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CLASSIFICATION OF QUADRATIC DIFFERENTIAL SYSTEMS WITH INVARIANT HYPERBOLAS ACCORDING TO THEIR CONFIGURATIONS OF INVARIANT HYPERBOLAS AND INVARIANT LINES

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Classification of quadratic differential systems with invariant hyperbolas according to their configurations of invariant hyperbolas and invariant lines

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Abstract

Let **QSH** be the whole class of non-degenerate planar quadratic differential systems possessing at least one invariant hyperbola. In this article, we study family $\mathbf{QSH}_{(\eta>0)}$ of systems in **QSH** which possess three distinct real singularities at infinity. We classify this family of systems, modulo the action of the group of real affine transformations and time rescaling, according to their geometric properties encoded in the configurations of invariant hyperbolas and invariant straight lines which these systems possess. The classification is given both in terms of algebraic geometric invariants and also in terms of invariant polynomials and it yields a total of 162 distinct such configurations. This last classification is also an algorithm which makes it possible to verify for any given real quadratic differential system if it has invariant hyperbolas or not and to specify its configuration of invariant hyperbolas and straight lines.

Key-words: quadratic differential systems, configuration, invariant hyperbolas and lines, affine invariant polynomials, group action

2000 Mathematics Subject Classification: 34C23, 34A34

1 Introduction and statement of the main results

A real planar polynomial differential system is a differential system of the form

$$dx/dt = p(x,y), dy/dt = q(x,y) \tag{1}$$

where $p(x,y), q(x,y)$ are polynomial in x, y with real coefficients ($p, q \in \mathbb{R}[x,y]$). We call degree of such a system the number $\max(\deg(p), \deg(q))$.

A real quadratic differential system is a polynomial differential system of degree 2, i.e.

$$\begin{aligned} \dot{x} &= p_0 + p_1(\tilde{a}, x, y) + p_2(\tilde{a}, x, y) \equiv p(\tilde{a}, x, y), \\ \dot{y} &= q_0 + q_1(\tilde{a}, x, y) + q_2(\tilde{a}, x, y) \equiv q(\tilde{a}, x, y) \end{aligned} \tag{2}$$

with $\max(\deg(p), \deg(q)) = 2$ and

$$\begin{aligned} p_0 &= a, \quad p_1(\tilde{a}, x, y) = cx + dy, \quad p_2(\tilde{a}, x, y) = gx^2 + 2hxy + ky^2, \\ q_0 &= b, \quad q_1(\tilde{a}, x, y) = ex + fy, \quad q_2(\tilde{a}, x, y) = lx^2 + 2mxy + ny^2. \end{aligned}$$

Here we denote by $\tilde{a} = (a, c, d, g, h, k, b, e, f, l, m, n)$ the 12-tuple of the coefficients of system (2). Thus a quadratic system can be identified with a points $\tilde{a}$ in $\mathbb{R}^{12}$.

We denote the class of all quadratic differential systems with **QS**.

Planar polynomial differential systems occur very often in various branchesb of applied mathematics, in modeling natural phenomena, for example, modeling the time evolution of conflicting species in biology and in chemical reactions and economics [13, 30], in astrophysics [6], in the equations of continuity describing the interactions of ions, electrons and neutral species in plasma physics [21]. Polynomial systems appear also in shock waves, in neural networks, etc. Such differential systems have also theoretical importance. Several problems on polynomial differential systems, which were stated more than one hundred years ago, are still open: the second part of Hilbert's 16th problem stated by Hilbert in 1900, the problem of algebraic integrability stated by Poincaré in 1891 [18], [19], the problem of the center stated by Poincaré in 1885 [17], and problems on integrability resulting from the work of Darboux [9] published in 1878. With the exception of the problem of the center, which was solved only for quadratic differential systems, all the other problems mentioned above, are still unsolved even in the quadratic case.

Our main motivation in this article comes from bintegrability problems related to the work of Darboux. Darboux showed that if a polynomial system has s invariant algebraic curves $f_i(x, y) = 0$ with $f_i \in \mathbb{C}[x, y]$ with $s \geq m(m+1)/2$ where m is the degree of the system, then the system is integrable. This condition is only sufficient for Darboux integrability (integrability in terms of invariant algebraic curves) and it is not also necessary. Thus the lower bound on the number of invariant curves sufficient for Darboux integrability stated in the theorem of Darboux is larger than necessary. Darboux' theory has been improved by including for example the multiplicity of the curves. Also, the number of invariant algebraic curves needed was reduced but by adding some conditions, in particular the condition that any two of the curves be transversal. But a deeper understanding about Darboux integrability is still lacking. We point out that algebraic integrability, which intervenes in the open problem of Poincaré, and which means the existence of a rational first integral for the system, is a particular case of Darboux integrability.

To advance knowledge on algebraic or more generally Darboux integrability it is necessary to have a large number of examples to analyze. In the literature scattered isolated examples were analyzed but a more systematic approach was still needed. Schlomiuk and Vulpe initiated a systematic program to construct such a data base for quadratic differential systems. Since the simplest case is of systems with invariant straight lines, their first works involved only lines (see [23], [25], [26], [27], [28]). In this work we study the class **QSH** of non-degenerate, i.e. p, q are relatively prime, quadratic differential systems having an invariant hyperbola. Such systems could also have some invariant lines and in many cases the presence of these invariant curves turns them into Darboux integrable systems. We always assume here that the systems (2) are non-degenerate because otherwise doing a time rescaling, they can be reduced to linear or constant systems. Under this assumption all the systems in **QSH** have a finite number of finite singular points.

On the class **QS** acts the group of real affine transformations and time rescaling and due to this, modulo this group action quadratic systems ultimately depend on five parameters. This group also acts on **QSH** and modulo this action the systems in this class depend on three parameters.

As we want this study to be intrinsic, independent of the normal form given to the systems, we use here invariant polynomials and geometric invariants for the classification.

An important ingredient in this work is the notion of *configuration of invariant curves* of a polynomial differential system. This notion appeared for the first time in [23], restricted for invariant lines.

Definition 1. *Consider a planar polynomial system of degree n. By **configuration of invariant algebraic curves** of this system we mean the set of (complex) invariant algebraic curves (which may have real coefficients) of the system, each one of these curves endowed with its own multiplicity and together with all the real singular points of this system located on these invariant curves, each one of these singularities endowed with its own multiplicity.*

We associate to each system in **QSH** its *configuration of invariant hyperbolas and straight lines*, in other words in the definition above we limit ourselves only to hyperbolas and lines.

Let us suppose that a polynomial differential system has an invariant algebraic curve $f(x,y)=0$ where $f(x,y)\in\mathbb{C}[x,y]$ is of degree n, $f(x,y)=a_0+a_{10}x+a_{01}y+...+a_{n0}x^n+a_{n-1,1}x^{n-1}y+...+a_{0n}y^n$ with $\hat{a}=(a_0,...,a_{0n})\in\mathbb{C}^N$ where $N=(n+1)(n+2)/2$. We note that the equation $\lambda f(x,y)=0$ where $\lambda\in\mathbb{C}^*$ and $\mathbb{C}^*=\mathbb{C}\backslash\{0\}$ yields the same locus of complex points in the plane as the the locus induced by $f(x,y)=0$. So that a curve of degree n defined by $\hat{a}$ can be identified with a point $[\hat{a}]=[a_0:a_{10}:...:a_{0n}]$ in $P_{N-1}(\mathbb{C})$. We say that a sequence of degree n curves $f_i(x,y)=0$ converges to a curve $f(x,y)=0$ if and only if the sequence of points $[a_i]$ converges to $[\hat{a}]=[a_0:a_{10}:...:a_{0n}]$ in the topology of $P_{N-1}(\mathbb{C})$.

Definition 2. *We say that an invariant curve $\mathcal{L}:f(x,y)=0$, $f\in\mathbb{C}[x,y]$ (respectively $\mathcal{L}_\infty:Z=0$) for a quadratic system S has **multiplicity** m if there exists a sequence of real quadratic systems S_k converging to S, such that each S_k has m (respectively $m-1$) distinct (complex) invariant curves $\mathcal{L}_{1,k}:f_{1,k}(x,y)=0,\dots,\mathcal{L}_{m,k}:f_k(x,y)=0$, converging to $\mathcal{L}$ (respectively to $\mathcal{L}_\infty$) as $k\to\infty$, in the topology of $P_{N-1}(\mathbb{C})$, and this does not occur for $m+1$ (respectively m).*

It is clear that the configuration of invariant curves of a system is an affine invariant. In particular the notion of multiplicity defined by Definition 2 is invariant under the group action, i.e. if a quadratic system S has an invariant curve $\mathcal{L}=0$ of multiplicity m, then each system S' in the orbit of S under the group action has a corresponding invariant line $\mathcal{L}'=0$ of the same multiplicity m.

In [15] the authors provide necessary and sufficient conditions for a non-degenerate quadratic differential system to have at least one invariant hyperbola and these conditions are expressed in terms of the coefficients of the systems.

The goal of this article is to produce a classification with respect to the configurations of invariant hyperbolas and invariant lines, of the whole class of non-degenerate quadratic differential systems possessing three real distinct singular points at infinity, i.e. when $\eta > 0$ (we denote this family by $\mathbf{QSH_{(\eta>0)}}$). This classification should $\mathbf{QSH_{(\eta>0)}}$ also be expressed in terms of invariant polynomials so that no matter how a system may be presented to us, we should be able to verify by using this classification whether the system has or does not have invariant hyperbolas and to detect its configuration.

Main Theorem. *Consider the class of all non-degenerate systems in $\boldsymbol{QSH_{(\eta>0)}}$ possessing an invariant hyperbola.*

(A) This family is classified according to the configurations of invariant hyperbolas and of invariant straight lines of the systems, yielding 162 distinct such configurations. This geometric classification appears in Diagrams 1 to 7. More precisely:

- *(A_1) There are exactly 3 configurations of systems possessing an infinite number of hyperbolas which are distinguished by the number and multiplicity of the invariant straight lines of such systems.*
- *(A_2) The remaining 159 configurations could have up to a maximum of 3 distinct invariant hyperbolas, real or complex, and up to 4 distinct invariant straight lines, real or complex, including the line at infinity.*

  Assuming we have m invariant hyperbolas $H_i : f_i(x, y) = 0$, m' invariant lines $L_j : g_j(x, y) = 0$, the geometry of the configurations is in part captured by the following invariants:

  - *(a) the type of the main divisor $\sum n(H_i)H_i + \sum n(L_j)L_j$ on the plane $P_2(\mathbb{R})$, where $n(H_i)$, $n(L_j)$ indicate the multiplicity of the respective invariant curve;*
  - *(b) the type of the zero-cycle $MS_{0C} = \sum l_iU_i + \sum m_js_j$ on the plane $P_2(\mathbb{R})$, where l_i, m_j indicate the multiplicity on the real projective plane, of the real singularities at infinity U_i and in the finite plane s_j of a system (2), located on the invariant lines and invariant hyperbolas;*
  - *(c) the number of the singular points of the systems which are smooth points of the curve: $T(X, Y, Z) = \prod F_i(X, Y, Z) \cdot \prod G_j(X, Y, Z) \cdot Z = 0$ where F_i, G_j's are the homogenizations of f_i's, g_j's respectively, where $f_i = 0$ are the invariant hyperbolas and $g_j = 0$ are the invariant straight lines, and by their positions on $T(X, Y, Z) = 0$. This position is expressed in the proximity divisor PD on the Poincaré disk of a system, defined in Section 2.*

  We have 120 configurations of systems with exactly one hyperbola which is simple:

  - *(i) 40 of them with no invariant line other than the line at infinity: 36 of them having only a simple line at infinity, 2 of them having a double line at infinity, and 2 of them having a triple line at infinity;*
  - *(ii) 46 of them with only one invariant line other than the line at infinity: 39 of them having only simple lines, 3 of them with a double finite line, and 4 of them with the line at infinity being double;*

(iii) 23 of them with two distinct simple affine invariant lines (real or complex) and a simple line at infinity;

(iv) 6 of them with three simple invariant straight lines other than the line at infinity;

(v) 2 of them with two simple lines and one double line: 1 of them with a double finite line and 1 of them with a double line at infinity;

(vi) 3 of them with four simple invariant straight lines other than the line at infinity.

We have exactly 35 configurations with hyperbolas of total multiplicity two:

(xi) 11 of them with no invariant straight line other than the line at infinity;

(xii) 5 of them with only one simple invariant straight line other than a simple line at infinity;

(xiii) 11 of them with exactly two invariant lines which are simple other than the line at infinity, which 2 of them with a double hyperbola;

(xiv) 3 of them with exactly one double line either in the finite plane or at infinity;

(xv) 5 of them with three simple invariant straight lines other than the line at infinity.

We have exactly 4 configurations with three distinct hyperbolas:

(xvi) 2 of them with only one invariant straight line other than the line at infinity;

(xvii) 2 of them with exactly two invariant lines which are simple other than the line at infinity.

*(B) The **affine classification** of these configurations is done in terms of invariant polynomials in Diagrams 8 and 9.* Dana

The diagrams Diagrams 8 and 9 give an algorithm to compute the configuration of a system with an invariant hyperbola for any system presented in any normal form and they are also the bifurcation diagrams of the configurations of such systems, done in the 12-parameter space of the coefficients of these systems.

Remark 1. *The polynomials $\chi_W^{(i)}$ in Diagrams 8 to 9, where $W = A, \ldots, G$ and $i = 1, \ldots, 8$, are introduced in Section 3 within the proof of part (B) of the Main Theorem whenever they become necessary, whereas other invariant polynomials (η, θ, μ_i, $\beta_j \ldots$ and so on) are introduced in Section 2.*

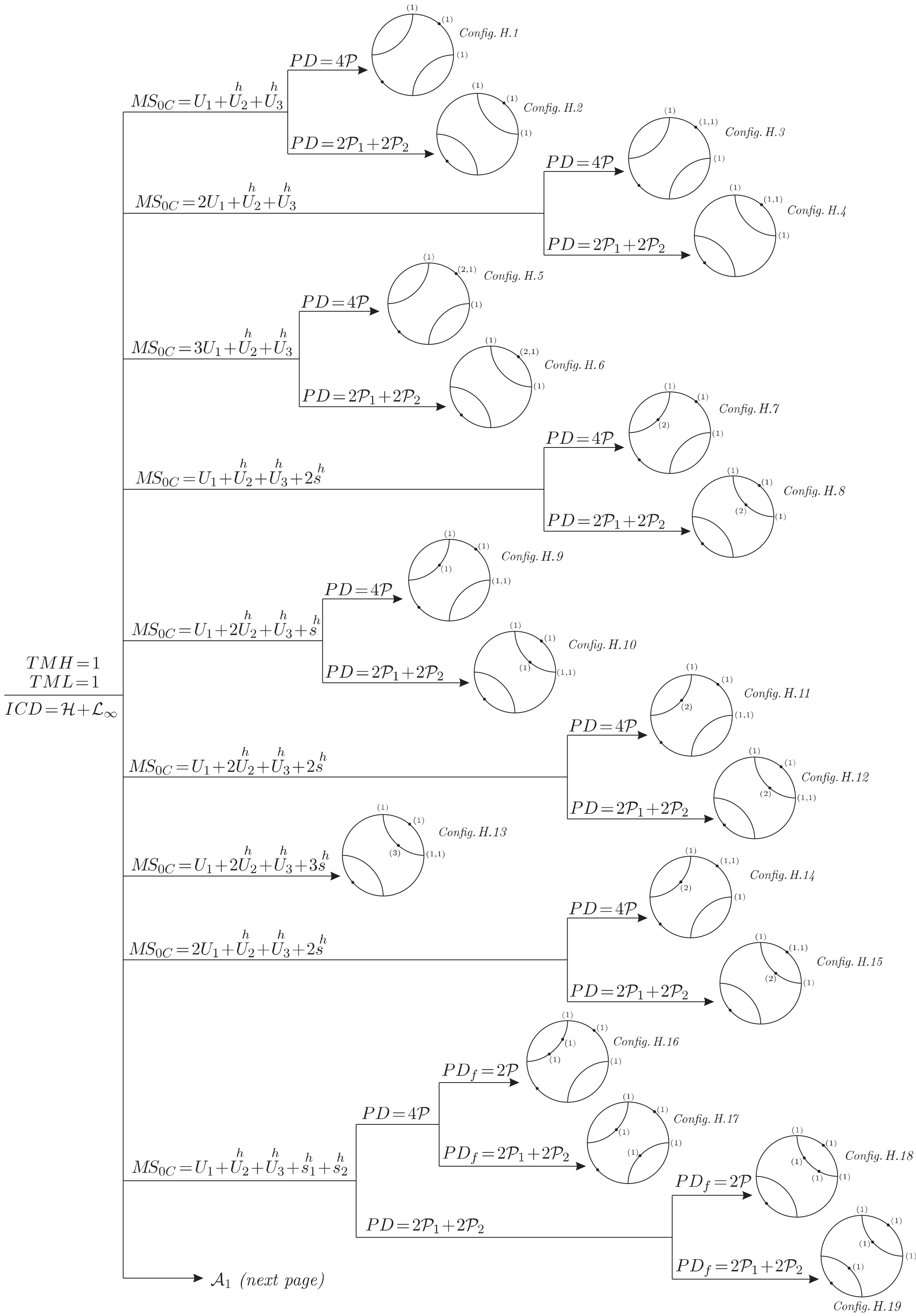

DIAGRAM 1: **Diagram of configurations with one hyperbola and $TML=1$**

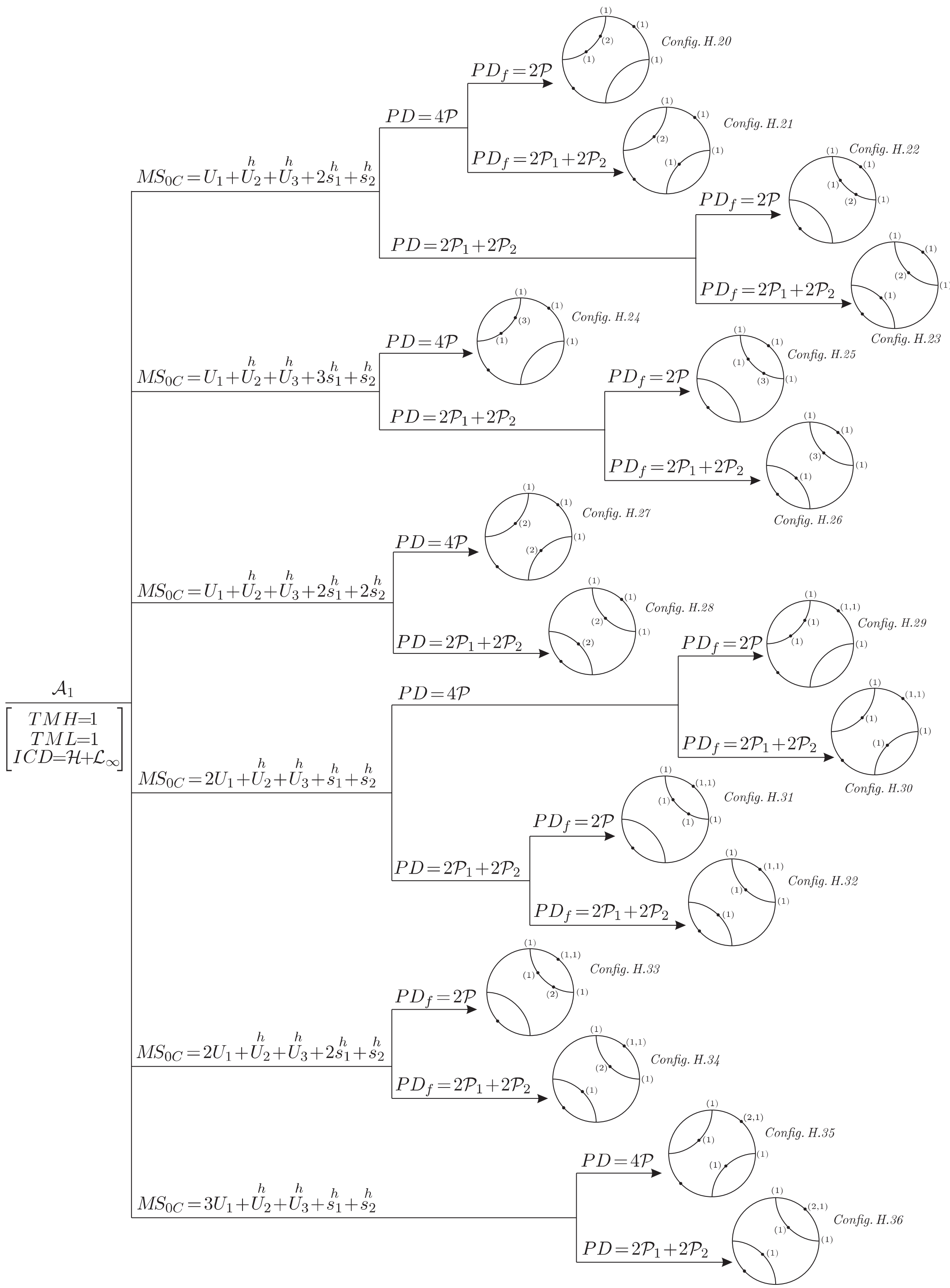

Diagram 1: (*Cont.*) **Diagram of configurations with one hyperbola and** $TML=1$

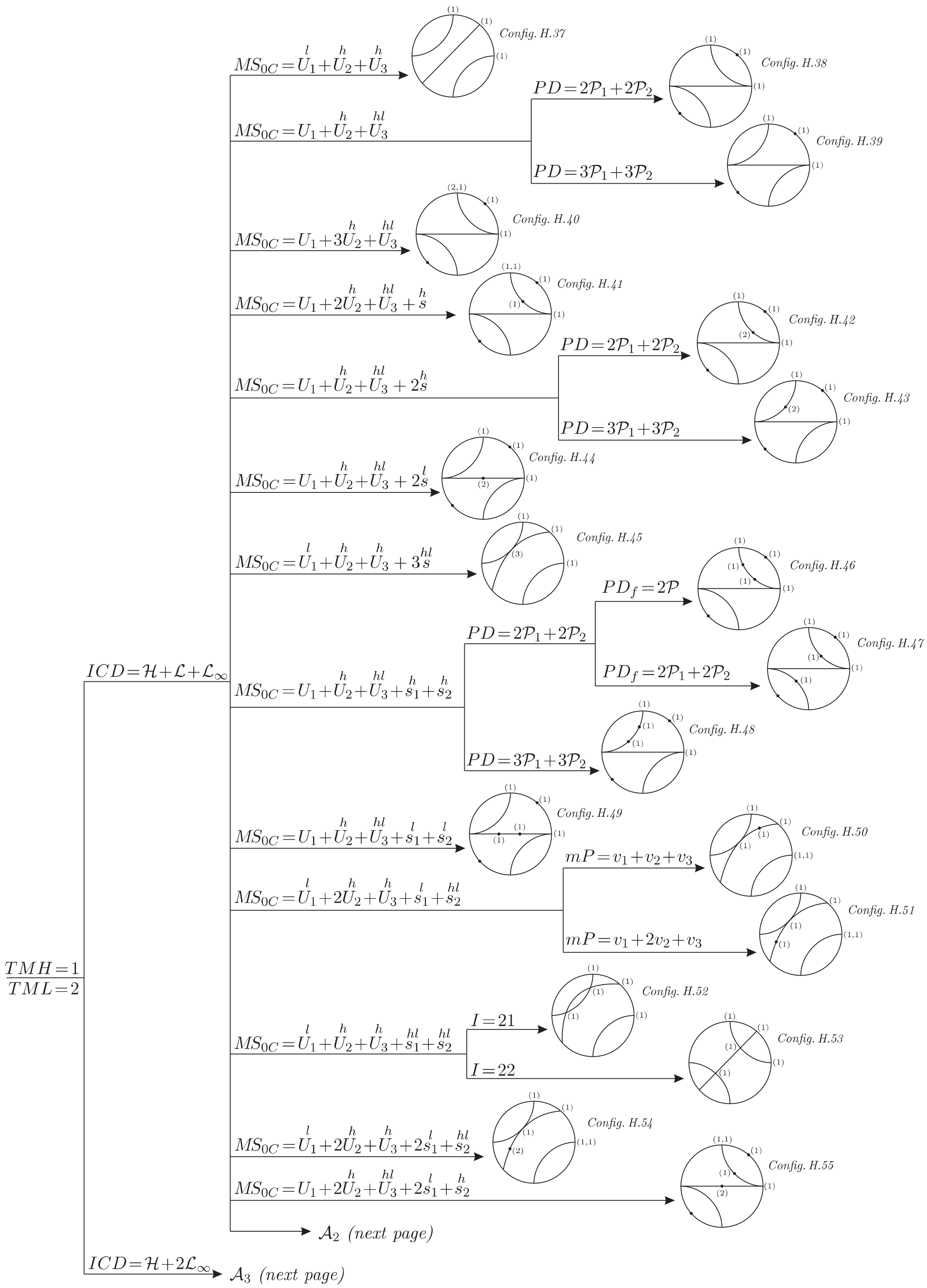

Diagram 2: **Diagram of configurations with one hyperbola and $TML = 2$**

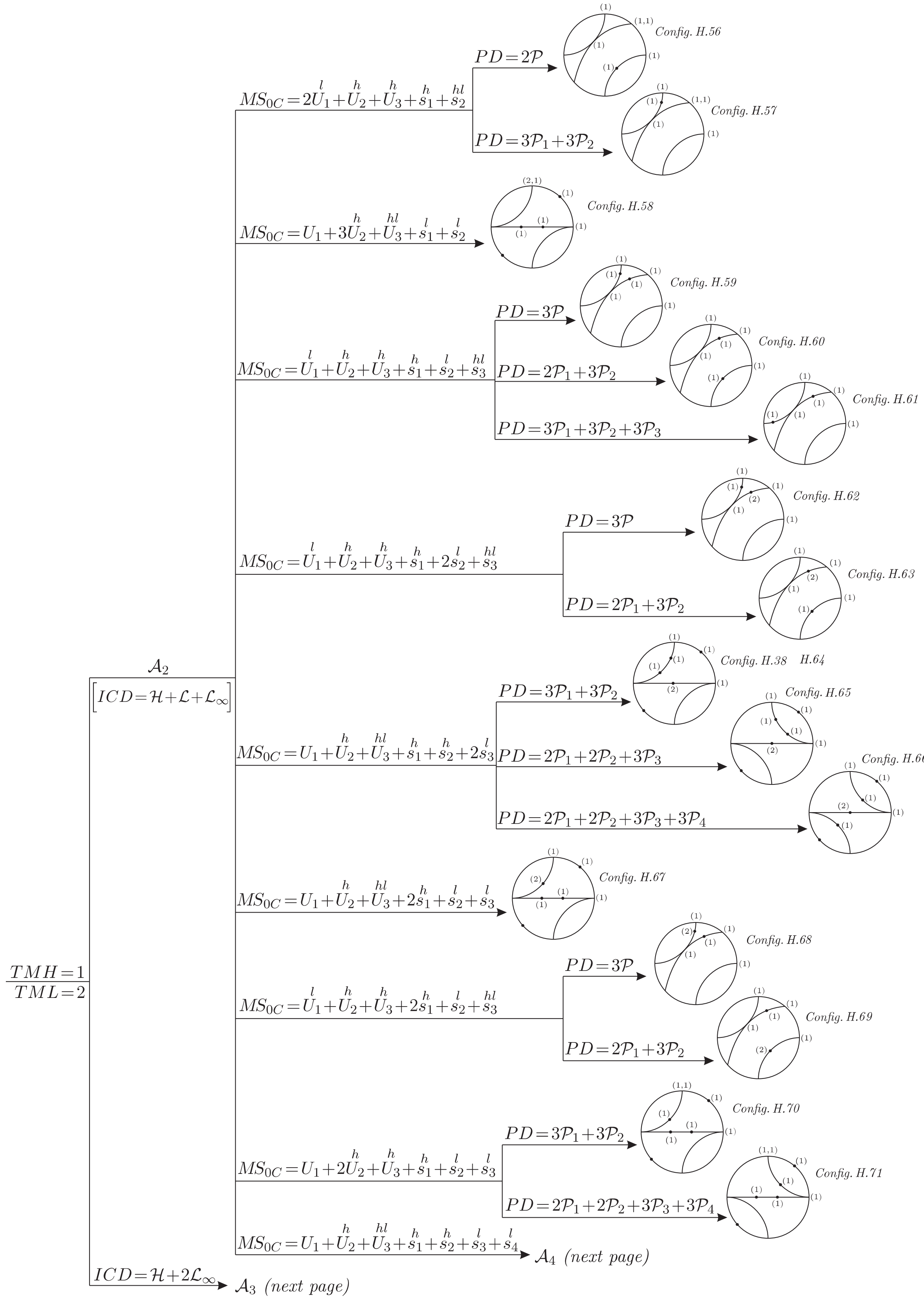

Diagram 2: (*Cont.*) **Diagram of configurations with one hyperbola and $TML = 2$**

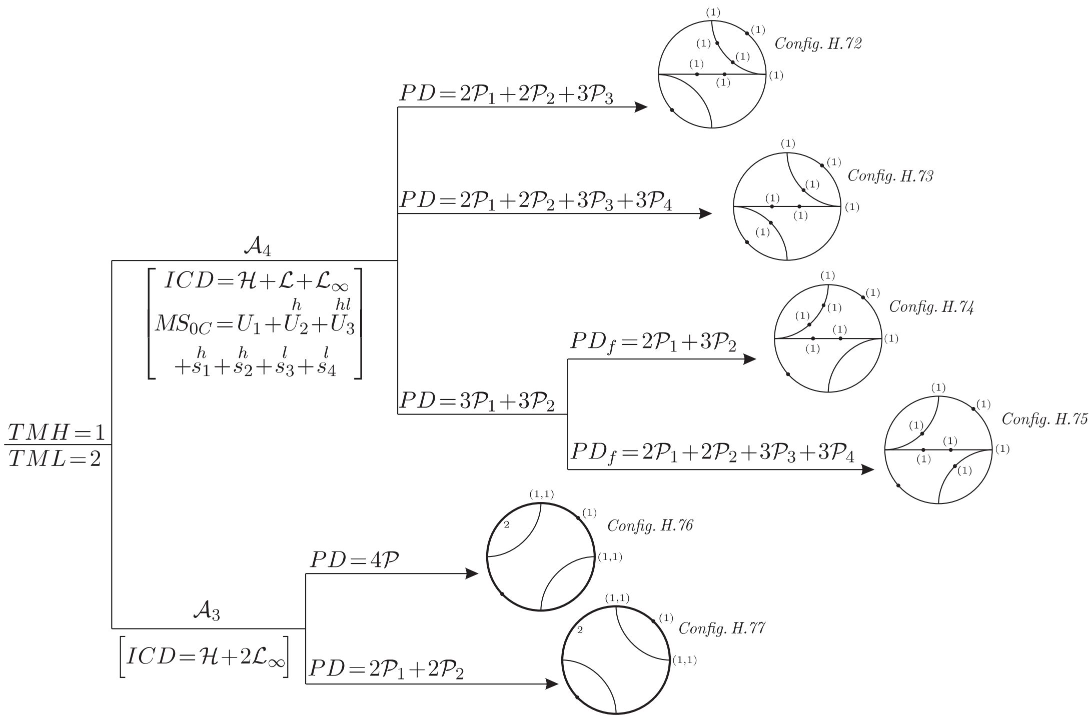

Diagram 2: (*Cont.*) **Diagram of configurations with one hyperbola and** $TML = 2$

2 Basic concepts, proof of part A of the Main Theorem and auxiliary results

In order to understand the classification stated in the Main Theorem we need to define all the invariants we used and to state some auxiliary results. A quadratic system possessing an invariant hyperbola could also possess invariant lines. We classified the systems possessing an invariant hyperbola in terms of their configurations of invariant hyperbolas and invariant lines. Each one of these invariant curve has a multiplicity as defined in [8]. We encode this picture in the multiplicity divisor of invariant hyperbolas and lines. We first recall the algebaic-geometric definition of an r-cycle on an irreducible algebraic variety of dimension n.

Definition 3. *Let V be an irreducible algebraic variety of dimension n over a field K. A cycle of dimension r or r-cycle on V is a formal sum $\Sigma_W n_W W$, where W is a subvariety of V of dimension r which is not contained in the singular locus of V, $n_W \in Z$, and only a finite number of n_W's are non-zero. We call degree of an r-cycle the sum Σn_W. An $(n-1)$-cycle is called a divisor.*

For polynomial differential systems (S) possessing a finite number of irreducible affine invariant algebraic curves $f_i(x, y) = 0$, each with multiplicity n_i, we may define the *multiplicity divisor* on the complex projective plane, of the invariant algebraic curves as being the divisor $ICD = \Sigma_{n_i} n_i F_i(X, Y, Z) + n_\infty Z$ where $F_i(X, Y, Z) = 0$ are the projective completions of $f_i(x, y) = 0$, n_i is the multiplicity of the curve $F_i = 0$ and n_∞ is the multiplicity of the line at infinity $Z = 0$.

In the case we consider here, we have a particular instance of this divisor because the invariant curves will be the invariant hyperbolas and invariant lines of a quadratic differential system, in case these are in finite number. Another ingredient of the configuration of invariant hyperbolas and

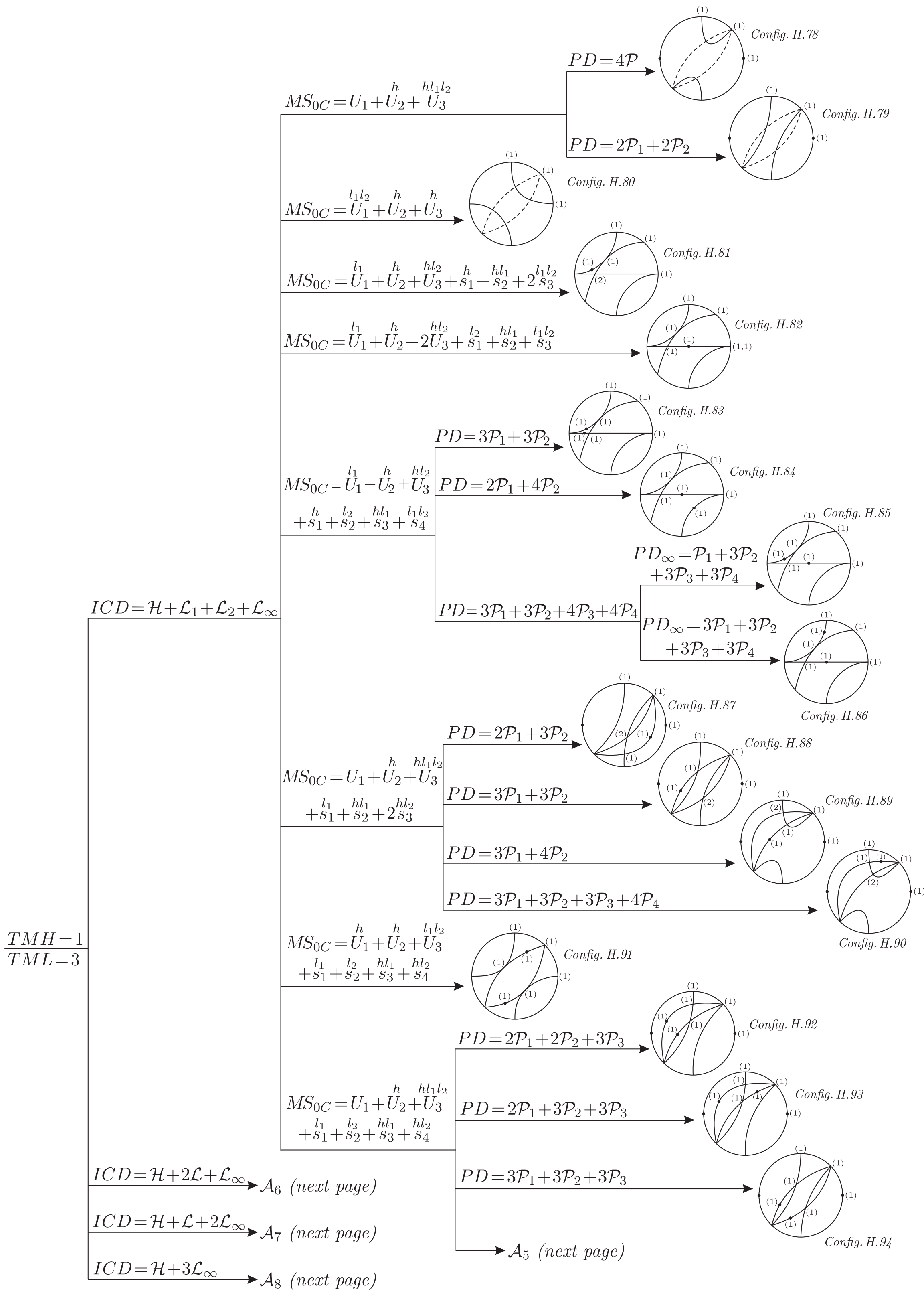

DIAGRAM 3: **Diagram of configurations with one hyperbola and $TML = 3$**

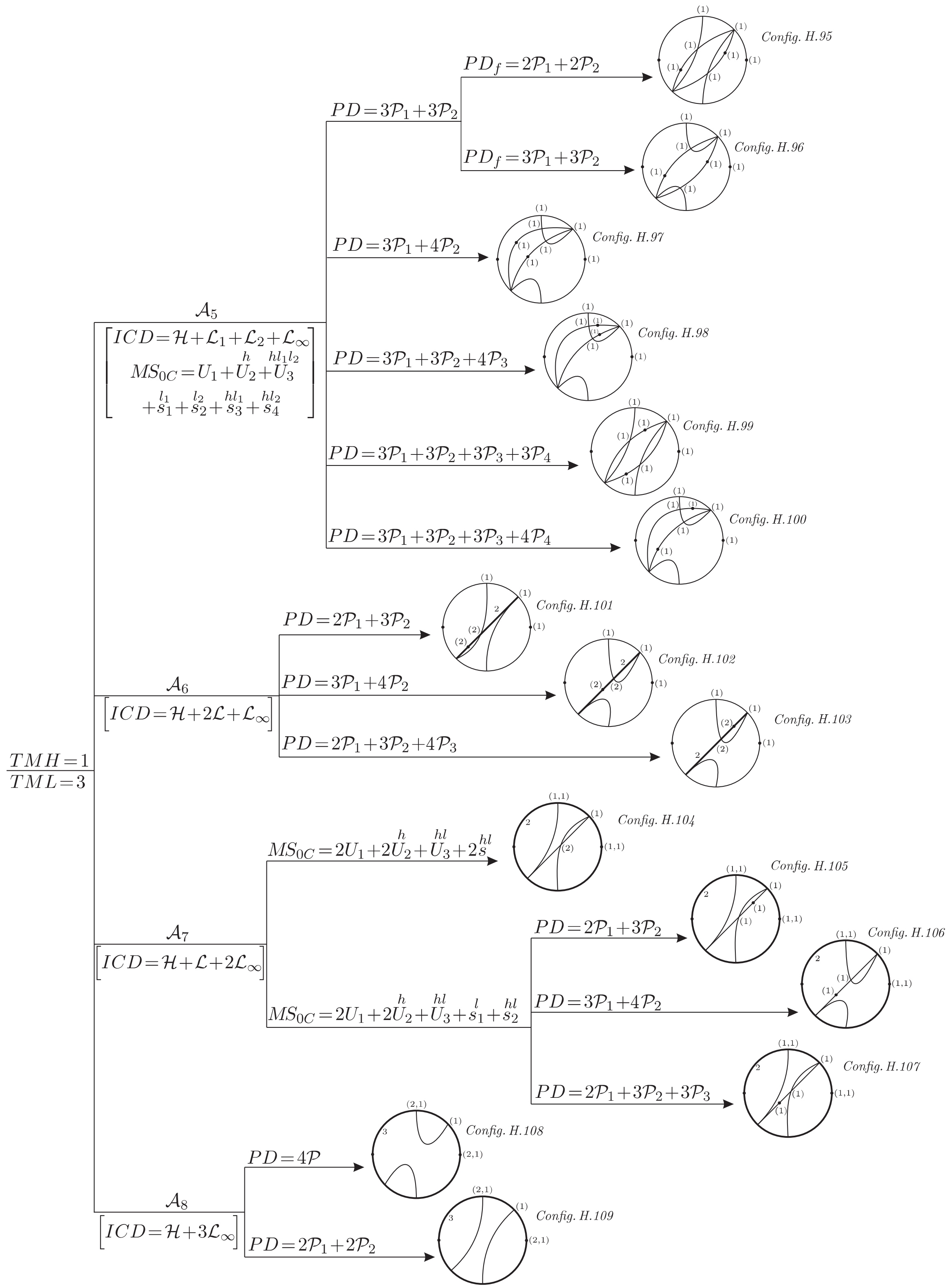

Diagram 3: (*Cont.*) **Diagram of configurations with one hyperbola and $TML = 3$**

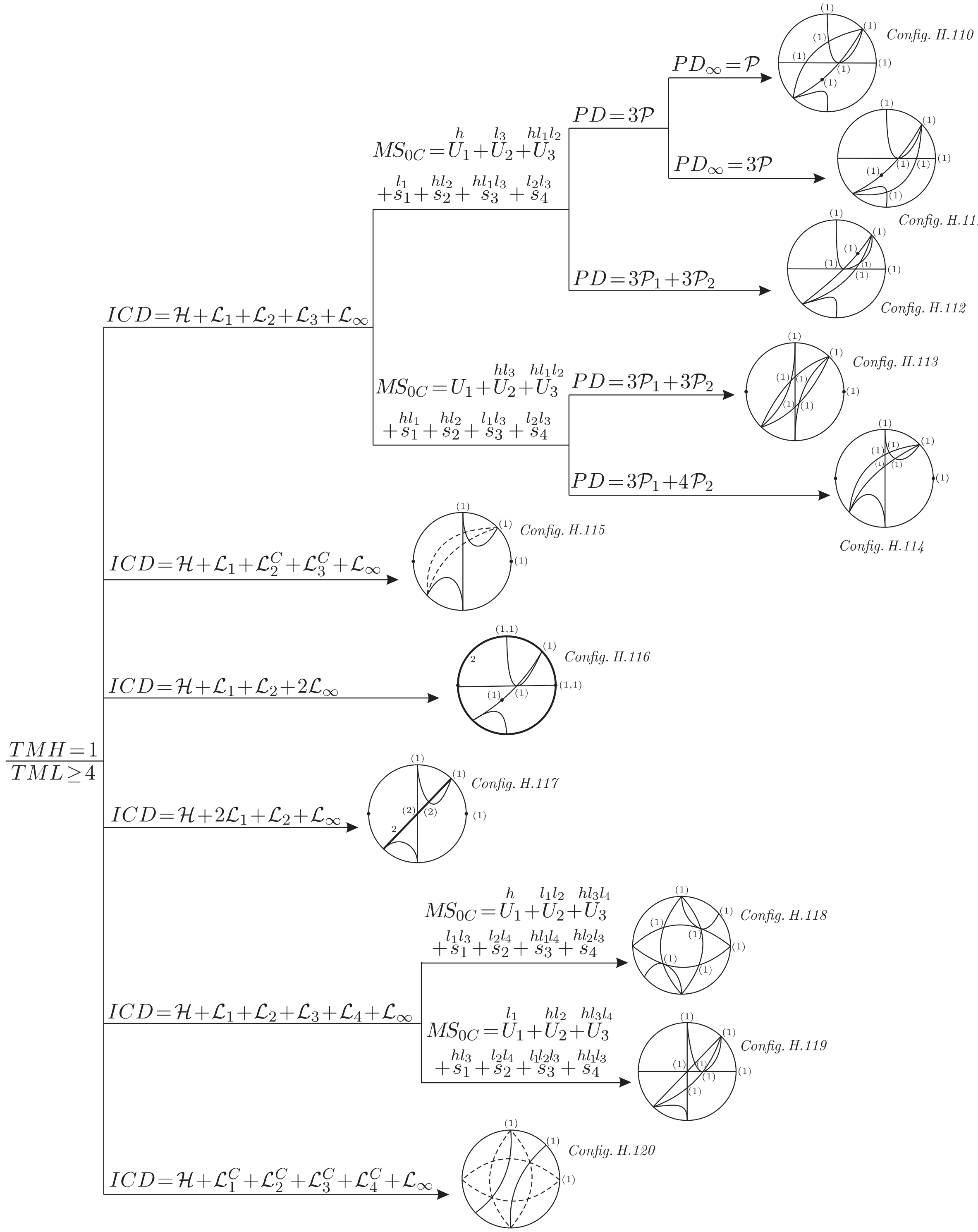

Diagram 4: **Diagram of configurations with one hyperbola and** $TML\geq 4$

invariant lines are the real singularities situated on these invariant curves. We therefore also need to use here the notion of multiplicity zero-cycle on the real projective plane of real singularities of a system.

