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GALOIS ALGEBRAS I: STRUCTURE THEORY

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ABSTRACT. We introduce a concept and develop a theory of Galois subalgebras in skew semigroup rings. Proposed approach has a strong impact on the representation theory, first of all the theory of Harish-Chandra modules, of many infinite dimensional algebras including the Generalized Weyl algebras, the universal enveloping algebras of reductive Lie algebras, their quantizations, Yangians etc. In particular, we show how some of these algebras can be embedded into skew (semi)group rings. As one of the applications of the developed technique we reprove the Gelfand-Kirillov conjecture for the universal enveloping algebra of gl_n and verify it for the Yangians of gl_2 and for the quantization of gl_2.

CONTENTS

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1. Introduction

Let Γ be an integral domain and $U \supset \Gamma$ an associative non-commutative algebra over a base field $\Bbbk$. A motivation for the study of such pairs "algebra-subalgebra" comes from the "non-commutative algebraic geometry", whose algebraic part studies the structure of certain important non-commutative rings, and from the representation theory of Lie algebras, where U is the universal enveloping algebra of a reductive finite dimensional Lie algebra and Γ is its commutative subalgebra. For instance, the case when Γ is the universal enveloping algebra of a Cartan subalgebra leads to the theory of Harish-Chandra modules with respect to this Cartan algebra (so-called weight modules). Even in this category a classification of irreducible modules is only known in the case of finite-dimensional weight spaces (cf. [Fe] and [Ma]).

A more general class of Gelfand-Tsetlin representations was studied in [DFO1]. This class is based on a natural generalization of Gelfand-Tsetlin basis for finite-dimensional representations of simple classical Lie algebras [GTs], [Zh], [M]. These representations are associated to a pair (U, Γ), where U is the universal enveloping algebra and Γ is a certain maximal commutative subalgebra of U, called *Gelfand-Tsetlin subalgebra*. Such pairs were considered in [FM] in the connection with the solutions of Euiler equation, in [Vi] in the connection with subalgebras of maximal Gelfand-Kirillov dimension in the universal enveloping algebra of a simple Lie algebra, in [KW] in the connection with quantum mechanics, and also in [Gr] in the connection with general hypergeometric functions on the Lie group $GL(n, \mathbb{C})$.

A similar approach was used by Okunkov and Vershik in their study of the representations of the symmetric group S_n [OV], with U being the group algebra of S_n and Γ being the maximal commutative subalgebra generated by the Jucys-Murphy elements

$$X_i = (1i) + \ldots + (i-1i), \quad i = 1, \ldots, n.$$

The elements of $\operatorname{Specm} \Gamma$ parametrize irreducible representations of U. Another recent advance in the representation theory of Yangians ([FMO]) is also based on similar techniques.

What is the intrinsic reason of the existence of Gelfand-Tsetlin formulae and of the successful study of Gelfand-Tsetlin representations of various classes of algebras? An attempt to understand the phenomena related the Gelfand-Tsetlin formulae was the paper [DFO2] where the notion of Harish-Chandra subalgebra of an associative algebra an the corresponding notion of a Harish-Chandra module were introduced.

Current paper can be viewed on one hand as a development of the ideas of [DFO2] in the "semi-commutative case" (non-commutative algebra and commutative subalgebra) and, on the other hand, as an attempt to understand the role of skew group algebras in the representation theory of infinite dimensional algebras (e.g., see [Ba]). We make an observation that the Gelfand-Tsetlin formulae for gl_n define an embedding of the corresponding universal enveloping algebra into a skew group algebra of a free abelian group over some field of rational functions L (see also [Kh]). The remarkable fact is that this field L is a Galois extension of the field of fractions of the corresponding Gelfand-Tsetlin subalgebra of the universal enveloping algebra.

This simple observation has some amazing consequences. We show that many properties of representations of enveloping algebras can be considered in a much broader situation. This leads to a concept of *Galois algebras* defined as certain subalgebras in skew group rings. These algebras can be viewed as *hidden skew group algebras.* They are endowed with a Gelfand-Tsetlin subalgebra and posses an analogue of Gelfand-Tsetlin formulae. In the framework of "non-commutative algebraic geometry" a class of non-commutative Galois algebras can be effectively studied using the techniques of affine geometry and commutative algebra.

Let Γ be a commutative finitely generated domain, K the field of fractions of Γ, $K \subset L$ a finite Galois extension, $G = G(L/K)$ the corresponding Galois group, $\mathcal{M} \subset \mathrm{Aut}\, L$ a submonoid. Assume that G belongs to the normalizer of $\mathcal{M}$ in $\mathrm{Aut}\, L$ and for $m_1, m_2 \in \mathcal{M}$ their double G-cosets coincide if and only if $m_1 = m_2^g$ for some $g \in G$, where $m_2^g = gmg^{-1}$. If $\mathcal{M}$ is a group the last condition can be rewritten as $\mathcal{M} \cap G = \{e\}$. The action of G on $\mathcal{M}$ skew commutes with its action on L, hence G acts on the skew group algebra $L * \mathcal{M}$ by isomorphisms: $g \cdot (am) = (g \cdot a)(g \cdot m)$. Let $L * \mathcal{M}^G$ be the subalgebra of G-invariants in $L * \mathcal{M}$.

We will say that an associative algebra U is a Γ*-algebra,* provided there is a fixed embedding $i : \Gamma \to U$. The Γ-algebra U will be an *algebra over* Γ if the image of i belongs to the center of U.

Definition 1. *A finitely generated* Γ*-subalgebra* $U \subset L * \mathcal{M}^G$ *is called a* Galois Γ-algebra *if* $KU = UK = L * \mathcal{M}^G$.

A concept of a Galois Γ-algebra can be viewed as a non-commutative version of a notion of Γ-order in $L * \mathcal{M}^G$.

Sometimes we will also say *Galois algebra with respect to* Γ in this case. If Γ is fixed then we simply say that U is a *Galois algebra.* In this case Γ is a maximal commutative subalgebra in U and the center of U coincides with $\mathcal{M}$-invariants in $U \cap K$ (Theorem 4.1). Moreover, the set $S = \Gamma \setminus \{0\}$ is an Ore multiplicative set (both from the left and from the right) and the corresponding localizations $U[S^{-1}]$ and $[S^{-1}]U$ are canonically isomorphic to $L * \mathcal{M}^G$ (Proposition 4.1).

If a Galois algebra U allows the left and the right skew-field of fractions $\mathcal{U}$ then the center of $\mathcal{U}$ coincides with the invariants $K^{\mathcal{M}}$ (Theorem 5.1).

The algebra $L * \mathcal{M}^G$ (and, hence, $U[S^{-1}]$, $[S^{-1}]U$) has the canonical decomposition into the sum of pairwise non-isomorphic finite dimensional left (or right) K-vector spaces. In particular, in the case of $U(\mathrm{gl}_n)$ these bimodules are parametrized by the orbits of the action the group $S_1 \times S_2 \times \cdots \times S_{n-1}$ on $\mathbb{Z}^1 \oplus \mathbb{Z}^2 \oplus \cdots \oplus \mathbb{Z}^{n-1}$, where S_i acts by permutation of coordinates in $\mathbb{Z}^i$, $i = 1, 2, \dots, n-1$.

How big is the class of Galois algebras? We note that any commutative algebra is Galois over itself. Moreover if $\Gamma \subset U \subset K$ and U is finitely generated over Γ, then U is a Galois Γ-algebra.

In Section 4.2 we define Galois algebras by generators and relations starting from so called *balanced* Γ-bimodules. This approach based on the bimodule theory allows to construct many natural examples of Galois algebras, all of which admit similar techniques, developed in the paper, to study their representations. This deep relation between the Galois algebras and balanced Γ-bimodules will be discussed in a subsequent paper.

Another important tool in the investigation of Galois algebras is their Gelfand-Kirillov dimension which is studied in Section 6. Using this technique we show in Section 7 that the following algebras are Galois subalgebras in the corresponding skew-group rings:

- Generalized Weyl algebras over integral domains with infinite order automorphisms which include many classical algebras, such as n-th Weyl algebra A_n, quantum plane, q-deformed Heisenberg algebra, quantized Weyl algebras, Witten-Woronowicz algebra among the others [Ba], [BavO];
- The universal enveloping algebra $U(\mathrm{gl}_n)$ is a Galois algebra with respect to its Gelfand-Tsetlin subalgebra;
- Restricted Yangians $\mathrm{Y}_p(\mathrm{gl}_2)$ for gl_2 with respect to its Gelfand-Tsetlin subalgebras [FMO];
- Quantized enveloping algebra $\breve{U}_q(\mathrm{gl}_2)$ with respect to Gelfand-Tsetlin subalgebra [KS].

If the skew group algebra $L * \mathcal{M}$ is a domain which satisfies the left and the right Ore conditions then the skew field of fractions of U coincides with G-invariants of the skew field of fractions of $L * \mathcal{M}$ (Corollary 5.2). In particular case of $U = U(\mathrm{gl}_n)$ it leads to the equivalence between the Gelfand-Kirillov conjecture and the noncommutative Noether's problem for the invariants in the Weyl algebra A_k under the action of the symmetric group S_k (Corollary 8.3). We then prove the validity of the Noether's problem and hence obtain a new proof of the Gelfand-Kirillov conjecture for gl_n. The key part is the Theorem 8.1 which describes the invariant differential operators over certain localized rings. Similarly we show that the Gelfand-Kirillov conjecture holds for restricted Yangians of gl_n (Corollary 8.5).

We emphasize that the theory of Galois algebras unifies the representation theories of universal enveloping algebras and generalized Weyl algebras. For example the Gelfand-Tsetlin formulae give an embedding of $U(\mathrm{gl}_n)$ into a certain localization of the Weyl algebra A_m for $m = n(n+1)/2$ (Remark 7.2, see also [Kh]). On the other hand the intrinsic reason for such unification is a similar hidden skew group structure of these algebras as Galois algebras.

We will discuss the representation theory of Galois algebras in part II of this paper.

2. Preliminaries

All fields in the paper contain the base algebraically closed field $\Bbbk$ of characteristic 0. All the algebras in the paper are $\Bbbk$-algebras. If K is a field then $\bar{K}$ will denote the algebraic closure of K. Unless the opposite is stated all bimodules are assumed to be $\Bbbk$-central.

2.1. **Skew (semi)group rings.** If a semigroup $\mathcal{M}$ acts on a set S, $\mathcal{M} \times S \xrightarrow{f} S$, from the left, then $f(m, s)$ will be denoted either by $m \cdot s$, or ms, or s^m. In particular $s^{mm'} = (s^{m'})^m$, $m, m' \in \mathcal{M}$, $s \in S$. By $S^{\mathcal{M}}$ we denote the subset of all $\mathcal{M}$-invariant elements in S.

Let R be a ring, $\mathcal{M}$ a semigroup and $f : \mathcal{M} \longrightarrow \mathrm{Aut}(R)$ a homomorphism. Then $\mathcal{M}$ acts naturally on R (from the left). In this case we will use the notation $r^g = f(g)(r)$ for $g \in \mathcal{M}, r \in R$.

