrace{\left(v_j(r, \theta) - i \int_0^\theta \hat{p}_j(s)\, ds \right)}_{:= \mu_j}$$

solves equation (3.3). Therefore

$$w(z) = \sum_{j=1}^{N} \gamma_j \log |z - z_j| + \mu(z),$$

with $\mu(z) = w_0(z) + \sum_{j=1}^{N} \mu_j(z)$ is the desired solution of equation (3.2). $\square$

With $w(z)$ given by in Lemma (3.1), the function

$$\mathrm{e}^{w(z)} = \left(\prod_{j=1}^{N} |z - z_j|^{\gamma_j} \right) \mathrm{e}^{\mu(z)} = M(z)\, \mathrm{e}^{\mu(z)}$$

is smooth in $\overline{\Omega}\backslash S$. A function $u(z)$ satisfies equation (2.3) if and only if the function

$$v(z) = \mathrm{e}^{-w(z)}u(z) = \mathrm{e}^{-\mu(z)}\,\frac{u(z)}{M(z)}$$

solves the equation

$$\frac{\partial v}{\partial \overline{z}} = \frac{B_1(z)}{L(z)}\overline{v} + F_1(z)\,, \tag{3.5}$$

with

$$B_1(z) = B(z)\mathrm{e}^{\overline{w(z)}-w(z)} \quad \text{and} \quad F_1(z) = \frac{\mathrm{e}^{-\mu(z)}F(z)}{M(z)}\,.$$

Note $|B_1(z)| = |B(z)|$ and that $F_1 \in E_{m+1,p}(\Omega)$ if and only if $\dfrac{F}{M} \in E_{m+1,p}(\Omega)$. Thanks to this reduction, from now on we will assume that $A = 0$ and consider the equation

$$\frac{\partial u}{\partial \overline{z}} = \frac{B(z)}{L(z)}\overline{u} + F(z)\,. \tag{3.6}$$

4. Properties of an associated integral operator

For $L(z)$ as given in (2.2) and $m \in \mathbb{Z}^+$, consider the integral operator $T_{L,m}$ defined by

$$T_{L,m}u(z) = \frac{-L(z)^m}{\pi}\int_{\Omega}\frac{B(\zeta)\overline{u(\zeta)}}{L(\zeta)^{m+1}(\zeta - z)}\,d\xi d\eta \tag{4.1}$$

where $\zeta = \xi + i\eta$. We have the following lemma.

Lemma 4.1. *For $p > 2$, the operator $T_{L,m} : E_{m+1,p}(\Omega) \longrightarrow E_{m,p}(\Omega)$ is bounded and $T_{L,m} : E_{m+1,p}(\Omega) \longrightarrow C^0(\Omega)$ is compact. Furthermore*

$$T_{L,m}\left(E_{m+1,p}(\Omega)\right) \subset C^{\alpha}(\Omega), \quad \textit{for} \quad \alpha = \frac{p-2}{p}.$$

Proof. The boundedness of $T_{L,m}$ is a consequence of estimates for the classical Cauchy-Pompeiu operator. Indeed, for $u \in E_{m+1,p}(\Omega)$ with $p > 2$, we have

$$\begin{aligned} |T_{L,m}u(z)| \;&\leq \frac{|L(z)|^m}{\pi}\int_{\Omega}\frac{|B(\zeta)|\,|u(\zeta)|}{|L(\zeta)|^{m+1}|\zeta - z|}\,d\xi d\eta \\ &\leq C|L(z)|^m\,\|B\|_{\infty}\,\|u\|_{m+1,p}\,, \end{aligned}$$

where C is a constant depending only on p and the size of the domain Ω. From this it also follows that $T_{L,m}u \in E_{m,p}(\Omega)$.