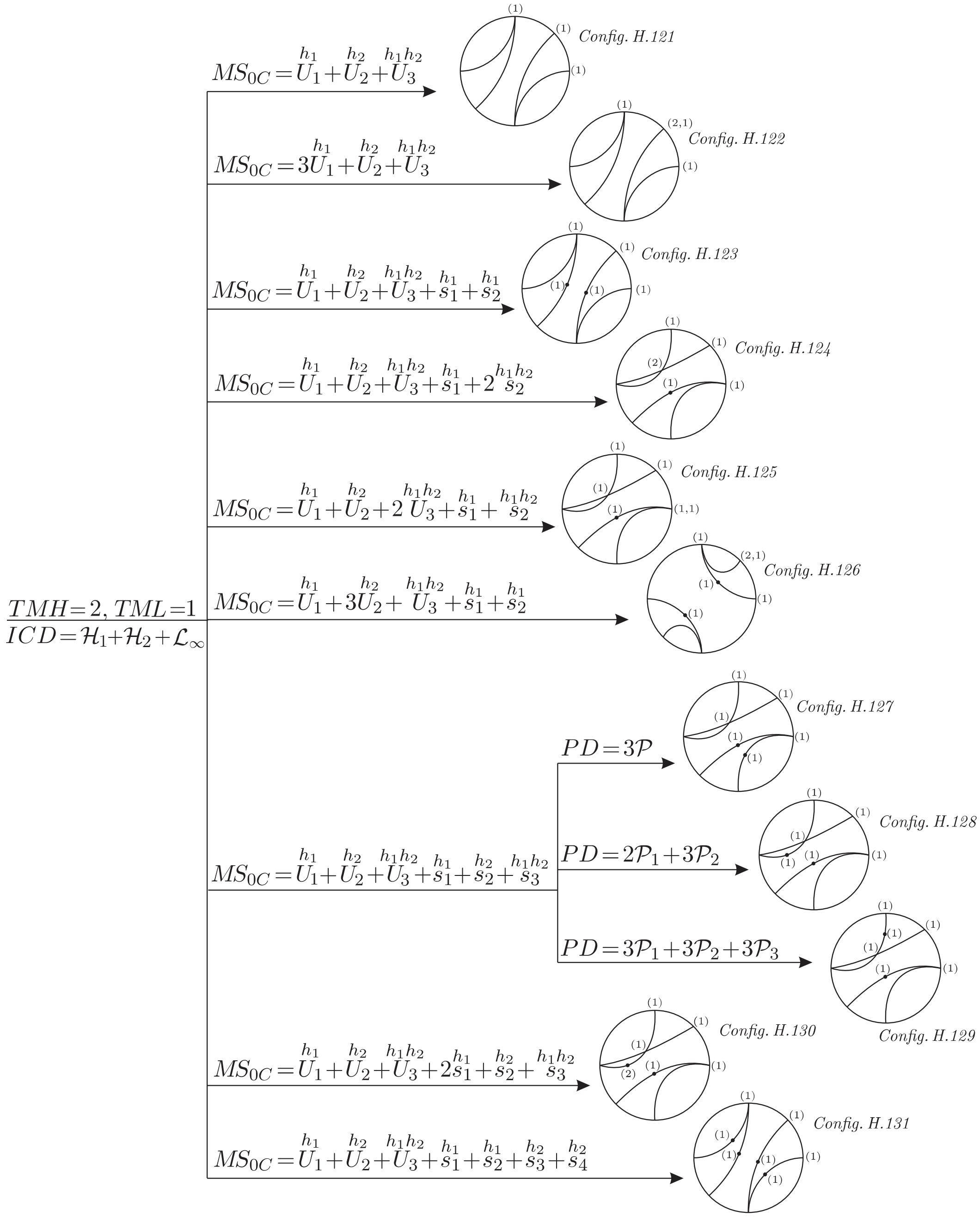

DIAGRAM 5: **Diagram of configurations with two hyperbolas and $TML = 1$**

Definition 4. *(1) Suppose a real polynomial system has a finite number of invariant hyperbolas and invariant lines. The divisor of invariant hyperbolas and invariant lines on the complex projective plane of such a polynomial system is the cycle defined as:*

$$ICD = n_1\mathcal{H}_1 + \ldots + n_k\mathcal{H}_l + m_\infty\mathcal{L}_\infty + m_1\mathcal{L}_1 + \ldots + m_k\mathcal{L}_k,$$

where $\mathcal{H}_j$ are the invariant hyperbolas, $\mathcal{L}_i$ are the invariant lines, n_j (respectively m_i) is the multiplicity of the hyperbola $\mathcal{H}_j$ (respectively of the line $\mathcal{L}_i$), $\mathcal{L}_\infty$ is the line at infinity and m_∞

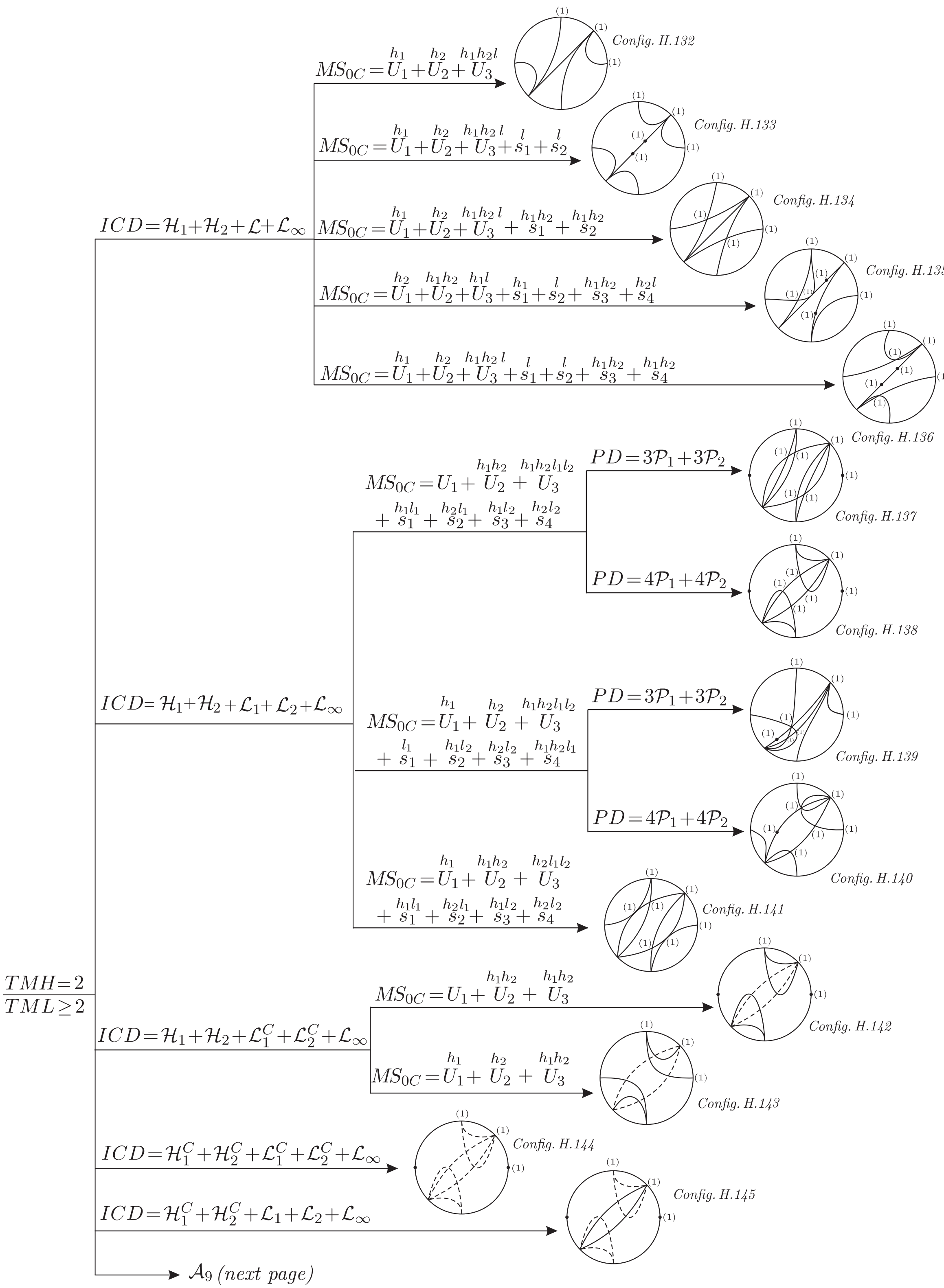

Diagram 6: **Diagram of configurations with two hyperbolas and $TML \geq 2$**

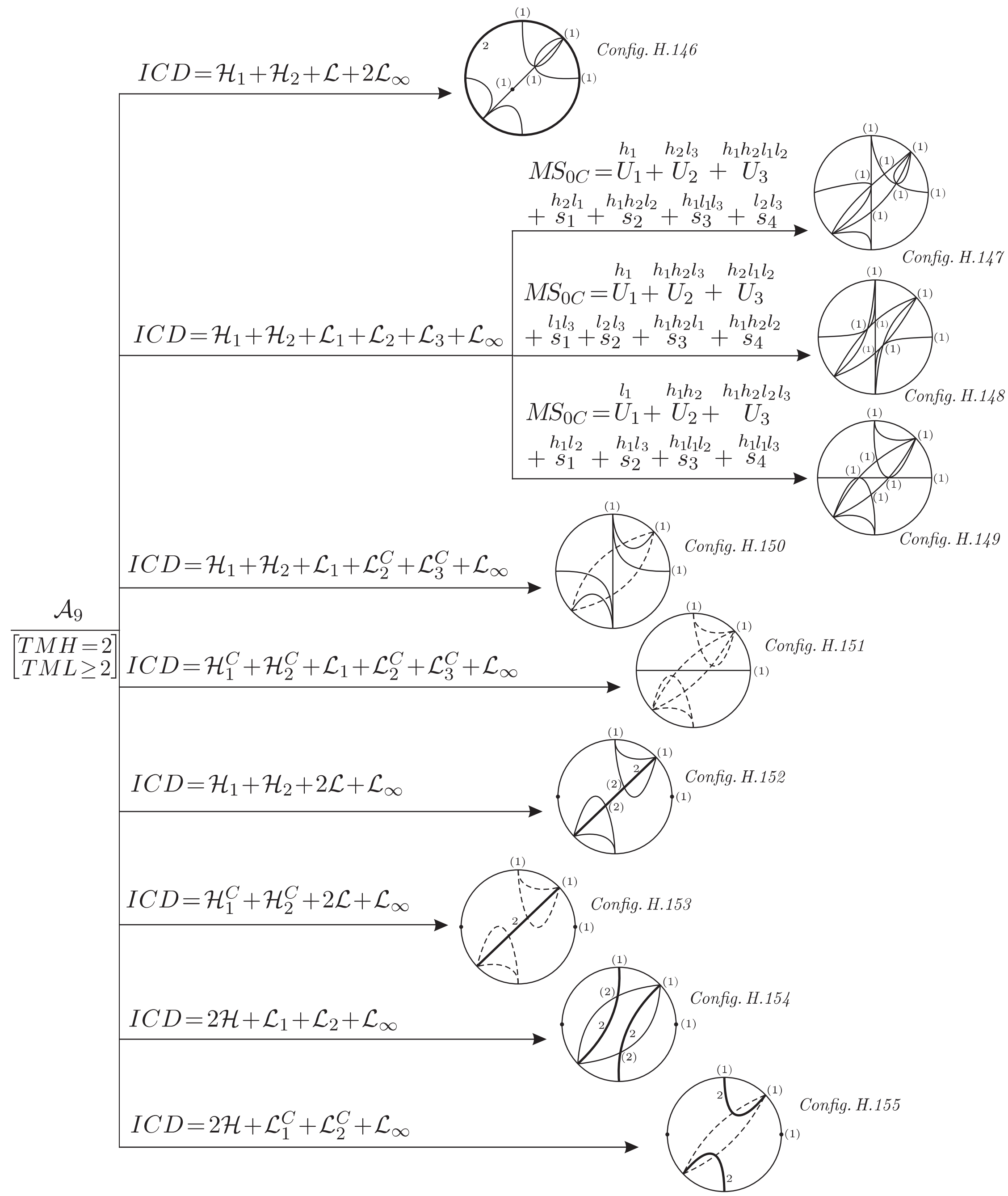

Diagram 6: (*Cont.*) **Diagram of configurations with two hyperbolas and $TML \geq 2$**

is its multiplicity. We also mark the complex invariant hyperbolas (respectively lines) denoting them by $\mathcal{H}_i^C$ (respectively $\mathcal{L}_i^C$). We define the total multiplicity TMH of invariant hyperbolas as the sum $\sum_i n_i$ and the total multiplicity TML of invariant line as the sum $\sum_i m_i$.

In case we have an infinite number of hyperbolas we define $ICD = m_\infty\mathcal{L}_\infty + m_1\mathcal{L}_1 + \ldots + m_k\mathcal{L}_k$;

(2) *The zero-cycle on the real projective plane, of real singularities of a system* (2) *located on the*

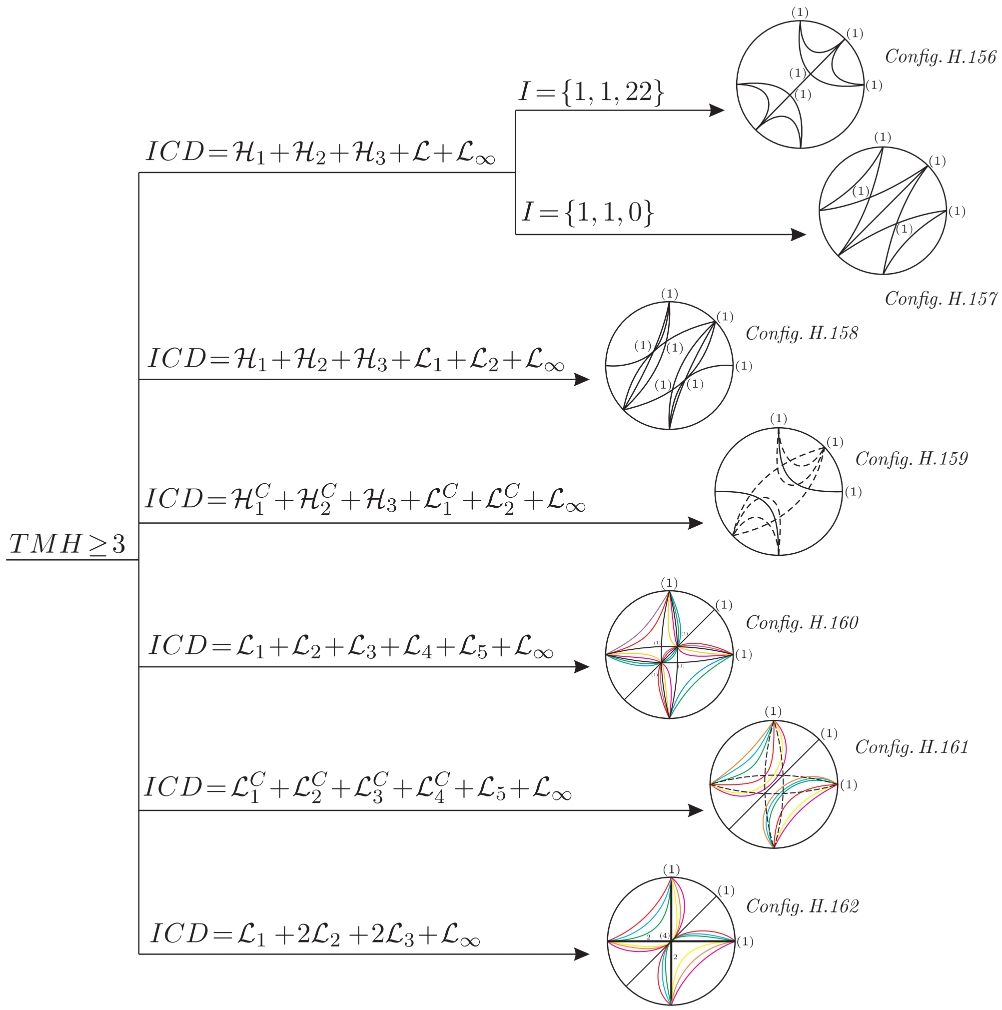

Diagram 7: **Diagram of configurations with three or more hyperbolas** ($TMH \geq 3$)

configuration of invariant lines and invariant hyperbolas, is given by:

$$MS_{0C} = l_1 U_1 + \ldots + l_k U_k + m_1 s_1 + \ldots + m_n s_n,$$

where U_i (respectively s_j) are all the real infinite (respectively finite) such singularities of the system and l_i (respectively m_j) are their corresponding multiplicities.

As the Main Theorem indicates, we also have three cases with an infinite number of hyperbolas but in these cases we have a finite number of invariant lines and the systems are classified by their configurations of invariant straight lines encoded in the invariant lines divisor.

The above defined divisor and zero-cycle contain several invariants such as the number of invariant lines and their total multiplicity TML, the number of invariant hyperbolas in case there are a finite number of them and their total multiplicity TMH, the number of "complex invariant hyperbolas" of a real system. This term requires some explanation. Indeed the term hyperbola is reserved for a real irreducible affine conic which has two real points at infinity. This distinguishes it from the other two irreducible real conics: the parabola with just one real point at infinity and the ellipse which has

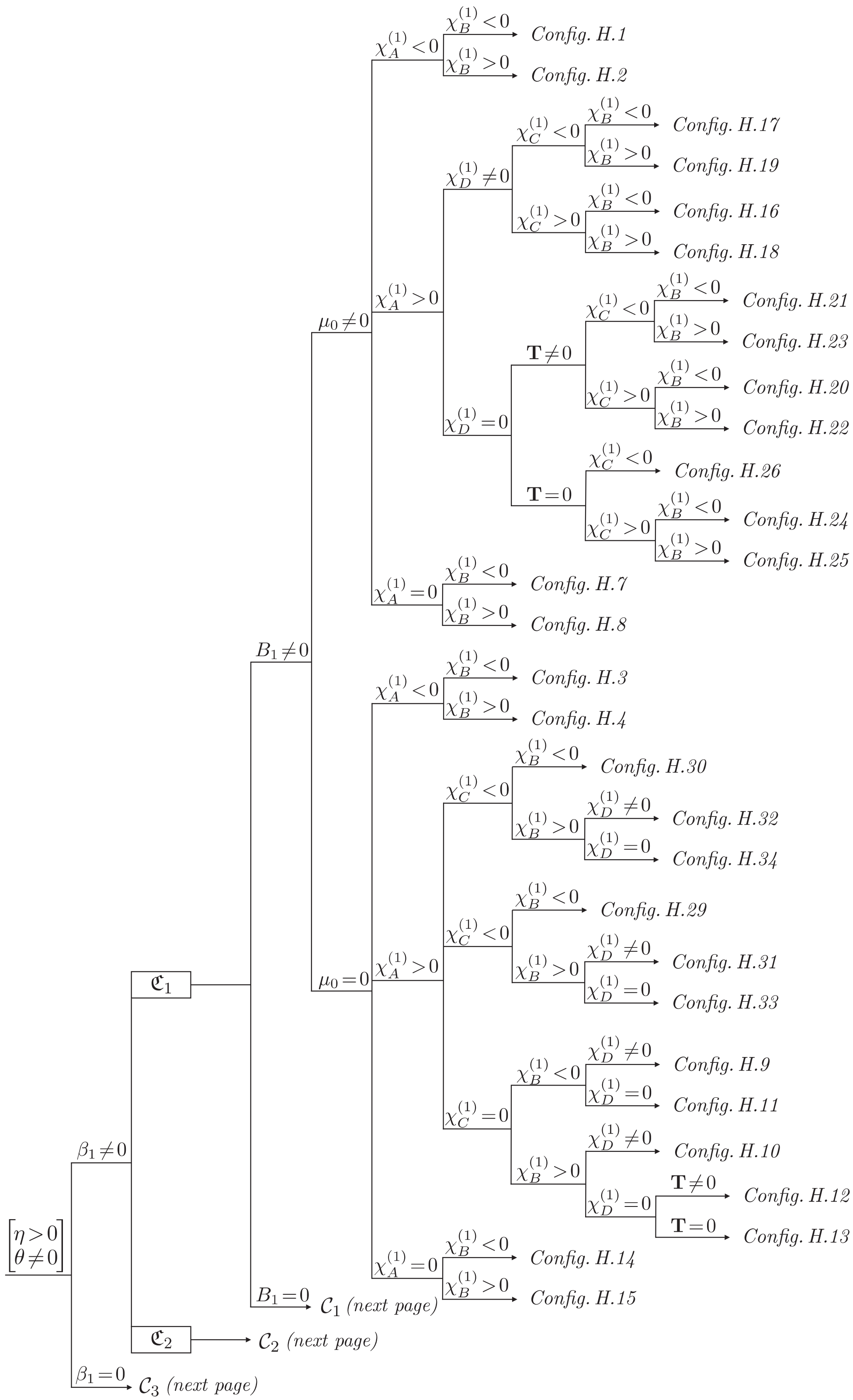

DIAGRAM 8: **Bifurcation diagram in $\mathbb{R}^{12}$ of the configurations: Case $\eta > 0$, $\theta \neq 0$**

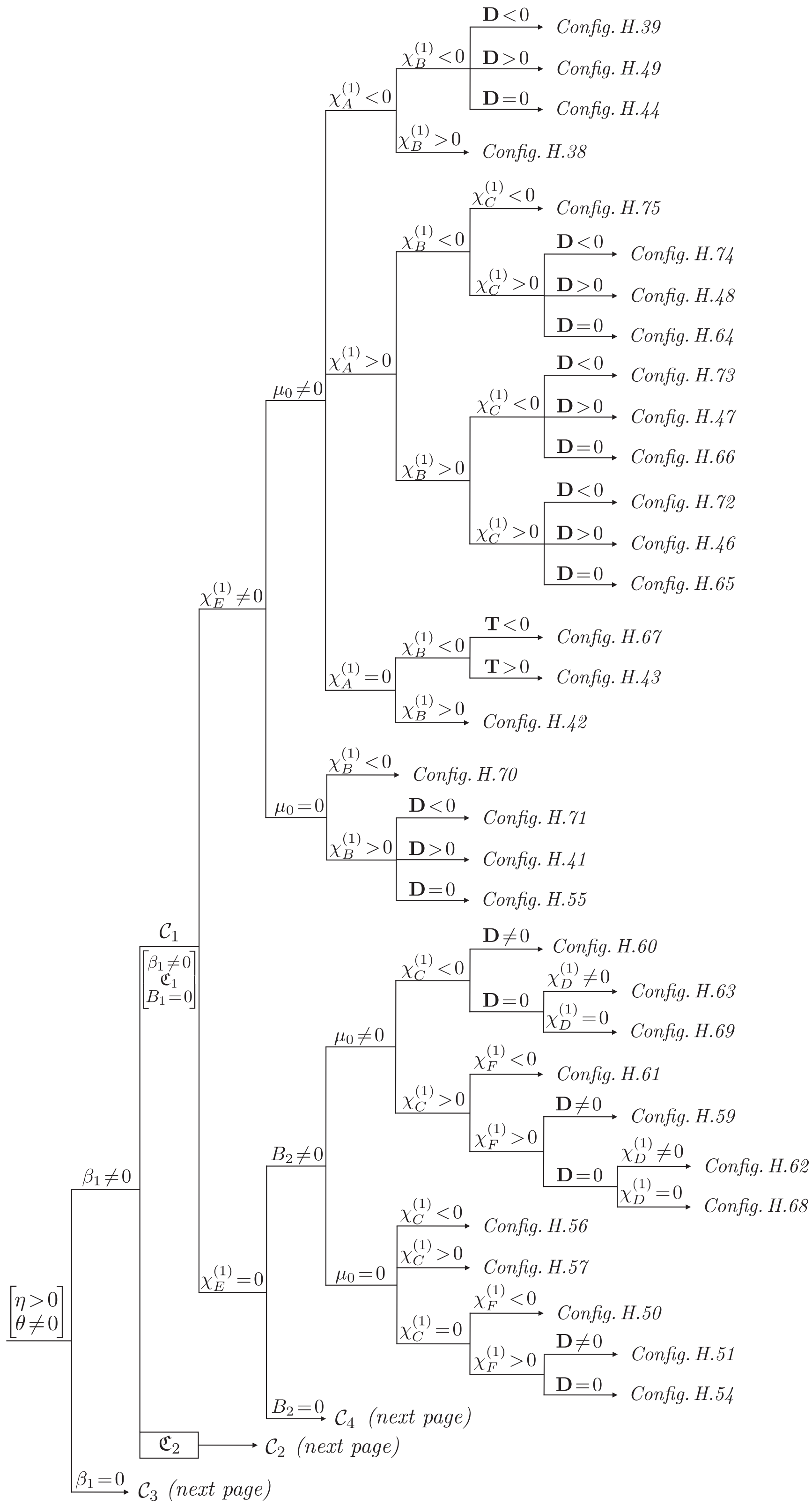

DIAGRAM 8: (*Cont.*) **Bifurcation diagram in $\mathbb{R}^{12}$ of the configurations: Case $\eta>0$, $\theta\neq 0$**

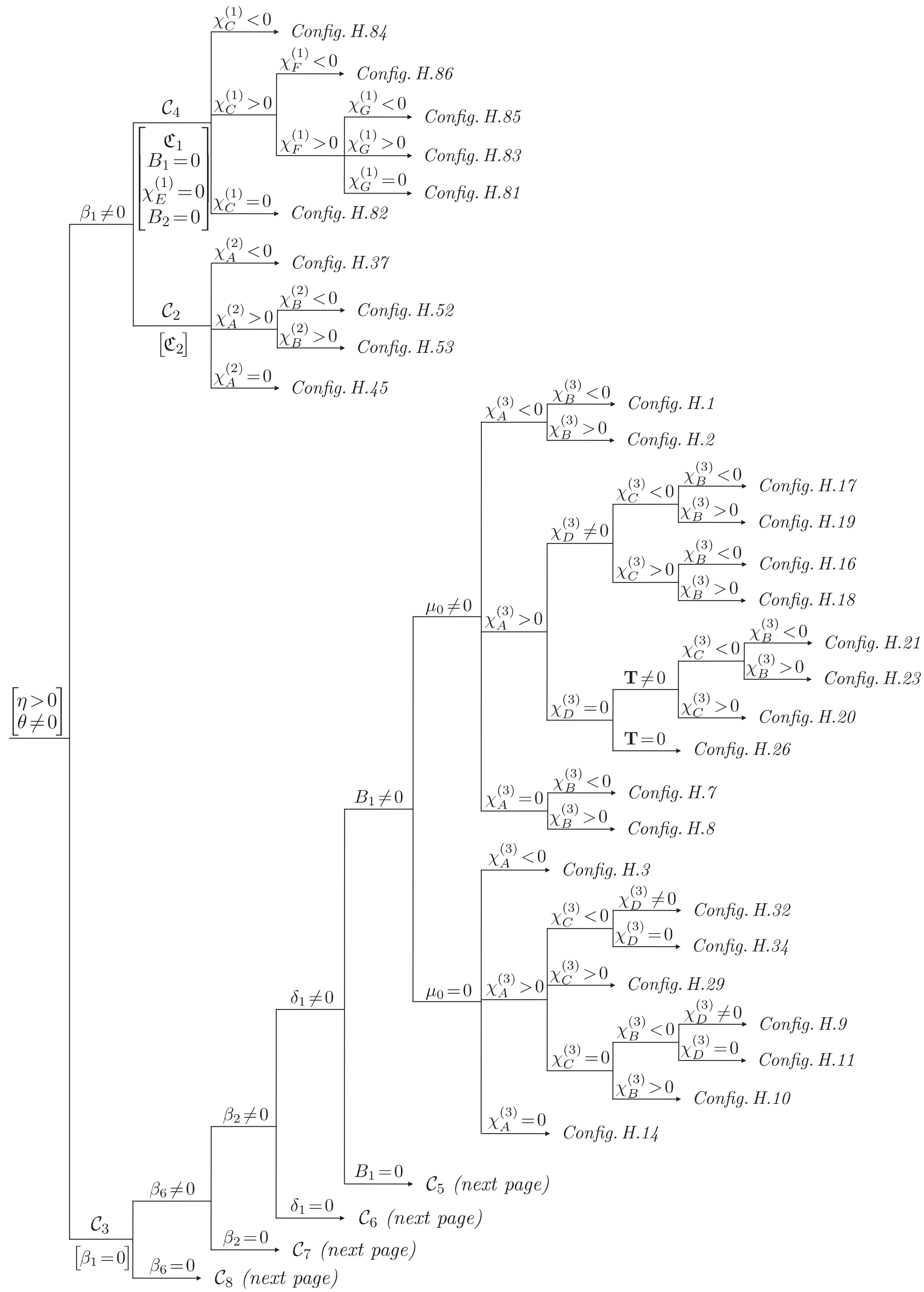

DIAGRAM 8: (*Cont.*) **Bifurcation diagram in $\mathbb{R}^{12}$ of the configurations: Case $\eta > 0$, $\theta \neq 0$**

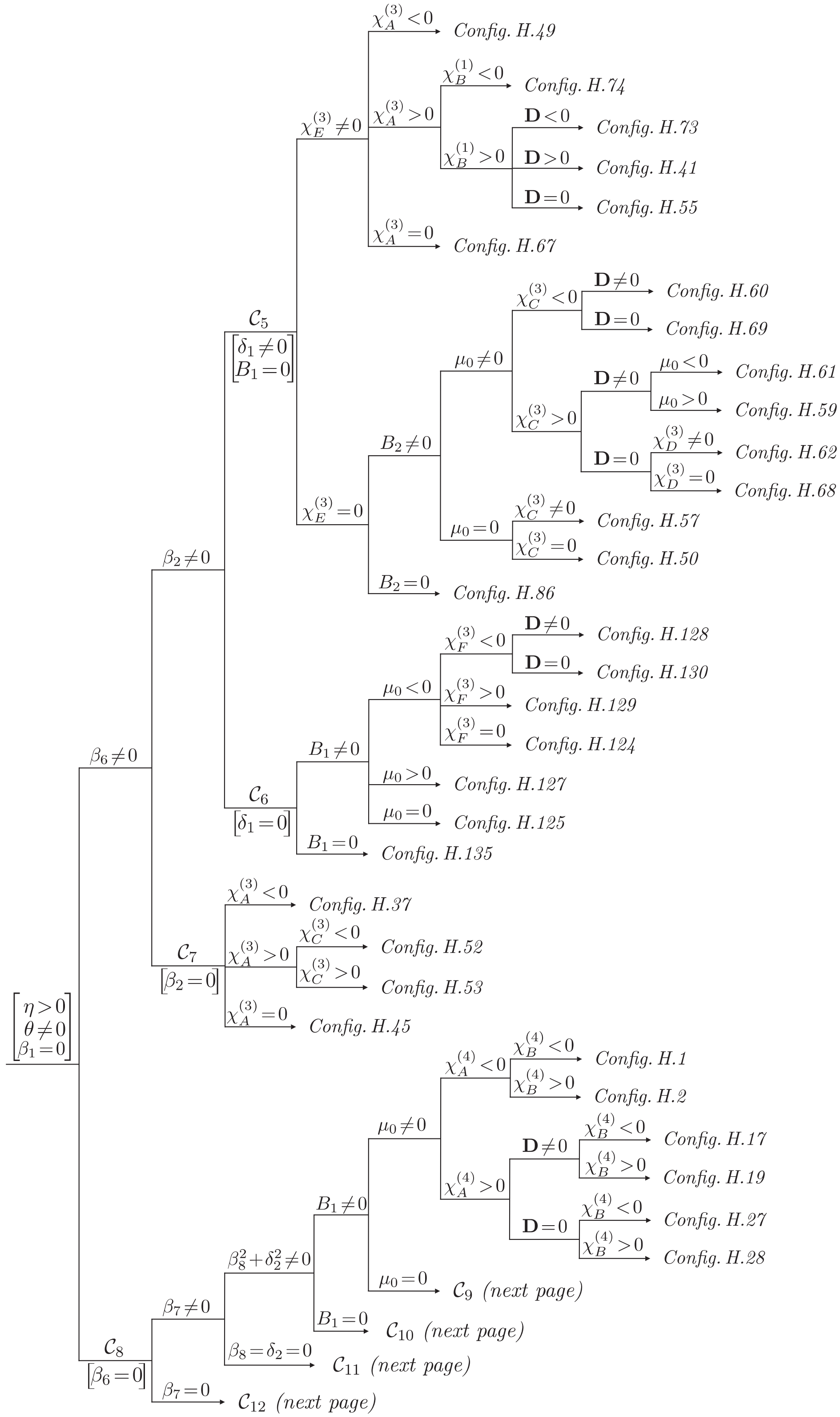

DIAGRAM 8: (*Cont.*) **Bifurcation diagram in $\mathbb{R}^{12}$ of the configurations: Case $\eta>0$, $\theta\neq 0$**

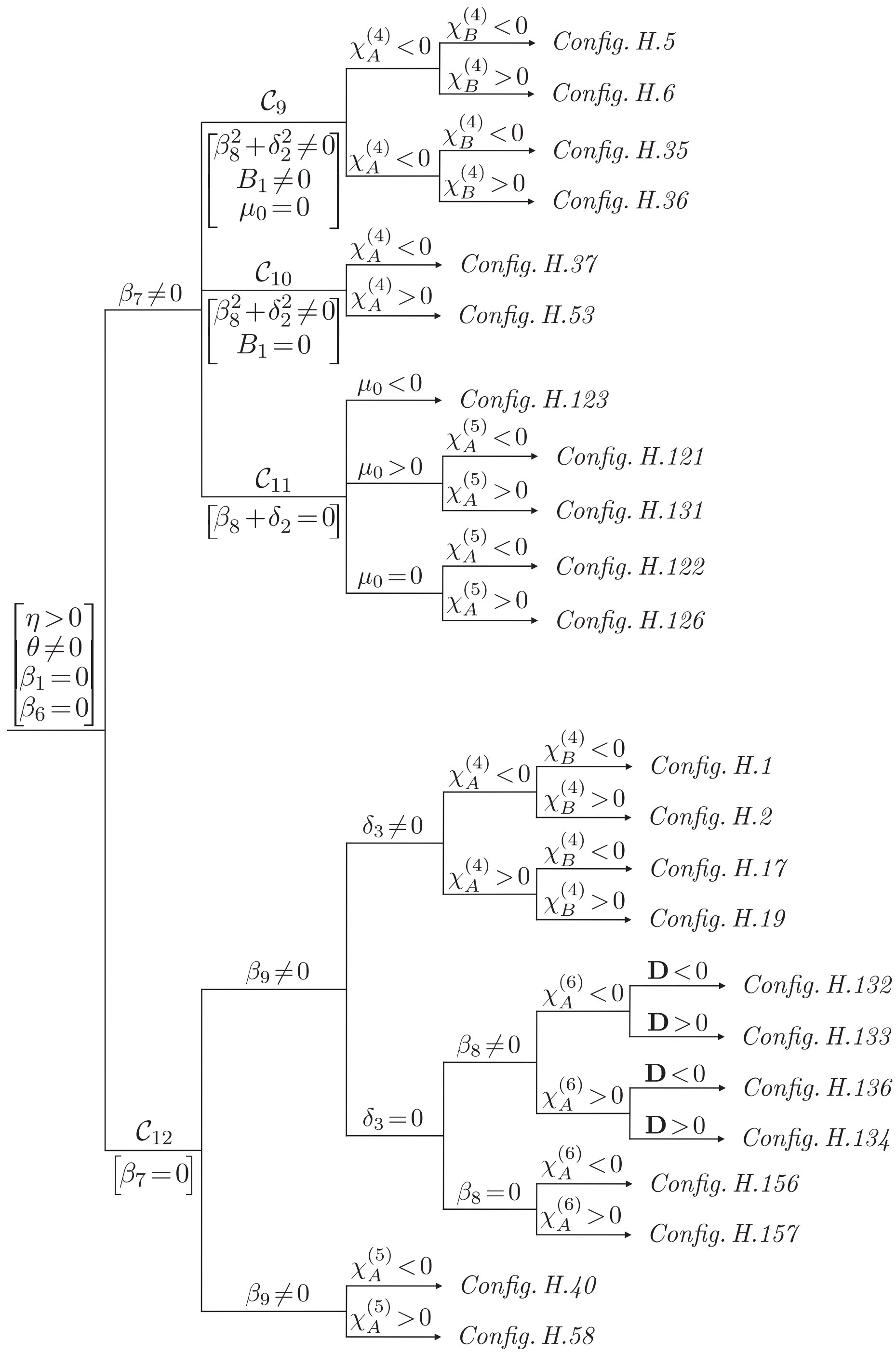

Diagram 8: (*Cont.*) **Bifurcation diagram in $\mathbb{R}^{12}$ of the configurations: Case $\eta > 0$, $\theta \neq 0$**

two complex points at infinity. We call "complex hyperbola" an irreducible affine conic $f_i(x, y) = 0$, with $f_i(x, y) = a_0 + a_{10}x + a_{01}y + a_{20}x^2 + 2a_{11}xy + a_{02}y^2$ over $\mathbb{C}$, such that there does not exist a non-zero complex number λ with $\lambda(a_0, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}) \in \mathbb{R}^6$ and in addition such that this conic has two real points at infinity.

Attached to an r-cycle C on an irreducible algebraic variety V we have an invariant, *the type of*

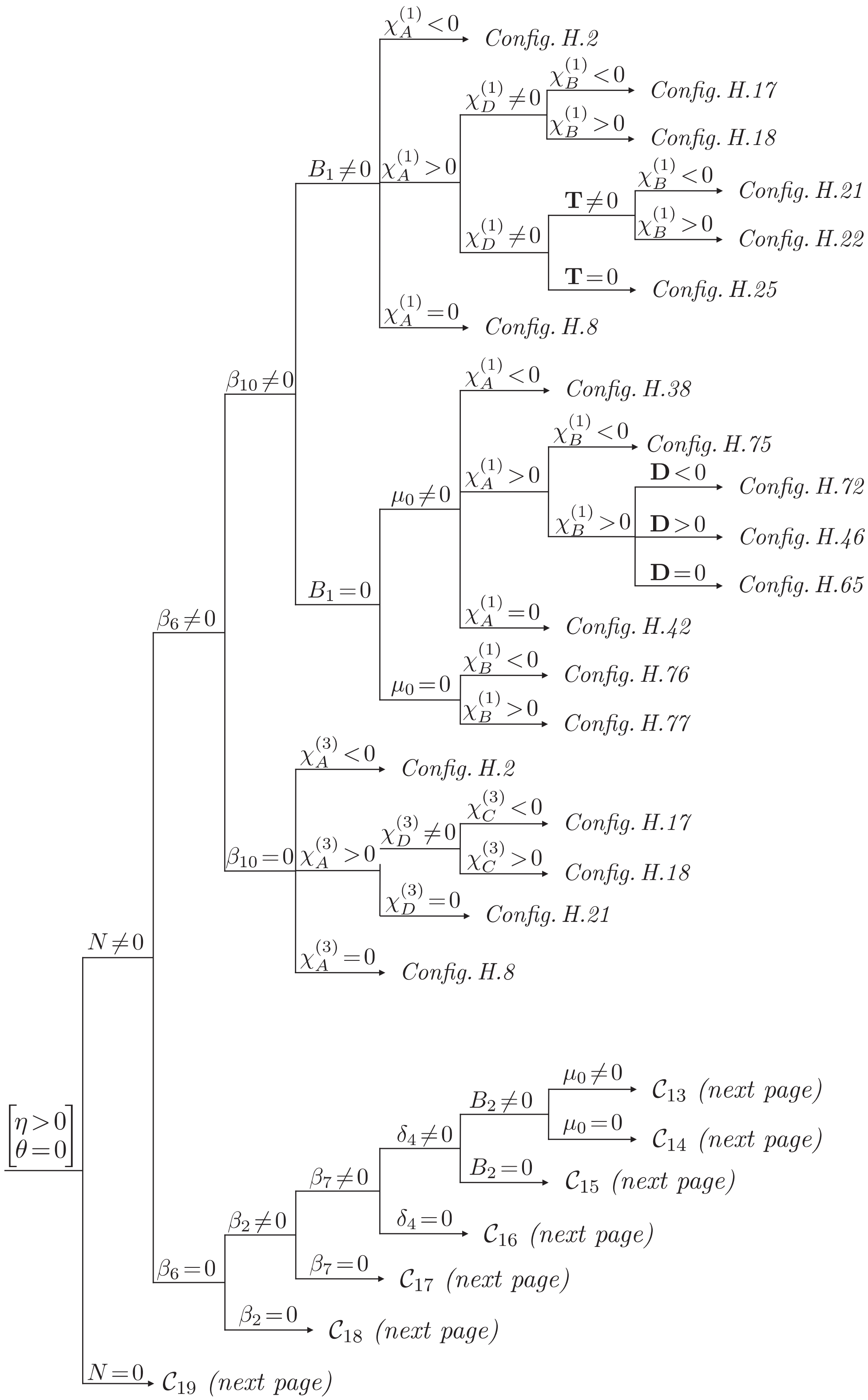

Diagram 9: **Bifurcation diagram in $\mathbb{R}^{12}$ of the configurations: Case $\boldsymbol{\eta > 0}$, $\boldsymbol{\theta = 0}$**

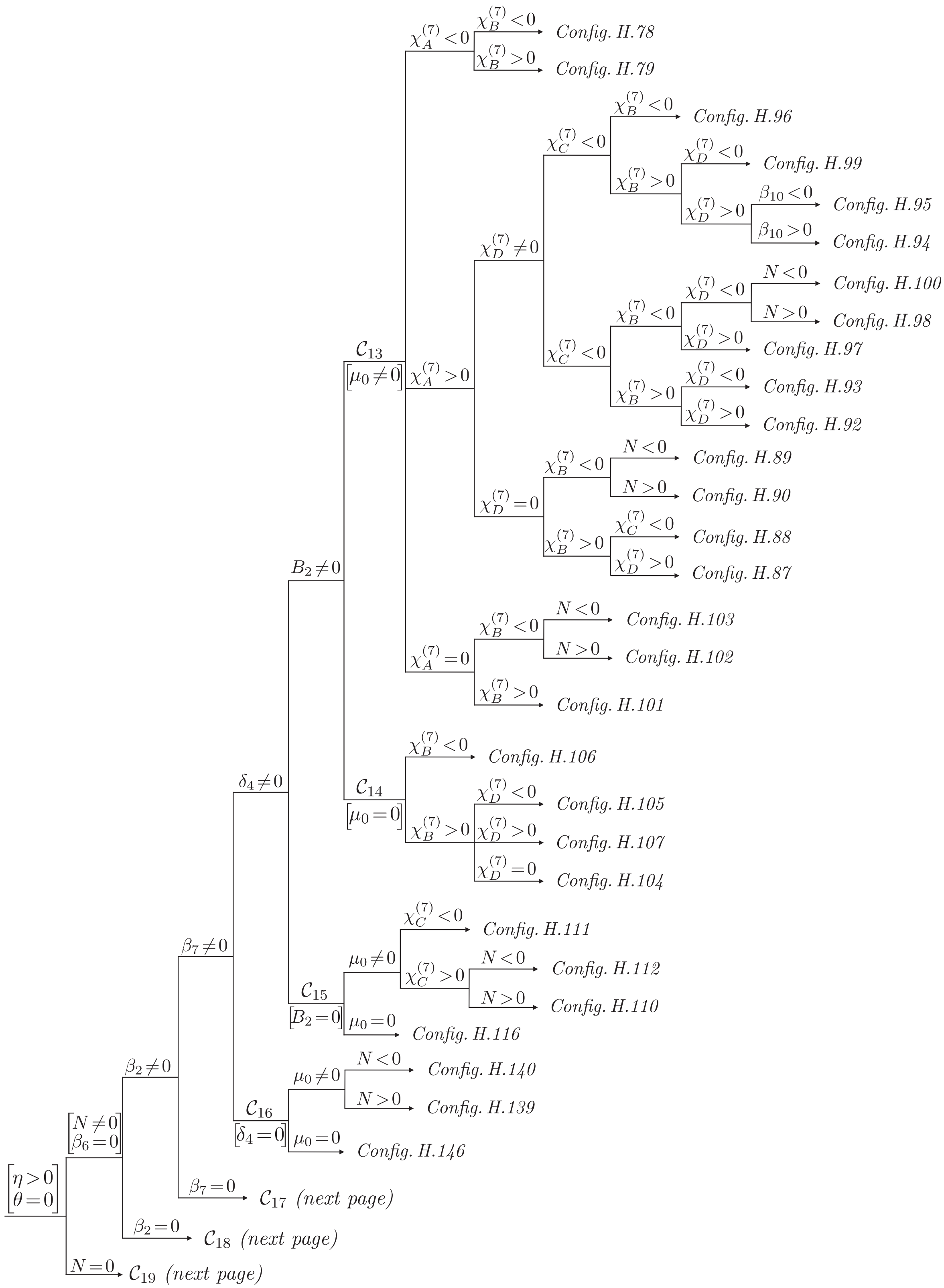

DIAGRAM 9: (*Cont.*) **Bifurcation diagram in $\mathbb{R}^{12}$ of the configurations: Case $\eta>0$, $\theta=0$**

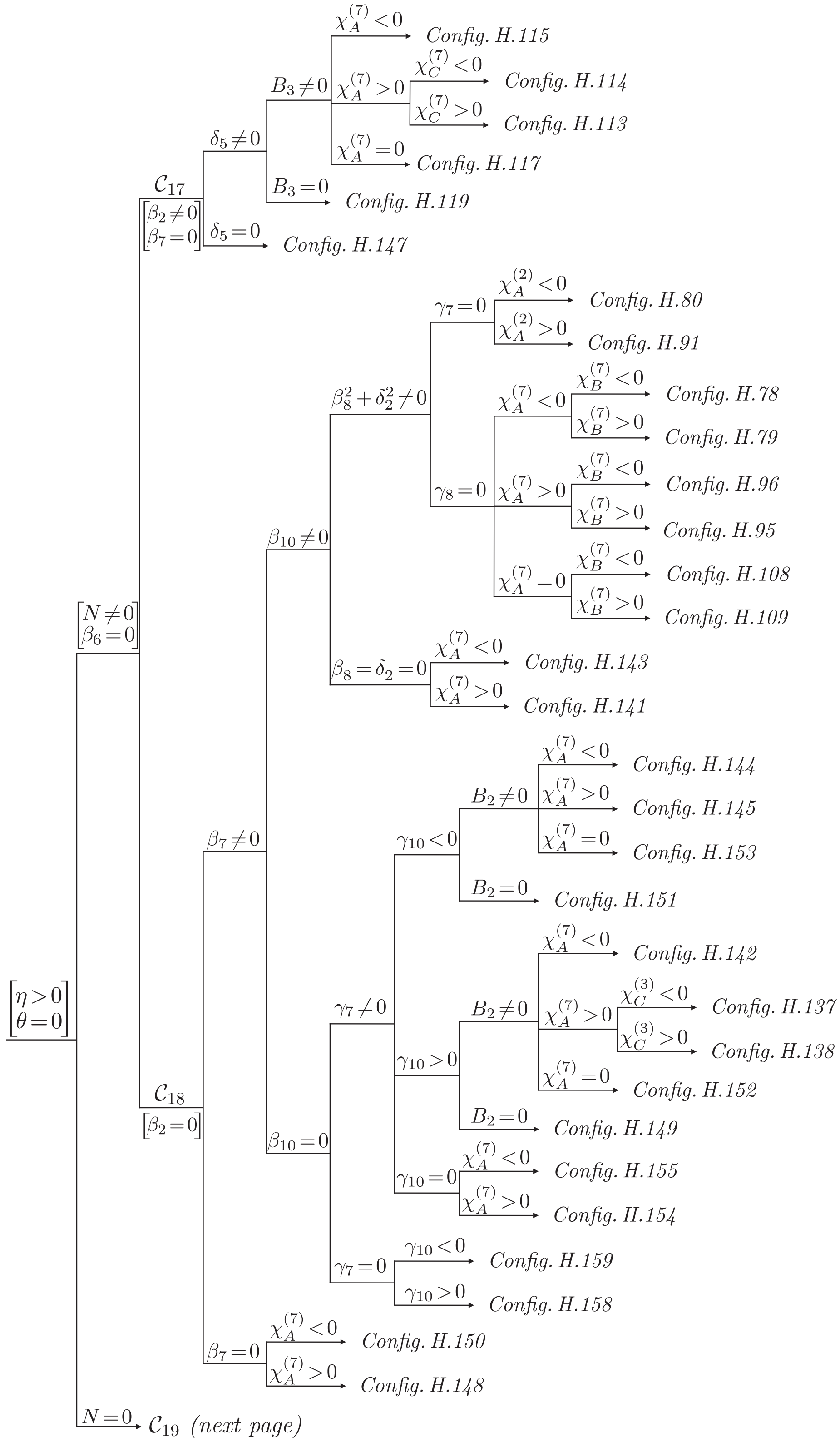

Diagram 9: (*Cont.*) **Bifurcation diagram in $\mathbb{R}^{12}$ of the configurations: Case $\eta>0$, $\theta=0$**

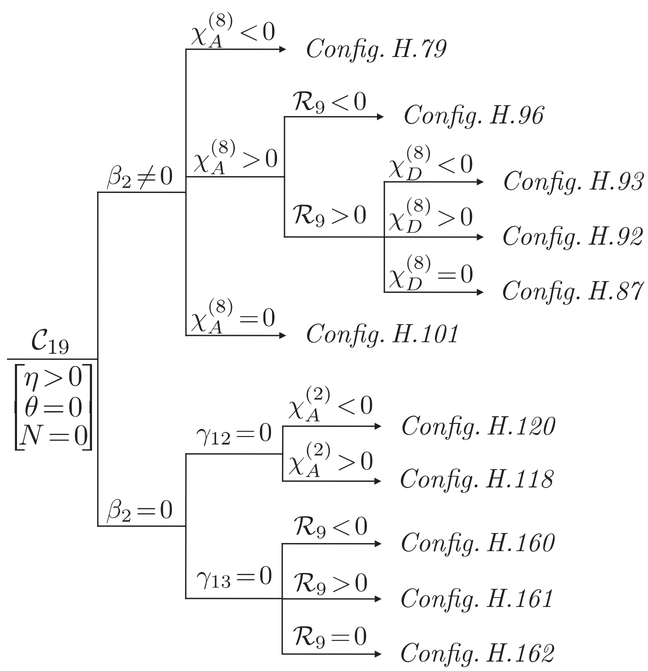

DIAGRAM 9: (*Cont.*) **Bifurcation diagram in $\mathbb{R}^{12}$ of the configurations: Case $\eta > 0$, $\theta = 0$**

the corresponding cycle defined as follows:

We call *type of an r-cycle C on an irreducible algebraic variety V*, the set of ordered couples $(s(m), m)$ where $s(m)$ is the number of the coefficients n_W in C which are equal to m and $1 \leq m \leq Max(C)$, where $Max(C)$ is the maximum of all n_W in C.