The *skew semigroup ring*, $R * \mathcal{M}$, associated with the left action of $\mathcal{M}$ on R, is a free left R-module, $\bigoplus_{m\in\mathcal{M}} Rm$, with a basis $\mathcal{M}$ and with the multiplication defined as follows

$$(r_1m_1) \cdot (r_2m_2) = (r_1r_2^{m_1})(m_1m_2), \quad m_1, m_2 \in \mathcal{M},\ r_1, r_2 \in R.$$

If $\mathcal{M}$ acts trivially on R then $R * \mathcal{M}$ coincides with the usual semigroup ring $R[\mathcal{M}]$. If $\mathcal{M}$ is finite and R is left noetherian then $R * \mathcal{M}$ is left noetherian. If $\mathcal{M}$ is a group and R is simple then the ring $R * \mathcal{M}$ is simple. If the ring R is commutative and $\mathcal{M}$ is a group then the ring $R * \mathcal{M}$ has a natural involution (antiisomorphism) $\varepsilon : R * \mathcal{M} \longrightarrow R * \mathcal{M}$, $rm \longmapsto r^{m^{-1}}m^{-1}$.

Assume, a finite group G acts by automorphisms on R and on $\mathcal{M}$.

Lemma 2.1. *If there holds the following commuting relations*

$$(1) \qquad g \cdot (m \cdot r) = ((g \cdot m),\ \textit{equivalently } (g \cdot r))(r^m)^g = (r^g)^{m^g},$$

for all $g \in G$, $m \in \mathcal{M}$, and $r \in R$, then the group G acts by automorphisms on $R * \mathcal{M}$:

$$(rm)^g = r^g m^g,\ r \in R,\ m \in \mathcal{M},\ g \in G.$$

Proof. Since the above action is additive we only need to check that

$$(r_1m_1)^g(r_2m_2)^g = (r_1m_1r_2m_2)^g.$$

Indeed in the left hand side we have

$$(r_1m_1)^g(r_2m_2)^g = r_1^g m_1^g r_2^g m_2^g = r_1^g (r_2^g)^{m_1^g} m_1^g m_2^g.$$

On the other hand, in the right hand side we have

$$(r_1r_2^{m_1}m_1m_2)^g = r_1^g (r_2^{m_1})^g (m_1m_2)^g.$$

□

Note that the commutativity condition 1 holds in the following important case

Lemma 2.2. *Assume a group G acts on the monoid $\mathcal{M}$ by conjugations. Then the conditions of* (1) *hold.*

Proof. $g(mr) = (gmg^{-1})gr$. □

We will assume that G act on $\mathcal{M}$ by conjugations.

If (1) holds then one can consider the subring of G-invariant elements in $R*\mathcal{M}$, denoted by $(R*\mathcal{M})^G$ or simply by $R*\mathcal{M}^G$. If R is simple and $\mathcal{M}$ is a finite outer subgroup then $R*\mathcal{M}^G$ is simple.

If $x \in R*\mathcal{M}$ then we write it in the form

$$x = \sum_{m\in\mathcal{M}} x_m m,$$

where only finitely many $x_m \in \mathcal{M}$ are nonzero. We call the finite set

$$\operatorname{supp} x = \{m \in \mathcal{M} | x_m \neq 0\}$$

the *support* of x. Hence $x \in R*\mathcal{M}^G$ if and only if $x_{m^g} = x_m^g$ for $m \in \mathcal{M}, g \in G$. If $x \in R*\mathcal{M}^G$ then $\operatorname{supp} x$ is a finite G-invariant subset in $\mathcal{M}$. For $\varphi \in \operatorname{Aut} R$ set

$$H_\varphi = \{h \in G | \varphi^h = \varphi\}. \tag{2}$$

Here we use the following agreement. If G is a finite group and H is its subgroup then the notation $F = \sum_{g\in G/H} F(g)$ means, that g runs a set of representatives of the quotient G/H and $F(g)$ does not depend on the choice of these representatives. In particular, the sum F is correctly defined.

Let $a \in R$. Then

$$\sum_{g\in G} a^g\varphi^g = \sum_{g\in G/H_\varphi} (\sum_{h\in H_\varphi} a^{gh})\varphi^{gh} = \sum_{g\in G/H_\varphi} (\sum_{h\in H_\varphi} a^h)^g \varphi^g.$$

Since every coefficient $\sum_{h\in H_\varphi} a^h$ belongs to the invariants $\sum R^{H_\varphi}$, we obtain a decomposition of $R*\mathcal{M}^G$ into a direct sum of left (right) R-submodules

$$\begin{aligned} R*\mathcal{M}^G &= \bigoplus_{\varphi\in G\backslash\mathcal{M}} (R*\mathcal{M})^G_\varphi, \quad \text{where} \\ (R*\mathcal{M})^G_\varphi &= \left\{ \sum_{g\in G/H_\varphi} a^g\varphi^g | a \in R^{H_\varphi} \right\} = \{[a\varphi] \,|\, a \in R^{H_\varphi}\}, \end{aligned} \tag{3}$$

where

$$[a\varphi] := \sum_{g\in G/H_\varphi} a^g\varphi^g \in R*\mathcal{M}^G, \tag{4}$$

for $\varphi \in \mathcal{M}$ and $a \in R^{H_\varphi}$. In particular $[a^g\varphi^g] = [a\varphi]$ for $g \in G$. Obviously there holds

$$\gamma\cdot[a\varphi] = [(a\gamma)\varphi],\ [a\varphi]\cdot\gamma = [(a\gamma^\varphi)\varphi],\ \gamma \in R^G. \tag{5}$$

Hence we have

$$[a_1\varphi_1][a_2\varphi_2] = \sum_{\tau\in\mathcal{O}_1\mathcal{O}_2} \left(\sum_{\substack{g_1\in G/H_{\varphi_1}, g_2\in G/H_{\varphi_2}, \\ \varphi_1^{g_1}\varphi_2^{g_2}=\tau}} a_1^{g_1} a_2^{\varphi_1^{g_1} g_2} \right)\tau,$$

where $\mathcal{O}_{\varphi_i} = \mathcal{M} \cdot \varphi_i$, $i = 1, 2$.

Analogously, one could also use the notation $[\varphi a]$, $\varphi \in \operatorname{Aut} R$, $a \in R$ for the generators of $R * \mathcal{M}^G$. We have

$$[a\varphi] = \sum_{g \in G/H_\varphi} a^g \varphi^g = \sum_{g \in G/H_\varphi} \varphi^g (g\varphi^{-1} g^{-1} g a) = \sum_{g \in G/H_\varphi} \varphi^g (a^{\varphi^{-1}})^g = [\varphi a^{\varphi^{-1}}].$$

Every $x \in R * \mathcal{M}^G$ can be uniquely presented as the sum $\sum_{\varphi \in \mathcal{M}\backslash G} [x_\varphi \varphi], x_\varphi \in R^{H_\varphi}$, which we call the *canonical (left) presentation* of x. In the same way, for $a, b \in R^{H_\varphi}$ denote

$$(6) \qquad [a\varphi b] = \sum_{g \in G/H_\varphi} a^g \varphi^g b^g, \text{ so for } \gamma \in R^G \text{ holds } [a\varphi] = [a\varphi 1],$$

$$\gamma[a\varphi b] = [(\gamma a)\varphi b] = [a\varphi(b\gamma^{\varphi^{-1}})], \ [a\varphi b]\gamma == [a\varphi(b\gamma)][(\gamma^\varphi a)\varphi b],$$

since $\varphi(R^G), \varphi^{-1}(R^G) \subset R^{H_\varphi}$. Note that the expression $[a\varphi b]$ is bilinear in a and b.

If $R = L$ is a field and $K = L^G$ then

$$(7) \qquad \dim_K^r (L * \mathcal{M})_\varphi^G = \dim_K^l (L * \mathcal{M})_\varphi^G = [L^{H_\varphi} : K] = |G : H_\varphi| = |\mathcal{O}_\varphi|.$$

Let $K \subset L$ be a finite Galois extension of fields, $G = G(L/K)$ the Galois group and $\imath$ the canonical embedding $K \hookrightarrow L$.

Definition 2. (1) *Monoid* $\mathcal{M} \subset \operatorname{Aut} L$ *is called* separating *(with respect to K) if for any $m_1, m_2 \in \mathcal{M}$ from*

$$m_1|_K = m_2|_K$$

follows $m_1 = m_2$.

(2) *An automorphism $\varphi : L \longrightarrow L$ is called* separating *(with respect to K) if the monoid generated by $\{\varphi^g \mid g \in G\}$ in* $\operatorname{Aut} L$ *is separating.*

Remark 2.1. *If $\mathcal{M}$ is separating then $\mathcal{M} \cap G = \{e\}$. Moreover, if $\mathcal{M}$ is a group, then these conditions are equivalent.*

Remark 2.2. *The following conditions are equivalent*

(1) *Monoid $\mathcal{M}$ is separating with respect to K.*
(2) *For any $m \in \mathcal{M}, m \neq e$ there exists $\gamma \in K$ such that $\gamma^m \neq \gamma$.*
(3) *If $Gm_1G = Gm_2G$ for some $m_1, m_2 \in \mathcal{M}$, then there exists $g \in G$ such that $m_1 = m_2^g$.*

Let $\jmath : K \hookrightarrow L$ be an embedding. Denote $\operatorname{St}_G(\jmath) = \{g \in G | g\jmath = \jmath\}$.

Lemma 2.3. *Let $\varphi, \varphi' \in \operatorname{Aut} L$. If $\jmath = \varphi\imath$, then $\operatorname{St}(\jmath) = G(L/\varphi(K)) \cap G(L/K)$, and $\jmath = \varphi'\imath$ only if $\varphi' \in \varphi G$. Besides, $L^{\operatorname{St}(\jmath)} = K \cdot \varphi(K)$.*

Proof.

$$g \in \operatorname{St}(\jmath) \Longleftrightarrow g\varphi\imath = \varphi\imath \Longleftrightarrow \varphi^{-1} g \varphi \in G \Longleftrightarrow g \in (\varphi G \varphi^{-1}) \cap G,$$

that proves the first statement. Second statement is obvious. □

Lemma 2.4. *Let $\varphi \in \operatorname{Aut} L$ be separating, $\jmath = \varphi\imath$. Then*

(1) $H_\varphi = \operatorname{St}_G(\jmath)$.
(2) $L^\jmath = K\varphi(K) = L^{H_\varphi}$.

Proof. Obviously $H_\varphi \subset \operatorname{St}_G(\jmath)$. Conversely, if $g\varphi\imath = \varphi\imath$, then $\varphi^{-1}g\varphi\imath = \imath$, hence $\varphi^{-1}g\varphi = g_1 \in G$ and $\varphi^{-1}(g\varphi g^{-1}) = g_1 g^{-1}$. Thus φ and $g\varphi g^{-1}$ coincide on K, implying $g\varphi g^{-1} = \varphi$ and (1). The statement (2) follows from (1) and Lemma 2.3. □

3. Balanced bimodules

Let $V = {}_K V_K$ be a K-bimodule and ${}_K V_L = V \otimes_K L$. Then the Galois group G acts naturally from the left on the $K - L$-bimodule ${}_K V_L$ and the stable elements of this action coincide with ${}_K V_K$. We assume that the right action of L on V is K-diagonalizable from the left. Hence ${}_K V_L$ splits in the sum of $K - L$-bimodules, one dimensional as a right L-module. If V is indecomposable then there exists an embedding $\jmath : K \hookrightarrow L$ such that

$${}_K V_L \simeq \bigoplus_{g \in G/H} L_{g\jmath},$$

where $H = \operatorname{St}_G \jmath$ and L_φ denotes a one-dimensional $K - L$-bimodule, which coincides with L as a right L-module, with $\lambda \cdot l = \varphi(\lambda) l$ for all $\lambda \in K$ and $l \in L$.

Denote ${}_L V_L = L \otimes_K {}_K V_L$.