Now for arbitrary z_1, z_2 distinct points in Ω , we have

$$T_{L,m}u(z_1) - T_{L,m}u(z_2) = \frac{-1}{\pi}\int_{\Omega}\frac{B(\zeta)\overline{u(\zeta)}}{L(\zeta)^{m+1}}\,\frac{\zeta K_1(z_1,z_2) + K_2(z_1,z_2)}{(\zeta - z_1)(\zeta - z_2)}\,d\xi d\eta, \tag{4.2}$$

where

(4.3)
$$K_1(z_1,z_2) = L(z_1)^m - L(z_2)^m = (z_1-z_2)P_m(z_1,z_2),$$
$$K_2(z_1,z_2) = L(z_1)^m(z_1-z_2) - z_1(L(z_1)^m - L(z_2)^m) = (z_1-z_2)Q_m(z_1,z_2),$$

where P_m and Q_m are polynomials in z_1, z_2 of degrees $Nm-1$ and Nm, respectively. It follows from (4.2) and (4.3)

(4.4)
$$\begin{aligned}|T_{L,m}u(z_1) - T_{L,m}u(z_2)| &\le C\left\|B\right\|_\infty |z_1-z_2| \int_\Omega \frac{|u(\zeta)|\, d\xi d\eta}{|L(\zeta)|^{m+1}|\zeta-z_1||\zeta-z_2|}\\ &\le C\left\|B\right\|_\infty |z_1-z_2|\, \|u\|_{m+1,p}\left(\int_\Omega \frac{d\xi d\eta}{||\zeta-z_1|^q|\zeta-z_2|^q|}\right)^{1/q}\\ &\le C'\left\|B\right\|_\infty \|u\|_{m+1,p}|z_1-z_2|^{(p-2)/p},\end{aligned}$$

where the constants depend only on p and Ω and q . In the last estimate, we used Hadamard's inequality (see [1] or [14]). The compactness of $T_{L,m}$ follows from the compactness of embedding into Hölder spaces.

□

Next we define the adjoint of $T_{L,m}$ with respect to the real bilinear form $\langle ., . \rangle$ given by by

$$\langle \phi, \psi\rangle = \mathrm{Re}(\phi,\psi) = \mathrm{Re}\left(\int_\Omega \phi(z)\overline{\psi(z)}\, dxdy\right).$$

If q is the Hölder conjugate of p, we set

$$T^*_{L,m} : X_{m,q}(\Omega) \longrightarrow X_{m+1,q}(\Omega),$$

defined in the space

$$X_{m,q}(\Omega) = \{v : \Omega \longrightarrow C : \; L(z)^m v(z) \in L^q(\Omega)\},$$

and given by

$$T^*_{L,m}v(\zeta) = \frac{-B(\zeta)}{\pi L(\zeta)^{m+1}}\int_\Omega \frac{L(z)^m\overline{v(z)}}{z-\zeta}\, dxdy\,. \tag{4.5}$$

Remark 4.1. The fact that we can take $X_{m+1,q}(\Omega)$ as the target space is a consequence of Theorem 1.2.6 of [14].

Now consider the operator

$$P_{L,m} : E_{m+1,p}(\Omega) \longrightarrow E_{m,p}(\Omega), \quad P_{L,m}u := u - T_{L,m}u,$$

and its adjoint

$$P^*_{L,m} : X_{m,q}(\Omega) \longrightarrow X_{m+1,q}(\Omega), \quad P^*_{L,m}v := v - T^*_{L,m}v.$$

We have the following lemmas

Lemma 4.2. *The spaces $Ker(P_{L,m})$ and $Ker(P^*_{L,m})$ are finite dimensional. Consequently, the operators $P_{L,m}$ and $P^*_{L,m}$ are Fredholm.*

Proof. First note that for $u \in \operatorname{Ker}(P_{L,m})$ we have $u = T_{L,m}u$ and since $p > 2$, then $u \in C^{\alpha}(\Omega) \subset L^2(\Omega)$ (where $\alpha = (p-2)/p$). We continue the proof by contradiction, suppose that $\dim(\operatorname{Ker}(P_{L,m})) = \infty$, then we can find an L^2-orthonormal basis $\{u_j\}_{j\in\mathbb{N}}$. It follows from the compactness of $T_{L,m}$ and from the fact that $u_j = T_{L,m}u_j$ that $\{u_j\}_{j\in\mathbb{N}}$ has a convergent subsequence. This is absurd, since $\|u_j - u_k\|_2 = \sqrt{2}$ for $j \neq k$.