In analogy with this definition we can then construct such types associated to the divisor and zero-cycle defined further above.

So although the cycles ICD and MS_{0C} are not themselves invariants, they are used in the classification because they explicitly contain these several specific invariants which actually turn out to classify the systems.

Given a system in **QSH**, consider its compactification in the Poincaré disk. In the compactified system the line at infinity of the affine plane is an invariant line. The system may have singular points located at infinity which are not points of intersection of invariant curves, points also denoted by U_r.

The points at infinity which are intersection point of two or more invariant algebraic curves we denote by $\overset{j}{U}_r$, where $j \in \{h, l, hh, hl, ll\}$. Here h (respectively l, hh, hl, ll) means that the intersection of the infinite line is with a hyperbola (respectively with a line, or with two hyperbolas, or with a hyperbola and a line, or with two lines).

In case we have a real finite singularity located on the invariant curves we denote it by $\overset{j}{s}_r$, where $j \in \{h, l, hh, hl, ll\}$. Here h (respectively l, hh, hl, ll) means that the singular point s_r is located on a hyperbola (respectively located on a line, on the intersection of two hyperbolas, on the intersection of a hyperbola and a line, on the intersection of two lines).

Suppose the real invariant hyperbolas and lines of a system (S) are given by equations $f_i(x, y) = 0$,

$i \in \{1,2,...,k\}$, $f_i \in \mathbb{R}[x,y]$. Let us denote by $F_i(X,Y,Z)=0$ the projection completion of the invariant curves $f_i=0$ in $P_2(\mathbb{R})$.

Definition 5. *We call* total curve of (S) *in* $P_2(\mathbb{C})$*, the curve* $\mathcal{T}(S) : \prod F_i(X,Y,Z)Z = 0$*.*

We use the above notion to define the *basic curvilinear polygons determined by the total curve* $\mathcal{T}(S)$. Consider the Poincaré disk and remove from it the (real) points of the total curve $\mathcal{T}(S)$. We are left with a certain number of 2-dimensional connected components.

Definition 6. *We call* basic polygon determined by $\mathcal{T}(S)$ *the closure of anyone of these components associated to* $\mathcal{T}(S)$*.*

Although a basic polygon is a 2-dimensional object, we shall think of it as being just its border.

Regarding the singular points of the systems situated on $T(S)$, they are of two kinds: those which are simple (or smooth) points of $\mathcal{T}(S)$ and those which are multiple points of $\mathcal{T}(S)$.

Remark 2. *To each singular point of the system we have its associated multiplicity as a singular point of the system. In addition, we also have the multiplicity of these points as points on the total curve. The simple points are those of multiplicity one. They are also the smooth points of this curve. Through a singular point of the systems there may pass several of the curves* $F_i=0$ *and* $Z=0$*. Also we may have the case when this point is a singular point of one or even of several of the curves in case we work with invariant curves with singularities. This leads to the multiplicity of the point as point of the curve* $\mathcal{T}(S)$*.*

The real singular points of the system which are simple points of $\mathcal{T}(S)$ are useful for defining some geometrical invariants, helpful in the geometrical classification, besides those which can be read from the zero-cycle defined further above.

Remark 3. *Here are some observations made from the list of configurations of systems with invariant hyperbolas:*

(a) The basic polygons could have one of its sides as a segment of the line at infinity (as for example in Config. H.1), or just two vertices on the line at infinity (as in Config. H.156) or just one vertex on the line at infinity (as in Config. H.95) or no vertex on the line at infinity (as in Config. H.118, i.e. the finite polygon with four vertices);

(b) A finite basic polygon has either three or four vertices. Examples of 3-vertices finite polygons are in Config. H.112 or Config. H.139; examples of 4-vertices polygons are in Config. H.147 or Config. H.118. Altogether we have 4 configurations with finite polygons with four vertices and 5 configurations with polygons with three vertices.

(c) There are 2 configurations (Config. H.119 and Config. H.149) which have two finite polygons both triangles.

We now introduce the notion of *minimal proximity polygon* of a singular point of the total curve. This notion plays a major role in the geometrical classification of the systems.

Definition 7. *Let p be a real singular point of a system, on $\mathcal{T}(S)$ on the Poincaré disk. Then p may belong to several basic polygons. We call* minimal proximity polygon of p *a basic polygon on which p is located and which has the minimum number of vertices, among the basic polygons to which p belongs. In case we have more than one polygon with the minimum number of vertices, we take all such polygons as being* minimal proximity polygons of p.

For a configuration C, consider for each real singularity p of the system which is a simple point of the curve $\mathcal{T}(S)$, its minimal proximity basic polygons. We construct some formal finite sums attached to the Poincaré disk, analogs of the algebraic-geometric notion of divisor on the projective plane. For this we proceed as follows:

We first list all real singularities of the systems on the Poincaré disk which are simple points of the total curve. In case we have such points U_i's located on the line at infinity, we start with those points which are at infinity. We obtain a list $U_1, \ldots, U_n, s_1, \ldots, s_k$, where s_i's are finite points. Associate to $U_1, \ldots, U_n$ their minimal proximity polygons $\mathcal{P}_1, \ldots, \mathcal{P}_n$. In case some of them coincide we only list once the polygons which are repeated. These minimal proximity polygons may contain some finite points from the list $s_1, \ldots, s_k$. We remove all such points from this list. Suppose we are left with the finite points $s'_1, \ldots, s'_r$. For these points we associate their corresponding minimal proximity polygons. We observe that for a point s'_j we may have two minimal proximity polygons in which case we consider only the minimal proximity polygon which has the maximum number of singularities s_j, simple points of the total curve. If the two polygons have the same number of ss points then we take the two of them. We obtain a list of polygons and we retain from this list only that polygon (or those polygons) which have the maximal number of ss points and add these polygons to the list $\mathcal{P}_1, \ldots, \mathcal{P}_n$. We remove all the ss points which appear in this list of polygons from the list of points $s'_1, \ldots, s'_r$ and continue the same process until there are no points left from the sequence $s_1, \ldots, s_k$ which have not being included. We thus end up with a list $\mathcal{P}_1, \ldots, \mathcal{P}_r$ of proximity polygons which we denote by $\mathcal{P}(C)$.

Definition 8. *We denote by PD the proximity "divisor" of the Poincaré disk*

$$PD = v_1\mathcal{P}_1 + \cdots + v_r\mathcal{P}_r,$$

over $P_2(\mathbb{R})$, associated to the list $\mathcal{P}(C)$ of the minimal proximity polygons of a configuration, where $\mathcal{P}_i$ are the minimal proximity polygons from this list and v_i are their corresponding number of vertices.

We used the word *divisor* in analogy with divisor on an algebraic curve and also thinking of polygons as the borders of the 2-dimensional polygons. The next divisor considers the proximity polygons in PD but only the ones attached to the finite singular points of the system which are simple points on the total curve. So this time we start with all such points $s_1, \ldots, s_k$ and build up the divisor like we did before. The result is called "*the proximity divisor of the real finite singular points of the systems, simple points of the total curve*" and we denote it by PD_f.

We also define a divisor on the Poincaré disk which encodes the way the minimal proximity polygons intersect the line at infinity.

Definition 9. *We denote by PD_∞ the "divisor" of the Poincaré disk encoding the way the proximity polygons in PD intersect the infinity and define it as*

$$PD_\infty = \sum_{\mathcal{P}} n_{\mathcal{P}}\mathcal{P},$$

where $\mathcal{P}$ is a proximity polygon in PD and $n_{\mathcal{P}}$ is 3 if $\mathcal{P}$ has one of its sides on the line at infinity, it is 2 if $\mathcal{P}$ has only two vertices on the line at infinity, it is 1 if only one of its vertices lies on the line at infinity and it is 0 if $\mathcal{P}$ is finite.

Definition 10. *For a proximity polygon $\mathcal{P}$ we introduce the multiplicity divisor*

$$m\mathcal{P} = \sum m(v)\ v,$$

where v is a vertex of $\mathcal{P}$ and $m(v)$ is the multiplicity of the singular point v of the system.

In case a configuration C has an invariant hyperbola $\mathcal{H}$ and an invariant line $\mathcal{L}$, we defined the following invariant I which helps us decide the type of their intersection.

Definition 11. *Suppose we have an invariant line $\mathcal{L}$ and an invariant hyperbola $\mathcal{H}$ of a polynomial differential system (S). We define the invariant I attached to the couple $\mathcal{L}, \mathcal{H}$ as being: 0 if and only if $\mathcal{L}$ intersects $\mathcal{H}$ in two complex non-real points; 1 if and only if $\mathcal{L}$ is tangent to $\mathcal{H}$; 21 if and only if $\mathcal{L}$ intersects $\mathcal{H}$ in two real points and both these points lie on only one branch of the hyperbola; 22 if and only if $\mathcal{L}$ intersects $\mathcal{H}$ in two real points and these points lie on distinct branches of the hyperbola. In case for a configuration C we have several hyperbolas $\mathcal{H}_i$, $i \in \{1, 2, ..., r\}$ and an invariant line $\mathcal{L}$, then $I = \{I(\mathcal{L}, \mathcal{H}_1), I(\mathcal{L}, \mathcal{H}_2), ..., I(\mathcal{L}, \mathcal{H}_r)\}$.*

We now indicate how the proof of part (A) of the Main Theorem is obtained, using its part (B) proved in Section 3.

Proof: We first need to make sure that the concepts introduced above gave us a sufficient number of invariants under the action of the affine group and time rescaling so as to be able to classify geometrically the class $\mathbf{QSH}_{(\boldsymbol{\eta}>0)}$ according to their configurations of their invariant hyperbolas and lines. Summing up all the concepts introduced, we end up with the list: ICD, MS_{0C}, TMH, TML, PD, PD_f, PD_∞, mP, I. From this list we clearly have that TMH, TML are invariants under the group action because the action conserves lines and the type of a conic as well as parallelism and it conserves singularities of the systems which are simple points on an invariant curve. The types of the divisor ICD on $P_2(\mathbb{C})$ and of the zero-cycle MS_{0C} on $P_2(\mathbb{R})$ are invariants under the group because the group conserves the multiplicities of the invariant curves as well as the multiplicities of the singularities. The number of vertices of a basic polygon is conserved under the group action basically because the number of intersection points of the various invariant curves is conserved. Furthermore the coefficients of mP are also conserved because multiplicities of the singularities are conserved. For analogous reasons the coefficients of PD, PD_f, PD_∞ are also conserved. The invariant I is also conserved because complex intersection points of a line with a hyperbola as well as intersection multiplicities are conserved. The concepts involved above yield all the invariants we need and we now prove that the 162 configurations obtained in Section 3 are distinct.

Fixing the values of TMH and TML, we first apply the main divisor ICD. In many cases, just using the invariants contained in ICD and the zero-cycle MS_{0C} (TMH, TML and the corresponding types) suffice for distinguishing the configurations in a group of configurations. In other cases more invariants are needed and we introduce the necessary additional invariants, to distinguish the configurations of the following groups. The result is seen in the Diagrams 1 to 7.

We finally obtain that the 162 geometric configurations obtained in Section 3 and displayed in Diagrams 1 to 7 are distinct, which yields the geometric classification of the class $\mathbf{QSH}_{(\eta>0)}$ according to the configurations of invariant hyperbolas and lines. This proves statement *(A)* of the Main Theorem, using its part (B), proved in Section 3. ■

A few more definitions and results which play an important role in the proof of the Main Theorem are needed. We do not prove these results here but we indicate where they can be found.

Consider the differential operator $\mathcal{L} = x \cdot \mathbf{L}_2 - y \cdot \mathbf{L}_1$ constructed in [2] and acting on $\mathbb{R}[\tilde{a}, x, y]$, where

$$\begin{aligned} \mathbf{L}_1 = \;& 2a_{00}\frac{\partial}{\partial a_{10}} + a_{10}\frac{\partial}{\partial a_{20}} + \tfrac{1}{2}a_{01}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{10}} + b_{10}\frac{\partial}{\partial b_{20}} + \tfrac{1}{2}b_{01}\frac{\partial}{\partial b_{11}}, \\ \mathbf{L}_2 = \;& 2a_{00}\frac{\partial}{\partial a_{01}} + a_{01}\frac{\partial}{\partial a_{02}} + \tfrac{1}{2}a_{10}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{01}} + b_{01}\frac{\partial}{\partial b_{02}} + \tfrac{1}{2}b_{10}\frac{\partial}{\partial b_{11}}. \end{aligned}$$

Using this operator and the affine invariant $\mu_0 = \operatorname{Res}_x\big(p_2(\tilde{a}, x, y), q_2(\tilde{a}, x, y)\big)/y^4$ we construct the following polynomials

$$\mu_i(\tilde{a}, x, y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\;\; i = 1, .., 4,$$

where $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$ and $\mathcal{L}^{(0)}(\mu_0) = \mu_0$.

These polynomials are in fact comitants of systems (2) with respect to the group $GL(2, \mathbb{R})$ (see [2]). Their geometrical meaning is revealed in the next lemma.

Lemma 1. ([1],[2]) *Assume that a quadratic system (S) with coefficients $\tilde{a}$ belongs to the family* (2)*. Then:*

(i) The total multiplicity of all finite singularities of this system equals $4 - \lambda$ if and only if for every $i \in \{0, 1, \ldots, \lambda - 1\}$ we have $\mu_i(\tilde{a}, x, y) = 0$ in the ring $\mathbb{R}[x, y]$ and $\mu_\lambda(\tilde{a}, x, y) \neq 0$. In this case, the factorization $\mu_\lambda(\tilde{a}, x, y) = \prod_{i=1}^{\lambda}(u_i x - v_i y) \neq 0$ over $\mathbb{C}$ indicates the coordinates $[v_i : u_i : 0]$ of those finite singularities of the system (S) which "have gone" to infinity. Moreover, the number of distinct factors in this factorization is less than or equal to three (the maximum number of infinite singularities of a quadratic system) and the multiplicity of each one of the factors $u_i x - v_i y$ gives us the number of the finite singularities of the system (S) which have collapsed with the infinite singular point $[v_i : u_i : 0]$.

(ii) The system (S) is degenerate (i.e. $\gcd(P, Q) \neq \text{const}$) if and only if $\mu_i(\tilde{a}, x, y) = 0$ in $\mathbb{R}[x, y]$ for every $i = 0, 1, 2, 3, 4$.

Proposition 1. ([31]) *The form of the divisor $\mathcal{D}_S(P, Q)$ for non-degenerate quadratic systems* (2) *is determined by the corresponding conditions indicated in* Table *1, where we write $p + q + r^c + s^c$*

Table 1

No.	Zero–cycle $\mathcal{D}_S(P,Q)$	Invariant criteria	No.	Zero–cycle $\mathcal{D}_S(P,Q)$	Invariant criteria
1	$p+q+r+s$	$\mu_0\neq 0, \mathbf{D}<0,$ $\mathbf{R}>0, \mathbf{S}>0$	10	$p+q+r$	$\mu_0=0, \mathbf{D}<0, \mathbf{R}\neq 0$
2	$p+q+r^c+s^c$	$\mu_0\neq 0, \mathbf{D}>0$	11	$p+q^c+r^c$	$\mu_0=0, \mathbf{D}>0, \mathbf{R}\neq 0$
3	$p^c+q^c+r^c+s^c$	$\mu_0\neq 0, \mathbf{D}<0, \mathbf{R}\leq 0$; $\mu_0\neq 0, \mathbf{D}<0, \mathbf{S}\leq 0$	12	$2p+q$	$\mu_0=\mathbf{D}=0, \mathbf{PR}\neq 0$
4	$2p+q+r$	$\mu_0\neq 0, \mathbf{D}=0, \mathbf{T}<0$	13	$3p$	$\mu_0=\mathbf{D}=\mathbf{P}=0, \mathbf{R}\neq 0$
5	$2p+q^c+r^c$	$\mu_0\neq 0, \mathbf{D}=0, \mathbf{T}>0$	14	$p+q$	$\mu_0=\mathbf{R}=0, \mathbf{P}\neq 0,$ $\mathbf{U}>0$
6	$2p+2q$	$\mu_0\neq 0, \mathbf{D}=\mathbf{T}=0,$ $\mathbf{PR}>0$	15	p^c+q^c	$\mu_0=\mathbf{R}=0, \mathbf{P}\neq 0,$ $\mathbf{U}<0$
7	$2p^c+2q^c$	$\mu_0\neq 0, \mathbf{D}=\mathbf{T}=0,$ $\mathbf{PR}<0$	16	$2p$	$\mu_0=\mathbf{R}=0, \mathbf{P}\neq 0,$ $\mathbf{U}=0$
8	$3p+q$	$\mu_0\neq 0, \mathbf{D}=\mathbf{T}=0,$ $\mathbf{P}=0, \mathbf{R}\neq 0$	17	p	$\mu_0=\mathbf{R}=\mathbf{P}=0,$ $\mathbf{U}\neq 0$
9	$4p$	$\mu_0\neq 0, \mathbf{D}=\mathbf{T}=0,$ $\mathbf{P}=\mathbf{R}=0$	18	0	$\mu_0=\mathbf{R}=\mathbf{P}=0,$ $\mathbf{U}=0, \mathbf{V}\neq 0$

if two of the finite points, i.e. r^c, s^c, *are complex but not real, and*

$$
\begin{aligned}
\mathbf{D} =&\Big[3\big((\mu_3,\mu_3)^{(2)},\ \mu_2\big)^{(2)} - \big(6\mu_0\mu_4 - 3\mu_1\mu_3 + \mu_2^2,\ \mu_4\big)^{(4)}\Big]/48,\\
\mathbf{P} =&12\mu_0\mu_4 - 3\mu_1\mu_3 + \mu_2^2,\\
\mathbf{R} =&3\mu_1^2 - 8\mu_0\mu_2,\\
\mathbf{S} =&\mathbf{R}^2 - 16\mu_0^2\mathbf{P},\\
\mathbf{T} =&18\mu_0^2(3\mu_3^2 - 8\mu_2\mu_4) + 2\mu_0(2\mu_2^3 - 9\mu_1\mu_2\mu_3 + 27\mu_1^2\mu_4) - \mathbf{PR},\\
\mathbf{U} =&\mu_3^2 - 4\mu_2\mu_4,\\
\mathbf{V} =&\mu_4.
\end{aligned}
\tag{3}
$$

The next result is stated in [15] and it gives us the necessary and sufficient conditions for the existence of at least one invariant hyperbolas for systems (2) and also their multiplicity.

Theorem 1. ([15]) ***(A)*** *The conditions* $\eta \geq 0$, $M \neq 0$ *and* $\gamma_1 = \gamma_2 = 0$ *are necessary for a non-degenerate quadratic system in* **QS** *to possess at least one invariant hyperbola.*

(B) *Assume that for a non-degenerate system in* **QS** *the condition* $\gamma_1 = \gamma_2 = 0$ *is satisfied.*

- ***(B₁)*** *If* $\eta > 0$, *then the necessary and sufficient conditions for this system to possess at least one invariant hyperbola are given in* Diagram *10, where we can also find the number and multiplicity of such hyperbolas.*
- ***(B₂)*** *In the case* $\eta = 0$ *and* $M \neq 0$ *the corresponding necessary and sufficient conditions for this system to possess at least one invariant hyperbola are given in* Diagram *11, where we can also find the number and multiplicity of such hyperbolas.*

(C) *The* DIAGRAMS *10 and 11 actually contain the global bifurcation diagram in the 12-dimensional space of parameters of the systems belonging to family of non-degenerate systems in* **QS***, which possess at least one invariant hyperbola. The corresponding conditions are given in terms of invariant polynomials with respect to the group of affine transformations and time rescaling.*

Remark 4. *An invariant hyperbola is denoted by $\mathcal{H}$ if it is real and by $\overset{c}{\mathcal{H}}$ if it is complex. In the case we have two such hyperbolas then it is necessary to distinguish whether they have parallel or non-parallel asymptotes in which case we denote them by $\mathcal{H}^p$ ($\overset{c}{\mathcal{H}}{}^p$) if their asymptotes are parallel and by $\mathcal{H}$ if there exists at least one pair of non-parallel asymptotes. We denote by $\mathcal{H}_k$ ($k = 2, 3$) a hyperbola with multiplicity k; by $\mathcal{H}_2^p$ a double hyperbola, which after perturbation splits into two $\mathcal{H}^p$; and by $\mathcal{H}_3^p$ a triple hyperbola which splits into two $\mathcal{H}^p$ and one $\mathcal{H}$.*

Following [15] we present here the invariant polynomials which according to DIAGRAMS 10 and 11 are responsible for the existence and the number of invariant hyperbolas which systems (2) could possess.

First we single out the following five polynomials, basic ingredients in constructing invariant polynomials for systems (2):

$$\begin{aligned} C_i(\tilde{a}, x, y) &= yp_i(x, y) - xq_i(x, y), \ (i = 0, 1, 2) \\ D_i(\tilde{a}, x, y) &= \frac{\partial p_i}{\partial x} + \frac{\partial q_i}{\partial y}, \ (i = 1, 2). \end{aligned} \tag{4}$$

As it was shown in [29] these polynomials of degree one in the coefficients of systems (2) are GL–comitants of these systems. Let $f,\ g \in \mathbb{R}[\tilde{a}, x, y]$ and

$$(f, g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h} \, \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

The polynomial $(f, g)^{(k)} \in \mathbb{R}[\tilde{a}, x, y]$ is called *the transvectant of index k of (f, g)* (cf. [11], [16])).

Theorem 2 (see [32])**.** *Any GL–comitant of systems* (2) *can be constructed from the elements* (4) *by using the operations: $+$, $-$, $\times$, and by applying the differential operation $(*, *)^{(k)}$.*

Remark 5. *We point out that the elements* (4) *generate the whole set of GL–comitants and hence also the set of affine comitants as well as the set of T-comitants and CT-comitants (see [23] for detailed definitions).*

We construct the following GL–comitants of the second degree with respect to the coefficients of the initial systems

$$\begin{array}{lll} T_1 = (C_0, C_1)^{(1)}, & T_2 = (C_0, C_2)^{(1)}, & T_3 = (C_0, D_2)^{(1)}, \\ T_4 = (C_1, C_1)^{(2)}, & T_5 = (C_1, C_2)^{(1)}, & T_6 = (C_1, C_2)^{(2)}, \\ T_7 = (C_1, D_2)^{(1)}, & T_8 = (C_2, C_2)^{(2)}, & T_9 = (C_2, D_2)^{(1)}. \end{array} \tag{5}$$

Using these GL–comitants as well as the polynomials (4) we construct the additional invariant polynomials. In order to be able to calculate the values of the needed invariant polynomials directly

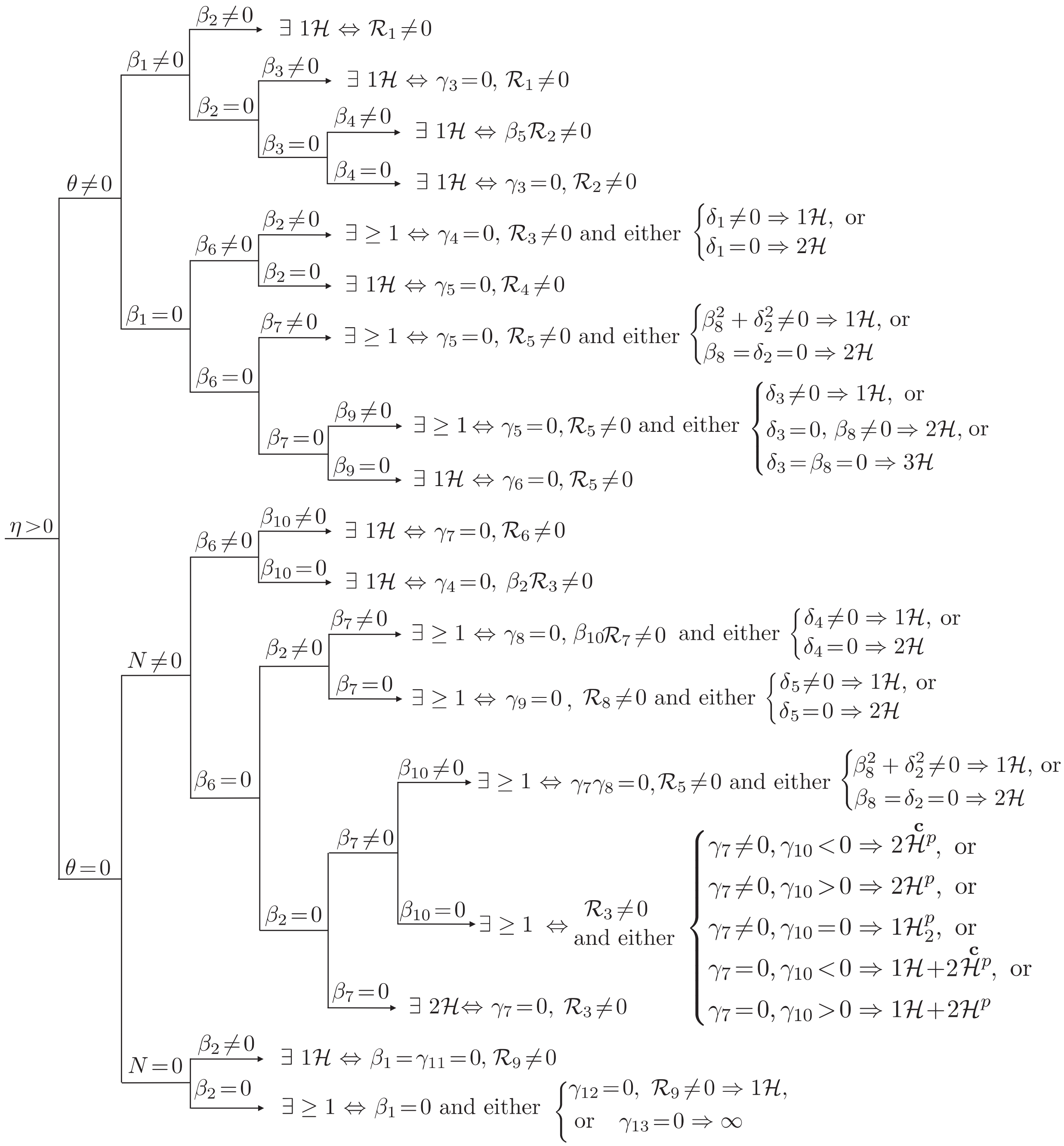

Diagram 10: **Existence of invariant hyperbolas: the case $\eta > 0$**

for every canonical system we shall define here a family of T–comitants expressed through C_i $(i =$

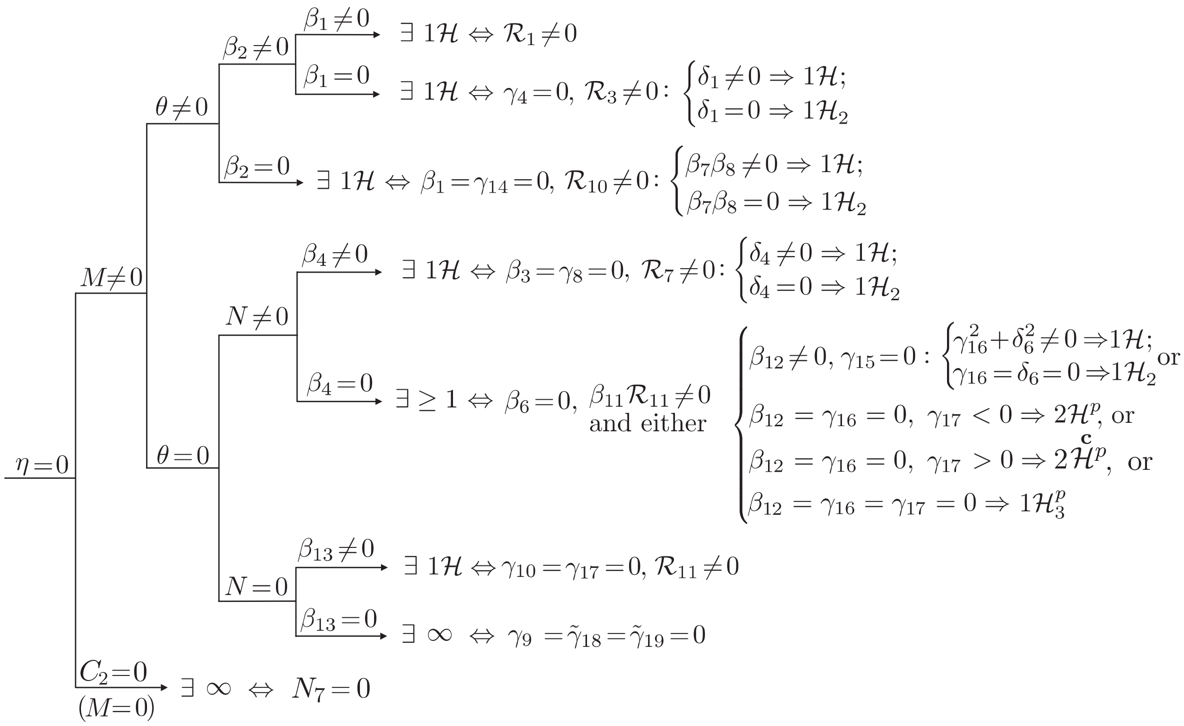

Diagram 11: **Existence of invariant hyperbolas: the case $\eta=0$**

$0,1,2)$ and D_j $(j=1,2)$:

$$\hat{A}=\left(C_1, T_8-2T_9+D_2^2\right)^{(2)}/144,$$

$$\widehat{D}=\Big[2C_0(T_8-8T_9-2D_2^2)+C_1(6T_7-T_6-(C_1,T_5)^{(1)}+6D_1(C_1D_2-T_5)-9D_1^2C_2\Big]/36,$$

$$\widehat{E}=\Big[D_1(2T_9-T_8)-3\,(C_1,T_9)^{(1)}-D_2(3T_7+D_1D_2)\Big]/72,$$

$$\begin{aligned}\widehat{F}=\big[&6D_1^2(D_2^2-4T_9)+4D_1D_2(T_6+6T_7)+48C_0\,(D_2,T_9)^{(1)}-9D_2^2T_4+288D_1\widehat{E}\\ &-24\left(C_2,\widehat{D}\right)^{(2)}+120\left(D_2,\widehat{D}\right)^{(1)}-36C_1\,(D_2,T_7)^{(1)}+8D_1\,(D_2,T_5)^{(1)}\Big]/144,\end{aligned}$$

$$\begin{aligned}\widehat{B}=\Big\{&16D_1\,(D_2,T_8)^{(1)}\,(3C_1D_1-2C_0D_2+4T_2)+32C_0\,(D_2,T_9)^{(1)}\,(3D_1D_2-5T_6+9T_7)\\ &+2\,(D_2,T_9)^{(1)}\left(27C_1T_4-18C_1D_1^2-32D_1T_2+32\,(C_0,T_5)^{(1)}\right)\\ &+6\,(D_2,T_7)^{(1)}\,[8C_0(T_8-12T_9)-12C_1(D_1D_2+T_7)+D_1(26C_2D_1+32T_5)+C_2(9T_4+96T_3)]\\ &+6\,(D_2,T_6)^{(1)}\,[32C_0T_9-C_1(12T_7+52D_1D_2)-32C_2D_1^2]+48D_2\,(D_2,T_1)^{(1)}\left(2D_2^2-T_8\right)\\ &-32D_1T_8\,(D_2,T_2)^{(1)}+9D_2^2T_4\,(T_6-2T_7)-16D_1\,(C_2,T_8)^{(1)}\left(D_1^2+4T_3\right)\\ &+12D_1\,(C_1,T_8)^{(2)}\,(C_1D_2-2C_2D_1)+6D_1D_2T_4\left(T_8-7D_2^2-42T_9\right)\\ &+12D_1\,(C_1,T_8)^{(1)}\,(T_7+2D_1D_2)+96D_2^2\Big[D_1\,(C_1,T_6)^{(1)}+D_2\,(C_0,T_6)^{(1)}\Big]-\\ &-16D_1D_2T_3\left(2D_2^2+3T_8\right)-4D_1^3D_2\left(D_2^2+3T_8+6T_9\right)+6D_1^2D_2^2\,(7T_6+2T_7)\\ &-252D_1D_2T_4T_9\Big\}/(2^8 3^3),\end{aligned}$$

$$\widehat{K}=(T_8+4T_9+4D_2^2)/72, \quad \widehat{H}=(8T_9-T_8+2D_2^2)/72.$$

These polynomials in addition to (4) and (5) will serve as bricks in constructing affine invariant polynomials for systems (2).

The following 42 affine invariants $A_1, \dots, A_{42}$ form the minimal polynomial basis of affine invariants up to degree 12. This fact was proved in [3] by constructing $A_1, \dots, A_{42}$ using the above bricks.

$$
\begin{array}{ll}
A_1 = \hat{A}, & A_{22} = \frac{1}{1152}\big[C_2, \widehat{D})^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)} D_2\big)^{(1)}, \\
A_2 = (C_2, \widehat{D})^{(3)}/12, & A_{23} = \big[\widehat{F}, \widehat{H})^{(1)}, \widehat{K}\big)^{(2)}/8, \\
A_3 = \big[C_2, D_2)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}/48, & A_{24} = \big[C_2, \widehat{D})^{(2)}, \widehat{K}\big)^{(1)}, \widehat{H}\big)^{(2)}/32, \\
A_4 = (\widehat{H}, \widehat{H})^{(2)}, & A_{25} = \big[\widehat{D}, \widehat{D})^{(2)}, \widehat{E}\big)^{(2)}/16, \\
A_5 = (\widehat{H}, \widehat{K})^{(2)}/2, & A_{26} = (\widehat{B}, \widehat{D})^{(3)}/36, \\
A_6 = (\widehat{E}, \widehat{H})^{(2)}/2, & A_{27} = \big[\widehat{B}, D_2)^{(1)}, \widehat{H}\big)^{(2)}/24, \\
A_7 = \big[C_2, \widehat{E})^{(2)}, D_2\big)^{(1)}/8, & A_{28} = \big[C_2, \widehat{K})^{(2)}, \widehat{D}\big)^{(1)}, \widehat{E}\big)^{(2)}/16, \\
A_8 = \big[\widehat{D}, \widehat{H})^{(2)}, D_2\big)^{(1)}/8, & A_{29} = \big[\widehat{D}, \widehat{F})^{(1)}, \widehat{D}\big)^{(3)}/96, \\
A_9 = \big[\widehat{D}, D_2)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}/48, & A_{30} = \big[C_2, \widehat{D})^{(2)}, \widehat{D}\big)^{(1)}, \widehat{D}\big)^{(3)}/288, \\
A_{10} = \big[\widehat{D}, \widehat{K})^{(2)}, D_2\big)^{(1)}/8, & A_{31} = \big[\widehat{D}, \widehat{D})^{(2)}, \widehat{K}\big)^{(1)}, \widehat{H}\big)^{(2)}/64, \\
A_{11} = (\widehat{F}, \widehat{K})^{(2)}/4, & A_{32} = \big[\widehat{D}, \widehat{D})^{(2)}, D_2\big)^{(1)}, \widehat{H}\big)^{(1)}, D_2\big)^{(1)}/64, \\
A_{12} = (\widehat{F}, \widehat{H})^{(2)}/4, & A_{33} = \big[\widehat{D}, D_2)^{(1)}, \widehat{F}\big)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}/128, \\
A_{13} = \big[C_2, \widehat{H})^{(1)}, \widehat{H}\big)^{(2)}, D_2\big)^{(1)}/24, & A_{34} = \big[\widehat{D}, \widehat{D})^{(2)}, D_2\big)^{(1)}, \widehat{K}\big)^{(1)}, D_2\big)^{(1)}/64, \\
A_{14} = (\widehat{B}, C_2)^{(3)}/36, & A_{35} = \big[\widehat{D}, \widehat{D})^{(2)}, \widehat{E}\big)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}/128, \\
A_{15} = (\widehat{E}, \widehat{F})^{(2)}/4, & A_{36} = \big[\widehat{D}, \widehat{E})^{(2)}, \widehat{D}\big)^{(1)}, \widehat{H}\big)^{(2)}/16, \\
A_{16} = \big[\widehat{E}, D_2)^{(1)}, C_2\big)^{(1)}, \widehat{K}\big)^{(2)}/16, & A_{37} = \big[\widehat{D}, \widehat{D})^{(2)}, \widehat{D}\big)^{(1)}, \widehat{D}\big)^{(3)}/576, \\
A_{17} = \big[\widehat{D}, \widehat{D})^{(2)}, D_2\big)^{(1)}, D_2\big)^{(1)}/64, & A_{38} = \big[C_2, \widehat{D})^{(2)}, \widehat{D}\big)^{(2)}, \widehat{D}\big)^{(1)}, \widehat{H}\big)^{(2)}/64, \\
A_{18} = \big[\widehat{D}, \widehat{F})^{(2)}, D_2\big)^{(1)}/16, & A_{39} = \big[\widehat{D}, \widehat{D})^{(2)}, \widehat{F}\big)^{(1)}, \widehat{H}\big)^{(2)}/64, \\
A_{19} = \big[\widehat{D}, \widehat{D})^{(2)}, \widehat{H}\big)^{(2)}/16, & A_{40} = \big[\widehat{D}, \widehat{D})^{(2)}, \widehat{F}\big)^{(1)}, \widehat{K}\big)^{(2)}/64, \\
A_{20} = \big[C_2, \widehat{D})^{(2)}, \widehat{F}\big)^{(2)}/16, & A_{41} = \big[C_2, \widehat{D})^{(2)}, \widehat{D}\big)^{(2)}, \widehat{F}\big)^{(1)}, D_2\big)^{(1)}/64, \\
A_{21} = \big[\widehat{D}, \widehat{D})^{(2)}, \widehat{K}\big)^{(2)}/16, & A_{42} = \big[\widehat{D}, \widehat{F})^{(2)}, \widehat{F}\big)^{(1)}, D_2\big)^{(1)}/16.
\end{array}
$$

In the above list, the bracket "[" is used in order to avoid placing the otherwise necessary up to five parentheses "(".