Definition 3. *A K-bimodule ${}_K V_K$ is called L-***balanced** *over a finite Galois extension $K \subset L$, if ${}_L V_L$ is a direct sum of one-dimensional L-bimodules. A K-bimodule ${}_K V_K$ is called* **balanced** *if it is L-***balanced** *over some finite Galois extension $K \subset L$.*

Proposition 3.1. *Let V be a simple L-balanced K-bimodule, $\imath : K \hookrightarrow L$ the canonical embedding. Then there exists $\varphi \in \operatorname{Aut} L$ such that*

$${}_L V_L \simeq \bigoplus_{g \in G/\operatorname{St}(\jmath)} \bigoplus_{\tilde{g} \in G} L_{g\varphi\tilde{g}},$$

where $\jmath = \varphi \circ \imath$, $\operatorname{St}_G(\jmath)$ the stabilizer of $\jmath$ in G. Moreover, all summands in this decomposition are non-isomorphic.

Proof. Let L_φ be any summand of ${}_L V_L$, $\jmath = \varphi\imath$. Consider two representatives g_1 and g_2 of different coclasses in $G/\operatorname{St}(\jmath)$ and assume that $g_1\varphi g' = g_2\varphi g''$, for some $g', g'' \in G$. Then $g_1\jmath = g_2\jmath$ and hence $g_1^{-1}g_2 \in \operatorname{St}(\jmath)$ which is a contradiction. Therefore all subscripts $g\varphi\tilde{g}$ in the decomposition are different and the corresponding summands are non-isomorphic as L-bimodules. □

Let $\varphi \in \operatorname{Aut} L$, $\jmath = \varphi\imath$. Consider the $K - L$-bimodule $L_\jmath$ and let $H = \operatorname{St}_G(\jmath)$. Then H acts on $L_\jmath$ from the left. Denote by $V(\varphi) = L_\jmath^H$ the set of H-invariant elements of $L_\jmath$. Then $V(\varphi)$ is obviously a right K-submodule in $L_\jmath$. But also $V(\varphi)$ is a left K-submodule since

$$(k \cdot l)^g = (l\,\jmath(k))^g = l^g\,\jmath(k)^g = l\,\jmath(k) = k \cdot l,$$

for all $k \in K$, $g \in H$, $l \in V(\varphi)$. Thus $V(\varphi)$ is a K-subbimodule of $L_\jmath$.

Theorem 3.1. (1) $V(\varphi)\otimes_K L \simeq \bigoplus_{g\in G/H} L_{g\varphi\imath}$.

(2) *$V(\varphi)$ is a simple K-bimodule.*
(3) *Let $\varphi, \varphi' \in \operatorname{Aut} L$. Then $V(\varphi) \simeq V(\varphi')$ if and only if $G\varphi|_K = G\varphi'|_K$ coincide, equivalently $G\varphi G = G\varphi' G$.*
(4) *Let $\varphi \in \mathcal{M}$ for a monoid $\mathcal{M} \subset \operatorname{Aut} L$, $a \in L^H$, $v = [a\varphi] = \sum_{g\in G/H} a^g\varphi^g \in L * \mathcal{M}^g$.*

Then $KvK \simeq V(\varphi)$ as K-bimodules.

Proof. Denote by K' the image $\varphi(K) \subset L$ and consider an induced isomorphism $\varphi' : K \to K'$. Then $W = K'_{\varphi'}$ can be viewed as a $K-K'$-bimodule, and we have a canonical isomorphism $L^H_\jmath \simeq W \otimes_{K'} L^H_{\imath'}$, via the map $l \longmapsto 1 \otimes l$, where $\imath' : K' \hookrightarrow L$ is a canonical embedding. Then we have the following chain of isomorphisms:

$$V(\varphi)\otimes_K L \simeq L^H_\jmath \otimes_K L \simeq W \otimes_{K'} (L^H_{\imath'} \otimes_K L) \simeq W \otimes_{K'} (\oplus_{g\in G/H} L_{g\imath'}) \simeq \oplus_{g\in G/H} L_{g\jmath}.$$

To prove simplicity of $V(\varphi)$ consider any nonzero $x \in L^H$. Then $KxK = x\varphi(K)K = xL^H = L^H$, that completes the proof.

Assume $V(\varphi) \simeq V(\varphi')$. Then $V(\varphi)\otimes_K L \simeq V(\varphi')\otimes_K L$. Hence from (1) we obtain $\varphi'\imath = g\varphi\imath$ for some $g \in G$. It proves that $G\varphi\imath = G\varphi'\imath$, equivalently $G\varphi|_K = G\varphi'|_K$. Thus $\varphi^{-1}g\varphi|_K = \imath$ implying that $\varphi^{-1}g\varphi \in G$. The converse statement easily follows.

Using the formulae (6) and Lemma 2.4, (2) we obtain $K[\varphi]K = [K\varphi(K)\varphi] = [L^H\varphi]$ which shows immediately that $[L^H\varphi] \simeq V(\varphi)$. □

4. Galois algebras

For the rest of the paper we will assume that Γ is an integral domain, K the field of fractions of Γ, $K \subset L$ is a finite Galois extension with the Galois group G, $\imath : K \to L$ is a natural embedding, $\mathcal{M} \subset \operatorname{Aut} L$ is a separating monoid on which G acts by conjugations, $\bar{\Gamma}$ is the integral closure of Γ in L.

We also fix a Galois algebra U with respect to Γ. Recall from the introduction that an associative non-commutative $\Bbbk$-algebra U containing Γ is called a *Galois Γ-algebra* if it is finitely generated Γ-subalgebra in $L * \mathcal{M}^G$ and $KU = UK = L * \mathcal{M}^G$. Note that following Lemma 4.1 below both equalities in this definition are equivalent.

Example 4.1. (1) *Let $U = \Gamma[x;\sigma]$ be the skew polynomial ring over Γ, where $\sigma \in \operatorname{Aut} K$, $x\gamma = \sigma(\gamma)x$, for all $\gamma \in \Gamma$. Denote $\mathcal{M} = \{\sigma^n, n = 0, 1, \ldots\} \subset \operatorname{Aut} K$. Then U is a Galois Γ-algebra in $K * \mathcal{M}$: $x \mapsto 1 * \sigma$ ($L = K, G = \{e\}$);*
(2) *Analogously the skew Laurent polynomial ring $U = \Gamma[x^{\pm1};\sigma]$ is a Galois algebra with $\mathcal{M} = \{\sigma^n \mid n \in \mathbb{Z}\}$ with trivial G.*
(3) Iterated Ore extensions. *Let*

$$R_n = \Bbbk[x_1][x_2;\sigma_2]\ldots[x_n;\sigma_n],\ \Gamma = \Bbbk[x_1],$$

$\mathcal{M}_i = \{\sigma_i^n, n = 0, 1, \ldots\}$, $i = 2, \ldots, n$, $\mathcal{M} = \mathcal{M}_2 \times \ldots \times \mathcal{M}_n \simeq \mathbb{Z}_+^{n-1}$. Then R_n is a Galois Γ-algebra, $R_n \subset \Bbbk(x_1) * \mathcal{M}$. Also R_n is a Galois algebra with respect to $\Bbbk$, $R_n \subset \Bbbk * \mathbb{Z}_+^n$. All R_n are noetherian domains.

An example of such ring is provided by the quantum torus which plays an important role in the theory of extended affine Lie algebras. Let $\mathfrak{q} = (q_{ij})_{n\times n}$ *be a complex matrix such that* $q_{ii} = 1$, $q_{ij} = q_{ji}^{-1}$. *The associated quantum torus*

$$A = \mathbb{C}_{\mathfrak{q}}[x_1, \dots, x_n] = \mathbb{C}[x_1][x_2;\sigma_2]\dots[x_n;\sigma_n],$$

where $\sigma_i(x_j) = q_{ij}x_j$, $1 \le j \le i-1$.

4.1. **Characterization of a Galois algebra.** A Γ-subbimodule of $L * \mathcal{M}^G$ which for every $m \in \mathcal{M}$ contains $[b_1 m], \dots, [b_k m]$ where $b_1, \dots, b_k$ is a K-basis in L^{H_m} will be called a Γ*-form* of $L * \mathcal{M}^G$. We will show that any Galois subalgebra in $L * \mathcal{M}^G$ is its Γ-form.

Lemma 4.1. *Let* $u \in U$ *be a nonzero element,* $T = \operatorname{supp} u$, $u = \sum_{m\in T}[a_m m]$. *Then*

$$K(\Gamma u\Gamma) = (\Gamma u\Gamma)K = KuK \simeq \bigoplus_{m\in T} V(m).$$

In particular U *is a* Γ*-form of* $L * \mathcal{M}^G$. *Besides,*

$$L(\Gamma u\Gamma) = (\Gamma u\Gamma)L = LuL = \sum_{m\in T} Lm \subset L * \mathcal{M}.$$

Proof. Note first that all $V(m)$ are pairwise non-isomorphic simple K-bimodules. Indeed, if $V(m) \simeq V(m')$ for some $m, m' \in T$, then $GmG = Gm'G$ by Theorem 3.1 and, thus, m and m' are conjugate (cf. Remark 2.2). Hence $[m] = [m']$. Since $K[m]K \simeq V(m)$, $m \in T$, we have

$$KuK \subset \sum_{m\in T} K[a_m m]K = \bigoplus_{m\in T} K[a_m m]K \simeq \bigoplus_{m\in T} K[m]K \simeq \bigoplus_{m\in T} V(m).$$

Since all $V(m)$ are simple, then the image of KuK in $W = \bigoplus_{m\in T} V(m)$ generates W as a K-bimodule. Hence $KuK \simeq W$ and therefore $K[a_m m]K \subset KuK$ for all $m \in T$. For the rest of the proof it is enough to consider $u = [am]$. Then $\Gamma[am]\Gamma = [\Gamma \cdot m(\Gamma)am]$ and $K\Gamma m(\Gamma) = Km(K)$. The first statement now follows from Lemma 2.4, (2).

Obviously $L[am]$ is a L-subbimodule in $\sum_{m\in T} Lm$. Since this sum is a direct sum of non-isomorphic simple L-bimodules, any its subbimodule has a form $\sum_{m\in T'} Lm, T' \subset T$. On the other hand $\operatorname{supp}[am] = T$, and thus $L[am] = \sum_{m\in T} Lm$. □

As an application of Lemma 4.1 we will prove that $L * \mathcal{M}^G$ is simple if $\mathcal{M}$ is a group. First we need

Lemma 4.2. *For* $x_1 = [a_1 m_1], x_2 = [a_2 m_2] \in L * \mathcal{M}^G$ *holds*

$$\operatorname{supp} x_1\Gamma x_2 = \bigcup_{\gamma\in\Gamma} \operatorname{supp} x_1\gamma x_2 = \operatorname{supp} x_1 \cdot \operatorname{supp} x_2. \tag{8}$$

Proof. Obviously $\operatorname{supp} x_1\Gamma x_2 \subset \operatorname{supp} x_1 \cdot \operatorname{supp} x_2$. On the other hand

$$\operatorname{supp} x_1\Gamma x_2 = \operatorname{supp} Kx_1\Gamma x_2 K = \operatorname{supp} x_1\Gamma x_2 K,$$

since the right and the left multiplications on non-zero elements from K do not change the support. But $Kx_1\Gamma = Kx_1K$ and $\Gamma x_2K = Kx_2K$. Hence it is enough to prove that $\operatorname{supp}[m_1]\cdot[m_2] = \operatorname{supp}[m_1]\cdot\operatorname{supp}[m_2]$. But this follows immediately, since the characteristics of $\Bbbk$ is 0. $\square$

A submonoid H of $\mathcal{M}$ is called an ideal of $\mathcal{M}$ if $\mathcal{M}H \subset H$ and $H\mathcal{M} \subset H$.