Now consider $v \in \operatorname{Ker}(P^*_{L,m})$; then

$$v(\zeta) = T^*_{L,m}v(\zeta) = -\frac{B(\zeta)}{\pi L(\zeta)^{m+1}} \int_{\Omega} \frac{L(z)^m \overline{v(z)}}{z-\zeta}\, dxdy\,. \tag{4.6}$$

Since $L(z)^m\overline{v(z)} \in L^q(\Omega)$ (with $q > 1$), it follows (Theorem 1.26 of [14]) that $v \in X_{m+1,2}(\Omega)$. With this, a similar argument as the one used for $\operatorname{Ker}(P_{L,m})$ shows that $\operatorname{Ker}(P^*_{L,m})$ is also finite dimensional. □

Lemma 4.3. *Let $H_{m,p}(\Omega) = E_{m,p}(\Omega)\cap\mathcal{H}(\Omega)$, where $\mathcal{H}(\Omega)$ denotes the space of holomorphic functions in Ω. Then*

$$E_{m,p}(\Omega) = \mathit{Range}(P_{L,m}) \,+\, H_{m,p}(\Omega).$$

Proof. To prove this result, it is enough to verify that if $v \in X_{m,q}(\Omega)$ is such that $v \in \operatorname{Range}(P_{L,m})^{\perp} \cap H_{m,p}(\Omega)^{\perp}$, then $v = 0$. Since

$$\operatorname{Range}(P_{L,m})^{\perp} = \operatorname{Ker}(P^*_{L,m}),$$

such a function v satisfies (4.6). Also it follows from $L(z)^m\overline{v(z)} \in L^q(\Omega)$, with $1 < q < 2$, that

$$\int_{\Omega} \frac{L(z)^m\overline{v(z)}}{z-\zeta} dxdy \,\in L^{\gamma}(\Omega),$$

for any $q < \gamma < \dfrac{2q}{2-q}$ (see [1] or [14] for properties of the Cauchy-Pompeiu operator). By repeating this argument, we find that in fact v is Hölder continuous in $\Omega\backslash S$.

The function $h_\zeta(z) = \dfrac{L(z)^m}{z-\zeta}$ is holomorphic in Ω for any $\zeta \notin \Omega$. Moreover, since h_ζ vanishes to order m on the set S, then $h_\zeta \in E_{m,p}(\Omega)$ for $\zeta \notin \overline{\Omega}$. Thus $h_\zeta \in H_{m,p}(\Omega)$ for $\zeta \notin \overline{\Omega}$. It follows from $v \in H_{m,p}(\Omega)^{\perp}$ that

$$\langle h_\zeta, v\rangle = \operatorname{Re}\left(\int_{\Omega} \frac{L(z)^m\overline{v(z)}}{z-\zeta}\, dxdy\right) = 0.$$

Similarly $\langle ih_\zeta, v\rangle = 0$. This means that $(h_\zeta, v) = 0$, which implies that $T^*_{L,m}v(\zeta) = 0$ for every $\zeta \notin \overline{\Omega}$. Hence $v(\zeta)$ is extends continuously to $\mathbb{C}\backslash S$ by taking it to be zero outside Ω. In particular $v = 0$ on the boundary $\partial\Omega$.

Let $Z = \{z \in \Omega :\ B(z) = 0\}$ and $\Omega_1 = \Omega\backslash Z$. It follows from (4.6) that $v = 0$ on the closed set Z. Since $v = 0$ on $\partial\Omega$, then $v = 0$ on $\partial\Omega_1$. By differentiating (4.6) with respect to $\overline{\zeta}$, we get

$$\frac{\partial v}{\partial \overline{\zeta}} = \frac{1}{B(\zeta)}\frac{\partial B(\zeta)}{\partial \overline{\zeta}}\, v - \frac{B(\zeta)}{L(\zeta)}\,\overline{v}$$

in $\Omega_1 \backslash S$. This equation is elliptic and it follows that locally v is similar to a holomorphic function. Since $v = 0$ on $\partial\Omega_1$, then $v = 0$ everywhere. This completes the proof of the lemma. □