Using the elements of the minimal polynomial basis given above we construct the affine invariant polynomials

$$\begin{aligned}
\gamma_1(\tilde{a}) =& A_1^2(3A_6+2A_7)-2A_6(A_8+A_{12}),\\
\gamma_2(\tilde{a}) =& 9A_1^2A_2(23252A_3+23689A_4)-1440A_2A_5(3A_{10}+13A_{11})-1280A_{13}(2A_{17}+A_{18}\\
&+23A_{19}-4A_{20})-320A_{24}(50A_8+3A_{10}+45A_{11}-18A_{12})+120A_1A_6(6718A_8\\
&+4033A_9+3542A_{11}+2786A_{12})+30A_1A_{15}(14980A_3-2029A_4-48266A_5)\\
&-30A_1A_7(76626A_1^2-15173A_8+11797A_{10}+16427A_{11}-30153A_{12})\\
&+8A_2A_7(75515A_6-32954A_7)+2A_2A_3(33057A_8-98759A_{12})-60480A_1^2A_{24}\\
&+A_2A_4(68605A_8-131816A_9+131073A_{10}+129953A_{11})-2A_2(141267A_6^2\\
&-208741A_5A_{12}+3200A_2A_{13}),\\
\gamma_3(\tilde{a}) =& 843696A_5A_6A_{10}+A_1(-27(689078A_8+419172A_9-2907149A_{10}-2621619A_{11})A_{13}\\
&-26(21057A_3A_{23}+49005A_4A_{23}-166774A_3A_{24}+115641A_4A_{24})).\\
\gamma_4(\tilde{a}) =& -9A_4^2(14A_{17}+A_{21})+A_5^2(-560A_{17}-518A_{18}+881A_{19}-28A_{20}+509A_{21})\\
&-A_4(171A_8^2+3A_8(367A_9-107A_{10})+4(99A_9^2+93A_9A_{11}+A_5(-63A_{18}-69A_{19}\\
&+7A_{20}+24A_{21})))+72A_{23}A_{24},\\
\gamma_5(\tilde{a}) =& -488A_2^3A_4+A_2(12(4468A_8^2+32A_9^2-915A_{10}^2+320A_9A_{11}-3898A_{10}A_{11}-3331A_{11}^2\\
&+2A_8(78A_9+199A_{10}+2433A_{11}))+2A_5(25488A_{18}-60259A_{19}-16824A_{21})\\
&+779A_4A_{21})+4(7380A_{10}A_{31}-24(A_{10}+41A_{11})A_{33}+A_8(33453A_{31}+19588A_{32}\\
&-468A_{33}-19120A_{34})+96A_9(-A_{33}+A_{34})+556A_4A_{41}-A_5(27773A_{38}+41538A_{39}\\
&-2304A_{41}+5544A_{42})),\\
\gamma_6(\tilde{a}) =& 2A_{20}-33A_{21},\\
\gamma_7(\tilde{a}) =& A_1(64A_3-541A_4)A_7+86A_8A_{13}+128A_9A_{13}-54A_{10}A_{13}-128A_3A_{22}+256A_5A_{22}\\
&+101A_3A_{24}-27A_4A_{24},\\
\gamma_8(\tilde{a}) =& 3063A_4A_9^2-42A_7^2(304A_8+43(A_9-11A_{10}))-6A_3A_9(159A_8+28A_9+409A_{10})\\
&+2100A_2A_9A_{13}+3150A_2A_7A_{16}+24A_3^2(34A_{19}-11A_{20})+840A_5^2A_{21}-932A_2A_3A_{22}\\
&+525A_2A_4A_{22}+844A_{22}^2-630A_{13}A_{33},\\
\gamma_9(\tilde{a}) =& 2A_8-6A_9+A_{10},\\
\gamma_{10}(\tilde{a}) =& 3A_8+A_{11},\\
\gamma_{11}(\tilde{a}) =& -5A_7A_8+A_7A_9+10A_3A_{14},\\
\gamma_{12}(\tilde{a}) =& 25A_2^2A_3+18A_{12}^2,\\
\gamma_{13}(\tilde{a}) =& A_2,\\
\gamma_{14}(\tilde{a}) =& A_2A_4+18A_2A_5-236A_{23}+188A_{24},\\
\gamma_{15}(\tilde{a},x,y) =& 144T_1T_7^2-T_1^3(T_{12}+2T_{13})-4(T_9T_{11}+4T_7T_{15}+50T_3T_{23}+2T_4T_{23}+2T_3T_{24}+4T_4T_{24}),\\
\gamma_{16}(\tilde{a},x,y) =& T_{15},
\end{aligned}$$

$$
\begin{aligned}
\gamma_{17}(\tilde a,x,y) =& -(T_{11}+12\,T_{13}),\\
\tilde\gamma_{18}(\tilde a,x,y) =& C_1(C_2,C_2)^{(2)}-2C_2(C_1,C_2)^{(2)},\\
\tilde\gamma_{19}(\tilde a,x,y) =& D_1(C_1,C_2)^{(2)}-((C_2,C_2)^{(2)},C_0)^{(1)},\\
\delta_1(\tilde a) =& 9A_8+31A_9+6A_{10},\\
\delta_2(\tilde a) =& 41A_8+44A_9+32A_{10},\\
\delta_3(\tilde a) =& 3A_{19}-4A_{17},\\
\delta_4(\tilde a) =& -5A_2A_3+3A_2A_4+A_{22},\\
\delta_5(\tilde a) =& 62A_8+102A_9-125A_{10},\\
\delta_6(\tilde a) =& 2T_3+3T_4,\\
\beta_1(\tilde a) =& 3A_1^2-2A_8-2A_{12},\\
\beta_2(\tilde a) =& 2A_7-9A_6,\\
\beta_3(\tilde a) =& A_6,\\
\beta_4(\tilde a) =& -5A_4+8A_5,\\
\beta_5(\tilde a) =& A_4,\\
\beta_6(\tilde a) =& A_1,\\
\beta_7(\tilde a) =& 8A_3-3A_4-4A_5,\\
\beta_8(\tilde a) =& 24A_3+11A_4+20A_5,\\
\beta_9(\tilde a) =& -8A_3+11A_4+4A_5,\\
\beta_{10}(\tilde a) =& 8A_3+27A_4-54A_5,\\
\beta_{11}(\tilde a,x,y) =& T_1^2-20T_3-8T_4,\\
\beta_{12}(\tilde a,x,y) =& T_1,\\
\beta_{13}(\tilde a,x,y) =& T_3,\\
\mathcal{R}_1(\tilde a) =& -2A_7(12A_1^2+A_8+A_{12})+5A_6(A_{10}+A_{11})-2A_1(A_{23}-A_{24})+2A_5(A_{14}+A_{15})\\
&+A_6(9A_8+7A_{12}),\\
\mathcal{R}_2(\tilde a) =& A_8+A_9-2A_{10},\\
\mathcal{R}_3(\tilde a) =& A_9,\\
\mathcal{R}_4(\tilde a) =& -3A_1^2A_{11}+4A_4A_{19},\\
\mathcal{R}_5(\tilde a,x,y) =& (2C_0(T_8-8T_9-2D_2^2)+C_1(6T_7-T_6)-(C_1,T_5)^{(1)}+6D_1(C_1D_2-T_5)-9D_1^2C_2),\\
\mathcal{R}_6(\tilde a) =& -213A_2A_6+A_1(2057A_8-1264A_9+677A_{10}+1107A_{12})+746(A_{27}-A_{28}),\\
\mathcal{R}_7(\tilde a) =& -6A_7^2-A_4A_8+2A_3A_9-5A_4A_9+4A_4A_{10}-2A_2A_{13},\\
\mathcal{R}_8(\tilde a) =& A_{10},\\
\mathcal{R}_9(\tilde a) =& -5A_8+3A_9,\\
\mathcal{R}_{10}(\tilde a) =& 7A_8+5A_{10}+11A_{11},\\
\mathcal{R}_{11}(\tilde a,x,y) =& T_{16}.
\end{aligned}
$$

$$\chi_A^{(1)}(\tilde{a}) = A_6(A_1A_2 - 2A_{15})(3A_1^2 - 2A_8 - 2A_{12}),$$

$$\begin{aligned}\chi_B^{(1)}(\tilde{a}) = &A_7\big[41A_1A_2A_3 + 846A_6A_9 - 252A_6A_{10} + 3798A_6A_{11} - 2A_7(6588A_1^2 - 830A_8 + 265A_{10} \\ &+ 366A_{11} - 156A_{12}) + 1098A_6A_{12} + 983A_3A_{14} - 1548A_4A_{14} - 365A_3A_{15} + 1350A_4A_{15} \\ &+ 1550A_2A_{16} - 1350A_1A_{23}\big],\end{aligned}$$

$$\begin{aligned}\chi_C^{(1)}(\tilde{a}) = &\,\theta\beta_1\beta_3\big[8A_1(42A_{23} - 24A_2A_3 + 59A_2A_5) + A_6(2196A_1^2 + 384A_9 + 24A_{10} + 360A_{11} \\ &- 432A_{12}) + 4A_7(123A_8 - 61A_{10} - 23A_{11} + 123A_{12}) + 8(2A_4A_{14} - 34A_5A_{15} - 19A_2A_{16})\big],\end{aligned}$$

$$\begin{aligned}\chi_D^{(1)}(\tilde{a}) = &\,5790A_1^2A_7 + A_7(-1531A_8 - 140A_9 + 177A_{10} + 947A_{11} - 2791A_{12}) + 2A_6(553A_8 \\ &+ 183A_9 - 100A_{10} - 39A_{11} + 144A_{12}) + A_4(467A_{14} + 922A_{15}) - A_1(461A_2A_4 \\ &- 183A_2A_5 + 296A_{22} - 122A_{24}),\end{aligned}$$

$$\begin{aligned}\chi_E^{(1)}(\tilde{a}) = &\,48A_6(65A_9 - 54A_{10} - 27A_{11}) - 16A_7(774A_1^2 - 382A_8 + 263A_{10} + 129A_{11} - 360A_{12}) \\ &+ 72A_4(23A14 + 3A15) - 16A_3(163A14 + 185A15) - 1792A_2A_{16} + 16A_1(54A_2A_5 \\ &- 173A_{22} + 27A_{24}),\end{aligned}$$

$$\chi_F^{(1)}(\tilde{a}) = \theta\beta_1\beta_3\big[A_7(2A_5 - A_4) - 2A_3A_6\big],$$

$$\chi_G^{(1)}(\tilde{a}) = 12A_3 - 7A_4,$$

$$\chi_A^{(2)}(\tilde{a}) = A_4(5A_8 - 18A_1^2 - A_{10} - 3A_{11} + 9A12),$$

$$\chi_B^{(2)}(\tilde{a}) = A_3(2A_8 - 6A_1^2 - A_9 + A_{10} - A_{11} + 3A_{12}),$$

$$\begin{aligned}\chi_A^{(3)}(\tilde{a}) = &\,49071656765835A_1^6 + 27A_1^4(1344257279043A_{11} - 1270094588593A_{12}) \\ &+ 3A_1^2(176071859457A_2^2A_4 + 2042424190056A_{11}^2 - 4553853105234A_{11}A_{12} \\ &+ 2056276619466A_{12}^2 + 221071597034A_5A_{18} - 539155411551A_5A_{19} \\ &+ 65833344676A_5A_{20} + 26464141896A_4A_{21} + 303070135713A_5A_{21} \\ &- 137515925820A_2A_{23}) + 1048(35846142A_2^2A_4A_{11} - 163576560A_{11}^3 - 21276288A_2^2A_4A_{12} \\ &- 195478380A_{11}^2A_{12} + 325223640A_{11}A_{12}^2 - 93862680A_{12}^3 + 782460A_4A_8A_{20} \\ &+ 26186136A_2A_8A_{22} + 42548200A_2A_9A_{22} - 2682720A_5^2A_{29} - 83946780A_2A_5A_{31} \\ &+ 429178020A_2A_5A_{32} - 204768603A_2A_4A_{34} - 125823390A_2A_5A_{34}),\end{aligned}$$

$$\begin{aligned}\chi_B^{(3)}(\tilde{a}) = &\,10687627614087A_1^6 - 36A_1^2A_{11}(57734730901A_{11} - 18520980346A_{12}) \\ &- 54A_1^4(29889576561A_{11} + 85579885241A_{12}) - 1848441298229A_4A_8A_{19} \\ &- 995417129104A_4A_{10}A_{19} + 139152650610A_5A_{10}A_{19} - 854619791782A_4A_{11}A_{19} \\ &- 234092667978A_5A_{11}A_{19} - 1064773031314A_4A_{12}A_{19} - 1538921088774A_5A_{12}A_{19} \\ &- 200109956062A_4A_8A_{20} - 33399158264A_4A_{10}A_{20} + 1182168636A_5A_{10}A_{20} \\ &- 33699561192A_4A_{11}A_{20} + 359794764A_5A_{11}A_{20} - 150658987068A_4A_{12}A_{20} \\ &- 97478758260A_5A_{12}A_{20} - 1043930677997A_4A_8A_{21} - 381285679090A_4A_{10}A_{21} \\ &- 266080146306A_5A_{10}A_{21} - 340140897016A_4A_{11}A_{21} - 373227206190A_5A_{11}A_{21} \\ &- 763104633190A_4A_{12}A_{21} - 470713035534A_5A_{12}A_{21},\end{aligned}$$

$$
\begin{aligned}
\chi_C^{(3)}(\tilde{a}) = & -(30838311945A_1^2A_2^2A_4 + 2760800121876A_1^2A_8^2 + 7697984307234A_1^2A_8A_9 \\
& + 3201113344320A_1^2A_9^2 - 1697507613684A_1^2A_8A_{10} + 31825111584A_2^2A_4A_{11} \\
& - 695990880A_1^2A_8A_{11} - 61410960A_1^2A_{11}^2 + 10245847104A_2^2A_4A_{12} \\
& - 24350953680A_4A_8A_{17} - 2913648480A_4A_9A_{17} - 2523363762580A_1^2A_5A_{18} \\
& - 29706323760A_4A_8A_{18} + 334082073870A_1^2A_5A_{19} + 142776946840A_1^2A_5A_{20} \\
& + 47764080A_4A_8A_{20} + 282210480A_1^2A_4A_{21} + 2047601391150A_1^2A_5A_{21} \\
& + 63016473792A_2A_8A_{22} + 77305513600A_2A_9A_{22} - 35441430120A_1^2A_2A_{23} \\
& - 42056705280A_2A_9A_{23} - 163762560A_5^2A_{29} - 94243374720A_2A_5A_{31} \\
& + 290822854080A_2A_5A_{32} - 150861290016A_2A_4A_{34} - 47162628000A_2A_5A_{34}), \\
\chi_D^{(3)}(\tilde{a}) = & (7815A_2^2A_4^2 - 1912260A_4A_8^2 - 3772362A_4A_8A_9 - 237900A_4A_8A_{10} - 178080A_2A_{10}A_{13} \\
& - 193248A_2A_{11}A_{13} - 1318176A_5^2A_{17} + 1194740A_4A_5A_{18} - 139104A_5^2A_{18} \\
& + 56706A_4A_5A_{19} + 702144A_5^2A_{19} - 56552A_4A_5A_{20} - 11040A_4^2A_{21} - 995070A_4A_5A_{21} \\
& - 32856A_2A_4A_{23} + 26112A_2A_5A_{24}), \\
\chi_E^{(3)}(\tilde{a}) = & 54A_1^2A_2 + 611A_2A_9 - 104A_2A_{11} - 140A_2A_{12} + 732A_1A_{14} - 243A_{31} - 234A_{33} + 245A_{34}, \\
\chi_F^{(3)}(\tilde{a}) = & -(11A_4 + 10A_5), \\
\chi_A^{(4)}(\tilde{a}) = & (-2A_2^2A_4 - 80A_8^2 + 64A_8A_9 - 80A_8A_{10} + 16A_9A_{10} - 9A_{10}^2 - 32A_8A_{11} + 48A_9A_{11} \\
& + 2A_{10}A_{11} + 23A_{11}^2 + 120A_5A_{17} + 24A_5A_{18} - 4A_5A_{19} + 6A_4A_{21} + 4A_5A_{21}) \\
& \times (264A_2^2A_8 - 112A_2^2A_9 - 56A_9A_{17} + 746A_{10}A_{17} + 1006A_{11}A_{17} + 424A_{10}A_{18} \\
& + 824A_{11}A_{18} + 1092A_8A_{19} - 384A_9A_{19} - 97A_{10}A_{19} + 153A_{11}A_{19} - 264A_8A_{20} \\
& + 168A_9A_{20} + 14A_{10}A_{20} - 14A_{11}A_{20} - 620A_8A_{21} + 81A_{10}A_{21} - 81A_{11}A_{21} \\
& + 126A_4A_{30} - 208A_2A_{31} - 112A_2A_{33}), \\
\chi_B^{(4)}(\tilde{a}) = & (-12(518A_8^2 - 16A_9(2A_{10} + 5A_{11}) + 2(A_{10} + 3A_{11})(31A_{10} + 69A_{11}) + A_8(369A_{10} \\
& + 871A_{11}) - 96A_3A_{17}) + 2A_5(552A_2^2 - 404A_{18} + 2271A_{19} - 316A_{20} - 1674A_{21}) \\
& - 135A_4A_{21} - 240A_2A_{23})(4A_2^2(6160A_9 - 60659A_{10} + 5565A_{11}) + 533574A_{10}A_{17} \\
& + 2120070A_{11}A_{17} + 365744A_{10}A_{18} + 657528A_{11}A_{18} - 713634A_{10}A_{19} + 8A_9(22484A_{17} \\
& + 10472A_{18} + 10911A_{19} - 2156A_{20}) + 121318A_{10}A_{20} - 11130A_{11}A_{20} + 522591A_{10}A_{21} \\
& - 357309A_{11}A_{21} + 72A_8(13247A_{17} + 1081A_{20} + 7084A_{21}) + 2079A_4A_{30} + 186520A_2A_{34}), \\
\chi_A^{(5)}(\tilde{a}) = & 95A_9 + 2A_{10}, \\
\chi_A^{(6)}(\tilde{a}) = & 4A_{11} - 4A_{10}, \\
\chi_B^{(6)}(\tilde{a}) = & (A_4 - 2A_5)(A_8 - 2A_{11}), \\
\chi_A^{(7)}(\tilde{a}) = & (A_3 - A_4)(A_8 - A_{10}), \\
\chi_B^{(7)}(\tilde{a}) = & -2A_8(6348A_9^2 - A_4(502073A_{18} + 250407A_{19} + 37072A_{20}) + 18A_2(720A_{22} + 8179A_{23})) \\
& + 3(640A_9^3 + 36A_7^2(3218A_{18} + 17721A_{19}) + 8A_9(7505A_4A_{18} + 37966A_2A_{23}) + 4A_2(A_{13} \\
& \times (74429A_{18} + 44574A_{19}) - 7A_{10}(5387A_{22} + 4741A_{23}) + 243552A_7A_{27}) \\
& + A_4(-341504A_{10}A_{18} - 78779A_7A_{25} + 234046A_2A_{33})),
\end{aligned}
$$

$$\chi_C^{(7)}(\tilde{a}) = 2484A_7^2(2A_{18}+9A_{19}) - 2A_8(276A_9^2 + A_4(-34111A_{18}+51231A_{19}-35504A_{20}) + 46794A_2A_{23}) + 3(4A_2(5403A_{13}A_{18} - 29222A_{13}A_{19} - 6123A_{10}A_{22} + 11444A_9A_{23} + 7131A_{10}A_{23} + 41384A_7A_{27}) + A_4(1080A_9A_{18} - 35328A_{10}A_{18} - 52173A_7A_{25} + 35842A_2A_{33})),$$

$$\chi_D^{(7)}(\tilde{a}) = (A_3 - A_4)(8A_7^2 - 44A_3A_8 + 27A_4A_8 + 4A_3A_9 + 22A_3A_{10} - 9A_4A_{10}),$$

$$\chi_A^{(8)}(\tilde{a}) = 5A_8 - A_9,$$

$$\chi_D^{(8)}(\tilde{a}) = 9A_9 - 25A_8.$$

Next we construct the following T-comitants which are responsible for the existence of invariant straight lines of systems (2):

Notation 1.

$$\begin{aligned} B_3(a,x,y) &= (C_2,D)^{(1)} = Jacob\,(C_2,D)\,, \\ B_2(a,x,y) &= (B_3,B_3)^{(2)} - 6B_3(C_2,D)^{(3)}, \\ B_1(a) &= \text{Res}_{\,x}\,(C_2,D)\,/y^9 = -2^{-9}3^{-8}\,(B_2,B_3)^{(4)}\,. \end{aligned} \tag{6}$$

Lemma 2 (see [23]). *For the existence of invariant straight lines in one (respectively 2; 3 distinct) directions in the affine plane it is necessary that $B_1 = 0$ (respectively $B_2 = 0$; $B_3 = 0$).*

At the moment we only have necessary and not necessary and sufficient conditions for the existence of an invariant straight line or for invariant lines in two or three directions.

Let us apply a translation $x = x' + x_0$, $y = y' + y_0$ to the polynomials $p(\tilde{a},x,y)$ and $q(\tilde{a},x,y)$. We obtain $\hat{p}(\hat{a}(a,x_0,y_0),x',y') = p(\tilde{a},x'+x_0,y'+y_0)$, $\hat{q}(\hat{a}(a,x_0,y_0),x',y') = q(\tilde{a},x'+x_0,y'+y_0)$. Let us construct the following polynomials

$$\Gamma_i(\tilde{a},x_0,y_0) \equiv \text{Res}_{\,x'}\Big(C_i\big(\hat{a}(\tilde{a},x_0,y_0),x',y'\big), C_0\big(\hat{a}(\tilde{a},x_0,y_0),x',y'\big)\Big)/(y')^{i+1},$$
$$\Gamma_i(\tilde{a},x_0,y_0) \in \mathbb{R}[\tilde{a},x_0,y_0],\ \ i=1,2.$$

Notation 2. *We denote by*

$$\tilde{\mathcal{E}}_i(\tilde{a},x,y) = \Gamma_i(\tilde{a},x_0,y_0)|_{\{x_0=x,\ y_0=y\}} \in \mathbb{R}[\tilde{a},x,y] \quad (i=1,2).$$

Observation 1. *We note that the polynomials $\tilde{\mathcal{E}}_1(a,x,y)$ and $\tilde{\mathcal{E}}_2(a,x,y)$ are affine comitants of systems* (2) *and are homogeneous polynomials in the coefficients a,b,c,d,e,f,g,h,k,l,m,n and non-homogeneous in x,y and $deg_{\tilde{a}}\tilde{\mathcal{E}}_1 = 3$, $deg_{(x,y)}\tilde{\mathcal{E}}_1 = 5$, $deg_{\tilde{a}}\tilde{\mathcal{E}}_2 = 4$, $deg_{(x,y)}\tilde{\mathcal{E}}_2 = 6$.*

Notation 3. *Let $\mathcal{E}_i(\tilde{a},X,Y,Z)$, $i=1,2$, be the homogenization of $\tilde{\mathcal{E}}_i(\tilde{a},x,y)$, i.e.*

$$\mathcal{E}_1(\tilde{a},X,Y,Z) = Z^5\tilde{\mathcal{E}}_1(\tilde{a},X/Z,Y/Z), \qquad \mathcal{E}_2(\tilde{a},X,Y,Z) = Z^6\tilde{\mathcal{E}}_1(\tilde{a},X/Z,Y/Z)$$

The geometrical meaning of these affine comitants is given by the following lemma (see [23]):

Lemma 3 (see [23]). *1) The straight line $\mathcal{L}(x,y) \equiv ux+vy+w=0$, $u,v,w \in \mathbb{C}$, $(u,v) \neq (0,0)$ is an invariant line for a quadratic system* (2) *if and only if the polynomial $\mathcal{L}(x,y)$ is a common factor of the polynomials $\tilde{\mathcal{E}}_1(\tilde{a},x,y)$ and $\tilde{\mathcal{E}}_2(\tilde{a},x,y)$ over $\mathbb{C}$, i.e.*

$$\tilde{\mathcal{E}}_i(\tilde{a},x,y) = (ux+vy+w)\widetilde{W}_i(x,y),\ \ i=1,2,$$

where $\widetilde{W}_i(x,y) \in \mathbb{C}[x,y]$.

2) If $\mathcal{L}(x,y) = 0$ *is an invariant straight line of multiplicity* λ *for a quadratic system* (2)*, then* $[\mathcal{L}(x,y)]^\lambda \mid \gcd(\tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}_2)$ *in* $\mathbb{C}[x,y]$*, i.e. there exist* $W_i(\tilde{a},x,y) \in \mathbb{C}[x,y]$*,* $i = 1,2$*, such that*

$$\tilde{\mathcal{E}}_i(\tilde{a},x,y) = (ux+vy+w)^\lambda W_i(\boldsymbol{a},x,y), \ \ i=1,2.$$

3) If the line $l_\infty : Z = 0$ *is of multiplicity* $\lambda > 1$*, then* $Z^{\lambda-1} \mid \gcd(\mathcal{E}_1, \mathcal{E}_2)$*.*

In order to detect the parallel invariant lines we need the following invariant polynomials:

$$N(\tilde{a},x,y) = D_2^2 + T_8 - 2T_9, \quad \theta(\tilde{a}) = 2A_5 - A_4 \ (\equiv \text{Discriminant}\big(N(a,x,y)\big)/1296).$$

Lemma 4 (see [23])**.** *A necessary condition for the existence of one couple (respectively two couples) of parallel invariant straight lines of a system* (2) *corresponding to* $\tilde{a} \in \mathbb{R}^{12}$ *is the condition* $\theta(\tilde{a}) = 0$ *(respectively* $N(\tilde{a},x,y) = 0$*).*

3 Proof of statement *(B)* of Main Theorem

In this section we provide the proof of statement *(B)* of our Main Theorem. In accordance with Theorem 1 stated in [15], we only investigate the case $\eta > 0$ (see Diagram 10). The case $\eta = 0$ will be considered in a future paper.

So in what follows we assume $\eta > 0$. In this case according to [23, Lemma 44] there exist an affine transformation and time rescaling which brings systems (2) to the systems

$$\frac{dx}{dt} = a + cx + dy + gx^2 + (h-1)xy, \quad \frac{dy}{dt} = b + ex + fy + (g-1)xy + hy^2, \tag{7}$$

with $\eta = 1$ and $\theta = -(g-1)(h-1)(g+h)/2$.

3.1 The subcase $\theta \neq 0$

Following Theorem 1 we assume that for a quadratic system (7) the conditions $\theta \neq 0$ and $\gamma_1 = 0$ are fulfilled. Then, as it was proved in [15], due to an affine transformation and time rescaling, this system could be brought to the canonical form

$$\frac{dx}{dt} = a + cx + gx^2 + (h-1)xy, \quad \frac{dy}{dt} = b - cy + (g-1)xy + hy^2, \tag{8}$$

for which we calculate

$$\begin{aligned} \gamma_2 =& -1575c^2(g-1)^2(h-1)^2(g+h)(3g-1)(3h-1)(3g+3h-4)\mathcal{B}_1, \\ \beta_1 =& -c^2(g-1)(h-1)(3g-1)(3h-1)/4, \\ \beta_2 =& -c(g-h)(3g+3h-4)/2, \quad \theta = -(g-1)(h-1)(g+h)/2, \end{aligned} \tag{9}$$

where $\mathcal{B}_1 = b(2h-1) - a(2g-1)$.

3.1.1 The possibility $\beta_1 \neq 0$

In this case the condition $\gamma_2 = 0$ is equivalent to $(3g+3h-4)\mathcal{B}_1 = 0$.

3.1.1.1 *The case $\beta_2 \neq 0$.* Then $3g+3h-4 \neq 0$ and we obtain $\mathcal{B}_1 = 0$. Since $c \neq 0$ due to the rescaling $(x, y, t) \mapsto (cx, cy, t/c)$ we may assume $c = 1$. Moreover as $(2g-1)^2 + (2h-1)^2 \neq 0$ due to $\beta_2 \neq 0$ (i.e. $g-h \neq 0$), the condition $\mathcal{B}_1 = 0$ could be written as $a = a_1(2h-1)$ and $b = a_1(2g-1)$. So setting the old parameter a instead of a_1, we arrive at the 3-parameter family of systems

$$\frac{dx}{dt} = a(2h-1) + x + gx^2 + (h-1)xy, \quad \frac{dy}{dt} = a(2g-1) - y + (g-1)xy + hy^2 \tag{10}$$

with the condition

$$a(g-1)(h-1)(g+h)(g-h)(3g-1)(3h-1)(3g+3h-4) \neq 0. \tag{11}$$

These systems possess the invariant hyperbola

$$\Phi(x, y) = a + xy = 0. \tag{12}$$

Remark 6. *We point out that for systems* (10) *the parameters g and h have the same significance, because we could replace g by h via the change $(x, y, t, a, g, h) \mapsto (-y, -x, -t, a, h, g)$, which keeps these systems.*

For systems (10) we calculate

$$B_1 = 2a^2(g-1)^2(h-1)^2(g-h)(2g-1)(2h-1)\big[a(g+h)^2 - 1\big]. \tag{13}$$

3.1.1.1.1 *The subcase $B_1 \neq 0$.* In this case by Lemma 2 we have no invariant lines. For systems (10) we calculate $\mu_0 = gh(g+h-1)$ and we consider two possibilities: $\mu_0 \neq 0$ and $\mu_0 = 0$.

a) *The possibility $\mu_0 \neq 0$.* Then by Lemma 1 the systems have finite singularities of total multiplicity 4. We detect that two of these singularities are located on the hyperbola, more exactly such singularities are $M_{1,2}(x_{1,2}, y_{1,2})$ with

$$x_{1,2} = \frac{-1 \pm \sqrt{Z_1}}{2g}, \quad y_{1,2} = \frac{1 \pm \sqrt{Z_1}}{2h}, \quad Z_1 = 1 - 4agh.$$

On the other hand for systems (10) we calculate the invariant polynomial

$$\chi_A^{(1)} = (g-1)^2(h-1)^2(g-h)^2(3g-1)^2(3h-1)^2\, Z_1,$$
$$\chi_B^{(1)} = -105a(g-1)^2(h-1)^2(g-h)^2(3g-1)^2(3h-1)^2/8$$

and by (11) we conclude that $\operatorname{sign}(\chi_A^{(1)}) = \operatorname{sign}(Z_1)$ (if $Z_1 \neq 0$) and $\operatorname{sign}(\chi_B^{(1)}) = -\operatorname{sign}(a)$. So we consider three cases: $\chi_A^{(1)} < 0$, $\chi_A^{(1)} > 0$ and $\chi_A^{(1)} = 0$.

a$_1$) *The case $\chi_A^{(1)} < 0$.* So we have no real singularities located on the invariant hyperbola and we arrive at the configurations of invariant curves given by *Config. H.1* if $\chi_B^{(1)} < 0$ and *Config. H.2* if $\chi_B^{(1)} > 0$.

a$_2$) *The case $\chi_A^{(1)} > 0$.* In this case we have two real singularities located on the hyperbola. We have the next result.

Lemma 5. *Assume that the singularities $M_{1,2}(x_{1,2}, y_{1,2})$ (located on the hyperbola) are finite. Then these singularities are located on different branches of the hyperbola if $\chi_C^{(1)} < 0$ and they are located on the same branch if $\chi_C^{(1)} > 0$.*

Proof: Since the asymptotes of the hyperbola (12) are the lines $x = 0$ and $y = 0$ it is clear that the singularities $M_{1,2}$ are located on different branches of the hyperbola if and only if $x_1 x_2 < 0$. We calculate

$$x_1 x_2 = \left[\frac{-1+\sqrt{Z_1}}{2g}\right]\left[\frac{-1-\sqrt{Z_1}}{2g}\right] = \frac{ah}{g},$$
$$\chi_C^{(1)} = 35agh(g-1)^4(h-1)^4(g-h)^2(g+h)^2(3g-1)^2(3h-1)^2/32 \tag{14}$$

and due to the condition (11) we obtain that $\operatorname{sign}(x_1x_2) = \operatorname{sign}(\chi_C^{(1)})$. This completes the proof of the lemma. ∎

Other two singular points of systems (10)are $M_{3,4}(x_{3,4}, y_{3,4})$ (generically located outside the hyperbola) with

$$x_{3,4} = \frac{(1-2h)\left[1 \pm \sqrt{Z_2}\right]}{2(g+h-1)}, \quad y_{3,4} = \frac{(2g-1)\left[1 \pm \sqrt{Z_2}\right]}{2(g+h-1)}, \quad Z_2 = 1 + 4a(1-g-h). \tag{15}$$

We need to determine the conditions when the singular points located outside the hyperbola coincide with its points (singular for the systems or not). In this order considering (12) we calculate

$$\Phi(x,y)|_{\{x=x_{3,4},\ y=y_{3,4}\}} = \frac{\tilde{A} - (2g-1)(2h-1)\left[1 \pm \sqrt{Z_2}\right]}{2(g+h-1)^2} \equiv \Omega_{3,4}(a,g,h),$$

where $\tilde{A} = 2a(g+h-1)(4gh-g-h)$. It is clear that at least one of the singular points $M_3(x_3, y_3)$ or $M_4(x_4, y_4)$ belongs to the hyperbola (12) if and only if

$$\Omega_3\Omega_4 = -\frac{a\,Z_3}{(g+h-1)^2} = 0, \quad Z_3 = (2g-1)(2h-1) - a(4gh-g-h)^2.$$

On the other hand for systems (10) we have

$$\chi_D^{(1)} = 105(g-h)(3g-1)(3h-1)\,Z_3/4$$

and clearly due to (11) the condition $\chi_D^{(1)} = 0$ is equivalent to $Z_3 = 0$. We examine two subcases: $\chi_D^{(1)} \neq 0$ and $\chi_D^{(1)} = 0$.

α) *The subcase $\chi_D^{(1)} \neq 0$.* Then $Z_3 \neq 0$ and on the hyperbola there are two simple real singularities (namely $M_{1,2}(x_{1,2}, y_{1,2})$). By Lemma 5 their position is defined by the invariant polynomial $\chi_C^{(1)}$ and we arrive at the following conditions and configurations:

- $\chi_C^{(1)} < 0$ and $\chi_B^{(1)} < 0 \;\Rightarrow\;$ *Config. H.17*;
- $\chi_C^{(1)} < 0$ and $\chi_B^{(1)} > 0 \;\Rightarrow\;$ *Config. H.19*;
- $\chi_C^{(1)} > 0$ and $\chi_B^{(1)} < 0 \;\Rightarrow\;$ *Config. H.16*;
- $\chi_C^{(1)} > 0$ and $\chi_B^{(1)} > 0 \;\Rightarrow\;$ *Config. H.18.*

$\boldsymbol{\beta}$) *The subcase* $\chi_D^{(1)} = 0$. In this case the conditions $Z_3 = 0$, $B_1 \neq 0$ (see (13)) and (11) implies $4gh - g - h \neq 0$ and we obtain $a = (2g-1)(2h-1)/(4gh-g-h)^2$. Then considering Proposition 1 we calculate

$$\mathbf{D} = 0, \quad \mathbf{T} = -3\big[g(g-1)(2h-1)x + h(h-1)(2g-1)y\big]^2\mathbf{P},$$
$$\mathbf{P} = \frac{(g-h)^2}{(4gh-g-h)^4}(2-3g-3h+4gh)^2(gx-hy)^2\big[(2g-1)x+(2h-1)y\big]^2.$$

$\boldsymbol{\beta_1}$) *The possibility* $\mathbf{T} \neq 0$. Then $\mathbf{T} < 0$ and according to Proposition 1 systems (10) possess one double and two simple real finite singularities. More exactly, we detect that one of the singular points $M_3(x_3, y_3)$ or $M_4(x_4, y_4)$ collapses with a singular point located on the hyperbola, whereas another one remains outside the hyperbola. Taking into consideration Lemma 5 we obtain the following conditions and configurations:

- $\chi_C^{(1)} < 0$ and $\chi_B^{(1)} < 0 \;\Rightarrow\;$ *Config. H.21*;
- $\chi_C^{(1)} < 0$ and $\chi_B^{(1)} > 0 \;\Rightarrow\;$ *Config. H.23*;
- $\chi_C^{(1)} > 0$ and $\chi_B^{(1)} < 0 \;\Rightarrow\;$ *Config. H.20*;
- $\chi_C^{(1)} > 0$ and $\chi_B^{(1)} > 0 \;\Rightarrow\;$ *Config. H.22.*

$\boldsymbol{\beta_2}$) *The possibility* $\mathbf{T} = 0$. In this case due to the conditions (11) and $\mu_0 \neq 0$ the equality $\mathbf{T} = 0$ holds if and only if $\mathbf{P} = 0$ which is equivalent to $2 - 3g - 3h + 4gh = 0$ (or equivalently $2 - 3g + h(4g-3) = 0$). Since $g - h \neq 0$ (see (11)), the condition $(4g-3)^2 + (4h-3)^2 \neq 0$ holds, then by Remark 6 we may assume $(4g-3) \neq 0$, i.e. $h = (3g-2)/(4g-3)$ and we obtain

$$\mathbf{D} = \mathbf{T} = \mathbf{P} = 0, \quad \mathbf{R} = \frac{3}{(4g-3)^4}(g-1)^2(2g-1)^2\big[g(4g-3)x + (2-3g)y\big]^2.$$

Since $\mathbf{R} \neq 0$, by Proposition 1 we obtain one triple and one simple singularities. More precisely the singular points M_3 and M_4 collapse with one of the singular points M_1 or M_2 and the last point becomes a triple one. In this case, we calculate

$$\chi_B^{(1)} = -\frac{105(g-1)^6(3g-1)^2(5g-3)^2}{8(4g-3)^5}, \;\; \chi_C^{(1)} = \frac{35\,g(3g-2)(g-1)^{10}(3g-1)^2(5g-3)^2(2g^2-1)^2}{8(4g-3)^{10}}.$$

We remark that the condition $\chi_C^{(1)} < 0$ implies $\chi_B^{(1)} > 0$. Indeed, if $\chi_C^{(1)} < 0$ then $g(3g-2) < 0$ (i.e. $0 < g < 2/3$) and for these values of g we have $4g - 3 < 0$, which is equivalent to $\chi_B^{(1)} > 0$. Taking into consideration Lemma 5 we obtain the following conditions and configurations:

- $\chi_C^{(1)} < 0 \;\Rightarrow\;$ *Config. H.26*;
- $\chi_C^{(1)} > 0$ and $\chi_B^{(1)} < 0 \;\Rightarrow\;$ *Config. H.24*;
- $\chi_C^{(1)} > 0$ and $\chi_B^{(1)} > 0 \;\Rightarrow\;$ *Config. H.25.*

***a*₃) *The case* $\chi_A^{(1)} = 0$. Due to the condition (11), the condition $\chi_A^{(1)} = 0$ implies $Z_1 = 0$ and it yields $a = 1/(4gh)$. In this case the points $M_{1,2}$ collapse and we have a double point on the hyperbola. So we calculate

$$\chi_B^{(1)} = -\frac{105}{32gh}(g-1)^2(h-1)^2(g-h)^2(3g-1)^2(3h-1)^2,$$
$$\chi_C^{(1)} = 35(g-1)^4(h-1)^4(g-h)^2(g+h)^2(3g-1)^2(3h-1)^2/128 > 0,$$
$$\chi_D^{(1)} = -\frac{105}{16gh}(g-h)^3(3g-1)(3h-1) \neq 0.$$

Since $\chi_C^{(1)} \neq 0$, no other point could coalesce with the double point on the hyperbola and we arrive at the configurations given by *Config. H.7* if $\chi_B^{(1)} < 0$ and *Config. H.8* if $\chi_B^{(1)} > 0$.

***b*)** *The possibility* $\mu_0 = 0$. Then by Lemma 1 at least one finite singular point has gone to infinity and collapsed with one of the infinite singular points $[1,0,0]$, $[0,1,0]$ or $[1,1,0]$. By the same lemma, a second point could go to infinity if and only if $\mu_1(x,y) = 0$. However, for systems (11) we have the following remark.

Remark 7. *If for a system* (10) *the condition* $\mu_0 = 0$ *holds then* $\mu_1 \neq 0$. *Moreover by* (3) *the condition* $\mathbf{R} = 3\mu_1^2 \neq 0$ *is fulfilled.*

Indeed for systems (10) we calculate

$$\mu_0 = gh(g+h-1) = 0, \quad \mu_1 = g(1-g-2gh)x + h(1-h-2gh)y. \tag{16}$$

We observe that in the case $g = 0$ (respectively $h = 0$; $g = 1-h$) we get $\mu_1 = h(1-h)y \neq 0$ (respectively $\mu_1 = g(1-g)y \neq 0$; $\mu_1 = h(h-1)(2h-1)(x-y) \neq 0$) due to the condition (11).

We consider the cases: $\chi_A^{(1)} < 0$, $\chi_A^{(1)} > 0$ and $\chi_A^{(1)} = 0$.

***b*₁)** *The case* $\chi_A^{(1)} < 0$. The points on the hyperbola are complex and, moreover, $1 - 4agh < 0$ implies $agh > 0$ and hence $\chi_C^{(1)} > 0$. Then we arrive at the configurations given by *Config. H.3* if $\chi_B^{(1)} < 0$ and *Config. H.4* if $\chi_B^{(1)} > 0$.

***b*₂)** *The case* $\chi_A^{(1)} > 0$. The points on the hyperbola are real and we observe that due to the condition (11) the equality $\chi_C^{(1)} = 0$ is equivalent to $gh = 0$. So we consider two subcases: $\chi_C^{(1)} \neq 0$ and $\chi_C^{(1)} = 0$.

***α*)** *The subcase* $\chi_C^{(1)} \neq 0$. Then the condition $\mu_0 = 0$ gives $g+h-1 = 0$, i.e. $g = 1-h$ and one finite singularity has gone to infinity and collapsed with the point $[1,1,0]$. Clearly that this must be a singular point located outside the hyperbola and hence on the finite part of the phase plane of systems (10) there are three singularities, two of which (M_1 and M_2) being located on the hyperbola.

Since the singular points on the hyperbola are real we have to decide when the third point will belong also to the hyperbola. For systems (10) with $g = 1-h$ we calculate

$$\chi_B^{(1)} = -105ah^2(h-1)^2(2h-1)^2(3h-1)^2(3h-2)^2/8,$$
$$\chi_D^{(1)} = 105(2h-1)^3(2-3h)(3h-1)\big[1 + a(2h-1)^2\big]/4.$$

We observe that the condition $\chi_B^{(1)} < 0$ implies $\chi_D^{(1)} \neq 0$. Indeed, supposing $\chi_D^{(1)} = 0$ and considering condition (11), we obtain $a = -1/(2h-1)^2$ and hence

$$\chi_B^{(1)} = 105h^2(h-1)^2(3h-1)^2(3h-2)^2/8 > 0.$$

So in the case $\chi_C^{(1)} < 0$ we get the following conditions and configurations:

- $\chi_B^{(1)} < 0 \;\Rightarrow\;$ *Config. H.30*;
- $\chi_B^{(1)} > 0$ and $\chi_D^{(1)} \neq 0 \;\Rightarrow\;$ *Config. H.32*;
- $\chi_B^{(1)} > 0$ and $\chi_D^{(1)} = 0 \;\Rightarrow\;$ *Config. H.34*;

whereas for $\chi_C^{(1)} > 0$ we get

- $\chi_B^{(1)} < 0 \;\Rightarrow\;$ *Config. H.29*;
- $\chi_B^{(1)} > 0$ and $\chi_D^{(1)} \neq 0 \;\Rightarrow\;$ *Config. H.31*;
- $\chi_B^{(1)} > 0$ and $\chi_D^{(1)} = 0 \;\Rightarrow\;$ *Config. H.33*.

$\boldsymbol{\beta}$**)** *The subcase* $\chi_C^{(1)} = 0$. Then $gh = 0$ and $g^2 + h^2 \neq 0$ due to $g - h \neq 0$. By Remark 6 we may assume $g = 0$ and then one of the singularities located on the hyperbola (12) has gone to infinity and collapsed with the point $[1, 0, 0]$. The calculations yield

$$\chi_B^{(1)} = -105ah^2(h-1)^2(3h-1)^2/8, \quad \chi_D^{(1)} = 105h(3h-1)(1-2h-ah^2)/4. \tag{17}$$

$\boldsymbol{\beta_1}$**)** *The possibility* $\chi_B^{(1)} < 0$. Then we have to analyze two cases: $\chi_D^{(1)} \neq 0$ and $\chi_D^{(1)} = 0$.

If $\chi_D^{(1)} \neq 0$, the finite singularities $M_{3,4}$ remain outside the hyperbola and we arrive at the configuration given by *Config. H.9*. In the case $\chi_D^{(1)} = 0$ (which yields $a = (1-2h)/h^2$), one of the singular points $M_{3,4}$ coalesces with the remaining singularity on the hyperbola. For this case we calculate

$$\mathbf{D} = 0,\; \mathbf{P} = (3h-2)^2y^2(x+y-2hy)^2,\; \mathbf{T} = -3h^2(h-1)^2y^2\mathbf{P}.$$

We observe that the condition $\chi_B^{(1)} > 0$ implies $\mathbf{T} \neq 0$. Indeed, the conditions $\chi_D^{(1)} = \mathbf{T} = 0$ imply $h = 2/3$ and $a = -3/4$, and hence $\chi_B^{(1)} > 0$.

Moreover, according to Remark 7, in the case $\mu_0 = 0$, the condition $\mathbf{R} \neq 0$ is satisfied for systems (10). Then, since $\mathbf{T} \neq 0$, we obtain $\mathbf{PR} \neq 0$, and by Proposition 1 we have a double singular point on the hyperbola and we arrive at *Config. H.11*.

$\boldsymbol{\beta_2}$**)** *The possibility* $\chi_B^{(1)} > 0$. We again analyze the cases $\chi_D^{(1)} \neq 0$ and $\chi_D^{(1)} = 0$. In the case $\chi_D^{(1)} \neq 0$, the finite singularities $M_{3,4}$ remain outside the hyperbola and we arrive at the configuration given by *Config. H.10*. If $\chi_D^{(1)} = 0$, we obtain the configurations shown in *Config. H.12* if $\mathbf{T} \neq 0$ and *Config. H.13* if $\mathbf{T} = 0$.

$\boldsymbol{b_3}$**)** *The case* $\chi_A^{(1)} = 0$. Due to the condition (11), the condition $\chi_A^{(1)} = 0$ implies $Z_1 = 0$ (then $gh \neq 0$) and hence $a = 1/(4gh)$. Therefore the condition $\mu_0 = 0$ yields $g = 1 - h$. In this case the singular points $M_{1,2}$ collapse and we have a double point on the hyperbola. For systems (10) with $g = 1-h$ and $a = 1/[4h(1-h)]$, we calculate

$$\begin{aligned}
\chi_B^{(1)} &= 105h(h-1)(2h-1)^2(3h-1)^2(3h-2)^2/32,\\
\chi_C^{(1)} &= \frac{35}{128}\,h^4(h-1)^4(2h-1)^2(3h-1)^2(3h-2)^2,\\
\chi_D^{(1)} &= \frac{105}{16h(h-1)}\,(2h-1)^3(3h-1)(3h-2),\\
\mathbf{D} = 0, &\quad \mathbf{T} = -3h^2(h-1)^2(2h-1)^2(x-y)^4(x+y)^2 \neq 0.
\end{aligned}$$

Since $\chi_D^{(1)} \neq 0$ (due to condition (11)), the singular point located outside the hyperbola could not collapse with this double point and we arrive at the configurations given by *Config. H.14* if $\chi_B^{(1)} < 0$ and *Config. H.15* if $\chi_B^{(1)} > 0$.

3.1.1.1.2 *The subcase $B_1 = 0$.* According to Lemma 2 the condition $B_1 = 0$ is necessary in order to exist an invariant line of systems (10). Considering the condition (11) we obtain that $B_1 = 0$ (see (13)) is equivalent to

$$(2g-1)(2h-1)\big[a(g+h)^2 - 1\big] = 0.$$

On the other hand, for these systems we calculate

$$\chi_E^{(1)} = -105(g-1)(h-1)(g-h)(3g-1)(3h-1)\,Z_4, \quad Z_4 = \big[a(g+h)^2 - 1\big],$$

and by (11) the condition $Z_4 = 0$ is equivalent to $\chi_E^{(1)} = 0$.

a) *The possibility $\chi_E^{(1)} \neq 0$.* In this case we get $g = 1/2$ and this leads to the systems

$$\frac{dx}{dt} = a(2h-1) + x + x^2/2 + (h-1)xy, \quad \frac{dy}{dt} = -y(2 + x - 2hy)/2, \tag{18}$$

for which the following condition holds (see (11)):

$$a(h-1)(2h-1)(2h+1)(3h-1)(6h-5) \neq 0. \tag{19}$$

We observe that besides the hyperbola (12) these systems possess the invariant line $y = 0$, which is one of the asymptotes of this hyperbola. For the above systems we calculate

$$\begin{aligned}
\mu_0 = h(2h-1)/4, &\quad \chi_E^{(1)} = -\frac{105}{8}\,(h-1)(2h-1)(3h-1)\,Z_4\big|_{\{g=1/2\}},\\
B_1 = 0, &\quad B_2 = -648a(h-1)^2(2h-1)^2y^4\,Z_4\big|_{\{g=1/2\}}.
\end{aligned}$$

Therefore we conclude that due to the conditions $\chi_E^{(1)} \neq 0$ and (19) we obtain $B_2 \neq 0$ and, by Lemma 2, we could not have an invariant line in a direction which is different from $y = 0$. Moreover, due to the condition $\theta \neq 0$ and according to Lemma 4, in the direction $y = 0$ we could not have either a couple of parallel invariant lines or a double invariant line.

a_1) *The case $\mu_0 \neq 0$.* Then $h(2h-1) \neq 0$ and considering the coordinates of the singularities $M_i(x_i, y_i)$ (i=1,2,3,4) mentioned earlier (see page 33) for $g = 1/2$ we have

$$\begin{aligned}
x_{1,2} = -1 \pm \sqrt{1-2ah}, &\quad y_{1,2} = -1 \mp \sqrt{1-2ah},\\
x_{3,4} = -1 \pm \sqrt{1+2a(1-2h)}, &\quad y_{3,4} = 0.
\end{aligned}$$

We recall that the singular points $M_{1,2}(x_{1,2}, y_{1,2})$ are located on the hyperbola. We also observe that the singularities $M_{3,4}(x_{3,4}, y_{3,4})$ are located on the invariant line $y = 0$.