Corollary 4.1. *There is one-to-one correspondence between the two-sided ideals in $L * \mathcal{M}^G$ and the G-invariant ideals in the monoid $\mathcal{M}$. This correspondence is given by the following bijection*

$$(9) \qquad I \longmapsto \mathcal{J} = \mathcal{J}(I) = \bigcup_{u \in I} \operatorname{supp} u, \quad \mathcal{J} \longmapsto I = I(\mathcal{J}) = \sum_{\varphi \in \mathcal{J}} K\varphi K,$$

where $I \subset L * \mathcal{M}^G$, $\mathcal{J} \subset \mathcal{M}$ are ideals, $I \neq 0$, $\mathcal{J}$ is G-invariant. In particular, if $\mathcal{M}$ is a group then $L * \mathcal{M}^G$ is a simple ring.

Proof. Let I be a nonzero ideal in $L * \mathcal{M}^G$. If $0 \neq u \in I$ then

$$KuK \simeq \sum_{\varphi \in \operatorname{supp} u/G} V(\varphi)$$

by Lemma 4.1, and $(K[m]K)(KuK) \subset I$. By Lemma 4.2 for every $m \in \mathcal{M}$ and $\varphi \in \operatorname{supp} u$ there exists $u' \in I$ such that $m\varphi \in \operatorname{supp} u'$ and there exists $u'' \in I$ such that $\varphi m \in \operatorname{supp} u''$. This gives the map $I \mapsto \mathcal{J}(I)$. Analogously, $I(\mathcal{J})$ is a two-sided ideal in $L * \mathcal{M}^G$ and both maps are mutually inverse. $\square$

Let $e \in \mathcal{M}$ be the unit element, $Le \subset L * \mathcal{M}$ and $U_e = U \cap Le$.

Theorem 4.1. *Let U be a Galois subalgebra in $L * \mathcal{M}$. Then*

1. *For every $x \in U$ holds $x_e \in K$ and $U_e \subset K$.*
2. *The $\Bbbk$-subalgebra in $L * \mathcal{M}$ generated by U and L coincides with $L * \mathcal{M}$.*
3. *$U \cap K$ is a maximal commutative $\Bbbk$-subalgebra in U.*
4. *The center $Z(U)$ of algebra U equals $U \cap K^{\mathcal{M}}$.*

Proof. Let $x \in U$ and $x_e = \lambda$, $\lambda \in L$. Then for any $g \in G$ holds $\lambda = x_e = (x^g)_e = \lambda^g$. Hence $\lambda \in L^G = K$.

To prove (2) consider any $m \in \mathcal{M}$ and $[am] \in \Gamma$, $a \neq 0$. Then $K[am]K \simeq V(m)$ and $K[am]L$ is a $K-L$-subbimodule in the sum of pairwise non-isomorphic simple $K-L$-bimodules $\sum_{g \in G/H_m} m^g L$. Besides, all $[am]_{m^g} = a^g \neq 0$, hence $[am]L$ coincides with this sum, hence $m \in [am]L \subset UL$.

Consider any $x \in L * \mathcal{M}$ such that $x\gamma = \gamma x$ for all $\gamma \in \Gamma$. Assume $x_g \neq 0$ for some $g \neq e$ and consider $\gamma \in \Gamma$ such that $\gamma^g \neq \gamma$. Then $(\gamma x)_g = \gamma x_g \neq \gamma^g x_g = (x\gamma)_g$ which is a contradiction. Hence $x \in U \cap Le = U_e \subset K$ which completes the proof of (3).

To prove (4) consider a nonzero $z \in Z(U)$. It follows from (3) that $z \in U \cap K$. Moreover, $z \in \Gamma \cap Z(U)$ if and only if for every $[a\varphi] \in U$ holds $z[a\varphi] = [a\varphi]z$, i.e. $z = z^\varphi$, (see (6)). $\square$

Theorem 4.1, (3) in particular shows that an associative algebra is never a Galois algebra with respect to its center.

Example 4.2. *Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra, H a Cartan subalgebra of $\mathfrak{g}$, $U(\mathfrak{g})$ and $U(H)$ are universal enveloping algebras of $\mathfrak{g}$ and H respectively. Since $U(H)$ is not maximal commutative subalgebra of $U(\mathfrak{g})$ then $U(\mathfrak{g})$ is not a Galois algebra with respect to $U(H)$ by Theorem 4.1,* (3).

Recall that if U is a domain then a multiplicative subset $S \subset U \setminus \{0\}$ satisfies a left (right) Ore condition if for any pair $u \in U$, $s \in S$ there exists $u' \in U$ and $s' \in S$ such that $us' = su'$ ($s'u = u's$ respectively).

Proposition 4.1. *Let U be a Galois algebra with respect to Γ, $S = \Gamma \setminus \{0\}$.*

(1) *The multiplicative set S satisfies both left and right Ore condition. Hence, there exist the classical rings of fractions $U[S^{-1}]$, $[S^{-1}]U$ (see* [St], *Chapter II).*
(2) $[S^{-1}]U \simeq U[S^{-1}] \simeq L * \mathcal{M}^G$.

Proof. Assume $s \in S, u \in U$. Following Lemma 4.1, U contains a right K-basis $u_1, \dots, u_k$ of KuK. Hence in KuK holds $s^{-1}u = \sum_{i=1}^{k} u_i \gamma_i s_i^{-1}$ for some $s_i \in S, \gamma_i \in \Gamma, i = 1, \dots, k$. Then in U holds $u \cdot (s_1 \dots s_k) = s \cdot (\sum_{i=1}^{k} u_i \gamma_i s_1 \dots s_{i-1} s_{i+1} \dots s_k)$. It shows (1). Following Lemma 4.1, the canonical embedding $U \hookrightarrow L * \mathcal{M}^G$ satisfies the conditions F1, F2, F3, [St], Chapter II, §1. Hence (2) follows. □

Corollary 4.2. *The canonical embedding $i : U \hookrightarrow L * \mathcal{M}^G$ induces an K-bimodule isomorphism $j : K \otimes_\Gamma U \otimes_\Gamma K \simeq L * \mathcal{M}^G$.*

Proof. Following Lemma 4.1, j is an epimorphism. If $x \in \operatorname{Ker} j$ then there exist $s_1, s_2 \in S$ such that $s_1 x s_2 \in U \cap \operatorname{Ker} i$. Hence $x = 0$. □

Theorem 4.2. *The tensor product of two Galois algebras in a Galois algebra.*

Proof. Let U_i be Galois subalgebra in skew-group algebra $L_i * \mathcal{M}_i$, over Γ_i with the fraction fields K_i, $G_i = G(L_i/K_i)$ $i = 1, 2$. Then $\mathcal{M}_1 \times \mathcal{M}_2$ acts on $L_1 \otimes_\Bbbk L_2$, $(m_1, m_2) \cdot (l_1 \otimes l_2) = (m_1 l_1, m_2 l_2)$. Since $\Bbbk$ is algebraically closed, $L_1 \otimes_\Bbbk L_2$ is a domain, hence $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ acts on the fraction field. Set $K \subset L$ the field of fraction of $K_1 \otimes_\Bbbk K_2$, which coincides with the field of fraction of $\Gamma_1 \otimes_k \Gamma_2$. The extension $K \subset L$ is a finite Galois extension with the Galois group $G = G_1 \times G_2$. consider the composition

$$(10) \qquad \imath : U_1 \otimes_\Bbbk U_2 \longrightarrow L_1 * \mathcal{M}_1 \otimes_\Bbbk L_2 * \mathcal{M}_2 \simeq (L_1 \otimes_\Bbbk L_2) * (\mathcal{M}_1 \times \mathcal{M}_2) \hookrightarrow L * \mathcal{M}.$$

The isomorphism above sends $L_1 * \mathcal{M}_1^{G_1} \otimes_k L_2 * \mathcal{M}_2^{G_2}$ into $L * \mathcal{M}^G$, which endows $U_1 \otimes_\Bbbk U_2$ with a structure of Galois algebra. To finish we shall to prove, that $K \cdot \imath(U_1 \otimes_\Bbbk U_2) = L * \mathcal{M}^G$. If $x \in L * \mathcal{M}^G$, then multiplying on $\alpha \in \Gamma_1 \otimes_\Bbbk \Gamma_2$ we can assume $x \in (L_1 \otimes_\Bbbk L_2) * \mathcal{M}$. But $(L_1 \otimes_\Bbbk L_2) * \mathcal{M}^G \simeq L_1 * \mathcal{M}_1^{G_1} \otimes_k L_2 * \mathcal{M}_2^{G_2} = KU_1 \otimes_k KU_2$. □

4.2. **Galois algebras via generators and relations.** It is natural to ask how big is the class of Galois algebras with respect to Γ. In this section we explain how Galois algebras are related to Γ-bimodules, providing a recipe to construct many natural examples of such algebras.

Let V be a torsion free Γ-bimodule such that ${}_KV_K$ is a L-balanced K-bimodule. Then by Proposition 3.1, there exists an isomorphism

(11)
$$ {}_LV_L \simeq \bigoplus_{\varphi \in S} \bigoplus_{g \in G/H_\varphi} \bigoplus_{\tilde{g} \in G} L_{g\varphi\tilde{g}}. $$

for a certain $S \subset \operatorname{Aut} L/G$, where G acts on $\operatorname{Aut} L$ by conjugations. Given $\varphi \in S$, denote

$$ \Omega_\varphi = \{\, g\varphi\tilde{g} \,|,\ g \in G/H_\varphi,\ \tilde{g} \in G\}. $$

The diagonal morphism $\Delta : G \longrightarrow G \times G$ induces the action of G on the bimodule ${}_LV_L$,

(12)
$$ g \cdot (l_1 \otimes_\Gamma v \otimes_\Gamma l_2) = l_1^g \otimes_\Gamma v \otimes_\Gamma l_2^g, $$

defining K-bimodule isomorphisms $T_g : {}_LV_L \longrightarrow {}_LV_L$, $g \in G$.

Lemma 4.3. *Let $g \in G$, ${}_LV_L$ be L-balanced K-bimodule, $\varphi \in \operatorname{Aut} L$. Assume that $L_\varphi \subset {}_LV_L$ is an L-subbimodule. Then $T_g(L_\varphi) \simeq L_{g\varphi g^{-1}}$.*

Proof. By definition $T_g(\lambda_1 x \lambda_2) = \lambda_1^g T_g(x) \lambda_2^g$, $\lambda_1, \lambda_2 \in L, x \in {}_LV_L$. Then

$$ \lambda T_g(x) = T_g(\lambda^{g^{-1}} x) = T_g(x \lambda^{\varphi g^{-1}}) = T_g(x)\lambda^{g\varphi g^{-1}},\ g \in G, \lambda \in L, $$

hence $T_g(L_\varphi) \simeq L_{g\varphi g^{-1}}$, which proves the statement. □

The group G acts on the set Ω_φ by conjugation. The orbits of G in Ω are $\mathcal{O}_\psi = \{\psi^g \,|\, g \in G\}$. Since all summands (11) are non-isomorphic, ${}_LV_L$ has a decomposition into a direct sum of G-invariant L-subbimodules:

$$ {}_LV_L = \bigoplus_{\varphi \in S} \bigoplus_{\psi \in \Omega_\varphi/G} V\{\varphi, \psi\}, $$

where $V\{\varphi,\psi\} = \sum_{g \in G/H_\psi} T_g(L_\psi)$, $\psi \in \Omega_\varphi/G$.

