THE CAUSALITY PROBLEM FOR LINEAR VOLTERRA INTEGRAL EQUATIONS

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We present here a problem with a very simple and elementary formulation: let E be a Banach space of numerical functions defined on $[0,1]$. For instance $E = \mathcal{C}([0,1])$; or $E = G([0,1])$ the space of regulated functions, i.e., functions that have only discontinuities of the first kind, endowed with the norm $\|x\| = \sup_{0 \le t \le 1}|x(t)|$; or $E = L_p([0,1])$, $1 \le p \le \infty$. Let

$$H: \Gamma = \{(t,s) \in [0,1] \times [0,1] \mid 0 \le s \le t\} \to \mathbb{R}$$

be a kernel such that

i) For every $x \in E$ we have $\mathcal{H} x \in E$, where

$$(\mathcal{H} x)(t) = \int_0^t H(t,s)x(s)ds, \; 0 \le t \le 1$$

and the integral is the Lebesgue one.

ii) For every $f \in E$ the linear Volterra integral equation

$$\text{(H)} \quad x(t) - \int_0^t H(t,s)x(s)ds = f(t), \; 0 \le t \le 1,$$

has one and one solution $x_f \in E$.

PROBLEM: <u>the operator</u> $f \in E \mapsto x_f \in E$ is <u>necessarely causal</u>?

(i.e., if $f \equiv 0$ on some interval $[0,c] \subset [0,1]$ does it follow that $x_f \equiv 0$ on the same interval?)

Remarks - 1) If $E = \mathcal{C}([0,1])$ and the kernel H is a continuous function, or, if $E = L_2([0,1]$ and $H \in L_2(\Gamma)$, it is well known that the answer is positive; see [2]; see also [3], p. 79, exerc. 3.15

to 3.18. The proof consists in showing that there exists a resolvent kernel $S: [0,1]\times[0,1] \to \mathbb{R}$ such that the solution x_f of (H) is given by

$$(\rho) \quad x_f(t) = f(t) + \int_0^t S(t,s)f(s)ds, \; 0 \le t \le 1;$$

the main point in the proof is to assure that there exists the Neumann series

$$\sum_{n=1}^{\infty} H^{(n)}(t,s) = S(t,s),$$

where $H^{(1)}(t,s) = H(t,s)$ and $H^{(n+1)}(t,s) = \int_s^t H^{(n)}(t,\sigma)H(\sigma,s)d\sigma$, since $H(t,s) = 0$ for $s>t$ it follows that $S(t,s) = 0$ for $s>t$, hence the causality.

2) In equation (H), in general, we may not have unicity of solutions: if $E = \mathcal{C}([0,1])$, every $\lambda \in]0,1]$ is an eigenvalue of the equation

$$\lambda x(t) - \frac{1}{t}\int_0^t x(s)ds = 0, \; 0 \le t \le 1,$$

with eigenfunction $x_\lambda(t) = t^{\frac{1}{\lambda}-1}$.

3) Equation (H) may have one and only one solution in $E = \mathcal{C}([0,1])$ but may have other solutions that are not in E even if H is continuous: see an example by Urysohn in [2], p. 18.

4) If we consider, more generally, a linear Volterra-Stieltjes integral equation

$$(K) \quad x(t) + \int_0^t d_sK(t,s).x(s) = f(t), \; 0 \le t \le 1,$$

the answer to the corresponding problem may be negative. So for

$E = \mathcal{C}([0,1])$ and $K(t,s) = \chi_{[t^2,t[}(s)$ (where $\chi_A(s) = \begin{cases}1 \text{ if } s \in A\\ 0 \text{ if } s \notin A\end{cases}$),

for every $f \in E$ there exists one and only one solution $x_f \in E$ of (K) and we have $x_f(t) = f(\sqrt{t})$, $0 \le t \le 1$, hence we do not have causality; see [4], the Example 2 that precedes the Theorem 3.4.

Next we bring some comments on the problem when $E = G([0,1])$

5) Take $E = G([0,1])$ and $H: \Gamma \to \mathbb{R}$; for every $x \in E$ we have $\mathcal{H}x \in E$ iff the kernel H has the properties

a) $\sup_{0 \le t \le 1} \int_0^t |H(t,s)|ds < \infty$.

b) For every $s \in [0,1]$ the function $t \in [s,1] \longmapsto \int_s^t H(t,\sigma)d\sigma \in \mathbb{R}$ is regulated.

From [1], exerc. 46 on p.517, follows an analogous characterization of the kernels when $E = \mathcal{C}([0,1])$.

We do not know a characterization of the operators $\mathcal{H} \in L[G([0,1])]$ defined by kernels H that satisfy a) and b) above.

6) When the answer to our problem is positive we do not know if the solution x_f is given by (ρ) with S having the properties a) and b) of the remark 5.

7) Equation (K) is a particular instance of equation (H): we take

$$K(t,s) = \int_s^t H(t,\sigma)d\sigma,\ 0 \le s \le t \le 1.$$

8) For properties equivalent to the causality of the solutions of equation (K), and hence of equation (H), see Theorem 3.4 of [4]; see also Théorème 3.4 of [7], or Theorem 1.7 [5] or Theorem 3.2 of [6].

9) The operator $\mathcal{H} \in L[G([0,1])]$ defined by the kernel H is compact iff the function $t \in [0,1] \longmapsto H^t \in L_1([0,1])$ is regulated, where $H^t(s) = \begin{cases} H(t,s) & \text{if } s \leq t \\ 0 & \text{if } t < s \end{cases}$

10) If the operator $\mathcal{H}$ defined by H is compact, or has compact power, then we have causality; for a proof see Théorème 3.23 of [7].

11) In general the proof of the causality of the solutions of (H) or of (K) reduces, directly or indirectly, to the proof of the convergence, in L(E), of the Neumann series defined by the kernel or by a modified kernel. But see the remark 10 and its proof, and , on the other hand, Coroll. 3.14 of [7].

12) The problem and the results above may easily be extended to functions with values in $\mathbb{R}^n$ or, more generally, in a Banach space. See [4].

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