5. Proof of Theorem 2.1

Let $\{w_1, \cdots, w_n\} \subset X_{m,q}(\Omega)$ be a basis of $\mathrm{Ker}(P^*_{L,m})$. It follows from Lemma 4.3 and from the fact that the operator $P_{L,m}$ is Fredholm that we can find $\{h_1, \cdots, h_n\} \subset H_{m,p}(\Omega)$ such that $\langle w_j, h_k\rangle = \delta_{jk}$, where δ_{jk} is the Kronecker symbol, and such that

$$E_{m,p}(\Omega) = \mathrm{Range}(P_{L,m}) + \mathrm{Span}\{h_1, \cdots, h_n\}\,. \tag{5.1}$$

Now given $F \in E_{m,p}(\Omega)$, define

$$f(z) = \frac{-L(z)^m}{\pi} \int_\Omega \frac{F(\zeta)}{L(\zeta)^m(\zeta - z)}\, d\xi d\eta\,. \tag{5.2}$$

Then $f \in E_{m,p}(\Omega)$ and $\dfrac{\partial f}{\partial \overline{z}} = F$. The decomposition (5.1) implies the existence of real constants $c_1, \cdots, c_n$ such that

$$f = g + \sum_{k=1}^{n} c_k h_k\,,$$

with $g \in \mathrm{Range}(P_{L,m})$. Let $u \in E_{m+1,p}(\Omega)$ such that $P_{L,m}u = g$. Thus,

$$P_{L,m}u = u - T_{L,m}u = g = f - \sum_{k=1}^{n} c_k h_k.$$

It follows from the definition of the operator $T_{L,m}$ in (4.1) and from the fact that $h_1, \cdots, h_n$ are holomorphic that

$$\frac{\partial u}{\partial \overline{z}} = \frac{\partial}{\partial \overline{z}}\left(f - \sum_{k=1}^{n} c_k h_k + T_{L,m}u\right) = F(z) + \frac{B(z)}{L(z)}\,\overline{u}.$$

Therefore the function u solves (3.6). This completes the proof of the Theorem 2.1.

Remark 5.1. It follows from the definition of the function f given in (5.2) that if the function F vanishes to infinite order at the singular points $z_1, \cdots, z_N$, then the solution u can be taken to vanish to any prescribed order at the points z_j. For it is enough to take m larger than the prescribed order of vanishing.

6. Proof of Theorem 2.3

To show the existence of nontrivial solutions in Ω of the homogeneous equation

$$\frac{\partial u}{\partial \overline{z}} = \frac{B(z)}{L(z)}\,\overline{u}\,, \tag{6.1}$$

we first recall results from [6] and [7] dealing with particular cases of (6.1).

Let $q(\theta)$ be a 2π-periodic and C^∞ function. It is proved in [6] that there exist a sequence of positive numbers

$$0<\lambda_1<\lambda_2<\,\cdots\,<\lambda_n<\,\cdots \quad \text{with} \quad \lim_{n\to\infty}\lambda_n=\infty$$

and a sequence of nonvanishing 2π-periodic and C^∞ functions $f_k(\theta)$ such for every $k\in\mathbb{Z}^+$, the function $v_k(r,\theta)=r^{\lambda_k}f_k(\theta)$ solves the equation

$$\frac{\partial v}{\partial \overline{z}} = \frac{q(\theta)}{r}\,\overline{v} \text{ where } z=r\mathrm{e}^{i\theta}\,. \tag{6.2}$$

Let $R>0$ and $g\in C^\infty(D(0,R)\backslash\{0\})$ such that $g=O(r^\alpha)$ for some $0<\alpha<1$. Then the proof of Theorem (4.1) of [7] shows that every solution of the equation

$$\frac{\partial v}{\partial \overline{z}} = \frac{q(\theta)+g(r,\theta)}{r}\,\overline{v} \tag{6.3}$$

is similar to a solution of (6.2) and vice versa. This means that for every $v_k(r,\theta)$ solution of (6.2) there exists a bounded function $s_k(r,\theta)$ in the disc $D(0,R)$ such that the function $w_k=v_k\mathrm{e}^{s_k}$ solves (6.3).

Now we turn back to equation (6.1) in the domain Ω. It follows from hypothesis (2.1) that the function B is of the form $q(\theta)+g(r,\theta)$ with $g=O(r^{\tau_j})$ in a neighborhood of the singular point z_j (here the polar coordinates are centered at z_j). It follows that equation (6.1) has a nontrivial solution v_j defined in a neighborhood of z_j and moreover, v_j can be chosen to vanish to any prescribed order at z_j. Thus such a function is class C^k near z_j (provided that the order of vanishing at z_j is large enough).