For each $\varphi \in S$ choose $\psi \in \Omega_\varphi$ and consider the submonoid $\mathcal{M} = \mathcal{M}(S)$ in $\operatorname{Aut} L$ generated by all the orbits $\mathcal{O}_\psi$. In the skew semigroup ring $L * \mathcal{M}$ consider the direct summand

$$ {}_LV_L(S) = \bigoplus_{\varphi \in S} {}_LV_L\{\varphi, \psi\} $$

of ${}_LV_L$.

Let $\pi_S : {}_LV_L \longrightarrow {}_LV_L(S)$ be the canonical projection. Fix $a_\varphi \in L$, denote $a(S) = \{a_\varphi \,|\, \varphi \in S\}$ and consider a G-equivariant L-bimodule monomorphism $\tau_\varphi : {}_LV_L(\varphi) \longrightarrow L * \mathcal{M}$, where

(13)
$$ L_{\varphi^g} \ni 1 \longmapsto a_\varphi^g \varphi^g,\ g \in G/H_\varphi,\ a_\varphi \in L^{H_\varphi}. $$

It gives the chain of Γ-bimodule homomorphisms

$$V \xrightarrow{i_\Gamma^K} {}_KV_K \xrightarrow{i_K^L} {}_LV_L \xrightarrow{\pi_L} {}_LV_L(S) \xrightarrow{\bigoplus_{\varphi\in S}\tau_\varphi} L * \mathcal{M}. \tag{14}$$

Denote by $\mathcal{G}_V$ the composition of Γ-bimodule homomorphisms above.

Lemma 4.4.

$$\mathcal{G}_V(V) \subset \sum_{\varphi\in S}[L\varphi] = \{\sum_{\varphi\in S}[L^{H_\varphi}\varphi] \,|\,\} \subset L * \mathcal{M}^G.$$

Proof. By (3) for every $v \in {}_KV_K$ holds $\mathcal{G}_V(v) = \sum_{\varphi\in S}[a_v\varphi]$ for some $a_v \in L^{H_\varphi}$. □

The chain of Γ-bimodule morphisms (14) induces the chain of $\Bbbk$-algebra homomorphisms of tensor algebras

$$T_\Gamma(V) \xrightarrow{\Gamma[i_\Gamma^K]} T_K({}_KV_K) \xrightarrow{K[i_K^L]} T_L({}_LV_L) \xrightarrow{L[\pi_S]} T_L({}_LV_L(S)) \xrightarrow{L[\bigoplus_{\varphi\in S}\tau_\varphi]} L * \mathcal{M}. \tag{15}$$

Denote by $\mathcal{G}$ the composition of these maps and consider its image

$$U = U(V, S, a(S)) = T_\Gamma(V)/\operatorname{Ker}\mathcal{G} \simeq \operatorname{Im}\mathcal{G} \subset L * \mathcal{M}.$$

Proposition 4.2. *Assume that the algebra $U \subset L*\mathcal{M}^G$ is generated over Γ by the elements $u_1, \dots, u_k \in U$ such that $\bigcup_{i=1}^k \operatorname{supp} u_i$ contains a set of generators of $\mathcal{M}$ as a semigroup. Then U is a Galois Γ-algebra.*

Proof. Consider a K-subbimodule $Ku_1K + \cdots + Ku_kK$ in $L * \mathcal{M}^G$. By Lemma 4.1, this bimodule contains the elements $[a_1\varphi_1], \dots, [a_N\varphi_N]$, where $\varphi_1, \dots, \varphi_N$ is a set of generators of $\mathcal{M}$. Then by Lemma 4.2 for every $m \in \mathcal{M}$ there exists nonzero $a_m \in L^{H_m}$ such that $[a_m m] \in U$. Since the bimodule $V(m)$ is L-balanced then $Lm = L \cdot a_m m \subset L[a_m m]$, hence $LU = L * \mathcal{M}$ by Lemma 4.1. □

This proposition shows that the construction above gives just the same class of algebras. Hence we will understand a Galois algebra U both as a quotient of $T_\Gamma(V)$ and as a $\Bbbk$-subalgebra in $L * \mathcal{M}$.

If V is the Γ-bimodule, then denote by V° the opposite bimodule

$$V^\circ = \{v^\circ \,|\, v \in V\},\ \gamma_1 v^\circ \gamma_2 = (\gamma_2 v \gamma_1)^\circ, \gamma_1, \gamma_2 \in \Gamma, v \in V.$$

Example 4.3. *Let V be an L-balanced Γ-bimodule with a simple ${}_KV_K = V(\varphi)$, and ${}_LV_L = \bigoplus_{g\in G/H_\varphi} L_{g\varphi}$. Then the opposite Γ-bimodule V° is also L-balanced and ${}_LV_L{}^\circ = \bigoplus_{g\in G/H_\varphi} L_{\varphi^{-1}g^{-1}}$. Fix $a, b \in L$. The Galois algebra $U(\Gamma, V \oplus V^\circ, \varphi, \varphi^{-1}, a, b)$ is generated over Γ by $[a\varphi]$ and $[b\varphi^{-1}]$. In particular, one-dimensional bimodule corresponds to the case of Generalized Weyl algebras, cf. Section 7.2.*

4.3. Galois algebras with involution. In this section we consider a class of Galois algebras generated by a bimodule $V \oplus V^\circ$. Then the corresponding monoid $\mathcal{M}$ is a group (cf. Example 4.3).

Let $U \subset L * \mathcal{M}$ is a such Galois algebra. Suppose there exists an anti-isomorphism of $\Bbbk$-algebras $\sigma : U \longrightarrow U$, such that $\sigma^2 = \mathrm{id}_U$, $\sigma|_\Gamma = \mathrm{id}_\Gamma$ and such that for $u \in U$, $h \in \operatorname{supp} u$ if and only if $h^{-1} \in \operatorname{supp} \sigma(u)$. In this case σ is called an *involution* of U.

Obviously, for an algebra with an involution holds $\operatorname{supp} u = (\operatorname{supp} \sigma(u))^{-1}$.

For every $\varphi \in \mathcal{M}$ choose nonzero $\lambda_\varphi \in L^{H_\varphi}$ and define a map $\circ : L * \mathcal{M} \to L * \mathcal{M}$ as follows:

$$a\varphi \longmapsto \varphi^{-1}\lambda_\varphi a,$$

for all $\varphi \in \mathcal{M}$ and $a \in L$. Then

$$(a\varphi b\psi)^\circ = \psi^{-1}\varphi^{-1}\lambda_{\varphi\psi} a b^\varphi$$

and

$$(b\psi)^\circ(a\varphi)^\circ = \psi^{-1}\varphi^{-1}\lambda_\psi^\varphi b^\varphi \lambda_\varphi a.$$

Hence $\circ$ is an anti-homomorphism if and only if the elements λ_φ satisfy the condition $\lambda_{\varphi\psi} = \lambda_\varphi \lambda_\psi^\varphi$.

We will define formally $\lambda_g = e$ for all $g \in G$. In particular, $e = \lambda_{\varphi\varphi^{-1}} = \lambda_\varphi \lambda_{\varphi^{-1}}^\varphi$ and thus $\lambda_\varphi^{-1} = \lambda_{\varphi^{-1}}^\varphi$. Moreover, with this condition the map $\circ$ becomes an involution of $L * \mathcal{M}$:

$$(a\varphi) \mapsto \varphi^{-1}\lambda_\varphi a = \lambda_\varphi^{\varphi^{-1}} a^{\varphi^{-1}} \varphi^{-1} \mapsto \varphi \lambda_{\varphi^{-1}} \lambda_\varphi^{\varphi^{-1}} a^{\varphi^{-1}} = \varphi a^{\varphi^{-1}} = a\varphi.$$

On the set of involutions of $L * \mathcal{M}$ acts the group $\operatorname{Aut}_{\mathcal{M}} L * \mathcal{M}$ of $\mathcal{M}$-graded automorphisms of $L * \mathcal{M}$:

$$f : \circ \longmapsto f \circ f^{-1},\ f \in \operatorname{Aut}_{\mathcal{M}} L * \mathcal{M}.$$

Consider a restriction of $\circ$ on $L * \mathcal{M}^G$. Since

$$[a\varphi]^\circ = \sum_{g \in G/H_\varphi} (g\varphi^{-1}g^{-1})\lambda_{g\varphi g^{-1}} a^g,$$

then $\circ$ induces an anti-homomorphism of $L * \mathcal{M}^G$ if and only if

$$\lambda_{g\varphi g^{-1}} = \lambda_\varphi^g, \text{ and then } [a\varphi]^\circ = [\varphi^{-1}\lambda_\varphi a].$$

Therefore, a set of nonzero $\lambda_\varphi \in L^{H_\varphi}$, $\varphi \in \mathcal{M}$, defines an involution on $L * \mathcal{M}^G$ if and only if $\lambda_{\varphi\psi} = \lambda_\varphi \lambda_\psi^\varphi$ for all $\varphi, \psi \in \mathcal{M}$ and $\lambda_{g\varphi g^{-1}} = \lambda_\varphi^g$ for all $\varphi \in \mathcal{M}$ and $g \in G$. We will call a set of such elements λ_φ, $\varphi \in \mathcal{M}$, *admissible*.

Proposition 4.3. *Let U be a Galois algebra associated with Γ-bimodule $V \oplus V^\circ$. If for an admissible set $\{\lambda_\varphi \in L^{H_\varphi}, \varphi \in \mathcal{M}\}$, $[a\varphi] \in V$ implies $[(a^{\varphi^{-1}}\lambda_{\varphi^{-1}}^{-1})\,\varphi^{-1}] \in V^\circ$, then it defines an involution on U.*

Proof. Let $\circ$ be the involution on $L * \mathcal{M}$ defined by the admissible set above. The conditions in the proposition show that $\circ$ induces an involution on the subspace $V \oplus V^\circ \subset U$, which generates U. The statement follows. ☐

Example 4.4. *Consider the case of GWA, realized as Galois algebras in subsection 7.2 (see Proposition 7.1). Then an involution on $K * \mathbb{Z}$ is obviously defined by $\lambda_\sigma \in K$ such that $(\sigma)^\circ = \sigma^{-1}\lambda_\sigma$. Then the condition, that $\circ$ interchanges bijectively ΛX and ΛY is equivalent to the condition $a\lambda_{\sigma^{-1}} \in \Lambda^*$. In particular, the canonical involution $X \leftrightarrow Y$ is obtained in the case $\lambda_{\sigma^{-1}} = a^{-1}$, equivalently $\lambda_\sigma = a^\sigma$.*

4.4. **Galois rings of finite rank.** Although all considered algebras are defined over the base field $\Bbbk$, the construction below allows to use the same approach in a more general situation, since the rings of the form $\mathcal{K} = L * \mathcal{M}^G$, where $\mathcal{M}$ is a group, allow a simple direct construction.

Let Λ be a commutative domain integrally closed in its fraction field L, $\mathcal{G} \subset \operatorname{Aut} L$ a subgroup, which splits into the semi-direct product of its subgroups $\mathcal{G} = G \ltimes \mathcal{M}$, where G is finite and $\mathcal{M}$ is finitely generated. Denote $\Gamma = \Lambda^G$ and $K = L^G$. Then Λ is just the integral closure of Γ in L and the action of G on $L * \mathcal{M}$ is defined as above. Consider the ring $\mathcal{K} = L * \mathcal{M}^G$ and a finitely generated Γ-subring $U \subset \mathcal{K}$ such that $KU = UK = \mathcal{K}$. Such subring is called a *Galois ring with respect to* Γ.