On the other hand, for systems (18) we calculate

$$\chi_A^{(1)} = 2^{-12}(h-1)^2(2h-1)^2(3h-1)^2(1-2ah), \quad \chi_B^{(1)} = -105a(h-1)^2(2h-1)^2(3h-1)^2/512,$$
$$\chi_C^{(1)} = 2^{-16}35ah(h-1)^4(2h-1)^2(2h+1)^2(3h-1)^2, \ \mathbf{D} = 3a^2(2h-1)^4\big[2a(2h-1)-1\big](1-2ah),$$

and it is clear that, due to the factors $1-2ah$ and $1+2a(1-2h)$, the invariant polynomials $\chi_A^{(1)}$ and $\mathbf{D}$ govern the types of the above singular points (i.e. are they real or complex or coinciding), whereas the invariant polynomials $\chi_B^{(1)}$ and $\chi_C^{(1)}$ are respectively responsible for the position of the hyperbola on the plane and for the location of the real singularities on the hyperbola (i.e. on the same branch or on the different ones).

$\boldsymbol{\alpha}$**)** *The subcase* $\chi_A^{(1)} < 0$. Then the singularities $M_{1,2}$ (located on the hyperbola) are complex, whereas the types of singularities $M_{3,4}$ (located on the invariant line $y = 0$) are governed by $\mathbf{D}$. We observe that clearly the condition $\chi_A^{(1)} < 0$ implies $\chi_C^{(1)} > 0$.

Furthermore, we see that $\chi_B^{(1)} > 0$ implies $\mathbf{D} < 0$. Indeed, the condition $\chi_B^{(1)} > 0$ yields $a < 0$ and, since $1-2ah < 0$ (i.e. $4ah > 2$), we have $2a(2h-1)-1 = 4ah-2a-1 > 0$; then $\mathbf{D} < 0$. So we arrive at the following conditions and configurations:

- $\chi_B^{(1)} < 0$ and $\mathbf{D} < 0 \ \Rightarrow$ *Config. H.39*;
- $\chi_B^{(1)} < 0$ and $\mathbf{D} > 0 \ \Rightarrow$ *Config. H.49*;
- $\chi_B^{(1)} < 0$ and $\mathbf{D} = 0 \ \Rightarrow$ *Config. H.44*;
- $\chi_B^{(1)} > 0 \ \Rightarrow$ *Config. H.38.*

$\boldsymbol{\beta}$**)** *The subcase* $\chi_A^{(1)} > 0$. In this case the singularities $M_{1,2}$ are real and we have to decide if they are located either on different branches or on the same branch and, moreover, the position of the hyperbola.

We observe that the conditions $\chi_B^{(1)} < 0$ and $\chi_C^{(1)} < 0$ imply $\mathbf{D} < 0$. Indeed, the conditions $\chi_B^{(1)} < 0$ and $\chi_C^{(1)} < 0$ yield $a > 0$ and $ah < 0$, respectively, and, since $1-2ah > 0$, we have $2a(2h-1)-1 = 4ah-2a-1 < 0$; then $\mathbf{D} < 0$.

So in the case $\chi_B^{(1)} < 0$ we get the following conditions and configurations:

- $\chi_C^{(1)} < 0 \ \Rightarrow$ *Config. H.75*;
- $\chi_C^{(1)} > 0$ and $\mathbf{D} < 0 \ \Rightarrow$ *Config. H.74*;
- $\chi_C^{(1)} > 0$ and $\mathbf{D} > 0 \ \Rightarrow$ *Config. H.48*;
- $\chi_C^{(1)} > 0$ and $\mathbf{D} = 0 \ \Rightarrow$ *Config. H.64*;

whereas for $\chi_B^{(1)} > 0$ we get

- $\chi_C^{(1)} < 0$ and $\mathbf{D} < 0 \;\Rightarrow\;$ *Config. H.73*; $\qquad \chi_C^{(1)} > 0$ and $\mathbf{D} < 0 \;\Rightarrow\;$ *Config. H.72*;
- $\chi_C^{(1)} < 0$ and $\mathbf{D} > 0 \;\Rightarrow\;$ *Config. H.47*; $\qquad \chi_C^{(1)} > 0$ and $\mathbf{D} > 0 \;\Rightarrow\;$ *Config. H.46*;
- $\chi_C^{(1)} < 0$ and $\mathbf{D} = 0 \;\Rightarrow\;$ *Config. H.66*; $\qquad \chi_C^{(1)} > 0$ and $\mathbf{D} = 0 \;\Rightarrow\;$ *Config. H.65.*

γ) *The subcase* $\chi_A^{(1)} = 0$. Due to the condition (11), the condition $\chi_A^{(1)} = 0$ implies $Z_1 = 0$ and hence $a = 1/(2h)$. In this case the points $M_{1,2}$ collapse and we have a double point on the hyperbola. For systems (10) with $a = 1/(2h)$ we calculate

$$\chi_C^{(1)} = \frac{35}{2^{17}}(h-1)^4(2h-1)^2(2h+1)^2(3h-1)^2,$$
$$\mathbf{T} = \frac{3}{2^{10}h}(h-1)(2h-1)^4y^2\big[x^2+4h(h-1)y^2\big]^2.$$

Due to (19), we have $\chi_C^{(1)} > 0$ and $\operatorname{sign}(\mathbf{T}) = \operatorname{sign}\big(h(h-1)\big)$, therefore according to Proposition 1, besides the double point on the hyperbola, we could have two simple points on the invariant line $y = 0$.

We observe that the condition $\chi_B^{(1)} > 0$ implies $\mathbf{T} > 0$. Indeed, if $\chi_B^{(1)} > 0$ we have $a < 0$ and, since $a = 1/(2h)$ (i.e. $h < 0$), we obtain $h(h-1) > 0$; then $\mathbf{T} > 0$.

So we arrive at the configuration *Config. H.67* if $\chi_B^{(1)} < 0$ and $\mathbf{T} < 0$; *Config. H.43* if $\chi_B^{(1)} < 0$ and $\mathbf{T} > 0$; and *Config. H.42* if $\chi_B^{(1)} > 0$.

a_2) *The case* $\mu_0 = 0$. Then $h(2h-1) = 0$ and considering the condition (19) we get $h = 0$. In this case one of the singular point located on the hyperbola has gone to infinity and collapsed with $[0,1,0]$ (since $\mu_1 = x/4$, see Lemma 1).The second singularity on the hyperbola has the coordinates $(-2, -a/2)$, whereas the coordinates of the singularities $M_{3,4}(x_{3,4}, y_{3,4})$ located on the invariant line $y = 0$ remain the same. Since for systems (18) with $h = 0$ we have $\mathbf{D} = -3a^2(2a+1)$ we obtain $\operatorname{sign}(\mathbf{D}) = \operatorname{sign}(2a+1)$.

We observe that in the case $\chi_B^{(1)} < 0$, we have $a > 0$ and hence $\mathbf{D} = 2a+1 > 0$, which implies the existence of two real simple singularities on $y = 0$ and we obtain the configuration shown in *Config. H.70*. Now, in the case $\chi_B^{(1)} > 0$, we obtain the following conditions and configurations: *Config. H.71* if $\mathbf{D} < 0$; *Config. H.41* if $\mathbf{D} > 0$; and *Config. H.55* if $\mathbf{D} = 0$.

b) *The possibility* $\chi_E^{(1)} = 0$. In this case we obtain $a = 1/(g+h)^2$ and this leads to the systems

$$\frac{dx}{dt} = \frac{2h-1}{(g+h)^2} + x + gx^2 + (h-1)xy, \quad \frac{dy}{dt} = \frac{2g-1}{(g+h)^2} - y + (g-1)xy + hy^2 \tag{20}$$

possessing the following invariant line and invariant hyperbola

$$x - y + 2/(g+h) = 0, \quad \Phi(x,y) = \frac{1}{(g+h)^2} + xy = 0. \tag{21}$$

We claim that the condition $\chi_E^{(1)} = 0$ implies $\mathbf{D} \le 0$ and $\chi_B^{(1)} < 0$. Indeed, if $\chi_E^{(1)} = 0$, then $a = 1/(g+h)^2$ and in this case we see that

$$\chi_B^{(1)} = -\frac{105(g-1)^2(h-1)^2(g-h)^2(3g-1)^2(3h-1)^2}{8(g+h)^2} < 0,$$
$$\mathbf{D} = -\frac{192(g-h)^6(g+h-2)^2(g+h-2gh)^2}{(g+h)^8} \le 0,$$

due to condition (19), and this proves our claim.

For the above systems we calculate

$$B_2 = -\frac{648}{(g+h)^4}\,(g-1)^2(h-1)^2(2g-1)(2h-1)(x-y)^4 \tag{22}$$

and by Lemma 2 for the existence of an invariant line in a direction different from $y = x$ it is necessary $B_2 = 0$.

***b*₁) *The case* $B_2 \neq 0$.** Since $\theta \neq 0$ by Lemma 4 we could not have a couple of parallel invariant lines in the direction $y = x$ and obviously the invariant line $y = x + 2/(g+h)$ is a simple one. As before we consider two subcases: $\mu_0 \neq 0$ and $\mu_0 = 0$.

α) *The subcase* $\mu_0 \neq 0$. Then $gh(g+h-1) \neq 0$ and systems (20) possess four real singularities $M_i(x_i, y_i)$ with the coordinates

$$\begin{aligned} &x_1 = -\frac{1}{g+h},\ y_1 = \frac{1}{g+h}; \qquad x_3 = -\frac{2h-1}{g+h},\ y_3 = \frac{2g-1}{g+h};\\ &x_2 = -\frac{h}{g(g+h)},\ y_2 = \frac{g}{h(g+h)};\quad x_4 = -\frac{2h-1}{(g+h)(g+h-1)},\ y_4 = \frac{2g-1}{(g+h)(g+h-1)}. \end{aligned} \tag{23}$$

It could be checked directly that the singularity $M_1(x_1, y_1)$ is a common (tangency) point of the invariant hyperbola and of the invariant line (21). Moreover, the singular point $M_2(x_2, y_2)$ (respectively $M_4(x_4, y_4)$) is located on the hyperbola (respectively on the invariant line), whereas the singularity $M_3(x_3, y_3)$ generically is located outside the invariant hyperbola as well as outside the invariant line.

For systems (20) we calculate

$$\begin{aligned} \chi_A^{(1)} &= -\frac{1}{64}(g-1)^2(h-1)^2(g-h)^2(3g-1)^2(3h-1)^2\, Z_1\big|_{\{a=1/(g+h)^2\}},\\ \chi_C^{(1)} &= \frac{35}{32}gh(g-1)^4(h-1)^4(g-h)^2(3g-1)^2(3h-1)^2,\\ \chi_D^{(1)} &= -\frac{105}{2(g+h)^2}(g-h)^3(3g-1)(3h-1)(g+h-2gh) \end{aligned} \tag{24}$$

and, due to (11), the condition $\chi_A^{(1)} = 0$ is equivalent to $Z_1 = -(g-h)^2/(g+h)^2 = 0$ and this contradicts the condition (11). So the singular points M_1 and M_2 could not collapse.

We consider two possibilities: $\chi_C^{(1)} < 0$ and $\chi_C^{(1)} > 0$.

α_1) *The possibility* $\chi_C^{(1)} < 0$. In this case the singularities $M_{1,2}$ are located on different branches of the hyperbola and we need to decide if the singular point M_3 coalesces with the singularities on the hyperbola, and this fact is governed by the polynomial $\mathbf{D}$. However, this last polynomial could vanish due to the factors $g+h-2$ and $g+h-2gh$. Then, according to (24), we need to disti guish the cases $\chi_D^{(1)} \neq 0$ and $\chi_D^{(1)} = 0$.

So we get the configurations *Config. H.60* if $\mathbf{D} \neq 0$; *Config. H.63* if $\mathbf{D} = 0$ and $\chi_D^{(1)} \neq 0$; and *Config. H.69* if $\mathbf{D} = 0$ and $\chi_D^{(1)} = 0$.

α_2) *The possibility* $\chi_C^{(1)} > 0$. Assume $\chi_C^{(1)} > 0$, i.e. $gh > 0$. Then, by Lemma 5, both singularities $M_{1,2}$ are located on the same branch of hyperbola. It is clear that the reciprocal position of the singularities M_2 (located on the hyperbola) and M_4 (located on the invariant line) with respect to the tangency point M_1 of the hyperbola and the invariant line (21), define different configurations. More exactly the type of the configuration depends on the sign of the expression:

$$(x_1 - x_2)(x_1 - x_4) = \frac{(g-h)^2}{g(g+h-1)(g+h)^2}$$

and hence we need $\operatorname{sign}\big(g(g+h-1)\big)$ when $gh > 0$. We calculate

$$\chi_F^{(1)} = (g+h)(g+h-1)(g-1)^4(h-1)^4(g-h)^2(3g-1)^2(3h-1)^2/256$$

and, since in the case $gh > 0$ we have $\operatorname{sign}(g) = \operatorname{sign}(g+h)$, we deduce that

$$\operatorname{sign}(\chi_F^{(1)}) = \operatorname{sign}\big((g+h)(g+h-1)\big) = \operatorname{sign}\big(g(g+h-1)\big).$$

We observe that the conditions $\chi_C^{(1)} > 0$ and $\chi_F^{(1)} < 0$ imply $\mathbf{D} \neq 0$ (i.e. $\mathbf{D} < 0$). Indeed, if we suppose $\mathbf{D} = 0$, then $(g+h-2)(g+h-2gh) = 0$. In the case $g = 2-h$, we have

$$\chi_F^{(1)} = (h-1)^{10}(3h-5)^2(3h-1)^2/32 > 0,$$

due to (19), which contradicts the condition $\chi_F^{(1)} < 0$. On the other hand, if $g = h/(2h-1)$, we have

$$\chi_F^{(1)} = \frac{1}{32(2h-1)^{10}}\, h^4(h-1)^{10}(h+1)^2(3h-1)^2(1-2h+2h^2) > 0,$$

due to (19), which again contradicts the condition $\chi_F^{(1)} < 0$. So we detect that in the case $\chi_F^{(1)} < 0$ we obtain the configuration *Config. H.61*.

In the case $\chi_F^{(1)} > 0$, the polynomial $\mathbf{D}$ could vanish and we need to detect to which singular points M_2 or M_4 the singularity M_3 collapses. So we get the following conditions and configurations: *Config. H.59* if $\mathbf{D} \neq 0$; *Config. H.62* if $\mathbf{D} = 0$ and $\chi_D^{(1)} \neq 0$; and *Config. H.68* if $\mathbf{D} = 0$ and $\chi_D^{(1)} = 0$.

$\boldsymbol{\beta}$**)** *The subcase* $\mu_0 = 0$. Then $gh(g+h-1) = 0$ and, by Lemma 1, at least one finite singularity has gone to infinity and collapsed with an infinite singular point. Since for systems (20) we have $\chi_C^{(1)} = 0$ if and only if $gh = 0$ (see (24)), we consider two possibilities: $\chi_C^{(1)} \neq 0$ and $\chi_C^{(1)} = 0$.

$\boldsymbol{\beta_1}$**)** *The possibility* $\chi_C^{(1)} \neq 0$. Then the condition $\mu_0 = 0$ implies $g+h-1 = 0$, i.e. $g = 1-h$ and considering the coordinates (23) of the finite singularities of systems (20) we observe that the singular point M_4 located on the invariant line has gone to infinity and collapsed with the singularity $[1, 1, 0]$. In this case calculation yields

$$\begin{aligned}
\chi_A^{(1)} &= h^2(h-1)^2(2h-1)^4(3h-1)^2(3h-2)^2/64, \\
\chi_B^{(1)} &= -105\, h^2(h-1)^2(2h-1)^2(3h-1)^2(3h-2)^2/8, \\
\chi_C^{(1)} &= 35\, h^5(1-h)^5(2h-1)^2(3h-1)^2(3h-2)^2/32, \\
\mathbf{D} &= -192(2h-1)^6(1-2h+2h^2)^2,
\end{aligned}$$

and by (19) we have $\chi_A^{(1)} > 0$, $\chi_B^{(1)} > 0$ and $\mathbf{D} < 0$. Moreover, since by Remark 7 the condition $\mathbf{R} \neq 0$ holds, then according to Proposition 1 all three finite singularities are distinct. This means that the singularities located on the hyperbola are simple and belong to different branches (respectively of the same branch) of the hyperbola if $\chi_C^{(1)} < 0$ (respectively $\chi_C^{(1)} > 0$). As a result we get configurations *Config. H.56* if $\chi_C^{(1)} < 0$ and *Config. H.57* if $\chi_C^{(1)} > 0$.

$\boldsymbol{\beta_2}$**)** *The possibility* $\chi_C^{(1)} = 0$. Then $gh = 0$ (this implies $\mu_0 = 0$) and we have $g^2 + h^2 \neq 0$ due to $g - h \neq 0$. Considering Remark 6, without loss of generality, we may assume $g = 0$. In this case, the singularity M_2 located on the hyperbola (21) has gone to infinity and collapsed with the point $[1, 0, 0]$. Since by Remark 7 we have $\mu_1 \neq 0$, then according to Lemma 1 other three finite singular points remain on the finite part of the phase plane.

It is clear that depending on the position of the singular point M_4 (located on the invariant line (21)) with respect to the vertical line $x = x_1$ we get different configurations. So this distinction is governed by the sign of the expression $x_4 - x_1 = 1/(1 - h)$. Moreover, since in this case we have the invariant line $x - y + 2/h = 0$, its position depends on the sign of h. Then we need to control the sign $\big(h(1 - h)\big)$. Thus, we calculate

$$\chi_F^{(1)} = h^3(h-1)^5(3h-1)^2/256, \quad \mathbf{D} = -192(h-2)^2$$

and we have sign $\big(h(1-h)\big) = -\text{sign}\,(\chi_F^{(1)})$.

It is clear that, in the case $\chi_F^{(1)} < 0$, we have $\mathbf{D} \neq 0$ and, since the condition $\mathbf{R} \neq 0$ holds (see Remark 7), Proposition 1 assures us that all three finite singularities are distinct if $\mathbf{D} \neq 0$. So we arrive at the cconfiguration given by *Config. H.50*.

Now, in the case $\chi_F^{(1)} > 0$, the polynomial $\mathbf{D}$ could vanish and we obtain the configuration *Config. H.51* if $\mathbf{D} \neq 0$ and *Config. H.54* if $\mathbf{D} = 0$.

$\boldsymbol{b_2}$**)** *The case* $B_2 = 0$. Considering (22) and the condition (11) we obtain $g = 1/2$ and this leads to the 1-parameter family of systems

$$\frac{dx}{dt} = \frac{4(2h-1)}{(2h+1)^2} + x + \frac{x^2}{2} + (h-1)xy, \quad \frac{dy}{dt} = -y(2 + x - 2hy)/2, \tag{25}$$

for which the condition $\theta\beta_1\beta_2 \neq 0$ gives

$$(h-1)(2h+1)(2h-1)(3h-1)(6h-5) \neq 0. \tag{26}$$

These systems possess two invariant lines and a hyperbola

$$x - y + \frac{4}{2h+1} = 0, \quad y = 0, \quad \Phi(x, y) = \frac{4}{(2h+1)^2} + xy = 0.$$

as well as the following singularities $M_i(x_i, y_i)$:

$$\begin{aligned} x_1 &= -\frac{2}{2h+1},\ y_1 = \frac{2}{2h+1}; \qquad x_3 = \frac{2(1-2h)}{2h+1},\ y_3 = 0;\\ x_2 &= -\frac{4h}{2h+1},\ y_2 = \frac{1}{h(2h+1)}; \quad x_4 = -\frac{4}{2h+1},\ y_4 = 0. \end{aligned} \tag{27}$$

We observe that due to the condition (26) all singularities are located on the finite part of the phase plane, except the singular point M_2 which could go to infinity in the case $h = 0$. For the above systems we calculate

$$\chi_C^{(1)} = 35h(h-1)^4(2h-1)^2(3h-1)^2/16384$$

and we analyze the subcases $\chi_C^{(1)} < 0$, $\chi_C^{(1)} > 0$ and $\chi_C^{(1)} = 0$.

α) *The subcase* $\chi_C^{(1)} < 0$. Then $h < 0$ and it implies

$$\mu_0 = h(2h-1)/4 \neq 0, \quad \mathbf{D} = -\frac{48}{(2h+1)^8}(2h-3)^2(2h-1)^6 \neq 0.$$

Since the singular points on the hyperbola are located on different branches, we arrive at the unique configuration *Config. H.84*.

β) *The subcase* $\chi_C^{(1)} > 0$. Then $h > 0$ (this implies again $\mu_0 \neq 0$) and the singularities on the hyperbola are located on the same branch. Thus, it is necessary to distinguish the position of M_2 on the hyperbola with relation to M_1, which is the intersection point of the hyperbola and the invariant line $x - y + 4/(2h+1) = 0$, and M_4, which is the intersection point of the two invariant lines, as well as the position of the singularities M_3 and M_4 on the invariant line $y = 0$. We calculate

$$(x_1 - x_2)(x_1 - x_4) = \frac{4(2h-1)}{(2h+1)^2}, \quad (x_4 - x_3) = \frac{2(2h-3)}{2h+1}$$

and hence sign $(2h-1)$ (respectively sign $(2h-3)$) will describe the position of the singularity M_2 on the hyperbola (respectively the position of the singularity M_3 on the invariant line $y = 0$). We calculate

$$\chi_F^{(1)} = 2^{-18}(2h-1)^3(2h+1)(h+1)^4(3h-1)^2, \quad \chi_G^{(1)} = (2h-3)(h+1)/8$$

and, due to (26) and since $h > 0$, we obtain sign $(2h-1) =$ sign $(\chi_F^{(1)})$ and sign $(2h-3) =$ sign $(\chi_G^{(1)})$.

We observe that the condition $\chi_G^{(1)} = 0$ yields $h = 3/2$ and this implies $\mathbf{D} = 0$. In this sense, we obtain the following conditions and configurations:

- $\chi_F^{(1)} < 0 \;\Rightarrow\;$ *Config. H.86*;
- $\chi_F^{(1)} > 0$ and $\chi_G^{(1)} < 0 \;\Rightarrow\;$ *Config. H.85*;
- $\chi_F^{(1)} > 0$ and $\chi_G^{(1)} > 0 \;\Rightarrow\;$ *Config. H.83*;
- $\chi_F^{(1)} > 0$ and $\chi_G^{(1)} = 0 \;\Rightarrow\;$ *Config. H.81*;

γ) *The subcase* $\chi_C^{(1)} = 0$. Then $h = 0$ (this implies $\mu_0 = 0$) and the singularity M_2 has gone to infinity and collapsed with $[0, 1, 0]$. As a result we get *Config. H.82*.

3.1.1.2 *The case* $\beta_2 = 0$. Since $\beta_1 \neq 0$ (i.e. $c \neq 0$) we get $(g-h)(3g+3h-4) = 0$. On the other hand, for systems (8) we have

$$\beta_3 = -c(g-h)(g-1)(h-1)/4$$

and we consider two possibilities: $\beta_3 \neq 0$ and $\beta_3 = 0$.

3.1.1.2.1 *The possibility* $\beta_3 \neq 0$. In this case we have $g-h \neq 0$ and the condition $\beta_2 = 0$ yields $3g+3h-4=0$, i.e. $g = 4/3-h$. In this case, for systems (8), we calculate

$$\gamma_3 = 7657c(h-1)^3(3h-1)^3\big[a(5-6h)-3b(2h-1)\big],$$
$$\beta_3 = -c(h-1)(3h-2)(3h-1)/18, \quad \mathcal{R}_1 = (a-b)c(h-1)^3(3h-1)^3/6.$$

Without loss of generality, we may assume $2h-1 \neq 0$, otherwise via the change $(x,y,t,a,b,c) \mapsto (y,x,t,b,a,-c)$ we could bring systems (8) with $h = 1/2$ to the same systems with $h = 5/6$. Therefore, due to $\beta_3 \neq 0$, the condition $\gamma_3 = 0$ yields $b = a(5-6h)/[3(2h-1)]$ and since $c \neq 0$ we may assume $c = 1$ due to the rescaling $(x,y,t) \mapsto (cx, cy, t/c)$.

We remark that the condition $\gamma_3 = 0$ could be written as $a = a_1(2h-1)$ and $b = a_1(5-6h)/3$. So setting the old parameter a instead of a_1, we arrive at the 2-parameter family of systems

$$\frac{dx}{dt} = 3a(2h-1) + x + \frac{4-3h}{3}x^2 + (h-1)xy, \quad \frac{dy}{dt} = \frac{a(5-6h)}{3} - y + \frac{1-3h}{3}xy + hy^2, \tag{28}$$

for which the condition $\theta\beta_1\beta_3\mathcal{R}_1 \neq 0$ is equivalent to the condition

$$a(h-1)(3h-1)(3h-2) \neq 0. \tag{29}$$

Moreover, these systems possess the same invariant hyperbola (12).

Observation 2. *We observe that the family of systems* (28) *is in fact a subfamily of systems* (10) *under the relation* $g = 4/3-h$. *Moreover, if we present the condition* (11) *in the form* $F(a,g,h)(3g+3h-4) \neq 0$, *then in the case* $g = 4/3-h$, *the condition* (29) *is equivalent to* $F(a,g,h) \neq 0$. *We also point out that the condition* $g = 4/3-h$ *does not imply the vanishing of any of the invariants* $\chi_A^{(1)}, \chi_B^{(1)}, \ldots, \chi_G^{(1)}$. *Hence, all the configurations of systems* (28) *are the configurations of systems* (10) *determined by the same invariant conditions.*

Considering this observation, we could join the conditions defining the family (10) (i.e. $\eta > 0$, $\theta\beta_1\beta_2 \neq 0$) with the conditions which define the subfamily (28) (i.e. $\eta > 0$, $\theta\beta_1 \neq 0$, $\beta_2 = 0$ and $\beta_3 \neq 0$). More precisely, the conditions defining both such families of systems are $\beta_2^2 + \beta_3^2 \neq 0$ and $(\mathfrak{C}_1)$, where

$$(\mathfrak{C}_1): \ (\beta_2\mathcal{R}_1 \neq 0) \cup (\beta_2 = \gamma_3 = 0 \cap \beta_3 \neq 0).$$

3.1.1.2.2 *The possibility* $\beta_3 = 0$. Due to $\beta_1 \neq 0$ (i.e. $(g-1)(h-1) \neq 0$), we get $g = h$. In this case, we calculate

$$\begin{aligned}\gamma_2 =& 6300h(h-1)^4(3h-2)(3h-1)^2\mathcal{B}_1,\\ \theta =& -h(h-1)^2, \quad \beta_1 = -(h-1)^2(3h-1)^2/4,\\ \beta_4 =& 2h(3h-2), \quad \beta_5 = -2h^2(2h-1).\end{aligned}$$

We shall consider two cases: $\beta_4 \neq 0$ and $\beta_4 = 0$.

a) *The case* $\beta_4 \neq 0$. So the condition $\gamma_2 = 0$ implies $\mathcal{B}_1 = 0$ and by Theorem 1 the condition $\beta_5 \neq 0$ is necessary for the existence of hyperbola. Hence, we arrive at the particular case of systems (10) when $g = h$, i.e. we get the systems

$$\frac{dx}{dt} = a(2h-1) + x + hx^2 + (h-1)xy, \quad \frac{dy}{dt} = a(2h-1) - y + (h-1)xy + hy^2 \tag{30}$$

with the condition

$$ah(h-1)(2h-1)(3h-1)(3h-2) \neq 0. \tag{31}$$

These systems possess the invariant line and hyperbola

$$1+h(x-y)=0, \quad \Phi(x,y)=a+xy=0.$$

Since $\mu_0 = h^2(2h-1) \neq 0$ (see (31)), then the systems have finite singularities $M_i\big(x_i, y_i\big)$ of total multiplicity 4:

$$x_{1,2} = -\frac{1 \pm \sqrt{1-4ah^2}}{2h}, \quad y_{1,2} = \frac{1 \mp \sqrt{1-4ah^2}}{2h},$$
$$x_{3,4} = \frac{-1 \pm \sqrt{1+4a-8ah}}{2}, \quad y_{3,4} = \frac{1 \mp \sqrt{1+4a-8ah}}{2},$$

We detect that the singularities $M_{1,2}$ are located on both the hyperbola and the straight line. These singular points are located on different branches (respectively on the same branch) of the hyperbola if only if $x_1x_2 < 0$ (respectively $x_1x_2 > 0$), where $x_1x_2 = a$. Moreover, these singularities are real if $1-4ah^2 > 0$, they are complex if $1-4ah^2 < 0$ and they coincide if $1-4ah^2 = 0$.

On the other hand, we calculate

$$\chi_A^{(2)} = 2h^2(2h-1)^2(3h-1)^2(1-4ah^2), \quad \chi_B^{(2)} = -a(h-1)^2(2h-1)^2(3h-1)^4/4$$

and, due to the condition (31), we have $\operatorname{sign}(1-4ah^2) = \operatorname{sign}(\chi_A^{(2)})$ (if $1-4ah^2 \neq 0$) and $\operatorname{sign}(x_1x_2) = -\operatorname{sign}(\chi_B^{(2)})$.

We observe that at least one of the singular points $M_{3,4}$ could be located either on the invariant hyperbola or on the invariant straight line. Next we determine the conditions for this to happen. We calculate

$$\Phi(x,y)|_{\{x=x_{3,4},\ y=y_{3,4}\}} = (-1+4ah \pm \sqrt{1+4a-8ah}) \equiv \Omega'_{3,4}(a,h),$$
$$[1+h(x-y)]|_{\{x=x_{3,4},\ y=y_{3,4}\}} = 1+h(-1 \pm \sqrt{1+4a-8ah}) \equiv \Theta_{3,4}(a,h).$$

So M_3 or M_4 could be located on the invariant hyperbola (respectively invariant line) if and only if $\Omega'_3\Omega'_4 = 0$ (respectively $\Theta_3\Theta_4 = 0$). So we have

$$\Omega'_3\Omega'_4 = -a(1-4ah^2) = 0, \quad \Theta_3\Theta_4 = (1-2h)(1-4ah^2) = 0$$

if and only $1-4ah^2 = 0$ (due to the condition (31)).

Thus, in the case $\chi_A^{(2)} \neq 0$ we arrive at the configuration given by *Config. H.37* if $\chi_A^{(2)} < 0$; *Config. H.52* if $\chi_A^{(2)} > 0$ and $\chi_B^{(2)} < 0$; and *Config. H.53* if $\chi_A^{(2)} > 0$ and $\chi_B^{(2)} > 0$.

Assume now $\chi_A^{(2)} = 0$, i.e. $1-4ah^2 = 0$. Due to the condition (31) we have $h \neq 0$ and hence $a = 1/(4h^2)$. It could be easily observed that in this case the singular points M_2 and M_3 coalesce with the singularity M_1 and this point becomes a triple point of contact of the invariant hyperbola and invariant line. We remark that this point of contact could not be of multiplicity 4 because in this case we have

$$\mu_0 = h^2(2h-1) \neq 0, \quad \mathbf{D} = \mathbf{T} = \mathbf{P} = 0, \quad \mathbf{R} = 3h^2(h-1)^2(2h-1)^2(x+y)^2 \neq 0,$$

due to the condition (31). Thus, in the case $\chi_A^{(2)} = 0$ we get the configuration given by *Config. H.45*.

b) *The case* $\beta_4 = 0$. Then, due to $\theta \neq 0$, we get $h = 2/3$ and we obtain a family of systems which is a subfamily of systems (30) setting $h = 2/3$. Since in this case we have

$$\chi_A^{(2)} = 8(9 - 16a)/729, \quad \chi_B^{(2)} = -a/324,$$

it is clear that we obtain again the same four configurations as for the family (31) with the same invariant conditions. As earlier we could join the cases $\beta_4 \neq 0$ and $\beta_4 = 0$. More precisely, the conditions defining the corresponding families of systems are

$$(\mathfrak{C}_2): \ (\beta_4\beta_5\mathcal{R}_2 \neq 0) \vee (\beta_4 = \gamma_3 = 0, \ \mathcal{R}_2 \neq 0).$$

3.1.2 The possibility $\beta_1 = 0$

Considering (9) and the condition $\theta \neq 0$, we get $c(3g - 1)(3h - 1) = 0$. On the other hand, for systems (8) we calculate

$$\beta_6 = -c(g - 1)(h - 1)/2$$

and we shall consider two cases: $\beta_6 \neq 0$ and $\beta_6 = 0$.

3.1.2.1 The case $\beta_6 \neq 0$. Then $c \neq 0$ (as before we could assume $c = 1$ due to a rescaling) and the condition $\beta_1 = 0$ implies $(3g - 1)(3h - 1) = 0$. Therefore, due to Remark 6, we may assume $h = 1/3$ and this leads to the following 3-parameter family of systems

$$\frac{dx}{dt} = a + x + gx^2 - 2xy/3, \quad \frac{dy}{dt} = b - y + (g - 1)xy + y^2/3, \tag{32}$$

which is a subfamily of (8).

For these systems we calculate

$$\gamma_4 = 16(g-1)^2(3g-1)^2\big[3a(2g-1)+b\big]\big[(3g+1)^2(b-2a+6ag)+6(1-3g)\big]/243,$$
$$\beta_6 = (g-1)/3, \quad \beta_2 = (1-g)(3g-1)/2, \quad \mathcal{R}_3 = a(3g-1)^3/18.$$

3.1.2.1.1 *The subcase* $\beta_2 \neq 0$. Then $3g - 1 \neq 0$ and, in order to have $\gamma_4 = 0$, we must have $\big[3a(2g-1)+b\big]\big[(3g+1)^2(b-2a+6ag)+6(1-3g)\big] = 0$.

We claim that systems (32) with $(3g+1)^2(b-2a+6ag)+6(1-3g) = 0$ (i.e. $b = 2(3g-1)(3 - a - 6ag - 9ag^2)/(3g+1)^2$) could be brought to the same systems with $b = 3a(1-2g)$ via an affine transformation. Indeed, due to $\theta \neq 0$ (i.e. $(3g+1)(g-1) \neq 0$), we may apply the affine transformation

$$x_1 = \frac{3g+1}{3(1-g)}x, \quad y_1 = \frac{3g+1}{3(1-g)}(x-y) + \frac{2}{1-g}, \quad t_1 = \frac{3(g-1)}{3g+1}t, \tag{33}$$

and we arrive at the systems

$$\frac{dx_1}{dt_1} = a_1 + x_1 + g_1x_1^2 - 2x_1y_1/3, \quad \frac{dy_1}{dt_1} = b_1 - y_1 + (g_1 - 1)x_1y_1 + y_1^2/3,$$

where $b_1 = -3a_1(2g_1 - 1)$, $a_1 = -a(3g+1)^2/[9(g-1)^2]$ and $g_1 = (2 - 3g)/3$. This completes the proof of our claim.

Thus, in what follows, we consider the following family of systems

$$\frac{dx}{dt} = a + x + gx^2 - 2xy/3, \quad \frac{dy}{dt} = -3a(2g-1) - y + (g-1)xy + y^2/3, \tag{34}$$

with the condition

$$a(g-1)(3g-1)(3g+1) \neq 0. \tag{35}$$

According to Theorem 1, these systems possess either one or two invariant hyperbolas if either $\delta_1 \neq 0$ or $\delta_1 = 0$, respectively, where $\delta_1 = (3g-1)\big[6(1-3g) + a(3g+1)^2\big]/18$.

***a*) *The possibility* $\delta_1 \neq 0$.** Then systems (34) possess the unique invariant hyperbola

$$\Phi(x,y) = 3a - xy = 0. \tag{36}$$

For systems (34) we calculate

$$B_1 = 8a^2(g-1)^2(2g-1)(3g-1)\big[3 + a(3g+1)^2\big]/27. \tag{37}$$

a_1) *The case* $B_1 \neq 0$. In this case, due to (35), we have $(2g-1)\big[3 + a(3g+1)^2\big] \neq 0$. For systems (34) we calculate $\mu_0 = g(3g-2)/9$ and we consider two possibilities: $\mu_0 \neq 0$ and $\mu_0 = 0$.

α) *The subcase* $\mu_0 \neq 0$. In this case the systems have finite singularities of total multiplicity 4 with coordinates $M_i(x_i, y_i)$:

$$x_{1,2} = \frac{-1 \pm \sqrt{1+4ag}}{2g}, \quad y_{1,2} = \frac{3(1 \pm \sqrt{1+4ag})}{2},$$
$$x_{3,4} = \frac{1 \pm \sqrt{1-8a+12ag}}{2(3g-2)}, \quad y_{3,4} = \frac{3(2g-1)(1 \pm \sqrt{1-8a+12ag})}{2(3g-2)}.$$

We detect that the singularities $M_{1,2}$ are located on the invariant hyperbola. More exactly, these singular points are located on different branches (respectively on the same branch) of the hyperbola if only if $x_1x_2 < 0$ (respectively $x_1x_2 > 0$), where $x_1x_2 = -a/g$. Moreover, these singularities are real if $1+4ag > 0$, complex if $1+4ag < 0$ or they coincide if $1+4ag = 0$.

On the other hand, we calculate

$$\chi_A^{(3)} = \frac{7713280}{243}(1+4ag)\big[6(1-3g) + a(3g+1)^2\big]^2,$$
$$\chi_B^{(3)} = \frac{164798932}{81}a(g-1)^2(3g-1)^2\big[6(1-3g) + a(3g+1)^2\big]^2,$$
$$\chi_C^{(3)} = -\frac{66560}{9}ag\big[6(1-3g) + a(3g+1)^2\big]^2,$$

and, due to the condition (35), we have $\text{sign}\,(\chi_A^{(3)}) = \text{sign}\,(1+4ag)$ (if $1+4ag \neq 0$) and $\text{sign}\,(\chi_C^{(3)}) = \text{sign}\,(x_1x_2)$.

We point out that at least one of the singular points $M_{3,4}$ could be located on the invariant hyperbola. Next we determine the conditions for this to happen. We calculate

$$\Phi(x,y)|_{\{x=x_{3,4},\ y=y_{3,4}\}} = \frac{3\big[2a(g-1)(3g-2) - (2g-1)(1 \pm \sqrt{1-8a+12ag})\big]}{2(3g-2)^2} \equiv \Omega''_{3,4}(a,g,h).$$

It is clear that at least one of the singular points M_3 or M_4 belongs to the hyperbola (36) if and only if $\Omega_3''\Omega_4'' = 0$. So we have

$$\Omega_3''\Omega_4'' = \frac{9a\big[a(g-1)^2 - 2g + 1\big]}{(3g-2)^2} = 0$$

and, since

$$\chi_D^{(3)} = \frac{736}{81}\,(3g-1)\big[a(g-1)^2 - 2g + 1\big]\big[6(1-3g) + a(3g+1)^2\big],$$

we deduce that at least one of the singular points $M_{3,4}$ belongs to the hyperbola if and only if $\chi_D^{(3)} = 0$.

$\boldsymbol{\alpha_1}$**)** *The possibility* $\chi_A^{(3)} < 0$. So we have no real singularities located on the invariant hyperbolas and we arrive at the configurations given by *Config. H.1* if $\chi_B^{(3)} < 0$ and *Config. H.2* if $\chi_B^{(3)} > 0$.

$\boldsymbol{\alpha_2}$**)** *The possibility* $\chi_A^{(3)} > 0$. In this case we have two real singularities located on the hyperbola and we need to decide if they are located either on different branches or on the same branch of the invariant hyperbola and also if at least one of the singular points $M_{3,4}$ will belong to the hyperbola.

***i*)** *The case* $\chi_D^{(3)} \neq 0$. Then $a(g-1)^2 - 2g + 1 \neq 0$ and on the hyperbola there are two simple real singularities (namely $M_{1,2}$) and we arrive at the conditions and configurations given by:

- $\chi_C^{(3)} < 0$ and $\chi_B^{(3)} < 0 \;\Rightarrow\;$ *Config. H.17*;
- $\chi_C^{(3)} < 0$ and $\chi_B^{(3)} > 0 \;\Rightarrow\;$ *Config. H.19*;
- $\chi_C^{(3)} > 0$ and $\chi_B^{(3)} < 0 \;\Rightarrow\;$ *Config. H.16*;
- $\chi_C^{(3)} > 0$ and $\chi_B^{(3)} > 0 \;\Rightarrow\;$ *Config. H.18.*

***ii*)** *The case* $\chi_D^{(3)} = 0$. In this case, due to $B_1 \neq 0$ and (35), we obtain $a = (2g-1)/(g-1)^2$. Then, considering Proposition 1, we calculate

$$\mathbf{D} = 0, \quad \mathbf{T} = -\frac{(5g-3)^2}{2187(g-1)^4}(3g-1)^2(3gx-y)^2\big[3(2g-1)x - y\big]^2\big[3g(g-1)x + 2(2g-1)y\big]^2.$$

***ii*.1)** *The subcase* $\mathbf{T} \neq 0$. Then $\mathbf{T} < 0$, $\chi_A^{(3)} > 0$ and, according to Proposition 1, in this case systems (10) possess one double and two simple real finite singularities. More exactly, we detect that one of the singular points M_3 or M_4 collapses with a singular point located on the hyperbola, whereas the other one remains outside the hyperbola. Then, we obtain the conditions and configurations as follow:

- $\chi_C^{(3)} < 0$ and $\chi_B^{(3)} < 0 \;\Rightarrow\;$ *Config. H.21*;
- $\chi_C^{(3)} < 0$ and $\chi_B^{(3)} > 0 \;\Rightarrow\;$ *Config. H.23*;
- $\chi_C^{(3)} > 0 \;\Rightarrow\;$ *Config. H.20*,

in which in the last case the condition $\chi_C^{(3)} > 0$ implies $\chi_B^{(3)} < 0$, because $\mathbf{T} < 0$ yields $0 < g < 1/2$ and, for these values of g combined with the condition $\chi_C^{(3)} > 0$, we have $a < 0$ and hence $\chi_B^{(3)} < 0$.

ii.2) *The subcase* $\mathbf{T} = 0$. In this case, due to the conditions (35) and $\mu_0 \neq 0$, the equality $\mathbf{T} = 0$ yields $g = 3/5$ and hence $\chi_C^{(3)} = -416000/3 < 0$, which leads to configuration given by *Config. H.26*.

$\boldsymbol{\alpha_3}$**)** *The possibility* $\chi_A^{(3)} = 0$. Due to (35), the condition $\chi_A^{(3)} = 0$ implies $1 + 4ag = 0$ and hence $a = -1/(4g)$. In this case the points $M_{1,2}$ collapse and we have a double point on the hyperbola. In this case we see that

$$\chi_D^{(3)} = \frac{46(3g-1)^3(9g-1)^2}{81g^2} \neq 0$$

and $\delta_1 \neq 0$, due to (35). So, as $\chi_A^{(3)} \neq 0$ no other point could collapse with the double point on the hyperbola, we arrive at the configuration *Config. H.7* if $\chi_B^{(3)} < 0$ and *Config. H.8* if $\chi_B^{(3)} > 0$.

$\boldsymbol{\beta}$**)** *The subcase* $\mu_0 = 0$. We consider the possibilities: $\chi_A^{(3)} < 0$, $\chi_A^{(3)} > 0$ and $\chi_A^{(3)} = 0$.

$\boldsymbol{\beta_1}$**)** *The possibility* $\chi_A^{(3)} < 0$. The singular points on the hyperbola are complex and, since $1+4ag < 0$ yields $ag < 0$, we have $g = 2/3$ and then $a < 0$, which is equivalent to $\chi_B^{(3)} < 0$. So we arrive at the configuration given by *Config. H.3*.

$\boldsymbol{\beta_2}$**)** *The possibility* $\chi_A^{(3)} > 0$. Analogously we have $g = 2/3$ and the points on the hyperbola are real. We observe that, due to the condition (35), the equality $\chi_C^{(3)} = 0$ is equivalent to $g = 0$. So we consider two subcases: $\chi_C^{(3)} \neq 0$ and $\chi_C^{(3)} = 0$.

i) *The case* $\chi_C^{(3)} \neq 0$. Then one finite singularity has gone to infinity and collapsed with the point $[1, 1, 0]$. As observed earlier, this must be a singular point located outside the hyperbola which goes to infinity and hence on the finite part of the phase plane of systems (38) there are three singularities, two of which (M_1 and M_2) being located on the hyperbola.

Since the singular points on the hyperbola are real, we have to decide when the third point will belong also to the hyperbola. For systems (34) with $g = 2/3$ we calculate

$$\chi_A^{(3)} = \frac{7713280}{81}(8a+3)(3a-2)^2, \quad \chi_B^{(3)} = \frac{164798932}{81}\,a(3a-2)^2,$$
$$\chi_C^{(3)} = -\frac{133120}{3}\,a(3a-2)^2, \quad \chi_D^{(3)} = \frac{736}{243}(a-3)(3a-2).$$

We observe that $\text{sign}\,(\chi_B^{(3)}) = -\text{sign}\,(\chi_C^{(3)})$ and, moreover, $\chi_D^{(3)} = 0$ (i.e. $a = 3$) implies $\chi_C^{(3)} < 0$. So we get the following conditions and configurations:

- $\chi_C^{(3)} < 0$ and $\chi_D^{(3)} \neq 0 \;\Rightarrow\;$ *Config. H.32*;
- $\chi_C^{(3)} < 0$ and $\chi_D^{(3)} = 0 \;\Rightarrow\;$ *Config. H.34*;
- $\chi_C^{(3)} > 0 \;\Rightarrow\;$ *Config. H.29*.

ii) *The case* $\chi_C^{(3)} = 0$. Then $g = 0$ and this implies

$$\chi_B^{(3)} = 164798932a(a+6)^2/81, \quad \chi_C^{(3)} = 0, \quad \chi_D^{(3)} = -736(a_1)(a+6)^2/81.$$

So we get the following conditions and configurations:

- $\chi_B^{(3)} < 0$ and $\chi_D^{(3)} \neq 0 \quad \Rightarrow \quad$ *Config. H.9*;
- $\chi_B^{(3)} < 0$ and $\chi_D^{(3)} = 0 \quad \Rightarrow \quad$ *Config. H.11*;
- $\chi_B^{(3)} > 0 \quad \Rightarrow \quad$ *Config. H.10*,

in which in the last case the condition $\chi_B^{(3)} > 0$ (i.e. $a > 0$) implies $\chi_D^{(3)} \neq 0$.