Let $\epsilon>0$ be such that $D(z_j,2\epsilon)\subset\Omega$ for every $j=1,\cdots,N$ and the discs are pairwise disjoint. For any fixed $k\in\mathbb{N}$ and each j, let $\phi_j\in C^k(\mathbb{C})$ with $\mathrm{Supp}(\phi_j)\subset D(z_j,2\epsilon)$ and $\phi_j\equiv 1$ in the disc $D(z_j,\epsilon)$. We assume that ϵ is small enough so that the functions v_j (described above) are defined in the disc $D(z_j,2\epsilon)$. We are going to construct a solution u of (6.1) in the whole domain Ω of the form

$$u(z)=w(z)+\sum_{j=1}^{N}\phi_j(z)v_j(z)\,. \tag{6.4}$$

In order for the function u to satisfy (6.1), the function w needs to solve

$$\frac{\partial w}{\partial \overline{z}} = \frac{B(z)}{L(z)}\,\overline{w}+F(z)\,, \tag{6.5}$$

with

$$F(z)=\sum_{j=1}^{N}\left(\frac{B(z)}{L(z)}\overline{\phi_j(z)v_j(z)}-\frac{\partial(\phi_j(z)v_j(z))}{\partial\overline{z}}\right). \tag{6.6}$$

Note that $F\equiv 0$ in the discs $D(z_j,\epsilon)$ for $j=1,\cdots,N$. It follows from Theorem 2.1 that equation (6.5) has a solution and that $w\in C^{k+1}(\Omega\backslash S)$ and that w can be chosen to vanish to any prescribed order at the singular points (Remark 5.1).

Now we need to verify that for some set of cut off functions $\{\phi_1,\cdots,\phi_N\}$ the constructed solution u is not trivial. By contradiction, suppose that for every $\{\phi_1,\cdots,\phi_N\}$, $u\equiv 0$. The corresponding function w satisfies

$$\begin{aligned} w &= -\sum_{j=1}^{N} v_j && \text{in } \textstyle\bigcup_{j=1}^{N} D(z_j,\epsilon),\\ w &= 0 && \text{in } \Omega_1=\Omega\backslash\left(\textstyle\bigcup_{j=1}^{N} D(z_j,\epsilon)\right). \end{aligned}$$

Let $u'(z)=w'(z)+\sum_{j=1}^{N}\phi'_j(z)v_j(z)\equiv 0$ be another such solution corresponding to another set $\{\phi'_1,\cdots,\phi'_N\}$ of cut off functions. Consider

$$\phi'_2=\phi_2,\ \cdots,\ \phi'_N=\phi_N \text{ and } \phi'_1\neq\phi_1.$$

Set $\psi=\phi'_1-\phi_1$ and $W=w'-w$. Hence $W\equiv 0$ everywhere except possibly on the annulus $A_\epsilon=D(z_1,2\epsilon)\backslash D(z_1,\epsilon)$ and it satisfies the equation

$$\frac{\partial W}{\partial\overline{z}}=\frac{B(z)}{L(z)}\overline{W}+G_\psi(z), \tag{6.7}$$

where

$$G_\psi(z)=\overline{\psi}(z)\,\frac{\partial v_1(z)}{\partial\overline{z}}-\frac{\partial(\psi(z)v_1(z))}{\partial\overline{z}}=\overline{\psi}(z)\,\frac{B(z)\overline{v_1(z)}}{L(z)}-\frac{\partial(\psi(z)v_1(z))}{\partial\overline{z}}. \tag{6.8}$$

Let $p_0\in A_\epsilon$ such that $B(p_0)\neq 0$ and $v_1(p_0)\neq 0$, and so $\dfrac{\partial v_1}{\partial\overline{z}}(p_0)\neq 0$, and let $\delta>0$ small enough so that $\dfrac{\partial v_1}{\partial\overline{z}}(z)\neq 0$ for all $z\in D(p_0,\delta)$ (the same is again true for B and v_1). We can assume after translation that $p_0=0$. In particular, we are in a situation where for any function

$$\psi=\phi'_1-\phi_1\in C^k(\overline{D(0,\delta)}) \text{ with } \mathrm{Supp}(\psi)\subset\overline{D(0,\delta)},$$

any solution of the equation (6.7) satisfies $W(z)=0$ for all $z\in\mathbb{C}$ with $|z|\geq\delta$. We are going to show that such a situation cannot happen.