If $\mathcal{M}$ is a finite group then a Galois algebra $U \subset K$ with respect to Γ will be called a *Galois ring of finite rank*. If in addition U is a $\Bbbk$-algebra then it will be called a *Galois algebra of finite rank.*

Proposition 4.4. *Let U be a Galois algebra of finite rank with respect to Γ and $E = L^{\mathcal{G}}$. Then $\mathcal{K}$ is a simple central algebra with respect to E and $\dim_E \mathcal{K} = |\mathcal{M}|^2$.*

Proof. Theorem 5.1 gives the statement about the center, while Corollary 4.1 gives the statement about the simplicity. Now from formulas (3) and (7) we obtain

$$\dim_K \mathcal{K} = \dim_K (L * \mathcal{M})^G = \sum_{\varphi \in G\backslash\mathcal{M}} \dim_K (K * \mathcal{M})^G_\varphi = \sum_{\varphi \in G\backslash\mathcal{M}} |\mathcal{O}_\varphi| = |\mathcal{M}| \tag{16}$$

both as a left and as a right K-space structure. On other hand, $\dim_E K = |\mathcal{M}|$, that completes the proof. □

5. Skew fields of fractions of Galois algebras

Let $U \subset L * \mathcal{M}^G$ be a Galois algebra with respect to a subalgebra Γ. Assume that U is a domain such that the multiplicative set $S = U \setminus \{0\}$ satisfies both left and right Ore conditions. Then U admits a skew field of fraction $\mathcal{U}$. In particular, any noetherian domain (e.g. iterated Ore extension) admits a skew field of fractions.

A natural question is what these skew fields look like. A knowledge of these rings gives a non-commutative version of "birational equivalence" for Galois algebras. Hence we will call two domains *rationally equivalent* if their skew fields of fractions are isomorphic.

It is a classical result that the operation of taking the invariants of a finite subgroup of automorphisms of a commutative domain commutes with the construction of the fraction field. In non-commutative case we recall the following standard result ([Co]).

Proposition 5.1. *If a non-commutative domain A satisfies the left and the right Ore conditions and H is a finite subgroup of automorphisms of A with invertible $|H|$ then A^H*

satisfies the left and the right Ore conditions and the skew field of fractions of A^H consists of H-invariants of the skew fraction field of A.

Since $\operatorname{char} \Bbbk = 0$ we can apply Proposition 5.1 in the case of $A = L * \mathcal{M}$.

Corollary 5.1. *Let $L * \mathcal{M}$ be a domain satisfying the left and the right Ore conditions. Then $L * \mathcal{M}^G$ satisfies the left and the right Ore conditions, the skew field of fractions $\mathcal{L}$ of $L * \mathcal{M}$ is endowed with the action of G and $\mathcal{L}^G$ coincides with the skew field of fractions of $L * \mathcal{M}^G$.*

Moreover, one can describe the skew field of fractions of the Galois algebras. Namely, one has

Corollary 5.2. *Let U be a Galois Γ-algebra. If the skew group algebra $L * \mathcal{M}$ allows a skew field of fractions $\mathcal{L}$ then $\mathcal{U} = \mathcal{L}^G$, where $\mathcal{U}$ is the skew field of fractions of U. In particular, all Galois subalgebras with respect to Γ in $L * \mathcal{M}^G$ are rationally equivalent.*

Proof. Clearly, $\mathcal{U} \subset \mathcal{L}^G$. Let $x \in \mathcal{L}$ be a G-invariant element. Then $x = s^{-1}u$, $s, u \in L * \mathcal{M}$, $s \neq 0$ and $sx = u$. By Corollary 5.1 there exist $\bar{s} \neq 0, \bar{u} \in L * \mathcal{M}^G$ such that $\bar{s}x = \bar{u}$ and $x = (\bar{s})^{-1}\bar{u}$. In fact, $\bar{s} = \sum_{g \in G} \lambda^g s^g \neq 0$, $\bar{u} = \sum_{g \in G} \lambda^g u^g$ for some $\lambda \in L$. The statement follows immediately from Proposition 4.1, (2). □

5.1. **Skew field centers of Galois algebras.** It is well-known that if $\mathcal{G}$ is a semisimple or nilpotent Lie algebra then the center of the skew field of fractions of $U(\mathcal{G})$ equals the field of fractions of the center of $U(\mathcal{G})$ ([D], 4.3.6). In particular it holds for $U(\mathrm{gl}_n)$ since gl_n is reductive. Here we have the following generalization of this fact for Galois algebras.

Let $U \subset L * \mathcal{M}^G$ be a Galois algebra with respect to Γ. By Theorem 4.1,(4), the center of U is $Z(U) = U \cap K^{\mathcal{M}}$. Since $UK = KU = L * \mathcal{M}^G$ the center of the skew group product $L * \mathcal{M}^G$ equals $K^{\mathcal{M}}$. Suppose that U allows the right and the left calculus of fractions. Denote by $\mathcal{U}$ the skew field of fractions of U and by $\mathcal{Z}$ the center of $\mathcal{U}$. A natural question is whether $\mathcal{Z}$ is isomorphic to the field of fractions of $Z(U)$.

Assume that $m^{-1}(\Gamma) \subset \bar{\Gamma}$ (respectively $m(\Gamma) \subset \bar{\Gamma}$) for all $m \in \mathcal{M}$. Let $S \subset \mathcal{M}$ be a finite G-invariant subset. Denote $U(S) = \{u \in U \mid \operatorname{supp} u \subset S\}$. Obviously, it is a Γ-subbimodule in U. It will be convenient to consider the Γ-bimodule structure on U as the $\Gamma \otimes_{\Bbbk} \Gamma$-module structure.

Let $f \in \Gamma$. Introduce the element $f_S^r \subset \Gamma \otimes_{\Bbbk} K$ (respectively $f_S^l \subset K \otimes_{\Bbbk} \Gamma$) as follows

$$f_S^r = \prod_{s \in S} (f \otimes 1 - 1 \otimes f^{s^{-1}}) = \sum_{i=0}^{|S|} f^{|S|-i} \otimes T_i, \ (T_0 = 1). \tag{17}$$

(respectively $f_S^l = \prod_{s \in S} (f^s \otimes 1 - 1 \otimes f)$).

From now on we will consider the properties of f_S^r. The case of f_S^l can be treated analogously.

Since S is G-invariant, then all T_i are G-invariant expressions, which are integral over Γ. Therefore $T_i \in \Gamma$ for all i and $f_S \in \Gamma \otimes \Gamma$.

We have the following lemma which describes the properties of f_S^Γ.

Lemma 5.1. *Let $S \subset \mathcal{M}$ be a G-invariant subset and $\mathcal{M}(\Gamma) \subset \bar{\Gamma}$. For any subset $X \subset \mathcal{M}$ set $f_X = f_X^\Gamma$. Then*

(1) *Let $u \in U$. Then $u \in U(S)$ if and only if $f_S \cdot u = 0$ for every $f \in \Gamma$.*
(2) *Let $u \in U$ and $T = \operatorname{supp} u \setminus S$. Then $f_T \cdot u \in U(S)$.*
(3) *Let $S \subset T$ be G-invariant subsets in $\mathcal{M}$, $f \in \Gamma$, $f_{T\setminus S} = \sum_{i=1}^{n} f_i \otimes g_i \in \Gamma \otimes_{\Bbbk} \Gamma$, $a \in L$, $m \in \mathcal{M}$. Then $f_{T\setminus S} \cdot [am] = [(\sum_{i=1}^{n} f_i g_i^m a)m]$.*
(4) *If $f \in \Gamma$, $S = \{e\}$ and $f_{T\setminus S} = \sum_{i=1}^{n} f_i \otimes g_i \in \Gamma \otimes_{\Bbbk} \Gamma$, then $f_{T\setminus S} \cdot u = (\sum_{i=1}^{n} f_i g_i)u$.*
(5) *Let S be a G-orbit. The Γ-bimodule homomorphism $P_S^T : U(T) \longrightarrow U(S)$, $u \mapsto f_{T\setminus S} \cdot u$ is either zero or $\operatorname{Ker} P_S^T = U(T \setminus S)$ (both cases are possible, cf. (1)).*
(6) *Let $S = S_1 \sqcup \cdots \sqcup S_n$ be the decomposition of S in G-orbits and $P_{S_i}^S : U(S) \longrightarrow U(S_i)$, $i = 1, \dots, n$ are defined in (5) nonzero homomorphisms. Then the homomorphism*

$$P^S : U(S) \longrightarrow \bigoplus_{i=1}^{n} U(S_i),\ P^S = (P_{S_1}^S, \dots, P_{S_n}^S), \tag{18}$$

is a monomorphism.

Proof. Consider any $m \in \mathcal{M}$, $s \in \operatorname{Aut} L$ and $a \in L$. Then

$$(f \otimes 1 - 1 \otimes f^s) \cdot [am] = [fam] - [amf^s] = [(f - f^{ms})am]$$

and

$$f_S \cdot [am] = \prod_{s \in S} (f \otimes 1 - 1 \otimes f^{s^{-1}}) \cdot [am] = [\prod_{s \in S} (f - f^{ms^{-1}})am].$$

If $m \in S$, then one of the factors $f - f^{ms^{-1}}$ is zero and, hence, $f_S \cdot [am] = 0$.

To prove the statement "if" it is enough to show, that for any $m \notin S$ there exists $f \in \Gamma$, such that $f \neq f^{ms^{-1}}$ for all $s \in S$. Since the action of $\mathcal{M}$ on L is separating, for every $m \in \mathcal{M}, m \neq e$ the space of m-invariants $\Gamma^m \neq \Gamma$. But over an infinite field the $\Bbbk$-vector space Γ can not be the union of a finite number of proper subspaces $\bigcup_{s \in S} \Gamma^{ms^{-1}}$, that completes the proof of (1).

By definition $u \in U(\operatorname{supp} u)$, hence $f_{\operatorname{supp} u} \cdot u = 0$ for any $f \in \Gamma$. Then the statement (2) follows from (1) and from the equality $f_{\operatorname{supp} u} = f_S f_T$.

The statement (3) follows from the formulas (6) (page 5) and (4) follows from (3).

To prove (5) note that by (3), $f_{T\setminus S} \neq 0$ if and only if $\sum_{i=1}^{n} f_i g_i^m \neq 0$, and in the last case $f_{T\setminus S}$ acts on $U(S)$ injectively. Finally, the statement (6) follows from (5), since $\bigcap_{i=1}^n \operatorname{Ker} P_{S_i}^S = 0$. □

Similar statements hold for the polynomial f_S^l. Now we are in the position to prove

Theorem 5.1. *Let $U \subset L * \mathcal{M}^G$ be a Galois algebra with respect to Γ that allows both left and right skew field of fractions $\mathcal{U}$ and $\mathcal{M}(\bar{\Gamma}) \subset \bar{\Gamma}$. Then $\mathcal{Z} = K^{\mathcal{M}}$.*

Proof. Let $z = s^{-1}u \in \mathcal{Z}$, $s, u \in U$, $s, u \neq 0$. We can assume that the unit of $\mathcal{M}$ belongs to $\operatorname{supp} u$. Indeed, let $u = [am] + \sum_{m' \neq m} [a_{m'}m']$, $a \neq 0$. Then following Lemma 4.2 there exists $[bm^{-1}] \in U$ such that $e \in \operatorname{supp}[bm^{-1}][am]$. On the other hand $e \notin \operatorname{supp}[bm^{-1}][a_{m'}m']$ and hence $e \in \operatorname{supp}[bm^{-1}]u$. Then we can change u to $[bm^{-1}]u$ and s to $[bm^{-1}]s$.