$\boldsymbol{\beta_3}$) *The possibility* $\chi_A^{(3)} = 0$. Due to (35), the condition $\mu_0 = \chi_A^{(3)} = 0$ implies $g(3g-2) = 1+4ag = 0$. Then this yields $g \neq 0$ and hence $g = 2/3$ and $a = -3/8$. In this case the singularities $M_{1,2}$ collapse and we have a double point on the hyperbola. For systems (34) with $a = -3/8$ we calculate

$$\chi_C^{(3)} = 162500 > 0, \quad \chi_D^{(3)} = 575/18 \neq 0.$$

Since $\chi_D^{(3)} \neq 0$, no other point could coalesce with the double point on the hyperbola and we arrive at the configuration *Config. H.14*.

$\boldsymbol{a_2}$) *The case* $B_1 = 0$. Thus, according to Lemma 2, the condition $B_1 = 0$ is necessary in order to exist an invariant line of systems (34). Considering (35), the condition $B_1 = 0$ (see (37)) is equivalent to

$$(2g-1)\big[3 + a(3g+1)^2\big] = 0.$$

On the other hand for these systems we calculate

$$\chi_E^{(3)} = (3g-1)\big[3 + a(3g+1)^2\big]\big[6(1-3g) + a(3g+1)^2\big]$$

and we examine two possibilities: $\chi_E^{(3)} \neq 0$ and $\chi_E^{(3)} = 0$.

$\boldsymbol{\alpha}$) *The subcase* $\chi_E^{(3)} \neq 0$. In this case we get $g = 1/2$ and this leads to the systems

$$\frac{dx}{dt} = a + x + x^2/2 - 2xy/3, \quad \frac{dy}{dt} = -y(1 + x/2 - y/3), \tag{38}$$

for which the following condition holds (see (35)):

$$a(25a - 12) \neq 0. \tag{39}$$

Since the family of systems (38) is a subfamily of (34) (setting $g = 1/2$), the invariant hyperbola remains the same as in (36). Besides this hyperbola, systems (38) possess the invariant line $y = 0$, which is one of the asymptotes of this hyperbola. For the above systems we calculate

$$\mu_0 = -1/36, \quad \chi_E^{(3)} = (25a+12)(25a-12)/192, \quad B_1 = 0, \quad B_2 = -8a(25a+12)y^4.$$

Therefore, we conclude that, due to the conditions $\chi_E^{(3)} \neq 0$ and (39), we obtain $B_2 \neq 0$ and by Lemma 2 we could not have another invariant line in a direction different from $y = 0$. Moreover, due to the condition $\theta \neq 0$ and according to Lemma 4, in the direction $y = 0$ we could not have either a couple of parallel invariant lines or a double invariant line.

Since $\mu_0 \neq 0$, systems (38) possess finite singular points of multiplicity 4 with coordinates $M_i(x_i, y_i)$ (i=1,2,3,4):

$$x_{1,2} = -1 \pm \sqrt{2a+1}, \quad y_{1,2} = 3(1 \pm \sqrt{2a+1})/2,$$
$$x_{3,4} = -1 \pm \sqrt{1-2a}, \quad y_{3,4} = 0.$$

We recall that the singular points $M_{1,2}$ are located on the hyperbola and that the singularities $M_{3,4}$ are located on the invariant line $y = 0$.

On the other hand for systems (38) we calculate

$$\chi_A^{(3)} = \frac{482080}{243}(2a+1)(25a-12)^2, \quad \mathbf{D} = a^2(2a-1)(2a+1)/3,$$
$$\chi_B^{(3)} = \frac{41199733}{5184}a(25a-12)^2, \quad \chi_C^{(3)} = -\frac{2080}{9}a(25a-12)^2$$

and then the invariant polynomials $\chi_A^{(3)}$ and $\mathbf{D}$ govern the types of the above singular points (i.e. are they real or complex or coinciding), whereas the invariant polynomials $\chi_B^{(3)}$ and $\chi_C^{(3)}$ are responsible respectively for the position of the hyperbola and the location of the real singularities on it (i.e. on the same branch or on the different ones).

$\boldsymbol{\alpha_1}$**)** *The possibility* $\chi_A^{(3)} < 0$. Then the singularities $M_{1,2}$ (located on the hyperbola) are complex. Since $\chi_A^{(3)} < 0$, we obtain $\chi_B^{(3)} < 0$ and $\mathbf{D} > 0$, and by Proposition 1 the singularities on the invariant line are real and distinct. So we get the configuration given by *Config. H.49*.

$\boldsymbol{\alpha_2}$**)** *The possibility* $\chi_A^{(3)} > 0$. In this case the singularities $M_{1,2}$ are real and they are located on different branches (respectively on the same branch) of the hyperbola if $\chi_C^{(3)} < 0$ (respectively $\chi_C^{(3)} > 0$). We observe that the conditions $\chi_A^{(3)} > 0$ and $\mathbf{D} \geq 0$ imply $a \geq 1/2$ and then $\chi_B^{(3)} > 0$ and $\chi_C^{(3)} < 0$. Moreover, the conditions $\chi_A^{(3)} > 0$ and $\chi_B^{(3)} < 0$ yield $-1/2 < a < 0$ and then $\mathbf{D} < 0$ and $\chi_C^{(3)} > 0$. Therefore, we arrive at the following conditions and configurations:

- $\chi_B^{(3)} < 0 \;\Rightarrow$ *Config. H.74*;
- $\chi_B^{(3)} > 0$ and $\mathbf{D} < 0 \;\Rightarrow$ *Config. H.73*;
- $\chi_B^{(3)} > 0$ and $\mathbf{D} > 0 \;\Rightarrow$ *Config. H.47*;
- $\chi_B^{(3)} > 0$ and $\mathbf{D} = 0 \;\Rightarrow$ *Config. H.66.*

$\boldsymbol{\alpha_3}$**)** *The possibility* $\chi_A^{(3)} = 0$. Due to the condition (39), the condition $\chi_A^{(3)} = 0$ implies $a = -1/2$. In this case the points $M_{1,2}$ collapse and we have a double point on the hyperbola. For systems (38) with $a = -1/2$ we calculate

$$\chi_B^{(3)} = 41199733/41472 > 0, \quad \mathbf{T} = -(3x-2y)^2(9x^2-24xy+8y^2)^2/4478976 < 0.$$

So, according to Proposition 1, besides the double point on the hyperbola, we have two simple real singular points on the invariant line $y = 0$ and we get the configuration given by *Config. H.67.*

$\boldsymbol{\beta}$**)** *The subcase* $\chi_E^{(3)} = 0$. In this case we obtain $a = -3/(3g+1)^2$ and this leads to the systems

$$\frac{dx}{dt} = -\frac{3}{(3g+1)^2} + x + gx^2 - 2xy/3, \quad \frac{dy}{dt} = \frac{9(2g-1)}{(3g+1)^2} - y + (g-1)xy + y^2/3 \qquad (40)$$

with the conditions

$$(g-1)(3g-1)(3g+1)(6g-1) \neq 0. \tag{41}$$

Moreover, systems (40) possess the following invariant line and invariant hyperbola

$$x - y + 6/(3g+1) = 0, \quad \Phi(x,y) = \frac{18}{(3g+1)^2} + 2xy = 0. \tag{42}$$

We observe that the condition $\chi_E^{(3)} = 0$ implies

$$\chi_A^{(3)} = \frac{7713280(3g-1)^2(6g-1)^2}{27(3g+1)^2} > 0, \quad \chi_B^{(3)} = -\frac{164798932(g-1)^2(3g-1)^2(6g-1)^2}{3(3g+1)^2} < 0,$$

due to (41). Therefore, the points on the hyperbola are real and distinct and the hyperbola assumes only one position.

For the above systems we calculate

$$B_2 = \frac{7776}{(3g+1)^4}(g-1)^2(2g-1)(x-y)^4 \tag{43}$$

and, by Lemma 2, for the existence of an invariant line in a direction different from $y = x$ it is necessary $B_2 = 0$.

***β*₁) *The possibility* $B_2 \neq 0$. Then $2g-1 \neq 0$ and, since $\theta \neq 0$, by Lemma 4 we could not have a couple of parallel invariant lines in the direction $y = x$ and obviously the invariant line $y = x + 6/(3g+1)$ is a simple one. As before, we consider two cases: $\mu_0 \neq 0$ and $\mu_0 = 0$.

***i*) *The case* $\mu_0 \neq 0$. Then $g(3g-2) \neq 0$ and systems (40) possess four real singularities $M_i(x_i, y_i)$ having the following coordinates:

$$\begin{aligned} x_1 &= -\frac{3}{3g+1},\ y_1 = \frac{3}{3g+1}; \qquad x_3 = -\frac{1}{3g+1},\ y_3 = \frac{3(2g-1)}{3g+1};\\ x_2 &= -\frac{1}{g(3g+1)},\ y_2 = \frac{9g}{3g+1}; \quad x_4 = -\frac{3}{(3g+1)(3g-2)},\ y_4 = \frac{9(2g-1)}{(3g+1)(3g-2)}. \end{aligned} \tag{44}$$

We could check directly that the singularity M_1 is a common (tangency) point of the invariant hyperbola and of line (42). Moreover, the singular point M_2 (respectively M_4) is located on the hyperbola (respectively on the invariant line), whereas the singularity M_3 is generically located outside the hyperbola as well as outside the invariant line.

For systems (20) we calculate

$$\begin{aligned} \mathbf{D} &= -\frac{64}{3(3g+1)^8}(g+1)^2(3g-5)^2(3g-1)^6, \quad \mu_0 = g(3g-2)/9,\\ \chi_C^{(3)} &= \frac{199680g(6g-1)^2}{(3g+1)^2}, \quad \chi_D^{(3)} = \frac{1472(g+1)(3g-1)^2(6g-1)}{27(3g+1)^2}. \end{aligned} \tag{45}$$

***i*.1) *The subcase* $\chi_C^{(3)} < 0$. Then $g < 0$ and the singular points $M_{1,2}$ are located on different branches of the hyperbola and we obtain the configuration *Config. H.60* if $\mathbf{D} \neq 0$ and *Config. H.69* if $\mathbf{D} = 0$.

i.2) *The subcase* $\chi_C^{(3)} > 0$. Then $g > 0$ and the singular points $M_{1,2}$ are located on the same branch of the hyperbola. It is clear that the reciprocal position of the singularities M_2 (located on the hyperbola) and M_4 (located on the invariant line) with respect to the tangency point M_1 of the hyperbola and the invariant line (42) defines different configurations. More exactly, the type of the configuration depends on the sign of the expression:

$$(x_1 - x_2)(x_1 - x_4) = \frac{3(3g-1)^2}{g(3g-2)(3g+1)^2}.$$

Hence, we observe that sign $\big((x_1 - x_2)(x_1 - x_4)\big) = \text{sign}\,(\mu_0)$. So, if $\mathbf{D} \neq 0$, we arrive at the configuration *Config. H.61* if $\mu_0 < 0$ and *Config. H.59* if $\mu_0 > 0$.

We consider now the case $\mathbf{D} = 0$. Then, due to the condition (20), we have $(g+1)(3g-5) = 0$ and clearly the invariant polynomial $\chi_D^{(3)}$ distinguishes which one of the two factors vanishes.

If $\chi_D^{(3)} \neq 0$, then $g + 1 \neq 0$ and we get $g = 5/3$. We observe that in this case the singularity M_3 collapses with the singular point M_4 located on the invariant line. On the other hand, we calculate

$$\mathbf{T} = -256(x-y)^2(5x-y)^2(5x+y)^2/177147 < 0$$

and, by Proposition 1, we have three distinct singularities (one of them being double). Now, assuming $g = 5/3$, for systems (40), we calculate

$$\chi_C^{(3)} = 748800 > 0, \quad (x_1 - x_2)(x_1 - x_4) = 4/15 > 0$$

and hence we arrive at the configuration given by *Config. H.62*.

In the case $\chi_D^{(3)} = 0$, we have $g = -1$ and then the singularity M_3 collapses with the singular point M_2 located on the hyperbola (but outside of the invariant line). Moreover, for $g = -1$ we have

$$\mathbf{T} = -256(x-y)^2(3x+y)^2(9x+y)^2/243 < 0$$

and again we conclude that systems (40) possess three distinct singularities (one double). In this case we have

$$\chi_C^{(3)} = -2446080 < 0, \quad (x_1 - x_2)(x_1 - x_4) = 12/5 > 0$$

and therefore we get the configuration given by *Config. H.68*.

ii) *The case* $\mu_0 = 0$. Then $g(3g-2) = 0$ and, by Lemma 1, at least one finite singularity has gone to infinity and coalesced with an infinite singular point. Since for systems (40) we have $\chi_C^{(3)} = 0$ if and only if $g = 0$ (see (45)), we consider two subcases: $\chi_C^{(3)} \neq 0$ and $\chi_C^{(3)} = 0$.

ii.1) *The subcase* $\chi_C^{(3)} \neq 0$. Then the condition $\mu_0 = 0$ implies $3g - 2 = 0$ (i.e. $g = 2/3$) and, considering the coordinates (44) of the finite singularities of systems (40), we observe that the singular point M_4 located on the invariant line has gone to infinity and collapsed with the singularity $[1, 1, 0]$. In this case calculation yields

$$\mathbf{D} = -1600/19683 < 0, \quad \chi_C^{(3)} = 133120 > 0,$$

and, since by Remark 7 the condition $\mathbf{R} \neq 0$ holds, according to Proposition 1, all three finite singularities are distinct. Moreover, due to $\chi_C^{(3)} > 0$, the singularities are located on the same branch of the hyperbola and we get the configuration given by *Config. H.57*.

ii.2) *The subcase* $\chi_C^{(3)} = 0$. Then $g = 0$ and in this case the singularity M_2 located on the hyperbola (42) has gone to infinity and collapsed with the point $[1, 0, 0]$. Since by Remark 7 we have $\mu_1 \neq 0$, according to Lemma 1 the other three finite singular points remain on the finite part of the phase plane.

Now, depending on the position of the singular point M_4 (located on the invariant line (42)) with respect to the vertical line $x = x_1$, we may get different configurations. This distinction is governed by the sign of the expression $x_4 - x_1$ and we calculate

$$\mathbf{D} = -1600/3 \neq 0, \quad x_4 - x_1 = 3/2 > 0.$$

Since by Remark 7 the condition $\mathbf{R} \neq 0$ holds, according to Proposition 1, all three finite singularities are distinct ($\mathbf{D} \neq 0$) and since $x_4 - x_1 > 0$, we arrive at the configuration given by *Config. H.50.*

β₂) *The possibility* $B_2 = 0$. Considering (43) and the condition (35), we obtain $g = 1/2$ and this leads to the system

$$\frac{dx}{dt} = -12/25 + x + x^2/2 - 2xy/3, \quad \frac{dy}{dt} = -y(1 + x/2 - y/3), \tag{46}$$

possessing the two invariant lines and the invariant hyperbola:

$$x - y + \frac{12}{5} = 0, \quad y = 0, \quad \Phi(x, y) = \frac{72}{25} + 2xy = 0.$$

as well as the following singularities $M_i(x_i, y_i)$ with the coordinates

$$x_1 = -\frac{6}{5},\ y_1 = \frac{6}{5}; \quad x_3 = \frac{2}{5},\ y_3 = 0; \quad x_2 = \ -\frac{4}{5},\ y_2 = \frac{9}{5}; \quad x_4 = -\frac{12}{5},\ y_4 = 0. \tag{47}$$

Hence, all singularities are located on the finite part of the phase plane since $\mu_0 = -1/36 \neq 0$. We calculate

$$\mathbf{D} = -2352/390625 < 0, \quad \chi_C^{(3)} = 319488/5 > 0.$$

Since $\chi_C^{(3)} > 0$, the singular points M_1 and M_2 are located on the same branch of the hyperbola and we need to detect the position of the singularity M_2 on the hyperbola. This fact is verified by the sign of the expression $(x_1 - x_2)(x_1 - x_4) = -12/25 < 0$. Then, we arrive at the configuration given by *Config. H.86.*

b) *The possibility* $\delta_1 = 0$. Due to condition (35) we get $a = 6(3g - 1)/(3g + 1)^2$ and we get the following 1-parameter family of systems

$$\frac{dx}{dt} = \frac{6(3g-1)}{(3g+1)^2} + x + gx^2 - \frac{2xy}{3}, \quad \frac{dy}{dt} = \frac{18(1-2g)(3g-1)}{(3g+1)^2} - y + (g-1)xy + \frac{y^2}{3}, \tag{48}$$

with the conditions

$$(g - 1)(3g - 1)(3g + 1) \neq 0. \tag{49}$$

Moreover, systems (48) possess two invariant hyperbolas:

$$\Phi_1(x, y) = \frac{36(1-3g)}{(3g+1)^2} + 2xy = 0, \quad \Phi_2(x, y) = \frac{36(3g-1)}{(3g+1)^2} + \frac{12}{3g+1}x + 2x(x - y) = 0. \tag{50}$$

We observe that the family of systems (48) is a subfamily of systems (34) and hence, via the transformation (33), systems (48) could be brought to systems of the same form (48) but with the new parameter $g_1 = 2/3 - g$. So, this transformation induces a transformation in the coefficient space which fixes the point $g = 1/3$ and sends the interval $(-\infty, 1/3]$ onto the interval $[1/3, +\infty)$. Thus, in what follows we shall consider only the values of the parameter g on the interval $(-\infty, 1/3]$.

In this sense, we get the next remark.

Remark 8. *Due to an affine transformation and a time rescaling, we could assume that the parameter g in systems* (48) *belongs to the interval* $(-\infty, 1/3]$.

For systems (49) we calculate

$$B_1 = \frac{32}{(3g+1)^4}(g-1)^2(3g-1)^3(2g-1)(6g-1) \tag{51}$$

and we analyze two subcases: $B_1 \neq 0$ and $B_1 = 0$.

***b*₁)** *The case $B_1 \neq 0$.* In this case due to (49) we have $(2g-1)(6g-1) \neq 0$. For systems (48) we calculate $\mu_0 = g(3g-2)/9$ and we consider two subcases: $\mu_0 \neq 0$ and $\mu_0 = 0$.

***α*)** *The subcase $\mu_0 \neq 0$.* Then the systems have finite singularities of total multiplicity 4 with coordinates:

$$\begin{aligned} &x_1 = \frac{3g-1}{g(3g+1)},\ y_1 = \frac{18g}{3g+1}; \qquad x_3 = \frac{3(3g-1)}{(3g+1)(3g-2)},\ y_3 = \frac{9(2g-1)(3g-1)}{(3g+1)(3g-2)};\\ &x_2 = -\frac{6}{3g+1},\ y_2 = \frac{3(1-3g)}{3g+1}; \quad x_4 = -\frac{2}{3g+1},\ y_4 = \frac{6(1-2g)}{3g+1}. \end{aligned} \tag{52}$$

We detect that the singularities $M_{1,2}$ are located on the first invariant hyperbola (88) and moreover the singularity M_2 is also located on the second hyperbola, i.e. M_2 is a point of intersection of these two hyperbolas on the finite part of the plane. The singular point M_3 belongs to the second hyperbola, whereas the singularity M_4 generically is located outside the hyperbolas.

For systems (88) we calculate

$$\chi_F^{(3)} = (9g-1)(9g-5)/9, \quad \mu_0 = g(3g-2)/9,$$
$$\mathbf{D} = -\frac{16}{3(3g+1)^8}(9g-1)^2(9g-5)^2(5g-1)^2(15g-7)^2.$$

On the other hand, we have

$$x_1x_2 = \frac{6(1-3g)}{g(3g+1)^2}, \quad \Phi_1(x_4,y_4) = \frac{12(1-5g)}{(3g-2)^2}, \quad \Phi_2(x_4,y_4) = \frac{4(15g-7)}{(3g-2)^2}.$$

We observe that the singular points $M_{1,2}$ are located on different branches (respectively on the same branch) of the first hyperbola if only if $x_1x_2 < 0$ (respectively $x_1x_2 > 0$), and this is governed by the $\text{sign}\,(x_1x_2) = -\text{sign}\,\big(g(3g-1)\big)$. Since by Remark 8 we have $g \in (-\infty, 1/3]$, we conclude that in this interval $\text{sign}\,(x_1x_2) = -\text{sign}\,(\mu_0)$.

Besides, we point out that the singular point $M_4(x_4, y_4)$ (which generically is located outside of the hyperbolas) could be located on one of these invariant hyperbolas if and only if the following condition holds:

$$\big[\Phi_1(M_4)\big]\big[\Phi_2(M_4)\big] = \left[\frac{12(1-5g)}{(3g+1)^2}\right]\left[\frac{4(15g-7)}{(3g+1)^2}\right] = \frac{48(1-5g)(15g-7)}{(3g+1)^4} = 0.$$

We observe that in the case $\chi_F^{(3)} \neq 0$ the condition $(5g-1)(15g-7) = 0$ is equivalent to $\mathbf{D} = 0$.

$\boldsymbol{\alpha_1}$**)** *The possibility* $\mu_0 < 0$. According to Remark 8, the condition $\mu_0 < 0$ is equivalent to $g > 0$ and the singular points $M_{1,2}$ are located on the same branch of the first hyperbola. We calculate

$$x_1 - x_2 = \frac{9g-1}{g(3g+1)}.$$

We observe that $\operatorname{sign}(x_1 - x_2) = \operatorname{sign}(\chi_F^{(3)})$ due to Remark 8. Then we consider the cases $\chi_F^{(3)} < 0$, $\chi_F^{(3)} > 0$ and $\chi_F^{(3)} = 0$.

***i*)** *The case* $\chi_F^{(3)} < 0$. Then $(9g-1)(9g-5) < 0$ and we consider two subcases: $\mathbf{D} \neq 0$ and $\mathbf{D} = 0$. If $\mathbf{D} \neq 0$ we have only simple singular points on the hyperbolas and we arrive at the configuration shown in *Config. H.128*. Otherwise, $\mathbf{D} = 0$ implies the existence of a double singular point on the first hyperbola and this point is characterized by the collision of the singular points M_1 and M_4, and we get the configuration given by *Config. H.130*.

***ii*)** *The case* $\chi_F^{(3)} > 0$. Then $(9g-1)(9g-5) > 0$ and we get the configuration given by *Config. H.129*.

***iii*)** *The case* $\chi_F^{(3)} = 0$. Then $(1-5g)(9g-5) = 0$ and, according to Remark 8, we get $g = 1/5$. In this case, the singularities M_1 and M_2 have collided and we obtain a double singular point at the intersection of the two hyperbolas (88) and hence we get the configuration given by *Config. H.124*.

$\boldsymbol{\alpha_2}$**)** *The possibility* $\mu_0 > 0$. In this case the singularities $M_{1,2}$ are located on different branches of the first hyperbola and we get the configuration given by *Config. H.127*.

$\boldsymbol{\beta}$**)** *The subcase* $\mu_0 = 0$. Then $g = 0$ and the point M_1 has coalesced with the point $[1, 0, 0]$ at infinity and we obtain the configuration shown in *Config. H.125*.

$\boldsymbol{b_2}$**)** *The case* $B_1 = 0$. Considering (49), the condition $B_1 = 0$ (see (51)) is equivalent to $(2g-1)(6g-1) = 0$. According to Remark 8, we have $g = 1/6$ and in this case, besides the hyperbola, we have the invariant line $x - y + 4 = 0$. Since $B_2 = -6400(x-y)^4/9 \neq 0$, the system could not possess another invariant line by Lemma 2. Moreover, we observe that the point M_1 is the point of intersection of the first hyperbola and the invariant line. Since $\mu_0 = -1/36 < 0$ and $\chi_{13} = -7/36 < 0$, we get the configuration given by *Config. H.135*.

3.1.2.1.2 *The subcase* $\beta_2 = 0$ Then $g = 1/3$ and we arrive at systems of the form

$$\frac{dx}{dt} = a + x + x^2/3 - 2xy/3, \quad \frac{dy}{dt} = b - y - 2xy/3 + y^2/3, \tag{53}$$

For systems (53) we calculate

$$\gamma_5 = 256ab(a-b)/81, \quad \mathcal{R}_4 = 128(a^2 - ab + b^2)/6561.$$

In order to have $\gamma_5 = 0$ we must have $ab(a-b) = 0$. We observe that i n the case $ab = 0$ we may assume $b = 0$ due to the change $(x, y, t) \mapsto (-y, -x, -t)$. On the other hand, the systems (53) with $b = 0$ could be brought to the same systems with $b = a$ via the change $(x, y, t) \mapsto (x, x-y+3, -t)$. Therefore, we consider the following family of systems

$$\frac{dx}{dt} = -a/3 + x + x^2/3 - 2xy/3, \quad \frac{dy}{dt} = -a/3 - y - 2xy/3 + y^2/3, \tag{54}$$

with the condition $a \neq 0$.

We observe that the above family of systems is a subfamily of systems (30) defined by the condition $h = 1/3$. For the family (30), it was shown that, due to the conditions (31) (i.e. $h \neq 1/3$), we have $\operatorname{sign}(\chi_A^{(2)}) = \operatorname{sign}(1-4ah^2)$ and $\operatorname{sign}(\chi_B^{(2)}) = \operatorname{sign}(x_1x_2)$. Clearly that for the subfamily (54) these invariants vanish and we need other invariant polynomials which are responsible for the $\operatorname{sign}(1-4ah^2)$ and $\operatorname{sign}(x_1x_2)$ in this particular case.

We calculate

$$(1-4ah^2)\big|_{\{h=1/3\}} = (9-4a)/9, \quad (x_1x_2)\big|_{\{h=1/3\}} = a.$$

On the other hand, for systems (54) we calculate

$$\chi_A^{(3)} = 123412480a^2(9-4a)/19683, \quad \chi_C^{(3)} = 1064960a^3/729$$

and hence $\operatorname{sign}(\chi_A^{(3)}) = \operatorname{sign}(9-4a)$ and $\operatorname{sign}(\chi_C^{(3)}) = \operatorname{sign}(x_1x_2)$.

Thus, considering the conditions and configurations for family (30), we get the configurations given by *Config. H.37* if $\chi_A^{(3)} < 0$; *Config. H.52* if $\chi_A^{(3)} > 0$ and $\chi_C^{(3)} < 0$; *Config. H.53* if $\chi_A^{(3)} > 0$ and $\chi_C^{(3)} > 0$ and *Config. H.45* if $\chi_A^{(3)} = 0$.

3.1.2.2 The case $\beta_6 = 0$.

The conditions $\beta_6 = -c(g-1)(h-1)/2 = 0$ and $\theta = (g-1)(h-1)(g+h)/2 \neq 0$ imply $c = 0$. Then for systems (8) with $c = 0$ we calculate

$$\beta_7 = 2(2g-1)(2h-1)(1-2g-2h), \quad \gamma_5 = -288(g-1)(h-1)(g+h)\mathcal{B}_1\mathcal{B}_2\mathcal{B}_3,$$

where

$$\mathcal{B}_1 \equiv b(2h-1) - a(2g-1); \quad \mathcal{B}_2 \equiv b(1-2h) + 2a(g+2h-1); \quad \mathcal{B}_3 \equiv a(1-2g) + 2b(2g+h-1).$$

We consider two subcases: $\beta_7 \neq 0$ and $\beta_7 = 0$.

Remark 9. *Considering systems* (8) *with* $c = 0$*, having the relation* $(2h-1)(2g-1)(1-2g-2h) = 0$ *(respectively* $(4h-1)(4g-1)(3-4g-4h) = 0$*), due to a change, we may assume any of the factors* $2h-1$*,* $2g-1$ *or* $1-2g-2h$ *(respectively* $4h-1$*,* $4g-1$ *or* $3-4g-4h$*) to be zero, for instance we could set* $2h-1 = 0$ *(respectively* $4h-1 = 0$*).*

Indeed, it is sufficient to observe that in the case $2g-1 = 0$ (respectively $4g-1 = 0$) we could apply the change

$$(x, y, a, b, g, h) \mapsto (y, x, b, a, h, g),$$

which conserves systems (8) with $c = 0$, whereas in the case $1-2g-2h = 0$ (respectively $3-4g-4h = 0$) we apply the change

$$(x, y, a, b, g, h) \mapsto (y-x, -x, b-a, -a, h, 1-g-h),$$

which also conserves systems (8) with $c = 0$.

3.1.2.2.1 The subcase $\beta_7 \neq 0$. According to Theorem 1, in this case for the existence of an invariant hyperbola, it is necessary and sufficient $\gamma_5 = 0$, which is equivalent to $\mathcal{B}_1\mathcal{B}_2\mathcal{B}_3 = 0$. We claim that, without loss of generality, we may assume $\mathcal{B}_1 = 0$, as other cases could be brought to this one via an affine transformation.

Indeed, assume first $\mathcal{B}_1 \neq 0$ and $\mathcal{B}_2 = 0$. Then we apply to systems (8) with $c = 0$ the linear transformation $x' = y - x,\ y' = -x$ and we get the systems

$$\frac{dx'}{dt} = a' + g'x'^2 + (h'-1)x'y', \quad \frac{dy'}{dt} = b' + (g'-1)x'y' + h'y'^2.$$

These systems have the following new parameters:

$$a' = b - a, \quad b' = -a, \quad g' = h, \quad h' = 1 - g - h.$$

A straightforward computation gives

$$\mathcal{B}_1' = b'(2h'-1) - a'(2g'-1) = b(1-2h) + 2a(-1+g+2h) = \mathcal{B}_2 = 0$$

and hence, the condition $\mathcal{B}_2 = 0$ we replace by $\mathcal{B}_1 = 0$ via a linear transformation.

Analogously in the case $\mathcal{B}_1 \neq 0$ and $\mathcal{B}_3 = 0$, via the linear transformation $x'' = -y,\ y'' = x - y$, we replace the condition $\mathcal{B}_3 = 0$ by $\mathcal{B}_1 = 0$ and this completes the proof of our claim.

Since $\beta_7 \neq 0$ (i.e. $2h - 1 \neq 0$) the condition $\mathcal{B}_1 = 0$ yields $b = a(2g-1)/(2h-1)$ and we arrive at the 3-parameter family of systems

$$\frac{dx}{dt} = a(2h-1) + gx^2 + (h-1)xy, \quad \frac{dy}{dt} = a(2g-1) + (g-1)xy + hy^2 \tag{55}$$

with the condition

$$a(g-1)(h-1)(2g-1)(2h-1)(g+h)(2g+2h-1) \neq 0. \tag{56}$$

These systems possess the invariant hyperbola

$$\Phi(x,y) = a + xy = 0. \tag{57}$$

For systems (55) we calculate

$$\begin{aligned} \beta_8 &= 2(4g-1)(4h-1)(3-4g-4h), \\ \delta_2 &= 2a(4g-1)(4h-1)\big[68(g^2+h^2) + 236gh - 79(g+h) - 144gh(g+h) + 22\big], \end{aligned} \tag{58}$$

According to Theorem 1, these systems possess either one or two invariant hyperbolas if either $\beta_8^2 + \delta_2^2 \neq 0$ or $\beta_8 = \delta_2 = 0$, respectively.

We claim that the condition $\beta_8 = \delta_2 = 0$ is equivalent to $(4g-1)(4h-1) = 0$. Indeed, assuming that $(4g-1)(4h-1) \neq 0$ and $\beta_8 = \delta_2 = 0$ we obtain

$$3 - 4g - 4h = 0, \quad 68(g^2+h^2) + 236gh - 79(g+h) - 144gh(g+h) + 22 = 0.$$

The first equation gives $g = 3/4 - h$ and then from the second one we obtain $(2h-1)(4h-1) = 0$, which contradicts the condition (56) and the assumption. This completes the proof of our claim.

***a*) *The possibility* $\beta_8^2 + \delta_2^2 \neq 0$.* Then this implies $(4g-1)(4h-1) \neq 0$ and systems (55) possess only one invariant hyperbola. For these systems we calculate

$$B_1 = 2a^3(g-1)^2(h-1)^2(2g-1)(2h-1)(g-h)(g+h)^2$$

and considering (56) we conclude that the condition $B_1 = 0$ is equivalent to $g - h = 0$. We examine two cases: $B_1 \neq 0$ and $B_1 = 0$.

***a*$_1$) *The case* $B_1 \neq 0$.* Then $g - h \neq 0$ and by Lemma 2 we have no invariant lines. For systems (55) we calculate $\mu_0 = gh(g+h-1)$ and we consider two subcases: $\mu_0 \neq 0$ and $\mu_0 = 0$.

$\boldsymbol{\alpha}$) *The subcase* $\mu_0 \neq 0$. In this case the systems have finite singularities of total multiplicity 4 with the following coordinates $M_i(x_i, y_i)$:

$$x_{1,2} = \pm\frac{\sqrt{-agh}}{g}, \quad y_{1,2} = \pm\frac{\sqrt{-agh}}{h},$$
$$x_{3,4} = \pm(2h-1)\frac{\sqrt{a(1-g-h)}}{g+h-1}, \quad y_{3,4} = \pm(2g-1)\frac{\sqrt{a(1-g-h)}}{g+h-1}.$$

We detect that the singularities $M_{1,2}$ are located on the invariant hyperbola. More exactly, these singular points are located on different branches (respectively on the same branch) of the hyperbola if only if $x_1x_2 < 0$ (respectively $x_1x_2 > 0$), where $x_1x_2 = ah/g$. Moreover, these singularities are real if $agh < 0$, they are complex if $agh > 0$ and they coincide if $agh = 0$.

On the other hand, we calculate

$$\chi_A^{(4)} = -16128\, a^5 gh(g-1)^2(h-1)^2(g+h)^4(2g-1)^4(2h-1)^4(4g-1)^2(4h-1)^2,$$
$$\chi_B^{(4)} = -4257792\, a^5(g+h)^4(2g-1)^6(2h-1)^6(4g-1)^2(4h-1)^2$$

and due to the condition (56) we have $\operatorname{sign}(\chi_A^{(4)}) = -\operatorname{sign}(agh) = -\operatorname{sign}(x_1x_2)$ and $\operatorname{sign}(\chi_B^{(4)}) = -\operatorname{sign}(a)$ (which corresponds to the position of the hyperbola). We observe that in the case the singular points M_1 and M_2 are real, they must be located on different branches of the hyperbola (we recall that systems (55) is symmetric with respect to the origin). Moreover, we could not have $\chi_A^{(4)} = 0$ due to $\mu_0 \neq 0$ and (56).

Besides, we point out that at least one of the singular points $M_{3,4}$ could be located on the invariant hyperbola and we determine the conditions for this to happen. We calculate

$$\Phi(x_3, y_3) = \Phi(x_4, y_4) = \frac{a(4gh-g-h)}{g+h-1}.$$

It is clear that both of the singular points M_3 and M_4 belong to the hyperbola (57) if and only if $4gh - g - h = 0$. Since

$$\mathbf{D} = -768a^4(4gh-g-h)^4\mu_0,$$

we deduce that both of the singular points $M_{3,4}$ belong to the hyperbola if and only if $\mathbf{D} = 0$.

$\boldsymbol{\alpha_1}$) *The possibility* $\chi_A^{(4)} < 0$. So we have no real singularities located on the invariant hyperbolas and we arrive at the configurations given by *Config. H.1* if $\chi_B^{(4)} < 0$ and *Config. H.2* if $\chi_B^{(4)} > 0$.

$\boldsymbol{\alpha_2}$) *The possibility* $\chi_A^{(4)} > 0$. In this case we have two real singularities located on the hyperbola and they are located on different branches. Now, we need to decide if both of the singular points $M_{3,4}$ will belong to the hyperbola.

i) *The case* $\mathbf{D} \neq 0$. Then $4gh - g - h \neq 0$ and on the hyperbola there are two simple real singularities and we obtain the configurations given by *Config. H.17* if $\chi_B^{(4)} < 0$ and *Config. H.19* if $\chi_B^{(4)} > 0$.

ii) *The case* $\mathbf{D} = 0$. Then $4gh - g - h = 0$ (i.e. $g = h/(4h-1)$) and in this case we calculate

$$\chi_A^{(4)} = -4128768\, a^5 h^{10}(h-1)^2(2h-1)^8(3h-1h)^2/(4h-1)^{11},$$
$$\mathbf{D} = \mathbf{T} = 0, \quad \mathbf{PR} = -256a^3h^8(2h-1)^8\big[x-(4h-1)y\big]^6/(4h-1)^{11}$$

and, due to $\chi_A^{(4)} > 0$, we have $\mathbf{PR} > 0$ and on the hyperbola there are two double real singularities (see Proposition 1) we arrive at the configurations given by *Config. H.27* if $\chi_B^{(4)} < 0$ and *Config. H.28* if $\chi_B^{(4)} > 0$.

$\boldsymbol{\beta}$) *The subcase* $\mu_0 = 0$. We consider the possibilities: $\chi_A^{(4)} < 0$, $\chi_A^{(4)} > 0$ and $\chi_A^{(4)} = 0$.

$\boldsymbol{\beta_1}$) *The possibility* $\chi_A^{(4)} < 0$. Then $gh \neq 0$ and the condition $\mu_0 = 0$ yields $g = 1 - h$. So we calculate

$$\mathbf{D} = 0, \quad \mu_1 = 0, \quad \mu_2 = ah(1-h)(2h-1)^2(x-y)^2 \neq 0.$$

Hence, two singular points go to infinity in the direction $y = x$ and we get the configurations *Config. H.5* if $\chi_B^{(4)} < 0$ and *Config. H.6* if $\chi_B^{(4)} > 0$.

$\boldsymbol{\beta_2}$) *The possibility* $\chi_A^{(4)} > 0$. As in the previous subcase, two singular points go to infinity in the direction $y = x$ and, moreover, the singularities $M_{1,2}$ are real. So we obtain the configurations *Config. H.35* if $\chi_B^{(4)} < 0$ and *Config. H.36* if $\chi_B^{(4)} > 0$.

$\boldsymbol{a}_2$) *The case* $B_1 = 0$. Then by conditions (56), we get $g = h$ and systems (55) possess the invariant line $x - y = 0$. For this case due to (56) we have

$$\mu_0 = h^2(2h-1) \neq 0, \quad \mathbf{D} = 12288a^4h^6(1-2h)^5 \neq 0.$$

$\boldsymbol{\alpha}$) *The subcase* $\chi_A^{(4)} < 0$. In this case the singularities $M_{1,2}$ are complex and, since

$$\chi_A^{(4)} = -258048\, a^5h^6(h-1)^4(2h-1)^8(4h-1)^4 < 0,$$

we have $\chi_B^{(4)} = -68124672\, a^5h^4(2h-1)^{12}(4h-1)^4 < 0$. So, we obtain the unique configuration *Config. H.37*.

$\boldsymbol{\beta}$) *The subcase* $\chi_A^{(4)} > 0$. In this case the singularities $M_{1,2}$ are real and analogously we have $\text{sign}\,(\chi_A^{(4)}) = \text{sign}\,(\chi_B^{(4)})$. So we get the unique configuration *Config. H.53*.

b) *The possibility* $\beta_8 = \delta_2 = 0$. Then this implies $(4g-1)(4h-1) = 0$ and, due to a change, we may assume $h = 1/4$, without loss of generality. In this case, systems (55) possess the two invariant hyperbolas

$$\Phi_1(x,y) = a + xy = 0, \quad \Phi_2(x,y) = a - x(x-y) = 0.$$

For these systems we calculate

$$\mu_0 = g(4g-3)/16, \quad B_1 = 9a^3(g-1)^2(2g-1)(4g-1)(4g+1)^2/1024$$

and, due to conditions (56), we verify that $B_1 \neq 0$. Then we consider two cases $\mu_0 \neq 0$ and $\mu_0 = 0$.

***b*$_1$) *The case* $\mu_0 \neq 0$.** Then $g(4g-3) \neq 0$ and the systems have finite singularities of total multiplicity 4 with the following coordinates $M_i(x_i, y_i)$:

$$x_{1,2} = \pm\frac{\sqrt{-ag}}{2g}, \quad y_{1,2} = \pm 2\sqrt{-ag}, \quad x_{3,4} = \pm\frac{\sqrt{-a(4g-3)}}{4g-3}, \quad y_{3,4} = \pm 2(2g-1)\frac{\sqrt{-a(4g-3)}}{4g-3}.$$

We detect that the singularities $M_{1,2}$ are located on the invariant hyperbola $\Phi_1(x,y) = 0$. More exactly, these singular points are located on different branches (respectively on the same branch) of the hyperbola if only if $x_1x_2 < 0$ (respectively $x_1x_2 > 0$), where $x_1x_2 = a/4g$. Moreover, these singularities are real if $ag < 0$, they are complex if $ag > 0$ and they coincide if $ag = 0$. We also point out that the position of the hyperbolas are governed by sign (a).

On the other hand, we calculate

$$\chi_A^{(5)} = -41\, a(8g-3)^3/128.$$

We observe that in the case the singular points M_1 and M_2 are real, they must be located on different branches of the hyperbola (we recall that systems (55) is symmetric with respect to the origin). Moreover, we could not have $\chi_A^{(5)} = 0$ due to $\mu_0 \neq 0$ and (56).

Moreover, we also detect that the singularities $M_{3,4}$ are located on the invariant hyperbola $\Phi_2(x,y) = 0$ and their position concerning which branch they are located is also governed by sign (a) and they will be complex, real or coinciding depending on the sign of the expression $a(4g-3)$ and hence the sign of μ_0 plays an important role in this analysis.

Besides, we point out that the singular points $M_{1,2}$ could not be located on the hyperbola $\Phi_2(x,y) = 0$ and, conversely, $M_{3,4}$ could not be located on the hyperbola $\Phi_1(x,y) = 0$, since we have

$$\Phi_2(x_{1,2}, y_{1,2}) = \frac{a}{4g} \neq 0, \quad \Phi_1(x_{3,4}, y_{3,4}) = \frac{a}{3-4g} \neq 0,$$

due to conditions (56).

We consider the case $\mu_0 < 0$ (i.e. $0 < g < 3/4$). Then, for these values of g, we have $8g-3 < 0$ and, independently of the sign of a, we get the unique configuration *Config. H.123*.

In the case $\mu_0 > 0$, we obtain the configuration *Config. H.121* if $\chi_A^{(5)} < 0$ and *Config. H.131* if $\chi_A^{(5)} > 0$.

***b*$_2$) *The case* $\mu_0 = 0$.** Then $g(4g-3) = 0$ and depending on which one of these two factors vanishes, we have different finite singular points coalescing with an infinite singular point. More precisely, if $4g-3 = 0$ then the singular points $M_{3,4}$ coalesce with $[1,1,0]$, and if $g = 0$ then the singular points $M_{1,2}$ coalesce with $[1,0,0]$.

However, we observe that, applying the change $(x,y,t,a) \mapsto (-x, y-x, t, -a)$, we could bring systems (56) with $h = 1/4$ and $g = 3/4$ to the same systems with $h = 1/4$ and $g = 0$. So, without loss of generality, we may assume $g = 0$.

Thus, we obtain the configurations given by *Config. H.122* if $\chi_A^{(5)} < 0$ and *Config. H.126* if $\chi_A^{(5)} > 0$.

3.1.2.2.2 The subcase $\beta_7 = 0$. We recall that the conditions $\beta_1 = \beta_6 = 0$ yields $c = 0$ and systems (8) with $c = 0$ becomes

$$\frac{dx}{dt} = a + gx^2 + (h-1)xy, \quad \frac{dy}{dt} = b + (g-1)xy + hy^2. \tag{59}$$

Without loss of generality, Remark 9 assures us that we may choose $g = 1/2 - h$ in order to have $\beta_7 = 2(2g-1)(2h-1)(1-2g-2h) = 0$.

Now, we calculate

$$\beta_9 = 4h(1-2h)$$

and we analyze two possibilities: $\beta_9 \neq 0$ and $\beta_9 = 0$.

a) *The possibility* $\beta_9 \neq 0$. As earlier, according to Theorem 1, in this case for the existence of at least one invariant hyperbola, it is necessary and sufficient $\gamma_5 = 0$, which is equivalent to $\mathcal{B}_1\mathcal{B}_2\mathcal{B}_3 = 0$ and, without loss of generality, we may assume $\mathcal{B}_1 = 0$, as other cases could be brought to this one via an affine transformation.

***a*$_1$)** *The case* $\delta_3 \neq 0$. In this case we have only one invariant hyperbola and the condition $\delta_3 \neq 0$ yields $a - b \neq 0$. Then, the condition $\gamma_5 = 0$ is equivalent to $b(1-2h) - 2ah = 0$, which could be rewritten as $a = a_1(2h-1)$ and $b = -2a_1h$. So, setting the old parameter a instead of a_1, we arrive at the 2-parameter family of systems

$$\frac{dx}{dt} = a(2h-1) + (1-2h)x^2/2 + (h-1)xy, \quad \frac{dy}{dt} = -2ah - (2h+1)xy/2 + hy^2, \tag{60}$$

with the condition

$$ah(h-1)(2h-1)(2h+1)(4h-1) \neq 0. \tag{61}$$

These systems possess the invariant hyperbola

$$\Phi(x,y) = a + xy = 0. \tag{62}$$

We observe that, due to (61),

$$B_1 = a^3h(h-1)^2(2h-1)(2h+1)^2(4h-1) \neq 0$$

and, hence, systems (60) possess no invariant line. Moreover, we have

$$\mu_0 = h(2h-1)/4 \neq 0, \quad \mathbf{D} = 12a^4h(1-2h)(1-4h+8h^2)^4 \neq 0,$$

due to the same conditions, and then all the finite singularities are in the finite part of the phase plane and none of them coalesces with other points. Considering the coordinates of these singularities $M_i(x_i, y_i)$ (i=1,2,3,4), we have

$$x_{1,2} = \pm\frac{\sqrt{2ah(2h-1)}}{2h-1}, \quad y_{1,2} = \mp\frac{\sqrt{2ah(2h-1)}}{2h}, \quad x_{3,4} = \pm(2h-1)\sqrt{2a}, \quad y_{3,4} = \pm 2h\sqrt{2a}.$$

After simple calculations, we obtain that $M_{1,2}$ are located on the hyperbola, whereas $M_{3,4}$ are located generically outside the hyperbola. Then, the singular points $M_{1,2}$ are complex if $ah(2h-1) <$

0 and they are real if $ah(2h-1)>0$. We point out that these two singularities could not coincide since $ah(2h-1)\neq 0$, due to (61). So, we need to control sign $\big(ah(2h-1)\big)$. Moreover, sign (a) gives the position of the hyperbola on the phase plane.