We use results from [14], Chapter III, Sections 8 and 10, dealing with representations of generalized analytic functions. In our case, we apply such representation to the solutions of equation (6.7) in the disc $\overline{D(0,\delta)}$. Thus, there exist kernels $K_1(\zeta,z)$ and $K_2(\zeta,z)$ (formulas 8.16, page 168 of [14]) depending only on the coefficient B/L in the disc such that

$$K_1(\zeta,z)=\frac{1}{\zeta-z}+C_1(\zeta,z) \quad \text{and} \quad K_2(\zeta,z)=C_2(\zeta,z), \tag{6.9}$$

where C_1, C_2 are C^∞ functions (because B/L is C^∞ in the annulus A_ϵ) such that any solution W of (6.7) has the representation

$$W(z)=\frac{-1}{\pi}\int_{D(0,\delta)}\Big(K_1(\zeta,z)G_\psi(\zeta)+K_2(\zeta,z)\overline{G_\psi(\zeta)}\Big)\,d\xi d\eta\,. \tag{6.10}$$

Note that the original formula of Vekua contains an additional term involving an integral over the boundary of the domain. But in our case the additional term is 0 because $W=0$ outside $D(0,\delta)$. It follows from (6.10) that

$$\int_{D(0,\delta)}\Big(K_1(\zeta,z)G_\psi(\zeta)+K_2(\zeta,z)\overline{G_\psi(\zeta)}\Big)\,d\xi d\eta=0 \quad \forall z\notin D(0,\delta)\,, \tag{6.11}$$

and this relation holds for any function ψ as above. We use (6.9) to rewrite (6.11) as

$$\int_{D(0,\delta)}\frac{G_\psi(\zeta)d\xi d\eta}{z-\zeta}=\int_{D(0,\delta)}\Big(C_1(\zeta,z)G_\psi(\zeta)+C_2(\zeta,z)\overline{G_\psi(\zeta)}\Big)\,d\xi d\eta\,, \tag{6.12}$$

for $|z|\geq\delta$. By using the expression of G_ψ given in (6.8) we have

$$\int_{D(0,\delta)}\frac{G_\psi(\zeta)d\xi d\eta}{z-\zeta}=-\int_{D(0,\delta)}\frac{\overline{\psi}(\zeta)\,B(\zeta)\overline{v_1(\zeta)}}{L(\zeta)(z-\zeta)}\,d\xi d\eta\,.$$

Now we select the function ψ as

$$\psi(\zeta)=(\delta^2-r^2)^{2k}\frac{\overline{L(\zeta)}}{\overline{B(\zeta)}\,v_1(\zeta)} \quad \text{for } \zeta=re^{i\theta}\in D(0,\delta)\,.$$

It follows that

$$G_\psi(\zeta)=(\delta^2-r^2)^{2k}-\frac{\partial}{\partial\overline{\zeta}}\left((\delta^2-r^2)^{2k}\frac{\overline{L(\zeta)}}{\overline{B(\zeta)}}\right)$$

and so

$$|G_\psi(\zeta)|\leq C_k\delta^{4k-1}, \quad \forall\zeta\in D(0,\delta), \tag{6.13}$$

where C_k is a constant which does not depend on δ. For such a choice of the function ψ the relation (6.12) becomes

$$\int_{D(0,\delta)}\frac{(\delta^2-r^2)^{2k}}{z-\zeta}\,d\xi d\eta=\int_{D(0,\delta)}\Big(C_1(\zeta,z)G_\psi(\zeta)+C_2(\zeta,z)\overline{G_\psi(\zeta)}\Big)\,d\xi d\eta\,, \tag{6.14}$$

for $|z|\geq\delta$. We evaluate the left hand side of (6.14) by using the series expansion $\dfrac{1}{z-\zeta}=\displaystyle\sum_{j=0}^{\infty}\frac{\zeta^j}{z^{j+1}}$, for $|z|>|\zeta|$, and by using polar coordinates to integrate. We find