Since $sz = zs$, we have $s^{-1}u = us^{-1}$. Then for any $x \in U$ holds $xus^{-1} = s^{-1}ux$, i.e. $sxu = uxs$. Therefore, for any $f_i, g_i \in \Gamma, i = 1, \dots, n$ holds

$$s \sum_{i=1}^{n} f_i u g_i = u \sum_{i=1}^{n} f_i s g_i.$$

In particular, $s_1 = \sum_{i=1}^{n} f_i s g_i = 0$ if and only if $s_2 = \sum_{i=1}^{n} f_i u g_i = 0$. It follows immediately from Lemma 5.1, (1) that $\operatorname{supp} s = \operatorname{supp} u$. Since $e \in \mathcal{M}$ belongs to this support and $U_e \subset K$, then applying Lemma 5.1, (6) we conclude that there exist $f_i, g_i \in \Gamma, i = 1, \dots, n$ such that both $s_1, s_2 \in K \setminus \{0\}$. Hence $s^{-1}u = s_1 s_2^{-1} \in K$ and thus $\mathcal{Z} \subset K$. Using the same reasoning as in the proof of Theorem 4.1, (4), we conclude that $\mathcal{Z} \cap K = K^{\mathcal{M}}$ completing the proof. □

Remark 5.1. *Note that $\mathcal{Z}$ in Theorem 5.1 is not necessarily isomorphic to the field of fractions of $Z(U)$.*

6. Gelfand-Kirillov dimension of Galois algebras

6.1. **Growth of group algebras.** Let $S_* = \{S_1 \subset S_2 \subset \dots \subset S_N \subset \dots\}$ be an increasing chain of finite sets. Then the growth of S_* is defined as

$$\text{growth}(S_*) = \overline{\lim_{N\to\infty}} \log_N |S_N|. \tag{19}$$

For $s \in S = \bigcup_{i=0}^{\infty} S_i$ we say that $\deg s = i$ provided that $s \in S_i \setminus S_{i-1}$, $i \geq 1$.

We will assume that the Gelfand-Kirillow dimension of Γ and the growth of $\mathcal{M}$ both are finite.

Let $\{\gamma_1, \dots, \gamma_k\}$ be a set of generators of Γ. For $N \in \mathbb{N}$ denote by $\Gamma_N \subset \Gamma$ the subspace of Γ generated by the products $\gamma_{i_1} \dots \gamma_{i_t}$, for all $t \leq N$, $i_1, \dots, i_t \in \{1, \dots, k\}$. Let $d_\Gamma(N) = \dim_{\Bbbk} \Gamma_N$ and $B_N(\Gamma)$ a basis in Γ_N ($B_1(\Gamma) = \{\gamma_1, \dots, \gamma_k\}$).

Fix a set of generators $\mathcal{M}$ of the form $\mathcal{M}_1 = \{G\cdot\varphi_1, \dots, G\cdot\varphi_n\}$. For $N \geq 1$, $\mathcal{M}_N$ is just the set of words $w \in \mathcal{M}$ such that $l(w) \leq N$, where l is the length of w, hence

$$\mathcal{M}_{N+1} = \mathcal{M}_N \bigcup \Big(\bigcup_{\varphi\in\mathcal{M}_1} \varphi\cdot\mathcal{M}_N \Big). \tag{20}$$

Note that all sets $\mathcal{M}_N$ are G-invariant. Denote the cardinality of $\mathcal{M}_N$ by $d_{\mathcal{M}}(N)$. Let $\mathcal{M}_* = \{\mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \subset \mathcal{M}_N \subset \dots\}$. Then the growth growth$(\mathcal{M})$ is by definition growth$(\mathcal{M}_*)$.

Without loss of generality we will assume that the Galois algebra U is generated over Γ by a set of generators $\mathcal{G} = \{[a_1\varphi_1], \dots, [a_n\varphi_n]\}$. Set $B_1(U) = B_1(\Gamma) \sqcup \mathcal{G}$. As above define the subspaces U_N and dimensions $d_U(N)$. For every $N \geq 1$ fix a basis $B_N(U)$ of U_N.

Let $\Gamma[\mathcal{M}]$ there is the group algebra. Then (in the notations above) the space $\Gamma[\mathcal{M}]_N$ has a G-invariant basis

$$B_N(\Gamma[\mathcal{M}]) = \bigsqcup_{i=0}^{N} \bigsqcup_{\substack{w\in\mathcal{M}_{N-i},\\ l(w)=N-i,}} B_i(\Gamma)w. \tag{21}$$

Then by definition GKdim $\Gamma[\mathcal{M}]$ = growth $B_*(\Gamma[\mathcal{M}])$. The group G acts on the chain $B_*(\Gamma[\mathcal{M}])$, this action is induced by its action on the generators of $\mathcal{M}_1$. Then the growth of the chain $B_*(\Gamma[\mathcal{M}])/G$ is equal to the growth growth $B_*(\Gamma[\mathcal{M}])$, since

$$|B_N(\Gamma[\mathcal{M}])| > |B_N(\Gamma[\mathcal{M}])/G| \geq \frac{1}{|G|}|B_N(\Gamma[\mathcal{M}])|.$$

Remark 6.1. *Consider a chain in* $L * \mathcal{M}^G$ *formed by*

$$B_N(\Gamma[[\mathcal{M}]]) = \bigsqcup_{i=0}^{N} \bigsqcup_{\substack{w\in\mathcal{M}_{N-i}/G,\\ l(w)=N-i,}} B_i(\Gamma)[w], \tag{22}$$

$N \geq 1$. *Then its growth equals* GKdim $\Gamma[\mathcal{M}]$. *It follows from the fact that*

$$\Big|\Big(\bigsqcup_{\substack{w\in\mathcal{M}_{N-i},\\ l(w)=N-i,}} B_i(\Gamma)w \Big)/G\Big| = \Big| \bigsqcup_{\substack{w\in\mathcal{M}_{N-i}/G,\\ l(w)=N-i,}} B_i(\Gamma)[w]\Big|. \tag{23}$$

The following formula is well known ([MCR])

$$\text{GKdim}\,\Gamma[\mathcal{M}] = \text{GKdim}\,\Gamma + \text{growth}(\mathcal{M}), \tag{24}$$

e.g. it follows from the formula (21).

6.2. **Gelfand-Kirillov dimension.** The goal of this section is to prove (under a certain condition) an analogue of the formula (24) for Galois algebras.

We will enforce the following restriction

Condition 1. *For every finite dimensional* $\Bbbk$*-vector space* $V \subset \bar{\Gamma}$ *the set* $\mathcal{M} \cdot V$ *is contained in a finite dimensional subspace in* $\bar{\Gamma}$.

Obviously, it is enough to check this condition in the case when V generates $\bar{\Gamma}$ over $\Bbbk$. The main result in this section is the following

Theorem 6.1. *Let* $U \subset L * \mathcal{M}$ *be a Galois* Γ*-algebra satisfying Condition 1,* $\mathcal{M}$ *a group of finite growth* growth$(\mathcal{M})$. *Then*

$$\text{GKdim}\, U \geq \text{GKdim}\, \Gamma + \text{growth}(\mathcal{M}). \tag{25}$$

The proof of this result is based on the following lemmas.

Lemma 6.1. *Suppose there exist* $p, q \in \mathbb{N}$ *and* $C \geq 0$ *such that for any* $N \in \mathbb{N}$ *holds*

$$d_U(pN + q) \geq C d_{\Gamma[\mathcal{M}]}(N). \tag{26}$$

Then $\text{GKdim}\, U \geq \text{GKdim}\, \Gamma[\mathcal{M}]$.

Proof.

$$\text{GKdim}\, \Lambda[\mathcal{M}] = \overline{\lim_{N\to\infty}} \log_N d_{\Gamma[\mathcal{M}]}(N) \leq \overline{\lim_{N\to\infty}} \log_N d_U(pN + q) = \tag{27}$$

$$\overline{\lim_{N\to\infty}} \log_{pN+q}(d_U(pN + q)) \frac{1}{\log_N(pN + q)} = \overline{\lim_{N\to\infty}} \log_{pN+q} d_U(pN + q) \leq \tag{28}$$

$$\overline{\lim_{N\to\infty}} \log_N d_U(N) = \text{GKdim}\, U. \tag{29}$$

□

Lemma 6.2. *Let* $\deg_\Lambda$ *be a degree on* Λ *defined by some set of generators* $\lambda_1, \dots, \lambda_s$. *Then for any* $d \geq 0$ *there exists* $e\,(= e(d))$, *such that, given* $\gamma \in \Gamma$, *from* $\deg_\Lambda \gamma \leq d$ *follows* $\deg \gamma \leq e$ *for the defined above* $\deg$ *in* Γ.

Proof. Since Λ_d is finite dimensional and $\bigcup_{i=0}^{\infty} \Gamma_i = \Gamma$ then there exists $e > 0$, such that $\Lambda_d \cap \Gamma = \Lambda_d \cap \Gamma_e$. □

Lemma 6.3. *There exists* $p' \geq 1$ *with the following property: if for some* $N, i \geq 0$ *the set* U_N *contains all the elements* $[b_m m]$, *for some* $b_m \neq 0$, *where* m *runs* $\mathcal{M}_i$, *then* $U_{N+p'}$ *contains* $[b_m m]$, *where* m *runs* $\mathcal{M}_{i+1}$ *with* $b_m \neq 0$.

Proof. Let $[a\varphi]$ be a standard generator of U and $[b_m m] \in U_N$ as in the lemma. Obviously,

$$\bigcup_{\substack{\varphi \in \{\varphi_1, \dots, \varphi_k\} \\ m \in \mathcal{M}_i \setminus \mathcal{M}_{i-1}}} \text{supp}[\varphi][m] = \mathcal{M}_{i+1}.$$

Step 1. We prove, that for some s (s does not depend on N)

$$\mathcal{M}_{i+1} \subset \{\text{supp}\, u \mid u \in U_{N+s}\}.$$

Let $m' \in \text{supp}[\varphi][m]$. We can assume without loss of generality that $m' = \varphi m$. Then the coefficient by φm in $[a\varphi][cm]$ equals

$$([a\varphi]\cdot[cm])_{\varphi m}=\sum_{\substack{g_1\in G/H_\varphi,g_2\in G/H_m\\ g_1\varphi g_1^{-1}g_2mg_2^{-1}=\varphi m}} a^{g_1}c^{g_1\varphi g_1^{-1}g_2}. \tag{30}$$

Note that if $g_1\in G/H_\varphi$ is fixed in the sum above, then condition $m^{g_2}=(\varphi^{-1})^{g_1}\varphi m$ defines a unique (if exists) $g_2\in G/H_m$. Hence g_2 is determined by g_1, $g_2=g_2(g_1)$, and we can rewrite the formula as

$$([a\varphi]\cdot[cm])_{\varphi m}=\sum_{g\in S\subset G/H_\varphi} a^g c^{\varphi^g g_2(g)}. \tag{31}$$

Since $[a\varphi]\gamma[cm]=[\gamma^\varphi a\varphi][cm]$ for any $\gamma\in\Gamma$, we obtain

$$([a\varphi]\gamma[cm])_{\varphi m}=\sum_{g\in S\subset G/H_\varphi}\gamma^{g\varphi}a^g c^{\varphi^g g_2(g)}. \tag{32}$$

All the automorphisms $g\varphi, g\in S$ are different. Hence there exists $\gamma_S\in\Gamma$ such that $g\varphi(\gamma_S), g\in S$, are mutually different. Let

$$X_S=\left((g\varphi((\gamma_S^j)))\right)_{g\in S,j=0,\dots,|S|-1},\quad v_S=(a^g c^{\varphi^g g_2(g)})_{g\in S}.$$

The Vandermond determinant $\det X_S$, is nonzero. Hence there exists j, $0\le j\le |S|-1$ such that

$$([a\varphi]\gamma^j[cm])_{\varphi m}=\sum_{g\in S,S\subset G/H_\varphi}(\gamma^{\varphi^g})^j a^g c^{\varphi^g g_2(g)}\neq 0, \tag{33}$$

which is just the j-th element of the vector $X_S\cdot v_S\neq 0$. Denote

$$s_1(\varphi,m)=\deg_U[a\varphi]\gamma^j,\quad s_2(\varphi,m)=\max_{g_1,g_2\in G}s_1(\varphi^{g_1},m^{g_2}),\ \text{and}\ s=\max_{\varphi\in\mathcal{M}_1,m\in\mathcal{M}_i}s_2(\varphi,m).$$

Hence, for every $m'\in\mathcal{M}_{i+1}$ there exists $u\in U_{N+s}$ such that $m'\in\operatorname{supp}u$ and $\operatorname{supp}u=\operatorname{supp}[\varphi]\cdot\operatorname{supp}[m]=\operatorname{supp}[\varphi][m]$ for some $\varphi\in\mathcal{M}_1$ and $m\in\mathcal{M}_i$.