On the other hand, we calculate

$$\chi_A^{(4)} = 2016\, a^5h^5(h-1)^2(2h-1)^5(2h+1)^2(4h-1)^4,$$
$$\chi_B^{(4)} = -17031168\, a^5h^6(2h-1)^6(4h-1)^4.$$

Therefore, we arrive at the following conditions and configurations:

- $\chi_A^{(4)}<0$ and $\chi_B^{(4)}<0 \;\Rightarrow\;$ *Config. H.1*;
- $\chi_A^{(4)}<0$ and $\chi_B^{(4)}>0 \;\Rightarrow\;$ *Config. H.2*;
- $\chi_A^{(4)}>0$ and $\chi_B^{(4)}<0 \;\Rightarrow\;$ *Config. H.17*;
- $\chi_A^{(4)}>0$ and $\chi_B^{(4)}>0 \;\Rightarrow\;$ *Config. H.19.*

$\boldsymbol{a_2}$**)** *The case* $\delta_3=0$. In this case, the conditions $\gamma_5=\delta_3=0$ yield $a-b=0$ (i.e. $b=a$) and systems

$$\frac{dx}{dt}=a+(1-2h)x^2/2+(h-1)xy,\quad \frac{dy}{dt}=a-(2h+1)xy/2+hy^2, \tag{63}$$

with the condition

$$ah(h-1)(2h-1)(2h+1)\neq 0, \tag{64}$$

possess at least two invariant hyperbolas. We calculate $\beta_8=-2(4h-1)^2$ and we analyze two subcases: $\beta_8\neq 0$ and $\beta_8=0$.

$\boldsymbol{\alpha}$**)** *The subcase* $\beta_8\neq 0$. Then $4h-1\neq 0$ and systems (63) possess two invariant hyperbolas:

$$\Phi_1(x,y)=-\frac{a}{2h-1}+x(x-y)=0,\quad \Phi_2(x,y)=\frac{a}{h}+2y(x-y)=0. \tag{65}$$

We observe that

$$B_1=0,\quad B_2=-162a^2(h-1)^2(2h+1)^2(x-y)^4\neq 0,$$

due to (64), and this implies that systems (63) possess only one invariant straight line, namely $x-y=0$.

Due to condition (64), we obtain

$$\mu_0=h(2h-1)/4\neq 0,\quad \mathbf{D}=-12a^4h(2h-1)\neq 0,$$

and then we have four distinct finite singularities $M_i(x_i,y_i)$ (i=1,2,3,4), where

$$x_{1,2}=\pm\frac{\sqrt{2ah(2h-1)}}{2h-1},\quad y_{1,2}=\pm\frac{\sqrt{2ah(2h-1)}}{2h},\quad x_{3,4}=\pm\sqrt{2a},\quad y_{3,4}=\pm\sqrt{2a}.$$

We observe that the singular points $M_{1,2}$ are located on the first hyperbola (65), whereas $M_{3,4}$ are located on the invariant line. Additionally, the singularities $M_{1,2}$ (respectively $M_{3,4}$) are complex if $ah(2h-1)<0$ (respectively $a<0$) and are real if $ah(2h-1)>0$ (respectively $a>0$).

So, we need to control sign $\big(ah(2h-1)\big)$ and sign (a). Moreover, sign $\big(h(2h-1)\big)$ gives the position of the hyperbolas on the phase plane.

On the other hand, we calculate

$$\chi_A^{(6)} = ah(2h-1).$$

If $\chi_A^{(6)} < 0$, then the singularities $M_{1,2}$ are complex and we get the configuration *Config. H.132* if $\mathbf{D} < 0$ and *Config. H.133* if $\mathbf{D} > 0$.

In the case $\chi_A^{(6)} > 0$, the singular points $M_{1,2}$ are real and we obtain the configuration *Config. H.136* if $\mathbf{D} < 0$ and *Config. H.134* if $\mathbf{D} > 0$.

***β*)** *The subcase* $\beta_8 = 0$. Then $h = 1/4$ and systems (63) possess three invariant hyperbolas, namely the two presented in (65) with $h = 1/4$ and

$$\Phi_3(x,y) = 2a - xy = 0.$$

In this case, we observe that $\mathbf{D} = 3a^4/2 > 0$ and we obtain the configuration *Config. H.156* if $\chi_A^{(6)} < 0$ and *Config. H.157* if $\chi_A^{(6)} > 0$.

***b*)** *The possibility* $\beta_9 = 0$. Then $h = 0$ (this yields $g = 1/2$) and systems (59) becomes

$$\frac{dx}{dt} = a + x^2/2 - xy, \quad \frac{dy}{dt} = b - xy/2. \tag{66}$$

According to Theorem 1, in this case for the existence of at least one invariant hyperbola, it is necessary and sufficient $\gamma_6 = 0$, which is equivalent to $(a-b)b = 0$. Without loss of generality, we may assume $b = 0$, since we could pass from the case $b = a$ to the case $b = 0$, via the affine transformation $(x, y, t) \mapsto (x, x - y, -t)$. Then, we arrive at the 1-parameter family of systems

$$\frac{dx}{dt} = a + x^2/2 - xy, \quad \frac{dy}{dt} = -xy/2. \tag{67}$$

with the condition $a \neq 0$.

The above family possesses the invariant hyperbola

$$\Phi(x,y) = a - xy = 0 \tag{68}$$

and, since $B_1 = 0$ and $B_2 = -162a^2y^4 \neq 0$, due to $a \neq 0$, systems (67) possess the only one invariant line $y = 0$.

We calculate

$$\mu_0 = \mu_1 = 0, \quad \mu_2 = ax^2/8, \quad \mathbf{D} = 0.$$

Then, two finite singular points has collapsed and gone to infinity and coalesced with $[0,1,0]$. Considering the remaining singularities on the finite part of the plane, their coordinates are $M_i(x_i, y_i)$ $(i = 1, 2)$:

$$x_{1,2} = \pm\sqrt{-2a}, \quad y_{1,2} = 0.$$

We point out that these two singularities are located on the invariant line and they are complex if $a > 0$ and are real if $a < 0$. So, we need to control sign (a), which also gives the position of the hyperbola on the phase plane.

On the other hand, we calculate

$$\chi_A^{(5)} = -a/16.$$

So, we obtain the configuration *Config. H.40* if $\chi_A^{(5)} < 0$ and *Config. H.58* if $\chi_A^{(5)} > 0$.

3.2 The subcase $\theta = 0$

For systems (2) we assume $\eta > 0$ and therefore we consider systems (7) for which we have

$$\theta = -(g-1)(h-1)(g+h)/2.$$

Since $\theta = 0$, we get $(g-1)(h-1)(g+h) = 0$ and we may assume $g = -h$, otherwise in the case $g = 1$ (respectively $h = 1$) we apply the change $(x, y, g, h) \mapsto (-y, x-y, 1-g-h, g)$ (respectively $(x, y, g, h) \mapsto (y-x, -x, h, 1-g-h)$) which preserves the quadratic parts of systems (7).

So, $g = -h$ and we arrive at the systems

$$\frac{dx}{dt} = a + cx - hx^2 + (h-1)xy, \quad \frac{dy}{dt} = b + fy + (h+1)xy + hy^2, \tag{69}$$

for which we calculate $N = 9(h^2-1)(x-y)^2$. We consider two possibilities: $N \neq 0$ and $N = 0$.

3.2.1 The possibility $N \neq 0$

For systems (69), we calculate

$$\gamma_1 = (c-f)^2(c+f)(h-1)^2(h+1)^2(3h-1)(3h+1)/64,$$
$$\beta_6 = (c-f)(h-1)(h+1)/4, \quad \beta_{10} = -2(3h-1)(3h+1).$$

According to Theorem 1, a necessary condition for the existence of hyperbolas for these systems is $\gamma_1 = 0$.

3.2.1.1 *The case $\beta_6 \neq 0$.* Then $c-f \neq 0$ and the condition $\gamma_1 = 0$ yields $(c+f)(3h-1)(3h+1) = 0$. So, we consider the subcases: $\beta_{10} \neq 0$ and $\beta_{10} = 0$.

3.2.1.1.1 *The subcase $\beta_{10} \neq 0$.* Then $(3h-1)(3h+1) \neq 0$ and we get $f = -c$ and obtain the following systems

$$\frac{dx}{dt} = a + cx - hx^2 + (h-1)xy, \quad \frac{dy}{dt} = b - cy - (h+1)xy + hy^2. \tag{70}$$

Now, in order to possess at least one hyperbola, it is necessary and sufficient that for the above systems the condition

$$\gamma_7 = 8(h-1)(h+1)\big[a(2h+1) + b(2h-1)\big] = 0$$

holds, and due to $N \neq 0$ this is equivalent to $a(2h+1) + b(2h-1) = 0$.

Since $\beta_6 = c(h-1)(h+1)/2 \neq 0$ (i.e. $c \neq 0$), we could apply the rescaling $(x, y, t) \mapsto (cx, cy, t/c)$ and assume $c = 1$. Moreover, since $(2h-1)^2 + (2h+1)^2 \neq 0$, the condition $a(2h+1) + b(2h-1) = 0$

could be written as $a = -a_1(2h-1)$ and $b = a_1(2h+1)$. So, setting the old parameter a instead of a_1, we arrive at the 2-parameter family of systems

$$\frac{dx}{dt} = a(2h-1) + x - hx^2 + (h-1)xy, \quad \frac{dy}{dt} = -a(2h+1) - y - (h+1)xy + hy^2, \tag{71}$$

with the conditions

$$a(h-1)(h+1)(3h-1)(3h+1) \neq 0. \tag{72}$$

We observe that the family of systems (71) is a subfamily of systems (10) with $g = -h$.

The above systems possess the invariant hyperbola

$$\Phi(x,y) = a + xy = 0 \tag{73}$$

and for them we calculate

$$B_1 = -4a^2h(h-1)^2(h+1)^2(2h-1)(2h+1). \tag{74}$$

a) *The possibility* $B_1 \neq 0$. Then $h(2h-1)(2h+1) \neq 0$ and by Lemma 2 systems (71) possess no invariant lines. Since $\mu_0 = h^2 \neq 0$, these systems have finite singularities $M_i(x_i, y_i)$ of total multiplicity 4, whose coordinates are

$$x_{1,2} = \frac{1 \pm \sqrt{1+4ah^2}}{2h}, \quad y_{1,2} = \frac{1 \mp \sqrt{1+4ah^2}}{2h},$$
$$x_{3,4} = \frac{(2h-1)(1 \pm \sqrt{1+4a})}{2}, \quad y_{3,4} = \frac{(2h+1)(1 \pm \sqrt{1+4a})}{2}.$$

We observe that the singular points $M_{1,2}$ are located on the hyperbola, whereas the singularities $M_{3,4}$ are generically located outside of it.

On the other hand, for systems (71), we calculate the invariant polynomials

$$\chi_A^{(1)} = h^2(h-1)^2(h+1)^2(3h-1)^2(3h+1)^2(1+4ah^2)/16,$$
$$\chi_B^{(1)} = -105\, ah^2(h-1)^2(h+1)^2(3h-1)^2(3h+1)^2/2$$

and, by the condition (72), we conclude that $\operatorname{sign}(\chi_A^{(1)}) = \operatorname{sign}(1+4ah^2)$ (if $1+4ah^2 \neq 0$) and $\operatorname{sign}(\chi_B^{(1)}) = -\operatorname{sign}(a)$. So, we consider three cases: $\chi_A^{(1)} < 0$, $\chi_A^{(1)} > 0$ and $\chi_A^{(1)} = 0$.

a_1) *The case* $\chi_A^{(1)} < 0$. Then $1+4ah^2 < 0$ yields $a < 0$ and hence $\chi_B^{(1)} > 0$. So, since the singular points located on the hyperbola are complex, we arrive at the configuration given by *Config. H.2.*

a_2) *The case* $\chi_A^{(1)} > 0$. In this case, we have two real singularities located on the hyperbola. We calculate $x_1x_2 = -a$ and, due to the condition (72), we obtain that $\operatorname{sign}(\chi_B^{(1)}) = \operatorname{sign}(x_1x_2)$, which defines the location of the singular points $M_{1,2}$ concerning the branches of the hyperbola (i.e. they are located either on different branches if $\chi_B^{(1)} < 0$ or on the same branch if $\chi_B^{(1)} > 0$).

However, we need to detect when the singularities $M_{3,4}$ also belong to the hyperbola. In this order, considering (73), we calculate

$$\Phi(x,y)|_{\{x=x_{3,4},\ y=y_{3,4}\}} = \frac{4h^2\left[1+2a \mp \sqrt{1+4a}\right] - 1 \pm \sqrt{1+4a}}{2} \equiv \Omega_{3,4}(a,g,h).$$

It is clear that at least one of the singular points M_3 or M_4 belongs to the hyperbola (73) if and only if

$$\Omega_3\Omega_4 = a(16ah^4 + 4h^2 - 1) = 0.$$

On the other hand, for systems (71), we have

$$\chi_D^{(1)} = -105h(3h-1)(3h+1)(16ah^4 + 4h^2 - 1)$$

and clearly, due to (72), the condition $\chi_D^{(1)} = 0$ is equivalent to $16ah^4 + 4h^2 - 1 = 0$. We examine two subcases: $\chi_D^{(1)} \neq 0$ and $\chi_D^{(1)} = 0$.

***α*)** *The subcase* $\chi_D^{(1)} \neq 0$. Then, on the hyperbola there only two simple real singularities and we obtain the configurations given by *Config. H.17* if $\chi_B^{(1)} < 0$ and *Config. H.18* if $\chi_B^{(1)} > 0$.

***β*)** *The subcase* $\chi_D^{(1)} = 0$. In this case, the condition $16ah^4+4h^2-1 = 0$ yields $a = -(2h-1)(2h+1)/(16h^4)$ and we calculate

$$\mathbf{D} = 0, \quad \mathbf{T} = -3(2h^2-1)^2(x+y)^2\big[(2h+1)x-(2h-1)y\big]^2\big[(h+1)(2h+1)x-(h-1)(2h-1)y\big]^2.$$

If $\mathbf{T} \neq 0$, then we have a double and a simple singular points on the hyperbola and we arrive at the configurations shown in *Config. H.21* if $\chi_B^{(1)} < 0$ and *Config. H.22* if $\chi_B^{(1)} > 0$. In the case $\mathbf{T} = 0$, we obtain $h = \pm\sqrt{2}/2$ and hence $\chi_B^{(1)} > 0$. Then, we have a triple and a simple singular points on the hyperbola and we obtain the configuration *Config. H.25*.

***a*$_3$)** *The case* $\chi_A^{(1)} = 0$. Then $a = -1/(4h^2)$ and hence $\chi_B^{(1)} > 0$. In this case, the singular points M_1 and M_2 coalesce and we get the configuration *Config. H.8*.

***b*)** *The possibility* $B_1 = 0$. Then $h(2h-1)(2h+1) = 0$ and we analyze the two cases: $\mu_0 \neq 0$ and $\mu_0 = 0$.

***b*$_1$)** *The case* $\mu_0 \neq 0$. Then $h \neq 0$ and the condition $B_1 = 0$ is equivalent to $(2h-1)(2h+1) = 0$. Without loss of generality, we may assume $h = -1/2$, otherwise we apply the change $(x, y, t, h) \mapsto (-y, -x, -t, a, -h)$, which keeps systems (71) and changes the sign of h.

So $h = 1/2$ and then systems (71) possess the invariant line $y = 0$ and the singularities $M_{3,4}$ are located on this line. In this case, we calculate

$$\chi_A^{(1)} = 225(a+1)/16384, \quad \chi_B^{(1)} = -23625\,a/2048, \quad \mathbf{D} = -48a^2(a+1)(4a+1).$$

***α*)** *The subcase* $\chi_A^{(1)} < 0$. Then $a+1 < 0$ implies $a < 0$ and hence $\chi_B^{(1)} > 0$. So, we obtain the configuration shown in *Config. H.38*.

***β*)** *The subcase* $\chi_A^{(1)} > 0$. Then $a > -1$ and we have real singularities on the hyperbola. So, we get the following conditions and configurations:

- $\chi_B^{(1)} < 0 \;\Rightarrow\;$ *Config. H.75*;
- $\chi_B^{(1)} > 0$ and $\mathbf{D} < 0 \;\Rightarrow\;$ *Config. H.72*;
- $\chi_B^{(1)} > 0$ and $\mathbf{D} > 0 \;\Rightarrow\;$ *Config. H.46*;

- $\chi_B^{(1)} > 0$ and $\mathbf{D} = 0 \Rightarrow$ *Config. H.65.*

γ) *The subcase* $\chi_A^{(1)} = 0$. Then $a = -1$ (consequently $\mathbf{D} = 0$ and $\chi_B^{(1)} > 0$) and this implies the existence of a double singular point on the hyperbola and the singularities on the invariant line are complex, obtaining the configuration *Config. H.42.*

b_2) *The case* $\mu_0 = 0$. Then $h = 0$ and we also have $\mu_1 = 0$ and $\mu_2 = -xy$, which means that the singular points $M_{1,2}$ have gone to infinity and coalesced with the singular points $[1,0,0]$ and $[0,1,0]$.

Considering Lemma 3 we detect that Z is a simple factor of $\mathcal{E}_1$ and $\mathcal{E}_2$. So, we deduce that the infinity line $Z = 0$ is a double invariant line for systems (71). Since $\chi_A^{(1)} = 1 > 0$, we obtain the configurations *Config. H.76* if $\chi_B^{(1)} < 0$ and *Config. H.77* if $\chi_B^{(1)} > 0$.

3.2.1.1.2 *The subcase* $\beta_{10} = 0$. Then $(3h-1)(3h+1) = 0$ and as earlier we may assume $h = 1/3$ and obtain the following systems

$$\frac{dx}{dt} = -\frac{a}{3} + x - \frac{x^2}{3} - \frac{2xy}{3}, \qquad \frac{dy}{dt} = -\frac{5a}{3} - y - \frac{4xy}{3} + \frac{y^2}{3}, \tag{75}$$

with the condition $a \neq 0$. We again remark that the family of systems (75) is a subfamily of systems (10) with $g = -h$ and $h = 1/3$.

These systems possess the invariant hyperbola

$$\Phi(x,y) = a + xy = 0. \tag{76}$$

and for them we calculate

$$\mu_0 = 1/9, \quad \mathbf{D} = -16(4a+1)(4a+9)(16a-45)/19683, \quad B_1 = 1280a^2/2187.$$

Since $B_1 \neq 0$, systems (75) do not possess invariant lines and the condition $\mu_0 \neq 0$ implies that the finite singularities $M_i(x_i, y_i)$ are of total multiplicity 4, and their coordinates are

$$x_{1,2} = \frac{3 \pm \sqrt{4a+9}}{2}, \quad y_{1,2} = \frac{3 \mp \sqrt{4a+9}}{2}, \quad x_{3,4} = \frac{-1 \pm \sqrt{4a+1}}{6}, \quad y_{3,4} = \frac{5(1 \mp \sqrt{4a+1})}{6}.$$

We observe that the singular points $M_{1,2}$ are located on the hyperbola, whereas the singularities $M_{3,4}$ are generically located outside of it.

Concerning the singular points $M_{1,2}$, we see that $x_1 x_2 = -a$ and the sign (a) will detect the location of these singularities on the same or different branches of the hyperbola as well as its position on the phase plane.

Moreover, we need to detect when the singularities $M_{3,4}$ also belong to the hyperbola. Considering (76), we calculate

$$\Phi(x,y)|_{\{x=x_{3,4},\ y=y_{3,4}\}} = \frac{8a \pm 5\sqrt{4a+1} - 5}{18} \equiv \Omega'_{3,4}(a,g,h)$$

and we observe that at least one of the singular points M_3 or M_4 belongs to the hyperbola (76) if and only if

$$\Omega'_3 \Omega'_4 = \frac{a(16a-45)}{18} = 0.$$

On the other hand, for systems (75), we calculate the invariant polynomials

$$\chi_A^{(3)} = \frac{123412480(4a+9)}{243}, \quad \chi_B^{(3)} = -\frac{168754106368\, a}{243},$$
$$\chi_C^{(3)} = -\frac{1064960\, a}{9}, \quad \chi_D^{(3)} = \frac{5888(16a-45)}{729}$$

and we conclude that $\operatorname{sign}(\chi_A^{(3)}) = \operatorname{sign}(4a+9)$ (if $4a+9 \neq 0$), $\operatorname{sign}(\chi_B^{(3)}) = \operatorname{sign}(\chi_C^{(3)}) = -\operatorname{sign}(a)$ and at least one of the singular points M_3 or M_4 belongs to the hyperbola if and only if $\chi_D^{(3)} = 0$.

We observe that the condition $\chi_A^{(3)} < 0$ implies $\chi_B^{(3)} > 0$ and $\chi_C^{(3)} > 0$, all the finite singular points are complex and we get the configuration *Config. H.2*.

In the case $\chi_A^{(3)} > 0$, the singularities $M_{1,2}$ are real and we arrive at the following conditions and configurations:

- $\chi_D^{(3)} \neq 0$ and $\chi_C^{(3)} < 0 \;\Rightarrow\;$ *Config. H.17*;
- $\chi_D^{(3)} \neq 0$ and $\chi_C^{(3)} > 0 \;\Rightarrow\;$ *Config. H.18*;
- $\chi_D^{(3)} = 0 \;\Rightarrow\;$ *Config. H.21*.

And in the case $\chi_A^{(3)} = 0$, the singular points $M_{1,2}$ have collapsed and $M_{3,4}$ are complex, obtaining the configuration *Config. H.8*.

3.2.1.2 *The case $\beta_6 = 0$.* Then $f = c$ and hence $\gamma_1 = 0$. We calculate

$$\beta_2 = c(h-1)(h+1)/2, \quad \beta_7 = -2(2h-1)(2h+1)$$

and we analyze two subcases: $\beta_2 \neq 0$ and $\beta_2 = 0$.

3.2.1.2.1 *The subcase $\beta_2 \neq 0$.* Then $c \neq 0$ and we obtain the systems

$$\frac{dx}{dt} = a + cx - hx^2 + (h-1)xy, \quad \frac{dy}{dt} = b + cy - (h+1)xy + hy^2. \tag{77}$$

a) *The possibility $\beta_7 \neq 0$.* Then $(2h-1)(2h+1) \neq 0$ and, according to Theorem 1, for the existence of at least one invariant hyperbola for systems (77), it is necessary and sufficient the conditions $\gamma_8 = 0$ and $\beta_{10}\mathcal{R}_7 \neq 0$. So, we calculate

$$\gamma_8 = 42(h-1)(h+1)\,\mathcal{E}_2\mathcal{E}_3, \quad \beta_{10} = -2(3h-1)(3h+1),$$
$$\mathcal{E}_2 = -2c^2(h-1)(2h-1) - 2a(h-1)(3h-1)^2 + b(2h-1)(3h-1)^2,$$
$$\mathcal{E}_3 = -2c^2(h+1)(2h+1) + 2b(h+1)(3h+1)^2 - a(2h+1)(3h+1)^2.$$

We observe that the condition $\gamma_8 = 0$ is equivalent to $\mathcal{E}_2\mathcal{E}_3 = 0$ and due to the change $(x, y, a, b, c, h) \mapsto (y, x, b, a, c, -h)$, we may assume that the condition $\mathcal{E}_2 = 0$ holds.

Since $\beta_7\beta_{10} \neq 0$, we could write the condition $\mathcal{E}_2 = 0$ as $c = c_1(3h-1)$, $b = b_1(h-1)$ and $a = (b_1 - 2c_1^2)(2h-1)/2$. Then, we apply the reparametrization $b_1 = ac_1^2$ and $a = 2a_1$. Finally, since

$c_1 \neq 0$ (due to $c \neq 0$), we could apply the rescaling $(x, y, t) \mapsto (c_1 x, c_1 y, t/c_1)$ and assume $c_1 = 1$. Thus, setting the old parameter a instead of a_1, we arrive at the 2-parameter family of systems

$$\begin{aligned}\frac{dx}{dt} &= (a-1)(2h-1) + (3h-1)x - hx^2 + (h-1)xy,\\ \frac{dy}{dt} &= 2a(h-1) + (3h-1)y - (h+1)xy + hy^2,\end{aligned} \tag{78}$$

with the conditions

$$(a-1)(h-1)(h+1)(2h-1)(2h+1)(3h-1)(3h+1) \neq 0. \tag{79}$$

These systems possess a couple of parallel invariant lines and an invariant hyperbola:

$$\begin{aligned}\mathcal{L}_{1,2}(x, y) &= h(x-y)^2 - (3h-1)(x-y) + 2h - a - 1 = 0,\\ \Phi(x, y) &= 1 - a - 2x + x(x-y) = 0.\end{aligned} \tag{80}$$

We remark that, since

$$\text{Discriminant}\big[\mathcal{L}_{1,2}(x, y), x-y\big] = (h-1)^2 + 4ah,$$

these lines are complex (respectively real) if $(h-1)^2 + 4ah < 0$ (respectively $(h-1)^2 + 4ah > 0$).

We calculate

$$\delta_4 = 3(h-1)(2h-1)\big[(h-1)^2(2h+1) + a(3h+1)^2\big]/2$$

and we consider two cases: $\delta_4 \neq 0$ and $\delta_4 = 0$.

$\boldsymbol{a_1}$**)** *The case* $\delta_4 \neq 0$. In this case we have $(h-1)^2(2h+1) + a(3h+1)^2 \neq 0$ and hence $\Phi(x, y) = 0$ (see (80)) is the unique invariant hyperbola. Since $B_1 = 0$ for systems (78), we calculate

$$B_2 = -1296\, a(a-1)(h-1)^3(h+1)^2(2h-1)(x-y)^4.$$

$\boldsymbol{\alpha}$**)** *The subcase* $B_2 \neq 0$. Then $a \neq 0$ and, since $\mu_0 = h^2$, we consider two possibilities: $\mu_0 \neq 0$ and $\mu_0 = 0$.

$\boldsymbol{\alpha_1}$**)** *The possibility* $\mu_0 \neq 0$. So we get $h \neq 0$ and the finite singularities of systems (78) are of multiplicity 4, and their coordinates are $M_i(x_i, y_i)$:

$$\begin{aligned}x_{1,2} &= \frac{h+1 \pm \sqrt{(h-1)^2 + 4ah}}{2}, \quad y_{1,2} = \frac{(h-1)\big[h-1 \pm \sqrt{(h-1)^2 + 4ah}\big]}{2h},\\ x_{3,4} &= \frac{(2h-1)\big[h+1 \pm \sqrt{(h-1)^2 + 4ah}\big]}{2h}, \quad y_{3,4} = h-1 \pm \sqrt{(h-1)^2 + 4ah}.\end{aligned}$$

We observe that the singular points $M_{1,2}$ are located on the hyperbola and on the invariant lines, whereas the singularities $M_{3,4}$ are located on the invariant lines.

Concerning the singular points $M_{1,2}$, we see that $x_1 x_2 = h(1-a)$ and hence sign $\big(h(a-1)\big)$ detects the location of these singularities on the same or different branches of the hyperbola. Moreover, the position of the hyperbola is governed by sign $(a-1)$.

In order to detect when the singularities $M_{3,4}$ also belong to the hyperbola, we consider (80) and we calculate

$$\Phi(x,y)|_{\{x=x_{3,4},\ y=y_{3,4}\}}=\frac{\tilde{A}\pm\big[(h+1)(2h-1)\sqrt{(h-1)^2+4ah}\big]}{2h^2}\equiv\Omega''_{3,4}(a,g,h)$$

where $\tilde{A}=2ah(1-3h)+(1-h)(1-h+2h^2)$, and we observe that at least one of the singular points M_3 or M_4 belongs to the hyperbola (80) if and only if

$$\Omega''_3\Omega''_4=\frac{(a-1)\big[a(3h-1)^2+2(h-1)^3\big]}{h^2}=0.$$

On the other hand, for systems (78), we calculate the invariant polynomials

$$\chi_A^{(7)}=(h-1)^2(h+1)^2\big[(h-1)^2+4ah\big]/16,$$
$$\chi_B^{(7)}=6480(a-1)(h-1)^2\big[(h-1)^2(2h+1)+a(3h+1)^2\big]^2,$$
$$\chi_C^{(7)}=2160\,h(1-a)(h-1)^2\big[(h-1)^2(2h+1)+a(3h+1)^2\big]^2,$$

and we conclude that sign $(\chi_A^{(7)})=$ sign $\big((h-1)^2+4ah\big)$ (if $(h-1)^2+4ah\neq 0$), sign $(\chi_B^{(7)})=$ sign $(a-1)$, sign $(\chi_C^{(7)})=$ sign $\big(h(1-a)\big)$ and at least one of the singular points M_3 or M_4 belongs to the hyperbola if and only if $a(3h-1)^2+2(h-1)^3=0$.

i) *The case* $\chi_A^{(7)}<0$. Then all the finite singular points are complex as well as the pair of invariant lines. Moreover, the condition $\chi_A^{(7)}<0$ (i.e. $(h-1)^2+4ah<0$) yields $ah<0$. Combining this inequality with $\chi_B^{(7)}<0$ (i.e. $a-1<0$) (respectively $\chi_B^{(7)}>0$ (i.e. $a-1>0$)), we obtain $h<0$ (respectively $h>0$) and hence $\chi_C^{(7)}<0$ (respectively $\chi_C^{(7)}>0$). So, we arrive at the configuration *Config. H.78* if $\chi_B^{(7)}<0$ and *Config. H.79* if $\chi_B^{(7)}>0$.

ii) *The case* $\chi_A^{(7)}>0$. Then all the finite singular points and the pair of invariant lines are real. In this sense, according to the position of the finite singular points on the hyperbola and on the invariant lines, we may have different configurations.

We calculate

$$(x_1-x_4)(x_2-x_3)=-\frac{a(3h-1)^2+2(h-1)^3}{h},$$
$$(x_1-x_4)-(x_2-x_3)=\frac{(3h-1)\sqrt{(h-1)^2+4ah}}{h},\quad (x_1-x_4)+(x_2-x_3)=\frac{(1-h)(h+1)}{h}$$

and we observe that sign $\big((x_1-x_4)(x_2-x_3)\big)$, sign $\big((x_1-x_4)-(x_2-x_3)\big)$ and sign $\big((x_1-x_4)+(x_2-x_3)\big)$ govern the position of the four finite singularities along the hyperbola and the invariant lines. More exactly, if $(x_1-x_4)(x_2-x_3)<0$ (respectively $(x_1-x_4)(x_2-x_3)>0$), then the sign $\big((x_1-x_4)-(x_2-x_3)\big)$ (respectively sign $\big((x_1-x_4)+(x_2-x_3)\big)$) distinguishes the position of M_3 and M_4 with respect to the hyperbola.

On the other hand, we calculate

$$\chi_D^{(7)}=3(h-1)^2(h+1)^2\big[a(3h-1)^2+2(h-1)^3\big]/8,$$
$$\beta_{10}=-2(3h-1)(3h+1),\quad N=9(h-1)(h+1)(x-y)^2.$$

We consider two subcases: $\chi_D^{(7)} \neq 0$ and $\chi_D^{(7)} = 0$.

***ii*.1)** *The subcase* $\chi_D^{(7)} \neq 0$. In this case the singularities $M_{3,4}$ do not belong to the hyperbola and we need to distinguish when the singular points $M_{1,2}$ are located on different or on the same branch.

***ii*.1.1)** *The possibility* $\chi_C^{(7)} < 0$. Then $M_{1,2}$ are located on different branches of the hyperbola and, if $\chi_B^{(7)} < 0$, we obtain $a < 0$ and $h < 0$, and hence $\chi_D^{(7)} < 0$. So, we get the configuration *Config. H.96*.

In the case $\chi_B^{(7)} > 0$, we observe that the condition $\chi_D^{(7)} < 0$ implies $N < 0$. So, we arrive at the following conditions and configurations:

- $\chi_D^{(7)} < 0 \;\Rightarrow\;$ *Config. H.99*;
- $\chi_D^{(7)} > 0$ and $\beta_{10} < 0 \;\Rightarrow\;$ *Config. H.95*;
- $\chi_D^{(7)} > 0$ and $\beta_{10} > 0 \;\Rightarrow\;$ *Config. H.94*.

***ii*.1.2)** *The possibility* $\chi_C^{(7)} > 0$. Then $M_{1,2}$ are located on the same branch of the hyperbola.

If $\chi_B^{(7)} < 0$, the condition $\chi_D^{(7)} > 0$ implies $\beta_{10} < 0$ and we obtain the following conditions and configurations:

- $\chi_D^{(7)} < 0$ and $N < 0 \;\Rightarrow\;$ *Config. H.100*;
- $\chi_D^{(7)} < 0$ and $N > 0 \;\Rightarrow\;$ *Config. H.98*;
- $\chi_D^{(7)} > 0 \;\Rightarrow\;$ *Config. H.97*.

In the case $\chi_B^{(7)} > 0$, the condition $\chi_D^{(7)} < 0$ implies $\beta_{10} < 0$. Moreover, if $\chi_D^{(7)} > 0$, independently of sign (N), we are led to the same configuration. So, considering the claim stated in the next paragraph, we arrive at the configuration *Config. H.93* if $\chi_D^{(7)} < 0$ and *Config. H.92* if $\chi_D^{(7)} > 0$.

We claim that, if $\chi_C^{(7)} > 0$ and $\chi_B^{(7)} > 0$ (i.e. the singular points $M_{1,2}$ are located on the same branch and the hyperbola is positioned in the sense of $\chi_B^{(7)} > 0$), we could not have the configuration with the singular points $M_{3,4}$ located inside the region delimited by both branches of the hyperbola.

Indeed, suppose the contrary, that this configuration is realizable. Then the conditions $\chi_A^{(7)} > 0$, $\chi_B^{(7)} > 0$ and $\chi_C^{(7)} > 0$ are necessary and these conditions are equivalent to

$$(h-1)^2 + 4ah > 0, \quad a - 1 > 0, \quad h < 0.$$

We assume that M_3 and M_4 are located inside the region delimited by both branches of the hyperbola. We observe that inside this region we also have the origin of coordinates (because $\Phi(0,0) = 1 - a < 0$). Therefore we must have $\Omega_3''\Omega_4'' > 0$ and $\text{sign}\,(\Omega_3'' + \Omega_4'') = \text{sign}\,(\tilde{A}) = \text{sign}\,(1-a)$. Hence the condition $\tilde{A} < 0$ must hold. However, the conditions $(h-1)^2 + 4ah > 0$ and $h < 0$ imply

$$\tilde{A} = 2ah(1-3h) + (1-h)(1-h+2h^2) \equiv \frac{1}{2}\big[(1-3h)[(h-1)^2 + 4ah] + (1-h)(h+1)^2\big] > 0,$$

and this proves our claim.

ii.2) *The subcase* $\chi_D^{(7)} = 0$. Then $a = -2(h-1)^3/(3h-1)^2$ and the singular points M_4 coalesces with the singularity M_1. We note that the hyperbola divides the plane into three regions: $\Phi(x,y) < 0$, $\Phi(x,y) > 0$ and $\Phi(x,y) = 0$, and the singular point M_3 could be located only in the first two regions. Moreover,

$$\Phi(M_3) = -\frac{(2h-1)(h-1)(h+1)^2}{h^2(3h-1)}$$

and, in this case, we have

$$\mathcal{L}_1 = x - y + \frac{3h-1-4h^2}{h(3h-1)} = 0, \quad \mathcal{L}_2 = x - y + \frac{3-5h}{3h-1} = 0.$$

We calculate

$$\chi_A^{(7)} = (h-1)^4(h+1)^4/(16(3h-1)^2), \quad \chi_B^{(7)} = -58320(2h-1)(h-1)^6(h+1)^6/(3h-1)^6,$$
$$\chi_C^{(7)} = 19440\, h(2h-1)(h-1)^6(h+1)^6/(3h-1)^6, \quad N = 9(h-1)(h+1)(x-y)^2.$$

Due to conditions (79), we have $\chi_A^{(7)} > 0$, $\text{sign}\,(\chi_B^{(7)}) = -\text{sign}\,(2h-1)$, $\text{sign}\,(\chi_C^{(7)}) = -\text{sign}\,\big(h(2h-1)\big)$ and $\text{sign}\,(N) = -\text{sign}\,\big((h-1)(h+1)\big)$. Moreover, $\mathcal{L}_1 - \mathcal{L}_2 = (h-1)(h+1)/\big[h(3h-1)\big]$.

If $\chi_B^{(7)} < 0$ (i.e $h > 1/2$), we have $\chi_C^{(7)} > 0$ and $\text{sign}\,\big(\Phi(M_3)\big) = -\text{sign}\,(\mathcal{L}_1 - \mathcal{L}_2) = -\text{sign}\,(N)$. Then we get the configuration *Config. H.89* if $N < 0$ and *Config. H.90* if $N > 0$.

In the case $\chi_B^{(7)} > 0$ (i.e $h < 1/2$), the condition $\chi_C^{(7)} < 0$ implies $N < 0$ (then $x_2 - x_3 < 0$), obtaining the configuration *Config. H.88*. If $\chi_C^{(7)} > 0$ (then $\Phi(M_3) > 0$), independently of the sign of N, we get the configuration *Config. H.87*.

iii) *The case* $\chi_A^{(7)} = 0$. Then we have two double singular points (namely $M_1 = M_2$ and $M_3 = M_4$) and a double invariant line. The condition $\chi_A^{(7)} = 0$ yields $a = -(h-1)^2/(4h)$ and hence $\chi_C^{(7)} > 0$ and $\text{sign}\,(\chi_B^{(7)}) = \text{sign}\,(\chi_D^{(7)}) = -\text{sign}\,(h)$.

We observe that, if $\chi_B^{(7)} > 0$, independently of $\text{sign}\,(\beta_{10})$ and $\text{sign}\,(N)$, we are conducted to the same configuration. Thus, we get the following conditions and configurations:

- $\chi_B^{(7)} < 0$ and $N < 0 \;\Rightarrow\;$ *Config. H.103*;
- $\chi_B^{(7)} < 0$ and $N > 0 \;\Rightarrow\;$ *Config. H.102*;
- $\chi_B^{(7)} > 0 \;\Rightarrow\;$ *Config. H.101.*

α_2) *The possibility* $\mu_0 = 0$. Then $h = 0$ and, since we also obtain $\mu_1 = 0$ and $\mu_2 = xy$, two finite singularities of systems (78) have gone to infinity and collapsed with $[1,0,0]$ and $[0,1,0]$. The remaining two finite singularities have the coordinates $M_i(x_i, y_i)$:

$$x_1 = 1,\; y_1 = -a, \qquad x_2 = a-1,\; y_2 = -2.$$

In this case, the invariant hyperbola remains the same, whereas one of the invariant lines (80) goes to infinity and hence the line of infinity $Z = 0$ becomes double (see Lemma 3). The remaining invariant line is expressed by $x - y - (a+1) = 0$.

We observe that the singular point M_1 is the intersection of the hyperbola and the straight line, whereas M_2 is generically located on the line and outside the hyperbola.

However, M_2 could be located on the hyperbola if and only if

$$\Phi(x_2, y_2) = (a-1)(a-2) = 0,$$

which is possible if and only if $a - 2 = 0$, due to conditions (79).

For systems (78) with $h = 0$, we calculate

$$\chi_B^{(7)} = 6480(a-1)(a+1)^2, \quad \chi_D^{(7)} = 3(a-2)/8.$$

We note that, if $\chi_B^{(7)} < 0$, then $a < 1$ and hence $\chi_D^{(7)} < 0$. So, we have the following conditions and configurations:

- $\chi_B^{(7)} < 0 \;\Rightarrow\;$ *Config. H.106*;
- $\chi_B^{(7)} > 0$ and $\chi_D^{(7)} < 0 \;\Rightarrow\;$ *Config. H.105*;
- $\chi_B^{(7)} > 0$ and $\chi_D^{(7)} > 0 \;\Rightarrow\;$ *Config. H.107*;
- $\chi_B^{(7)} > 0$ and $\chi_D^{(7)} = 0 \;\Rightarrow\;$ *Config. H.104*.

β) *The subcase* $B_2 = 0$. Then $a = 0$ and we arrive at the family of systems

$$\frac{dx}{dt} = 1 - 2h + (3h-1)x - hx^2 + (h-1)xy, \quad \frac{dy}{dt} = (3h-1)y - (h+1)xy + hy^2, \tag{81}$$

with the conditions

$$(h-1)(h+1)(2h-1)(2h+1)(3h-1)(3h+1) \neq 0. \tag{82}$$

These systems possess three invariant lines and an invariant hyperbola

$$\begin{gathered} \mathcal{L}_1(x,y) = x - y - 1 = 0, \quad \mathcal{L}_2(x,y) = h(x-y) + 1 - 2h = 0, \\ \mathcal{L}_3(x,y) = y = 0, \quad \Phi(x,y) = 1 - 2x + x(x-y) = 0. \end{gathered} \tag{83}$$

Since $\mu_0 = h^2$, we consider again the possibilities: $\mu_0 \neq 0$ and $\mu_0 = 0$.

β_1) *The possibility* $\mu_0 \neq 0$. Then $h \neq 0$ and the finite singularities of systems (81) are of multiplicity 4, and their coordinates are $M_i(x_i, y_i)$:

$$x_1 = 1,\; y_1 = 0, \quad x_2 = h,\; y_2 = \frac{(h-1)^2}{h}, \quad x_3 = 2h-1,\; y_3 = 2(h-1), \quad x_4 = \frac{2h-1}{h},\; y_4 = 0.$$

We observe that the singular points $M_{1,2}$ are located on the hyperbola, M_1 is located on the lines $\mathcal{L}_1 = 0$ and $\mathcal{L}_3 = 0$, M_2 is located on the line $\mathcal{L}_2 = 0$, M_3 is located on the line $\mathcal{L}_1 = 0$ and M_4 is located on the lines $\mathcal{L}_2 = 0$ and $\mathcal{L}_3 = 0$.

Concerning the position of these singularities with relation to the invariant lines and the invariant hyperbola, we have:

- the location of M_1 and M_2 on the branches of the hyperbola: $\operatorname{sign}(x_1x_2) = \operatorname{sign}(h)$;
- M_3 and M_4 could not belong to the hyperbola, since $\Phi(x_3, y_3) = 2(1-h) \neq 0$ and $\Phi(x_4, y_4) = (h-1)^2/h^2 \neq 0$, due to conditions (82);
- the position of the line $\mathcal{L}_2 = 0$ with respect to the line $\mathcal{L}_1 = 0$: $\operatorname{sign}(\mathcal{L}_1 - \mathcal{L}_2) = \operatorname{sign}\big(h(h-1)\big)$;
- the position of M_1 and M_4 on $\mathcal{L}_3 = 0$: $\operatorname{sign}(x_1 - x_4) = \operatorname{sign}\big(h(1-h)\big)$;
- the position of M_2 and M_4 on $\mathcal{L}_2 = 0$: $\operatorname{sign}(x_2 - x_4) = \operatorname{sign}(h)$;
- the position of M_1 and M_3 on $\mathcal{L}_1 = 0$: $\operatorname{sign}(x_1 - x_3) = \operatorname{sign}(1-h)$.

On the other hand, for systems (81), we calculate the invariant polynomials

$$\chi_C^{(7)} = 2160\, h(h-1)^6(2h+1)^2, \quad N = 9(h-1)(h+1)(x-y)^2.$$

We observe that the condition $\chi_C^{(7)} < 0$ implies that $\operatorname{sign}(h-1)$ is controlled and we have the unique configuration given by *Config. H.111*.

In the case $\chi_C^{(7)} > 0$, we obtain the configuration *Config. H.112* if $N < 0$ and *Config. H.110* if $N > 0$.

$\boldsymbol{\beta_2}$**)** *The possibility* $\mu_0 = 0$. Then $h = 0$ and, since we also obtain $\mu_1 = 0$ and $\mu_2 = xy$, two finite singularities of systems (78) have gone to infinity and collapsed with $[1, 0, 0]$ and $[0, 1, 0]$. The remaining two finite singularities have the coordinates $M_i(x_i, y_i)$:

$$x_1 = -1,\ y_1 = -2, \qquad x_2 = 1,\ y_2 = 0.$$

In this case, the invariant hyperbola remains the same (since it does not depend on h), whereas the invariant line $\mathcal{L}_2 = 0$ goes to infinity and hence the line of infinity $Z = 0$ becomes double and we obtain only one configuration given by *Config. H.116*.

$\boldsymbol{a_2}$**)** *The case* $\delta_4 = 0$. In this case, the condition $(h-1)^2(2h+1) + a(3h+1)^2 = 0$ yields $a = -(h-1)^2(2h+1))/(3h+1)^2$, which leads to the family of systems

$$\begin{aligned} \frac{dx}{dt} &= \frac{2(h+1)^3(1-2h)}{(3h+1)^2} + (3h-1)x - hx^2 + (h-1)xy, \\ \frac{dy}{dt} &= \frac{2(1-h)^3(2h+1)}{(3h+1)^2} + (3h-1)y - (h+1)xy + hy^2, \end{aligned} \tag{84}$$

with the conditions

$$(h-1)(h+1)(2h-1)(2h+1)(3h-1)(3h+1) \neq 0. \tag{85}$$

These systems possess two invariant lines and two invariant hyperbolas

$$\begin{gathered} \mathcal{L}_1(x, y) = x - y - \frac{4h}{3h+1} = 0, \quad \mathcal{L}_2(x, y) = x - y - \frac{5h^2 - 1}{h(3h+1)} = 0, \\ \Phi_1(x, y) = \frac{2(h+1)^3}{(3h+1)^2} - 2x + x(x-y) = 0, \quad \Phi_2(x, y) = \frac{2(1-h)^3}{(3h+1)^2} + \frac{2(3h-1)}{3h+1}x - y(x-y) = 0. \end{gathered} \tag{86}$$

Since $\mu_0 = h^2$, we consider again the possibilities: $\mu_0 \neq 0$ and $\mu_0 = 0$.