$$\frac{\pi\delta^{4k+2}}{(2k+1)z} = \int_{D(0,\delta)} \Big(C_1(\zeta,z)G_\psi(\zeta) + C_2(\zeta,z)\overline{G_\psi(\zeta)} \Big)\; d\xi d\eta\,,$$

for $|z| > \delta$. By differentiating with respect to z and using (6.13), we find the estimate

$$\begin{aligned}\frac{\pi\delta^{4k+2}}{(2k+1)|z|^2} \;&\le \int_{D(0,\delta)} \left(\left|\frac{\partial C_1(\zeta,z)}{\partial z}\right| + \left|\frac{\partial C_2(\zeta,z)}{\partial z}\right| \right)\; |G_\psi(\zeta)|\, d\xi d\eta \\ &\le C_k\delta^{4k-1} \int_{D(0,\delta)} \left(\left|\frac{\partial C_1(\zeta,z)}{\partial z}\right| + \left|\frac{\partial C_2(\zeta,z)}{\partial z}\right| \right)\; d\xi d\eta\,.\end{aligned}$$

Since the functions inside the integral are bounded, by possibly increasing C_k we have

$$\frac{\pi\delta^{4k+2}}{(2k+1)|z|^2} \le \pi C_k \delta^{4k+1},$$

where C_k does not depend on δ when it is small. If we take $|z| = \dfrac{3\delta}{2}$, it follows that

$$\frac{\pi\delta^{4k+2}}{(2k+1)} \le \pi C_k \delta^{4k+3} \quad\Rightarrow\; 1 \le (2k+1)C_k\delta.$$

This is clearly a contradiction if we reduce δ sufficiently, which completes the proof of the theorem.

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References

[1] H. Begehr, *Complex analytic methods for partial differential equations. An introductory text.* World Scientific Publishing Co., Inc., River Edge, NJ, (1994).

[2] H. Begehr and D. Dai, *On continuous solutions of a generalized Cauchy-Riemann system with more than one singularity*, J. Differential Equations **196** (2004), no. 1, 67–90.

[3] L. Bers, *An outline of the theory of pseudoanalytic functions*, Bull. Amer. Math. Soc. **62** (1956), 291-331

[4] B. de Lessa Victor and A. Meziani, *Infinitesimal bendings for classes of two-dimensional surfaces* (2021), Preprint.

[5] V. Kravchenko, *Applied pseudoanalytic function theory.* Frontiers in Mathematics. Birkhäuser Verlag, Basel, (2009).

[6] A. Meziani, *Generalized CR equation with a punctual singularity: the model case*, Complex Var. Theory Appl. **48** (2003), no. 6, 495–512.

[7] A. Meziani, *Generalized CR equation with a singularity*, Complex Var. Theory Appl. **48** (2003), no. 9, 739–752.

[8] A. Meziani, *Representation of solutions of a singular Cauchy-Riemann equation in the plane*, Complex Var. Elliptic Equ. **53** (2008), no. 12, 1111–1130.

[9] A. Rasulov and A. Soldatov, *Boundary value problem for a generalized Cauchy-Riemann equation with singular coefficients*. Translation of Differ. Uravn. **52** (2016), no. 5, 637–650. Differ. Equ. **52** (2016), no. 5, 616–629.

[10] A. Rasulov and A. Soldatov, *Generalized Cauchy–Riemann Equations with Power-law Singularities in Coefficients of Lower Order*, Problems and Applications of Operator Theory and Harmonic Analysis, Springer (2020), 535-548.

[11] Y. Rodin, *Generalized analytic functions on Riemann surfaces*, Lecture Notes in Math, **1288**, Springer-Verlag (1980).

[12] Z. Usmanov, *On characteristics of solutions to a model generalized Cauchy-Riemann system degenerating on a part of the boundary*, Complex Var. Elliptic Equ. **51** (2006), no. 8-11, 825–830.

[13] Z. Usmanov, *Generalized Cauchy-Riemann systems with a singular point*, Pitman Monographs and Surveys in Pure and Applied Mathematics, **85**. Longman, Harlow (1997).

[14] I. Vekua, *Generalized analytic functions*, Pergamon Press, (1962).

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