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AN ELEMENTARY PROOF OF CALABI'S THOEREMS ON HOLOMORPHIC CURVES

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§1 - INTRODUCTION

In the study of minimal surfaces in euclidean spaces, Calabi's theory on holomorphic curves has profound influences. Some of its applications can be found in [2,3,5,6]. Calabi's original presentation [1] is, at least to the author, too general to be swallowed. Some alternative versions have been given by Lawson [7] and Griffiths [4]. In this paper we would like to present some simple and elementary proofs for some theorems of Cabali in order to complete Lawson's alternative presentation [7]. We hope, therefore that a quite complete and direct access to the Calabi's theory concerning minimal surfaces is provided.

§2 - RIGIDITY OF HOLOMORPHIC CURVES IN $P^n(\mathbb{C})$

In the complex projective space $P^n(\mathbb{C})$ we use the following normalized Fubini-Study metric:

$$ds^2 = \frac{2|z \wedge dz|^2}{|z|^4} \tag{1}$$

Let $\phi(\zeta)$ and $\psi(\zeta)$ be two holomorphic curves in $P^n(\mathbb{C})$, defined on some connected Riemann surface M. We say that ϕ and ψ are isometric if the induced metrics are the same.

THEOREM 1 - If ϕ and ψ are isometric then there exists an

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isometry A on $P^n(\mathbb{C})$ such that $A\circ\phi = \psi$.

PROOF - By analyticity we may assume that M is a disk D in the complex plane. And, without loss of generality, we may also assume ϕ and ψ are regular and that their homogeneous coordinates $\phi_0(\zeta),\ldots,\phi_n(\zeta)$; $\psi_0(\zeta),\ldots,\psi_n(\zeta)$ are holomorphic functions.

From the hypothesis, we have

$$(2) \qquad \frac{|\phi\wedge\phi'|^2}{|\phi|^4} = \frac{|\psi\wedge\psi'|^2}{|\psi|^4}$$

Together with the identity

$$(3) \qquad \partial\bar{\partial}\log|\phi|^2 = \frac{|\phi\wedge\phi'|^2}{|\phi|^4},$$

we get

$$(4) \qquad \partial\bar{\partial}\log\frac{|\phi|}{|\psi|} = 0,$$

where $\partial = \frac{\partial}{\partial\zeta}$, $\bar{\partial} = \frac{\partial}{\partial\bar{\zeta}}$.

Hence there exists a holomorphic function $f(\zeta)$ such that

$$(5) \qquad |\phi(\zeta)| = |e^{f(\zeta)}|\,|\psi(\zeta)|$$

Now set

$$\tilde{\psi}(\zeta) = e^{f(\zeta)}(\psi_0(\zeta),\ldots,\psi_n(\zeta)).$$

We have clearly

$$(6) \qquad |\phi(\zeta)| \equiv |\tilde{\psi}(\zeta)|$$

Use a theorem of Calabi (See [7,p.149]), we know that there exists a unitary transformation $U\colon \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1}$ such that $\tilde{\psi} = U\circ\phi$. Passing to $P^n(\mathbb{C})$, we get our desired result. Q.E.D.

§3 - HOLOMORPHIC CURVES IN $P^n(\mathbb{C})$ WITH CONSTANT GAUSSIAN CURVATURE

We shall show the theory in 3 steps:

THEOREM 2 - Let $\phi: D \longrightarrow P^n(\mathbb{C})$ be a holomorphic curve with constant curvature $K \equiv c$. Then $c = 2/k$ for some positive integer k.

PROOF - Let $\psi = [\phi, -i\phi]$, which is a holomorphic curve isometric to ϕ in $P^{2n+1}(\mathbb{C})$. ψ represents the generalized Gauss map of the holomorphic curve

(7) $$\Gamma = \int \phi(\zeta) d\zeta$$

in $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$. From a theorem of Calabi (See [7,p.163]),we know that the function

(8) $$F(\zeta) = |\phi(\zeta)|^2$$

has to satisfy the recursive formula

(9) $$F_{j+1} = \frac{F_j^2}{F_{j-1}} \partial\bar{\partial} \log F_j$$

with $F_{k+1} \equiv 0$ for some $k > 0$, where $F_0 = 1$, $F_1 = F$.

On the other hand, from the observation of Lawson [6] about minimal surfaces with constantly curved Gauss map, we see that

$$K \equiv c \iff \partial\bar{\partial}\log|\psi\wedge\psi'|^2 = (2-c)\frac{|\psi\wedge\psi'|^2}{|\psi|^4} \iff \partial\bar{\partial}\log[(-\tilde{K})^{\frac{1}{1+c}}|\psi|^2] \equiv 0 \iff$$

(10) $$\partial\bar{\partial}\log[(-\tilde{K})^{\frac{1}{1+c}} F] \equiv 0$$

where $\tilde{K}$ is the Gaussian curvature of Γ. A detailed proof be found in [3].

From (9), (10) and

(11) $$\tilde{K} = -\frac{2\partial\bar{\partial}\log F}{F}$$

we get

$$F_{j+1} = a_j(-\tilde{K})^{\frac{j(j+1)}{2}} F^{\frac{(j+1)(j+2)}{2}} \tag{12}$$

whenever F_{j+1} is well-defined, where $a_j \geq 0$ is some absolute constant. Comparing (10), (12) and the fact $F_{k+1} \equiv 0$ for some $k > 0$, we see that

$$1+c = \frac{(k+1)(k+2)}{2} \Big/ \frac{k(k+1)}{2}$$

or

$$c = 2/k$$

for some $k > 0$.

Q.E.D.

Further, by a direct computation using (11), one can prove

THEOREM 3 - The curve $\tilde{c}_n \colon D \longrightarrow P^n(\mathbb{C})$ given by

$$\zeta \longmapsto [1, \sqrt{n}\zeta, \ldots, \sqrt{\tbinom{n}{j}}\zeta^j, \ldots, \zeta^n] \tag{13}$$

has constant curvature $2/n$.

Putting theorems 1,2,3 together with the fact that surfaces of the same constant Gaussian curvature are locally isometric, we get

THEOREM 4-

i) The curve $c_n \colon P^1(\mathbb{C}) \longrightarrow P^n(\mathbb{C})$ given in homogeneous coordinates by

$$[z_0, z_1] \longmapsto [z_0^n, \sqrt{n}\, z_0^{n-1} z_1, \ldots, \sqrt{\tbinom{n}{j}}\, z_0^{n-j} z_1^j, \ldots, z_1^n] \tag{14}$$

has constant curvature $2/n$.

ii) Any holomorphic curve $\psi \colon D \longrightarrow P^n(\mathbb{C})$ which has constant curvature and does not lie in any linear subspace of $P^n(\mathbb{C})$ must be unitarily equivalent to the curve $\tilde{c}_n \colon D \longrightarrow P^n(\mathbb{C})$ given in (13)

iii) If $\psi \colon D \longrightarrow P^n(\mathbb{C})$ has constant curvature K, then $K = 2/k$ for some integer in the range $1 \leq k \leq n$. Any two such curves are unitarily equivalent. Furthermore, $K = 2/k$ if and only if ψ takes values in some k-dimensional linear subspace of $P^n(\mathbb{C})$ and is equivalent to

$$\zeta \longmapsto [1, \sqrt{k}\,\zeta, \ldots, \sqrt{\tbinom{k}{j}}\,\zeta^j, \ldots, \zeta^k, 0, \ldots, 0] \tag{15}$$

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