$\boldsymbol{\alpha}$) *The subcase* $\mu_0 \neq 0$. Then $h \neq 0$ and the four finite singularities of systems (84) have coordinates $M_i(x_i, y_i)$, where:

$$x_1 = \frac{(h+1)^2}{3h+1},\ y_1 = \frac{(h-1)^2}{3h+1}, \qquad x_2 = \frac{2h(h+1)}{3h+1},\ y_2 = \frac{(2h+1)(h-1)^2}{h(3h+1)},$$
$$x_3 = \frac{2(h+1)(2h-1)}{3h+1},\ y_3 = \frac{2(h-1)(2h+1)}{3h+1}, \qquad x_4 = \frac{(2h-1)(h+1)^2}{h(3h+1)},\ y_4 = \frac{2h(h-1)}{3h+1}.$$

We observe that the singular point M_1 is located on both hyperbolas and on the line $\mathcal{L}_1 = 0$, M_2 is located on the hyperbola $\Phi_1 = 0$ and on the line $\mathcal{L}_2 = 0$, M_3 is located on the line $\mathcal{L}_1 = 0$ and M_4 is located on the hyperbola $\Phi_2 = 0$ and on the line $\mathcal{L}_2 = 0$.

Concerning the position of the singular points on the lines and hyperbolas, we observe that the position of M_1 and M_3 on $\mathcal{L}_1 = 0$ is governed by $\operatorname{sign}(x_1 - x_3) = \operatorname{sign}\big((h-1)(h+1)(3h+1)\big)$ and the position of M_2 and M_4 on $\mathcal{L}_2 = 0$ is governed by $\operatorname{sign}(x_2 - x_4) = \operatorname{sign}\big(h(h-1)(h+1)(3h+1)\big)$. Moreover, the position of the hyperbolas is governed by $\operatorname{sign}\big((h-1)(h+1)\big)$.

We observe that, in the case $(h-1)(h+1) < 0$, we have $-1 < h < 1$. Then, analyzing the sign of the expression $h(3h+1)$, we verify that all the possible configurations for these values of the parameter coincide. Analogously, we obtain the same configurations by analyzing the sign of $h(3h+1)$ subjected to $(h-1)(h+1) > 0$. So, it is sufficient to only study $\operatorname{sign}\big((h-1)(h+1)\big)$.

Thus, we conclude that $\operatorname{sign}(N) = \operatorname{sign}\big((h-1)(h+1)\big)$ and we arrive at the configuration given by *Config. H.140* if $N < 0$ and *Config. H.139* if $N > 0$.

$\boldsymbol{\beta}$) *The subcase* $\mu_0 = 0$. Then $h = 0$ and two finite singular points have gone to infinity and coalesced with $[1, 0, 0]$ and $[0, 1, 0]$, since $\mu_1 = 0$ and $\mu_2 = xy$. The remaining two finite singularities have the coordinates $M_i(x_i, y_i)$, where

$$x_1 = -2,\ y_1 = -2, \qquad x_2 = 1,\ y_2 = 1.$$

In this case, both invariant hyperbolas remain the same (since they do not depend on h), whereas the invariant line $\mathcal{L}_2 = 0$ goes to infinity and hence the line of infinity $Z = 0$ becomes double (see Lemma 3) and we obtain only one configuration given by *Config. H.146*.

b) *The possibility* $\beta_7 = 0$. We recall that the conditions $\beta_6 = 0$ and $\beta_2 \neq 0$ imply $f = c \neq 0$, and then we arrive at systems (77). As earlier, via a time rescaling, we may assume $c = 1$. Moreover, the condition $\beta_7 = 0$ implies $(2h-1)(2h+1) = 0$ and, without loss of generality, we could choose $h = 1/2$, otherwise we apply the change $(x, y, t, a, b, h) \mapsto (-y, -x, -t, b, a, -h)$, which keeps the systems (77) and changes the sign of h.

Now, according to Theorem 1, for the existence of at least one hyperbola for systems (77), it is necessary and sufficient the conditions $\gamma_9 = 0$ and $\mathcal{R}_8 \neq 0$. So, we calculate $\gamma_9 = 3a/2$ and, setting $a = 0$, we obtain the 1-parameter family of systems

$$\frac{dx}{dt} = x - x^2/2 - xy/2, \quad \frac{dy}{dt} = b + y - 3xy/2 + y^2/2, \tag{87}$$

with the condition $b + 4 \neq 0$.

These systems possess three invariant lines (two of them being parallel) and an invariant hyperbola

$$\begin{gathered}\mathcal{L}_{1,2}(x,y)=(x-y)^2-2(x-y)+2b=0,\quad \mathcal{L}_3(x,y)=x=0,\\ \Phi(x,y)=4+b-4x+x(x-y)=0.\end{gathered}\tag{88}$$

We remark that, since Discriminant $\big[\mathcal{L}_{1,2}(x,y),x-y\big]=4(1-2b)$, these lines are complex (respectively real) if $2b-1<0$ (respectively $2b-1>0$).

We calculate $\delta_5=3(8-25b)/2$ and we consider two cases: $\delta_5\neq 0$ and $\delta_5=0$.

b_1) *The case* $\delta_5\neq 0$. In this case we have $25b-8\neq 0$ and hence $\Phi(x,y)=0$ (see (88)) is the unique invariant hyperbola. Since $B_1=B_2=0$ for systems (87), we calculate

$$B_3=-27\,b\,x^2(x-y)^2/4.$$

α) *The subcase* $B_3\neq 0$. Then $b\neq 0$ and, since $\mu_0=1/4$, the finite singularities $M_i(x_i,y_i)$ of systems (87) are of total multiplicity 4, and their coordinates are

$$x_{1,2}=\frac{3\pm\sqrt{1-2b}}{2},\ y_{1,2}=\frac{1\mp\sqrt{1-2b}}{2},\qquad x_{3,4}=0,\ y_{3,4}=-1\pm\sqrt{1-2b}.$$

We observe that the singular points $M_{1,2}$ are located on the hyperbola and on the invariant lines $\mathcal{L}_{1,2}=0$, whereas the singularities $M_{3,4}$ are located on the intersections of the couple of parallel invariant lines with the third one.

Considering the singular points $M_{1,2}$, we see that $x_1x_2=(b+4)/2$ and hence sign$\,(b+4)$ detects the location of these singularities on the same or different branches of the hyperbola. Moreover, the position of the hyperbola is governed by sign$\,(b+4)$.

In order to detect when the singularities $M_{3,4}$ also belong to the hyperbola, we consider (88) and we calculate

$$\big[\Phi(x_3,y_3)\big]\big[\Phi(x_4,y_4)\big]=(b+4)^2\neq 0,$$

otherwise the hyperbola splits into two lines. Thus none of the singular points M_3 or M_4 could belong to the hyperbola (88).

On the other hand, for systems (87), we calculate the invariant polynomials

$$\chi_A^{(7)}=9(1-2b)/256,\quad \chi_C^{(7)}=135(b+4)(25b-8)^2/8$$

and we conclude that sign$\,(\chi_A^{(7)})=$ sign$\,(1-2b)$ (if $2b-1\neq 0$) and due to $\delta_5\neq 0$ (i.e. $25b-8\neq 0$) we have sign$\,(\chi_C^{(7)})=$ sign$\,(b+4)$.

α_1) *The possibility* $\chi_A^{(7)}<0$. Then all four finite singularities are complex as well as the invariant lines $\mathcal{L}_{1,2}=0$ and we get the configuration shown in *Config. H.115*.

α_2) *The possibility* $\chi_A^{(7)}>0$. Then all four finite singularities and the invariant lines $\mathcal{L}_{1,2}=0$ are real and we obtain the configuration *Config. H.114* if $\chi_C^{(7)}<0$ and *Config. H.113* if $\chi_C^{(7)}>0$.

α_3) *The possibility* $\chi_A^{(7)}=0$. Then we have two double finite singular points (namely, $M_1{=}M_2$ and $M_3{=}M_4$) and also the invariant lines $\mathcal{L}_{1,2}=0$ collapse and we obtain a double invariant line. So, we arrive at the configuration *Config. H.117*.

β) *The subcase* $B_3 = 0$. Then $b = 0$ and we obtain a specific system possessing a fourth invariant line, namely $\mathcal{L}_4 = y = 0$. Then, we obtain the unique configuration *Config. H.119*.

b$_2$) *The case* $\delta_5 = 0$. Then $b = 8/25$ and again we obtain a concrete system, but now possessing a second hyperbola, namely $\Phi_2(x, y) = -4/25 - 4y/5 + y(x - y) = 0$. Moreover, we observe that, for systems (87) with $b = 8/25$, we have $B_3 = -54x^2(x - y)^2/25 \neq 0$ and hence there are no more invariant lines rather than the ones given in (88). So, we arrive at the unique configuration *Config. H.147*.

3.2.1.2.2 *The subcase* $\beta_2 = 0$. Then $c = 0$ and we obtain the systems

$$\frac{dx}{dt} = a - hx^2 + (h - 1)xy, \quad \frac{dy}{dt} = b - (h + 1)xy + hy^2. \tag{89}$$

a) *The possibility* $\beta_7 \neq 0$. Then $(2h - 1)(2h + 1) \neq 0$ and, since $\beta_{10} = -2(3h - 1)(3h + 1)$, we consider two cases: $\beta_{10} \neq 0$ and $\beta_{10} = 0$.

a$_1$) *The case* $\beta_{10} \neq 0$. Then $(3h - 1)(3h + 1) \neq 0$ and, according to Theorem 1, for the existence of at least one invariant hyperbola for systems (89), it is necessary and sufficient the conditions $\gamma_7\gamma_8 = 0$ and $\mathcal{R}_5 \neq 0$. So, we calculate

$$\gamma_7 = 8(h - 1)(h + 1)\,\mathcal{E}_1, \quad \gamma_8 = 42(h - 1)(h + 1)(3h - 1)^2(3h + 1)^2\,\mathcal{E}_2\mathcal{E}_3,$$
$$\mathcal{E}_1 = a(2h + 1) + b(2h - 1), \quad \mathcal{E}_2 = 2a(1 - h) + b(2h - 1), \quad \mathcal{E}_3 = 2b(h + 1) - a(2h + 1).$$

We observe that we could pass from the condition $\mathcal{E}_2 = 0$ to the condition $\mathcal{E}_3 = 0$ via the change $(x, y, a, b, h) \mapsto (y, x, b, a, -h)$, and any of these conditions is equivalent to $\gamma_8 = 0$. However, the condition $\mathcal{E}_1 = 0$ could not be replaced. So, we need to analyze the possibility $\gamma_7 = 0$ and then the possibility $\gamma_8 = 0$.

We calculate

$$\beta_8 = -6(4h - 1)(4h + 1), \quad \delta_2 = 2\big[(a + b)(128h^2 - 11) + (a - b)h(400h^2 - 49)\big].$$

α) *The subcase* $\beta_8^2 + \delta_2^2 \neq 0$. By Theorem 1 (see DIAGRAM 10 in this case systems (89) possess a single invariant hyperbola if and only if $\gamma_7\gamma_8 = 0$ and $\mathcal{R}_5 \neq 0$. We consider the cases $\gamma_7 = 0$ and $\gamma_8 = 0$ separately.

α₁) *The possibility* $\gamma_7 = 0$. Then $\mathcal{E}_1 = 0$ and we obtain a subfamily of systems (71) with $c = 0$. So, we arrive at the 2-parameter family of systems

$$\frac{dx}{dt} = a(2h - 1) - hx^2 + (h - 1)xy, \quad \frac{dy}{dt} = -a(2h + 1) - (h + 1)xy + hy^2, \tag{90}$$

for which $h \neq 0$, otherwise we get degenerate systems, and considering the condition $N\beta_7\beta_{10}\mathcal{R}_5(\beta_8^2 + \delta_2^2) \neq 0$, we have

$$ah(h - 1)(h + 1)(2h - 1)(2h + 1)(3h - 1)(3h + 1)(4h - 1)(4h + 1) \neq 0. \tag{91}$$

These systems possess two parallel invariant lines and the invariant hyperbola

$$\mathcal{L}_{1,2} = (x - y)^2 - 4a = 0, \quad \Phi(x, y) = a + xy = 0. \tag{92}$$

Since $\mu_0 = h^2 \neq 0$, these systems possess all four finite singularities on the finite part of the phase plane and their coordinates are $M_i(x_i, y_i)$, where

$$x_{1,2} = \pm\sqrt{a},\ y_{1,2} = \mp\sqrt{a}, \qquad x_{3,4} = \pm(2h-1)\sqrt{a},\ y_{3,4} = \pm(2h+1)\sqrt{a}.$$

We observe that the singular points $M_{1,2}$ are located on the hyperbola and on the invariant lines $\mathcal{L}_{1,2} = 0$, whereas the singularities $M_{3,4}$ are located only on the invariant lines.

Considering the singular points $M_{1,2}$, we see that $x_1x_2 = -a$ and hence sign (a) detects the location of these singularities on the same or different branches of the hyperbola. Moreover, the position of the hyperbola is also governed by sign (a).

We point out that the singularities $M_{3,4}$ could not belong to the hyperbola since

$$\big[\Phi(x_3, y_3)\big]\big[\Phi(x_4, y_4)\big] = 16a^2h^4 \neq 0,$$

due to conditions (91). On the other hand, we calculate $\chi_A^{(2)} = 80ah^6$ and we note that sign $(\chi_A^{(2)}) =$ sign (a). So, we arrive at the configurations given by *Config. H.80* if $\chi_A^{(2)} < 0$ and *Config. H.91* if $\chi_A^{(2)} > 0$.

$\boldsymbol{\alpha_2}$**)** *The possibility* $\gamma_8 = 0$. Then $\mathcal{E}_2 = 0$ and this is equivalent to the relations $a = a_1(2h-1)$ and $b = 2a_1(h-1)$, where a_1 is a new parameter. So, setting this reparametrization in (89) and replacing the old parameter a instead of a_1, we arrive at the 2-parameter family of systems

$$\frac{dx}{dt} = a(2h-1) - hx^2 + (h-1)xy, \quad \frac{dy}{dt} = 2a(h-1) - (h+1)xy + hy^2, \tag{93}$$

with the conditions

$$a(h-1)(h+1)(2h-1)(2h+1)(3h-1)(3h+1)(4h-1)(4h+1) \neq 0. \tag{94}$$

These systems possess two parallel invariant lines and the invariant hyperbola

$$\mathcal{L}_{1,2} = (x-y)^2 - a/h = 0, \quad \Phi(x, y) = a - x(x-y) = 0. \tag{95}$$

We consider the coordinates $M_i(x_i, y_i)$ of the finite singular points of systems (93):

$$x_{1,2} = \pm\sqrt{ah},\ y_{1,2} = \pm\frac{(h-1)\sqrt{ah}}{h}, \qquad x_{3,4} = \pm\frac{(2h-1)\sqrt{ah}}{h},\ y_{3,4} = \pm2\sqrt{ah}.$$

We observe that the singular points $M_{1,2}$ are located on the hyperbola and on the invariant lines $\mathcal{L}_{1,2} = 0$, whereas the singularities $M_{3,4}$ are located only on the invariant lines.

Considering the singular points $M_{1,2}$, we see that $x_1x_2 = -ah$ and hence sign (ah) detects the location of these singularities on the same or different branches of the hyperbola. Moreover, the position of the hyperbola is governed by sign (a).

We remark that the singular points $M_{3,4}$ could not belong to the hyperbola since

$$\big[\Phi(x_3, y_3)\big]\big[\Phi(x_4, y_4)\big] = \frac{a^2(3h-1)^2}{h^2} \neq 0,$$

due to conditions (94). On the other hand, we calculate

$$\chi_A^{(7)} = ah(h-1)^2(h+1)^2/4, \quad \chi_B^{(7)} = 6480\, a^3(h-1)^2(3h+1)^4$$

and we note that sign $(\chi_A^{(7)}) = $ sign (ah) and sign $(\chi_B^{(7)}) = $ sign (a).

If $\chi_A^{(7)} \neq 0$ (i.e. $h \neq 0$), we obtain the following conditions and configurations:

- $\chi_A^{(7)} < 0$ and $\chi_B^{(7)} < 0 \Rightarrow$ *Config. H.78*;
- $\chi_A^{(7)} < 0$ and $\chi_B^{(7)} > 0 \Rightarrow$ *Config. H.79*;
- $\chi_A^{(7)} > 0$ and $\chi_B^{(7)} < 0 \Rightarrow$ *Config. H.96*;
- $\chi_A^{(7)} > 0$ and $\chi_B^{(7)} > 0 \Rightarrow$ *Config. H.95.*

In the case $\chi_A^{(7)} = 0$ (i.e. $h = 0$), then we have $\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0$ and $\mu_4 = a^2x^2y^2 \neq 0$. Thus, the four finite singularities have gone to infinity and two of them coalesced with $[1, 0, 0]$ and the other two of them coalesced with $[0, 1, 0]$. Moreover, the two invariant lines $\mathcal{L}_{1,2} = 0$ have also gone to infinity and hence the line of infinity $Z = 0$ is a triple invariant line for the system, because Z^2 is a double factor of the polynomials $\mathcal{E}_1$ and $\mathcal{E}_2$ (see Lemma 3).

Now, according to the sign (a) we have different position of the hyperbola and consequently distinct configurations. So, we get the configurations shown by *Config. H.108* if $\chi_B^{(7)} < 0$ and by *Config. H.109* if $\chi_B^{(7)} > 0$.

β) *The subcase $\beta_8 = \delta_2 = 0$.* Then the condition $\beta_8 = 0$ gives $(4h-1)(4h+1) = 0$ and, without loss of generality, we may assume $h = 1/4$ due to the change $(x, y, a, b, h) \mapsto (y, x, b, a, -h)$.

We calculate

$$\delta_2 = 6(b - 3a), \quad \gamma_7 = -15(3a - b)/4, \quad \gamma_8 = 15435(3a - 5b)(3a - b)/8192$$

and hence the condition $\delta_2 = 0$ yields $b = 3a$ and then $\gamma_7 = \gamma_8 = 0$. So we obtain the 1-parameter family of systems

$$\frac{dx}{dt} = a - x^2/4 - 3xy/4, \quad \frac{dy}{dt} = 3a - 5xy/4 + y^2/4, \tag{96}$$

with the condition $a \neq 0$.

These systems possess two parallel invariant lines and two invariant hyperbolas

$$\mathcal{L}_{1,2} = (x-y)^2 + 8a = 0, \quad \Phi_1(x, y) = 2a - xy = 0, \quad \Phi_2(x, y) = 2a + x(x - y) = 0. \tag{97}$$

Since $\mu_0 = 1/16 \neq 0$, all the four finite singularities are on the finite part of the phase plane and their coordinates are $M_i(x_i, y_i)$:

$$x_{1,2} = \pm\sqrt{-2a},\ y_{1,2} = \mp\sqrt{-2a}, \qquad x_{3,4} = \pm\frac{\sqrt{-2a}}{2},\ y_{3,4} = \mp\frac{3\sqrt{-2a}}{2}.$$

We observe that the singular points $M_{1,2}$ are located on the first hyperbola $\Phi_1 = 0$, whereas the singularities $M_{3,4}$ are located on the second hyperbola $\Phi_2 = 0$. All singular points are located on the invariant lines $\mathcal{L}_{1,2} = 0$.

Considering the singular points $M_{1,2}$ (respectively $M_{3,4}$), we see that $x_1x_2 = 2a$ (respectively $x_3x_4 = a/2$) and hence $\operatorname{sign}(a)$ detects the location of these singularities to be on the same or different branches of the hyperbolas that they are located on. Moreover, the position of the hyperbola is also governed by $\operatorname{sign}(a)$.

We remark that the singular points $M_{1,2}$ (respectively $M_{3,4}$) could not belong to the hyperbola $\Phi_2 = 0$ (respectively $\Phi_1 = 0$) since

$$\big[\Phi_2(x_1, y_1)\big]\big[\Phi_2(x_2, y_2)\big] = 4a^2 \neq 0, \quad \big[\Phi_1(x_3, y_3)\big]\big[\Phi_1(x_4, y_4)\big] = a^2/4 \neq 0,$$

due to $a \neq 0$.

On the other hand, we calculate

$$\chi_A^{(7)} = -225a/2048$$

and we note that $\operatorname{sign}(\chi_A^{(7)}) = -\operatorname{sign}(a)$. So, we get the configurations shown by *Config. H.143* if $\chi_A^{(7)} < 0$ and *Config. H.141* if $\chi_A^{(7)} > 0$.

$\boldsymbol{a_2}$) *The case* $\beta_{10} = 0$. Then $(3h-1)(3h+1) = 0$ and, without loss of generality, we may assume $h = 1/3$, since the case $h = -1/3$ could be brought to the case $h = 1/3$ via the change $(x, y, a, b, h) \mapsto (y, x, b, a, -h)$. So, we arrive at the systems

$$\frac{dx}{dt} = a - x^2/3 - 2xy/3, \quad \frac{dy}{dt} = b - 4xy/3 + y^2/3. \tag{98}$$

with the condition $a \neq 0$, possessing a pair of parallel invariant lines and a couple of invariant hyperbolas with parallel asymptotes

$$\mathcal{L}_{1,2}(x, y) = (x-y)^2 - 3(a-b) = 0, \quad \Phi_{1,2}(x, y) = 3a \pm \sqrt{3(4a-b)}x + x(x-y) = 0. \tag{99}$$

In accordance to Theorem 1, we have to analyze the following subcases: $\gamma_7 \neq 0$ and $\gamma_7 = 0$ and we calculate

$$\gamma_7 = -65(5a-b)/27, \quad \gamma_{10} = 8(4a-b)/27.$$

$\boldsymbol{\alpha}$) *The subcase* $\gamma_7 \neq 0$. Thenb we could not have other invariant hyperbolas rather than the ones in (99). Moreover, the hyperbolas (99) are complex if $\gamma_{10} < 0$, real if $\gamma_{10} > 0$ and they coincide if $\gamma_{10} = 0$. Then, we consider two possibilities: $\gamma_{10} < 0$ and $\gamma_{10} \geq 0$.

$\boldsymbol{\alpha_1}$) *The possibility* $\gamma_{10} < 0$. bThen the hyperbolas (99) are complex. In this case, we set a new parameter $v \neq 0$ satisfying $4a - b = -3v^2$, which yields $b = 4a + 3v^2$ and we obtain the 2-parameter family of systems

$$\frac{dx}{dt} = a - x^2/3 - 2xy/3, \quad \frac{dy}{dt} = 4a + 3v^2 - 4xy/3 + y^2/3, \tag{100}$$

with the condition $av \neq 0$, possessing the invariant lines and invariant hyperbolas

$$\mathcal{L}_{1,2}(x, y) = (x-y)^2 + 9(a+v^2) = 0, \quad \Phi_{1,2}(x, y) = 3a \pm 3ivx + x(x-y) = 0. \tag{101}$$

We calculate

$$\mu_0 = 1/9, \quad B_1 = 0, \quad B_2 = -512\,a(4a+3v^2)(x-y)^4$$

and we consider two cases: $B_2 \neq 0$ and $B_2 = 0$.

i) *The case* $B_2 \neq 0$. Then there are no other invariant lines rather than $\mathcal{L}_{1,2} = 0$ in (101). We calculate

$$\mu_0 = 1/9 \neq 0, \quad \mathbf{D} = -4096\, v^4(a+v^2)^2/3, \quad \mathbf{S} = 256\, v^2(a+v^2)(x-y)^2(2x+y)^2/2187,$$
$$\mathbf{R} = -16\big[(4a+5v^2)x^2 + 2(2a+v^2)xy + (a+2v^2)y^2\big]/81, \quad \mathbf{T} = -81\mathbf{R}\,\mathbf{S}/32.$$

We claim that all four finite singular points are complex. Indeed, if $a + v^2 > 0$, we observe that

$$\text{Discriminant}\,\big[\mathbf{R}, x\big] = -1024v^2(a+v^2)y^2/729 < 0, \quad \text{Coefficient}\,\big[\mathbf{R}, y^2\big] = -16(a+2v^2)/81 < 0$$

and hence $\mathbf{R} < 0$. Since $\mathbf{D} < 0$, by Proposition 1 all four finite singularities of systems (100) are complex.

Now, if $a + v^2 < 0$, then $\mathbf{D} < 0$ and $\mathbf{S} < 0$, and by Proposition 1 all four finite singularities of systems (100) are complex.

Finally, if $a + v^2 = 0$, then $\mathbf{D} = \mathbf{T} = 0$ and we have two collisions of finite singular points, i.e. we have two double singular points. As in any case we have only complex singularities, these double singular points are also complex. So, our claim is proved.

We calculate $\chi_A^{(7)} = -16(a+v^2)/81$ and we note that $\text{sign}\,(\chi_A^{(7)}) = -\text{sign}\,(a+v^2)$.

If $\chi_A^{(7)} < 0$, then the invariant lines are also complex and we get the configuration *Config. H.144*. In the case $\chi_A^{(7)} > 0$ the invariant lines are real and we arrive at the configuration *Config. H.145*. If $\chi_A^{(7)} = 0$, then the invariant lines collapse and become double, which leads to configuration *Config. H.153*.

ii) *The case* $B_2 = 0$. Then $4a + 3v^2 = 0$ and systems (100) have a third invariant line $y = 0$ and the lines $\mathcal{L}_{1,2} = 0$ are complex. So, we get the configuration *Config. H.151*.

$\boldsymbol{\alpha_2}$**)** *The possibility* $\gamma_{10} > 0$. In this case, we set the new parameter $v \neq 0$ satisfying $4a - b = 3v^2$, which yields $b = 4a - 3v^2$ and we obtain the 2-parameter family of systems

$$\frac{dx}{dt} = a - x^2/3 - 2xy/3, \quad \frac{dy}{dt} = 4a - 3v^2 - 4xy/3 + y^2/3, \tag{102}$$

with the condition $a \neq 0$, possessing the invariant lines and invariant hyperbolas

$$\mathcal{L}_{1,2}(x,y) = (x-y)^2 + 9(a-v^2) = 0, \quad \Phi_{1,2}(x,y) = 3a \pm 3vx + x(x-y) = 0. \tag{103}$$

Remark 10. *We remark that, the condition* $v = 0$ *for systems* (102) *is equivalent to* $\gamma_{10} = 0$.

We calculate

$$\mu_0 = 1/9, \quad B_1 = 0, \quad B_2 = -512\, a(4a-3v^2)(x-y)^4$$

and we consider two cases: $B_2 \neq 0$ and $B_2 = 0$.

i) *The case* $B_2 \neq 0$. Then there is no other invariant line rather than $\mathcal{L}_{1,2} = 0$ in (103). Since $\mu_0 \neq 0$, all four finite singularities of systems (102) are on the finite part of the phase plane and their coordinates are $M_i(x_i, y_i)$, where

$$x_{1,2} = -v \pm \sqrt{v^2-a},\ y_{1,2} = -v \mp 2\sqrt{v^2-a}, \quad x_{3,4} = v \pm \sqrt{v^2-a},\ y_{3,4} = v \mp 2\sqrt{v^2-a}.$$

We observe that the singular points $M_{1,2}$ are located on the first hyperbola $\Phi_1 = 0$ and on the invariant lines $\mathcal{L}_{1,2} = 0$, whereas the singularities $M_{3,4}$ are located on the second hyperbola $\Phi_2 = 0$ and on the invariant lines $\mathcal{L}_{1,2} = 0$.

Considering the pairs of singular points $M_{1,2}$ and $M_{3,4}$, we see that $x_1x_2 = x_3x_4 = a$ and hence $\operatorname{sign}(a)$ detects the location of these singularities to be on the same or different branches of the respective hyperbola they are located on.

We remark that the singular points $M_{1,2}$ (respectively $M_{3,4}$) could belong to the hyperbola $\Phi_2 = 0$ (respectively $\Phi_1 = 0$) if and only if

$$\big[\Phi_2(x_1, y_1)\big]\big[\Phi_2(x_2, y_2)\big] = 36av^2 = 0, \qquad \big[\Phi_1(x_3, y_3)\big]\big[\Phi_1(x_4, y_4)\big] = 36av^2 = 0,$$

which are equivalent to $v = 0$. However by Remark 10 and the condition $\gamma_{10} > 0$ we have $v \neq 0$.

On the other hand, we calculate

$$\chi_A^{(7)} = 16(v^2 - a)/81, \quad \chi_C^{(3)} = 17039360\, a(a + 3v^2)^2/9$$

and we conclude that $\operatorname{sign}(\chi_A^{(7)}) = \operatorname{sign}(v^2 - a)$ and $\operatorname{sign}(\chi_C^{(3)}) = \operatorname{sign}(a)$.

Since $v \neq 0$, the invariant hyperbolas $\Phi_{1,2} = 0$ are distinct. We observe that the condition $\chi_A^{(7)} \leq 0$ implies $a > 0$ (as $v \neq 0$) and consequently, $\chi_C^{(3)} > 0$. Moreover, if $\chi_A^{(7)} = 0$, then both invariant lines coalesce and we obtain the double invariant line $(x - y)^2 = 0$. So, we arrive at the following conditions and configurations:

- $\chi_A^{(7)} < 0 \;\Rightarrow\;$ *Config. H.142*;
- $\chi_A^{(7)} > 0$ and $\chi_C^{(3)} < 0 \;\Rightarrow\;$ *Config. H.137*;
- $\chi_A^{(7)} > 0$ and $\chi_C^{(3)} > 0 \;\Rightarrow\;$ *Config. H.138*;
- $\chi_A^{(7)} = 0 \;\Rightarrow\;$ *Config. H.152.*

ii) *The case* $B_2 = 0$*.* Then $a = 3v^2/4$ and we have a third invariant line $\mathcal{L}_3(x, y) = y = 0$ and the previous two lines could be factored as $\mathcal{L}_1(x, y) = 2x - 2y + 3v = 0$ and $\mathcal{L}_2(x, y) = 2x - 2y - 3v = 0$.

Since $a > 0$, we have

$$\chi_A^{(7)} = 4v^2/81 > 0, \quad \chi_C^{(3)} = 19968000v^2 > 0$$

and we obtain the unique configuration *Config. H.149.*

$\boldsymbol{\alpha_3}$) *The possibility* $\gamma_{10} = 0$. In this case according to Remark 10 we have $v = 0$, and then $\chi_A^{(7)} = -16a/81 \neq 0$. In this case, both hyperbola collapse and we get a double hyperbola. Furthermore, the singularities collapse two by two and we have two double singular points (namely $M_1 = M_3$ and $M_2 = M_4$).

It remains to observe that the condition $\chi_A^{(7)} < 0$ (respectively $\chi_A^{(7)} > 0$) implies $\chi_C^{(3)} > 0$ (respectively $\chi_C^{(3)} < 0$). So, we get the configuration *Config. H.155* if $\chi_A^{(7)} < 0$ and *Config. H.154* if $\chi_A^{(7)} > 0$.

$\boldsymbol{\beta}$) *The subcase* $\gamma_7 = 0$. Then $b = 5a$ and we arrive at the 1-parameter family of systems

$$\frac{dx}{dt} = a - x^2/3 - 2xy/3, \quad \frac{dy}{dt} = 5a - 4xy/3 + y^2/3, \tag{104}$$

with the condition $a \neq 0$.

These systems possess a couple of parallel invariant lines, a pair of invariant hyperbolas with parallel asymptotes presented in (99) and a third hyperbola

$$\begin{gathered}\mathcal{L}_{1,2}(x,y) = (x-y)^2 + 12a = 0,\\ \Phi_{1,2}(x,y) = 3a \pm \sqrt{-3a}\,x + x(x-y) = 0, \quad \Phi_3(x,y) = xy - 3a = 0.\end{gathered} \tag{105}$$

Since $B_1 = 0$ and $B_2 = -2560a^2(x-y)^4 \neq 0$, systems (104) could not possess other invariant lines rather than the ones in (105). Moreover, we have $\mu_0 = 1/9 \neq 0$ and all the four singularities are on the finite part of the phase plane with coordinates $M_i(x_i, y_i)$, where

$$x_{1,2} = \pm\sqrt{-3a},\ y_{1,2} = \mp\sqrt{-3a}, \quad x_{3,4} = \pm\frac{\sqrt{-3a}}{3},\ y_{3,4} = \mp\frac{5\sqrt{-3a}}{3}.$$

We observe that all four singular points are located on the invariant lines and also: M_1 is located on the hyperbolas $\Phi_2 = 0$ and $\Phi_3 = 0$, M_2 is located on the hyperbolas $\Phi_1 = 0$ and $\Phi_3 = 0$, M_3 is located on the hyperbola $\Phi_1 = 0$ and M_4 is located on the hyperbola $\Phi_2 = 0$.

Concerning the position of the singularities on the hyperbolas, we have

- the position of M_2 and M_3 on $\Phi_1(x,y) = 0$ is controlled by $\text{sign}\,(x_2x_3) = \text{sign}\,(a)$;
- the position of M_1 and M_4 on $\Phi_2(x,y) = 0$ is controlled by $\text{sign}\,(x_1x_4) = \text{sign}\,(a)$;
- the position of M_1 and M_2 on $\Phi_3(x,y) = 0$ is controlled by $\text{sign}\,(x_2x_3) = \text{sign}\,(3a)$.

We also point out that due to $a \neq 0$, the singularities could be located on the hyperbolas only as it is described above.

We remark that, if $a > 0$, then the four singularities are complex as well as the pair of invariant hyperbolas $\Phi_{1,2}(x,y) = 0$ and the couple of invariant lines $\mathcal{L}_{1,2}(x,y) = 0$.

On the other hand, we calculate $\gamma_{10} = -8a/27$ and we conclude that $\text{sign}\,(\gamma_{10}) = -\text{sign}\,(a)$. So, we arrive at the configuration *Config. H.159* if $\gamma_{10} < 0$ and *Config. H.158* if $\gamma_{10} > 0$.

b) *The possibility* $\beta_7 = 0$. Then $(2h-1)(2h+1) = 0$ and, without loss of generality as earlier, we may assume $h = 1/2$. So, we obtain the systems

$$\frac{dx}{dt} = a - x^2/2 - xy/2, \quad \frac{dy}{dt} = b - 3xy/2 + y^2/2. \tag{106}$$

According to Theorem 1, the condition $\gamma_7 = 0$ is necessary and sufficient for the existence of invariant hyperbolas for systems (106). Moreover, this condition implies the existence of two such hyperbolas.

We calculate $\gamma_7 = -12a = 0$ and we obtain the 1-parameter family of systems

$$\frac{dx}{dt} = -x^2/2 - xy/2, \quad \frac{dy}{dt} = b - 3xy/2 + y^2/2. \tag{107}$$

with the condition $b \neq 0$.

These systems possess three invariant lines and two invariant hyperbolas

$$\begin{aligned} \mathcal{L}_{1,2}(x,y) &= (x-y)^2+2b=0, \quad \mathcal{L}_3(x,y)=x=0,\\ \Phi_1(x,y) &= b-2xy=0, \quad \Phi_2(x,y)=b+x(x-y)=0. \end{aligned} \tag{108}$$

For systems (107) we calculate $B_1=B_2=0$ and $B_3=-27bx^2(x-y)^2/4\neq 0$ and therefore by Lemma 2 systems (107) could not posses other invariant lines rather than the ones in (108). Since $\mu_0=1/4\neq 0$, these systems have finite singularities of total multiplicity 4 with coordinates $M_i(x_i,y_i)$, where

$$x_{1,2}=\pm\frac{\sqrt{-2b}}{2},\ y_{1,2}=\mp\frac{\sqrt{-2b}}{2}, \quad x_{3,4}=0,\ y_{3,4}=\pm\sqrt{-2b}.$$

We observe that the singular points $M_{1,2}$ are located on the two hyperbolas and on the lines $\mathcal{L}_{1,2}=0$ and the singularities $M_{3,4}$ are located on the three invariant lines.

Moreover, due $b\neq 0$ we deduce that the singular points $M_{3,4}$ could not belong to the hyperbolas. By the same argument the singular points $M_{1,2}$ could not belong to the invariant line $\mathcal{L}_3=0$.

Since $x_1x_2=b/2$, the position of the singular points $M_{1,2}$ on the hyperbola is governed by sign (b), as well as the position of the invariant hyperbolas.

We calculate $\chi_A^{(7)}=-9b/128$ and we conclude that sign $(\chi_A^{(7)})=$ sign (b).

It is worth mentioning that, if $b>0$, then all four singular points are complex as well as the couple of invariant lines $\mathcal{L}_{1,2}=0$. So, we get the configuration *Config. H.150* if $\chi_A^{(7)}<0$ and *Config. H.148* if $\chi_A^{(7)}>0$.

3.2.2 The possibility $N=0$

Since for systems (7) we have $\theta=-(g-1)(h-1)(g+h)/2=0$, we observe that the condition

$$N=(g-1)(g+1)x^2+2(g-1)(h-1)xy+(h-1)(h+1)y^2=0$$

implies the vanishing of two factors of θ. Then, without loss of generality, we may assume $g=1=h$, otherwise in the case $g+h=0$ and $g-1\neq 0$ (respectively $h-1\neq 0$), we apply the change $(x,y,g,h)\mapsto(-y,x-y,1-g-h,g)$ (respectively $(x,y,g,h)\mapsto(y-x,-x,h,1-g-h)$) which preserves the form of such systems.

So, $g=h=1$ and due to an additional translation we arrive at the systems

$$\frac{dx}{dt}=a+dy+x^2, \quad \frac{dy}{dt}=b+ex+y^2, \tag{109}$$

for which we calculate

$$\beta_1=4de, \quad \beta_2=-2(d+e).$$

According to Theorem 1, a necessary condition for the existence of hyperbolas for these systems is $\beta_1=0$. This condition is equivalent to $de=0$ and, without loss of generality, we may assume $e=0$, due to the change $(x,y,a,b,d,e)\mapsto(y,x,b,a,e,d)$.

Then $\beta_2=-2d$ and we analyze two cases: $\beta_2\neq 0$ and $\beta_2=0$.

3.2.2.1 *The case $\beta_2 \neq 0$.* Then $d \neq 0$ and via the rescaling $(x, y, t) \mapsto (4d\,x, 4d\,y, t/(4d))$, we may assume $d = 4$.

In this case, since $\beta_1 = 0$, according to Theorem 1 the conditions $\gamma_{11} = 0$ and $\mathcal{R}_9 \neq 0$ are necessary and sufficient for the existence of one invariant hyperbola.

We calculate $\gamma_{11} = -64(a - 4b + 1)$ and, setting $a = 4b - 1$, we obtain the 1-parameter family of systems

$$\frac{dx}{dt} = 4b - 1 + 4y + x^2, \quad \frac{dy}{dt} = b + y^2, \tag{110}$$

for which $\mathcal{R}_9 = 40(b+1) \neq 0$.

These systems possess the invariant lines and the invariant hyperbola

$$\mathcal{L}_{1,2}(x, y) = y^2 + b = 0, \quad \Phi(x, y) = b - 1 - x + 3y + y(x - y) = 0. \tag{111}$$

Since $B_1 = 0$ and $B_2 = -124416(b+1)y^4 \neq 0$, systems (110) could not possess other invariant lines rather than the ones in (111). Moreover, $\mu_0 = 1 \neq 0$ implies that these systems possess finite singularities $M_i(x_i, y_i)$ of total multiplicity four and their coordinates are

$$x_{1,2} = -1 \pm 2\sqrt{-b},\ y_{1,2} = \pm\sqrt{-b}, \quad x_{3,4} = 1 \pm 2\sqrt{-b},\ y_{3,4} = \mp\sqrt{-b}.$$

We observe that the singular points $M_{1,2}$ are located on the hyperbola and on the lines, whereas the singularities $M_{3,4}$ are located on the invariant lines.

Moreover, at least one of the singular points $M_{3,4}$ could belong to the hyperbola if an only if

$$\big[\Phi(x_3, y_3)\big]\big[\Phi(x_4, y_4)\big] = 4(b+1)(4b+1) = 0,$$

i.e. if and only if $4b + 1 = 0$.

Since $x_1 x_2 = 4(4b+1)$, the position of the singular points $M_{1,2}$ on the hyperbola is governed by $\text{sign}\,(4b+1)$, while the position of the invariant hyperbola is governed by $\text{sign}\,(b)$.

We calculate

$$\chi_A^{(8)} = -80\,b, \quad \chi_D^{(8)} = 80(4b+1), \quad \mathcal{R}_9 = 40(b+1)$$

and we conclude that $\text{sign}\,(\chi_A^{(8)}) = -\text{sign}\,(b)$ and $\text{sign}\,(\chi_D^{(8)}) = \text{sign}\,(4b+1)$.

We observe that, if $b > 0$, then all four singularities and the invariant lines are complex. So, we arrive at the unique configuration *Config. H.79* if $\chi_A^{(8)} < 0$.

If $\chi_A^{(8)} > 0$, we get the following conditions and configurations:

- $\mathcal{R}_9 < 0$ *Config. H.96*;
- $\mathcal{R}_9 > 0$ and $\chi_D^{(8)} < 0 \Rightarrow$ *Config. H.93*;
- $\mathcal{R}_9 > 0$ and $\chi_D^{(8)} > 0 \Rightarrow$ *Config. H.92*;
- $\mathcal{R}_9 > 0$ and $\chi_D^{(8)} = 0 \Rightarrow$ *Config. H.87.*

If $\chi_A^{(8)} = 0$, then $b = 0$ and the invariant lines collapse and become double. Moreover, the singularity M_1 coalesces with M_3, and so does M_2 with M_4, and we have two double singular points, leading us to the configuration *Config. H.101.*

3.2.2.2 *The case* $\beta_2 = 0$. Then $d = 0$ and, according to Theorem 1, the condition $\gamma_{12} = 0$ leads to the existence of only one invariant hyperbola, whereas the condition $\gamma_{13} = 0$ leads to the existence of an infinite number of such hyperbolas.

We calculate

$$\gamma_{12} = -128(a-4b)(4a-b), \quad \gamma_{13} = 4(a-b).$$

3.2.2.2.1 *The subcase* $\gamma_{12} = 0$. Then $(a-4b)(4a-b) = 0$ and, via the change $(x, y, a, b) \mapsto (y, x, b, a)$, we may assume $b = 4a$ and we arrive at the 1-parameter family of systems

$$\frac{dx}{dt} = a + x^2, \quad \frac{dy}{dt} = 4a + y^2, \tag{112}$$

with the condition $a \neq 0$.

These systems possess two couples of parallel invariant lines and the invariant hyperbola

$$\mathcal{L}_{1,2}(x,y) = x^2 + a = 0, \quad \mathcal{L}_{3,4}(x,y) = y^2 + 4a = 0, \quad \Phi(x,y) = a - x(x-y) = 0. \tag{113}$$

Since $B_1 = B_2 = 0$ and $B_3 = 36ax^2y^2 \neq 0$, systems (112) could not possess other invariant lines rather than the ones in (113). Moreover as $\mu_0 = 1 \neq 0$, by Lemma 1 the above systems possess finite singularities $M_i(x_i, y_i)$ of total multiplicity four and their coordinates are

$$x_{1,2} = \pm\sqrt{-a},\ y_{1,2} = \pm 2\sqrt{-a}, \quad x_{3,4} = \pm\sqrt{-a},\ y_{3,4} = \mp 2\sqrt{-a}.$$

We observe that all four singularities belong to the lines $\mathcal{L}_{1,2,3,4} = 0$. Moreover, the singular points $M_{1,2}$ are located on the hyperbola, whereas the singular points $M_{3,4}$ could not belong to the hyperbola due to $a \neq 0$.

Since $x_1x_2 = 4a$, the position of the singular points $M_{1,2}$ on the hyperbola is governed by $\mathrm{sign}\,(a)$, as well as the position of the invariant hyperbola.

We calculate $\chi_A^{(2)} = -80a$ and we conclude that $\mathrm{sign}\,(\chi_A^{(2)}) = -\mathrm{sign}\,(a)$.

Since in the case $a > 0$ all four singularities and the invariant lines are complex, we arrive at the configuration *Config. H.120* if $\chi_A^{(2)} < 0$ and *Config. H.118* if $\chi_A^{(2)} > 0$.

3.2.2.2.2 *The subcase* $\gamma_{13} = 0$. Then $b = a$ and we arrive at the 1-parameter family of systems

$$\frac{dx}{dt} = a + x^2, \quad \frac{dy}{dt} = a + y^2, \tag{114}$$

with the condition $a \neq 0$.

These systems possess five invariant lines and the family of invariant hyperbolas

$$\begin{gathered}\mathcal{L}_{1,2}(x,y) = x^2 + a = 0, \quad \mathcal{L}_{3,4}(x,y) = y^2 + a = 0, \quad \mathcal{L}_5(x,y) = x - y = 0,\\ \Phi(x,y) = 2a - r(x-y) + 2xy = 0,\ r \in \mathbb{C}.\end{gathered} \tag{115}$$

Since $\mu_0 = 1 \neq 0$ the above systems possess finite singularities $M_i(x_i, y_i)$ of total multiplicity four and their coordinates are

$$x_{1,2} = \pm\sqrt{-a},\ y_{1,2} = \pm\sqrt{-a}, \quad x_{3,4} = \pm\sqrt{-a},\ y_{3,4} = \mp\sqrt{-a}.$$

We observe that all four singularities belong to the lines $\mathcal{L}_{1,2,3,4} = 0$. Moreover, the singular points $M_{1,2}$ are located on the hyperbolas for any $r \in \mathbb{C}$ and on the line $\mathcal{L}_5 = 0$.

The sign (a) distinguishes if the singularities are either real, or complex, or coinciding (if $a = 0$). Since $\mathcal{R}_9 = 16a$, we conclude that sign $(\mathcal{R}_9) =$ sign (a).

In the case $a \neq 0$, we could assume $a = 1$ if $a > 0$ and $a = -1$ if $a < 0$, due to a rescaling. So, we arrive at the configuration *Config. H.160* if $\mathcal{R}_9 < 0$, *Config. H.161* if $\mathcal{R}_9 > 0$ and *Config. H.162* if $\mathcal{R}_9 = 0$.

The proof of statement *(B)* of the Main Theorem is completed because because all the cases have been examined. ∎

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