Step 2. We prove, that for every $k\ge 0$ and every $u\in U_j$ with $|\operatorname{supp}u|\le k$, there exists $t=t(k)$ such that for any $\psi\in\operatorname{supp}u$, U_{j+t} contains an element of the form $[b\psi]$ (t depends only on k).

If $\operatorname{supp}u=G\cdot\psi$ then $u=[b\psi]$, and there is nothing to prove. Assume $u=[b'\psi]+\dots$. For $f\in\Lambda_1$ consider the polynomial

$$f_S=\prod_{s\in S}(f^s\otimes 1-1\otimes f)=\sum_{i=0}^{|S|}T_i\otimes f^{|S|-i},\ (T_0=1), \tag{34}$$

where $S = \operatorname{supp} u \setminus G\cdot\psi$. Applying Lemma5.1 we obtain

$$f_S\cdot u = f_S\cdot[b'\psi] = \sum_{g\in G/H_\psi} b'^g \prod_{s\subset S}(f^s - f^{\psi^g})\psi^g = [b'\prod_{s\in S}(f^s - f^\psi)\psi].$$

Since $G\psi \cap S = \varnothing$, there exists $f \in \Gamma_1$ such that all factors in the last product are non-zero. Indeed, if $f^s = f^\psi$ for every $f \in \Gamma_1$, then s and ψ differ by an element of G which is a contradiction.

Fix such f and denote

$$(35)\qquad [b\psi] := f_S\cdot u = \sum_{i=0}^{|S|} T_i u f^{|S|-i}, \text{ where } T_i = \sum_{\substack{T\subset S,\\ T=\{t_1,\dots,t_i\},|T|=i}} f^{t_1}\dots f^{t_i} \in \Gamma.$$

Let $C_1 = \max_{\gamma\in\mathcal{M}\cdot\Gamma_1} \deg_\Lambda \gamma$ (C_1 is finite due to Condition 1). Then $\deg_\Lambda T_i \le iC_1 \le kC_1$ and $\deg_\Lambda f^{|S|-i} \le kC_1$. Hence, $\deg T_i \le e(kC_1)$ and $\deg f^{|S|-i} \le e(kC_1)$ by Lemma 6.2. Thus we can chose $t(k) = 2e(kC_1)$.

Step 3. Fix s from Step 1 and $t = t(|G|^2)$ from Step 2. Set $p' = s + t(|G|^2)$. Suppose that $[cm] \in U_N$ for some $m \in \mathcal{M}_i$. Then by Step 1 given $\varphi \in \mathcal{M}_1$ there exists $u \in U_{N+s}$ such that $\varphi m \in \operatorname{supp} u$. Moreover u can be chosen in the form $u = [a\varphi]\gamma[cm]$ for some $\gamma\in\Gamma$, in particular $\operatorname{supp} u \subset \operatorname{supp}[\varphi][m] \le |G|^2$. Applying Step 2 we conclude that $U_{N+p'}$ contains an element of the form $[b\varphi m]$ which completes the proof. □

Now we are in the position to prove Theorem 6.1. The space U_1 contains elements of the form $[a_i\varphi_i]$, where φ_i runs over $\mathcal{M}_1/G$. Hence, by Lemma 6.3, $U_{p'(N-1)+1}$ contains the set $\tilde{\mathcal{M}}_N = \{[c_m m] \mid m \in \mathcal{M}_N, c_m \neq 0\}$. On the other hand $B_N(\Gamma) \subset U_{p'N+1}$ and hence $U_{(p'+1)N+1}$ contains $\Gamma_N\tilde{\mathcal{M}}_N$. Moreover all elements from the last product are linearly independent. But the set $B_N(\Gamma[[\mathcal{M}]])$ is embedded into $\Gamma_N\tilde{\mathcal{M}}_N$ by setting $\gamma[w] \mapsto \gamma[c_w w]$, $\gamma\in\Gamma_N$, $w \in \mathcal{M}_{N+1}$. Therefore,

$$d_U((p'+1)N+1) \ge |B_N(\Gamma[[\mathcal{M}]])| \ge \frac{1}{|G|}|B_N(\Gamma[\mathcal{M}])|.$$

It remains to set $p = p'+1, q = 1$, $C = \frac{1}{|G|}$ and apply Lemma 6.1. Theorem is proved.

7. Examples of Galois algebras

7.1. Commutative case. Even though our goal was to introduce a class of non-commutative Galois algebras, this concept has a natural interpretation in the commutative case as well.

Let U be a Galois subalgebra with respect to Γ in $L * \mathcal{M}^G$. If U is commutative, then $\mathcal{M}$ acts trivially on K, hence $\mathcal{M} = \{e\}$. Therefore

$$\Gamma \subset U \subset L * \mathcal{M}^G = L^G = K.$$

On the other hand any finitely generated over Γ subring in K is a Galois Γ-algebra.

7.2. **Generalized Weyl algebras.** In this section we realize the Generalized Weyl algebras (GWA) with infinite order automorphisms as Galois algebras. We recall the definition of GWA from [Ba].

Let D be a ring, $Z(D)$ its center, $\sigma = (\sigma_1, \dots, \sigma_n)$, $\sigma_i \in \operatorname{Aut} D$, $\sigma_i\sigma_j = \sigma_j\sigma_i$, $a = (a_1, \dots, a_n)$, $a_i \in Z(D)$, $i = 1, \dots, n$, such that $\sigma_i(a_j) = a_j$, if $i \neq j$. The generalized Weyl algebra $D(\sigma, a)$ of degree n is the ring generated by D and $X_1, \dots, X_n$, $Y_1, \dots, Y_n$ subject to the relations:

$$Y_iX_i = a_i, \quad X_iY_i = \sigma_i(a_i),$$
$$X_id = \sigma_i(d)X_i, \quad Y_id = \sigma_i^{-1}(d)Y_i,$$

for all $d \in D$, and

$$[X_i, X_j] = [Y_i, Y_j] = [X_i, Y_j] = 0,$$

for all $i \neq j$.

For simplicity we consider only degree 1 generalized Weyl algebras.

Let Λ be an integral domain with the field of fractions K, $\sigma : \Lambda \longrightarrow \Lambda$ is an automorphism of Λ of infinite order. If $\varphi \in \operatorname{Aut}\Lambda$ then denote by Λ_φ a Λ-bimodule Λv such that $\lambda \cdot v = v\varphi(\lambda)$ and $v \cdot \lambda = \varphi^{-1}(\lambda)v$ for all $\lambda \in \Lambda$.

Let X and Y be generators of the bimodules $\Lambda_{\sigma^{-1}}$ and Λ_σ respectively, and let $V = \Lambda_{\sigma^{-1}} \oplus \Lambda_\sigma$. As a splitting field for the bimodule V we can choose K itself. Then any structure homomorphism $\tau : \Lambda[V] \longrightarrow K* < \sigma, \sigma^{-1} >$ has a form

$$X \longmapsto a_X b_X^{-1}\sigma^{-1}, \quad Y \longmapsto a_Y b_Y^{-1}\sigma,$$

for some $a_X, b_X, a_Y, b_Y \in \Lambda \setminus \{0\}$. Indeed, suppose $\tau(X) = \alpha\sigma + \beta\sigma^{-1}$ for some $\alpha, \beta \in K$. Then for any $\lambda \in \Lambda$,

$$\lambda\tau(X) = \lambda(\alpha\sigma + \beta\sigma^{-1}) = \alpha\sigma\sigma(\lambda) + \beta\sigma^{-1}\sigma^{-1}(\lambda) = \tau(\lambda X) =$$
$$\tau(X\sigma^{-1}(\lambda)) = (\alpha\sigma + \beta\sigma^{-1})\sigma^{-1}(\lambda).$$

Hence, $\sigma^2(\lambda) = e$ for all $\lambda \in \Lambda$, which is a contradiction.

Without loss of generality we can assume $a_X = b_X = 1$. Let U be a corresponding Galois algebra. The element $a = a_Y b_Y^{-1}$ defines a 2-cocycle $\xi : \mathbb{Z} \times \mathbb{Z} \to K$, such that $\xi(-1, 1) = a$.

Proposition 7.1. *Let $a = a_Y b_Y^{-1} \in \Lambda$. Then U is isomorphic to the generalized Weyl algebra generated over Λ by X, Y subject to the relations*

$$X\lambda = \lambda^\sigma X, \quad \lambda Y = Y\lambda^\sigma, \quad \lambda \in \Lambda; \qquad\qquad (36)$$
$$YX = a, \quad XY = a^\sigma.$$

Proof. Let A be the GWA, defined in (36). Then there exists the canonical epimorphism of rings $\pi : A \longrightarrow U$. On the other hand, the algebra U has the following decomposition as Λ-bimodule:

$$U = \Lambda \oplus (\bigoplus_{i=1}^{\infty}(\Lambda X^i \oplus \Lambda Y^i)).$$

It implies that π is a monomorphism and $A \simeq U$. □

Proposition 7.1 immediately implies

Corollary 7.1. *Generalized Weyl algebra A is a Galois subalgebra of $K * \mathbb{Z}$. Moreover, if the cocycle ξ is invertible then A is isomorphic to $\Lambda * \mathbb{Z}$.*

Remark 7.1. *Note, that in the case if a finite order automorphism σ the corresponding GWA is not a Galois algebra.*

Recall that GWA is endowed with the canonical involution $X \leftrightarrow Y$ (cf. 4.4).

7.3. **PBW algebras.** Let U be an associative algebra over $\Bbbk$, endowed with an increasing filtration $\{U_i\}_{i\in\mathbb{Z}}$, $U_{-1} = \{0\}$, $U_0 = \Bbbk$, $U_iU_j \subset U_{i+j}$. Let $\overline{U} = \operatorname{gr} U$ be the associated graded algebra $\overline{U} = \bigoplus_{i=0}^{\infty} U_i/U_{i-1}$.

Recall that algebra U is called a PBW algebra if any element of U can be written uniquely as a linear combination of ordered monomials in some fixed generators of U. We will assume for that U is a PBW algebra and that $\operatorname{gr} U$ is a polynomial algebra in n variables. A well known result of Gelfand and Kirillov states that U is an Ore domain and the field of fractions has dimension n. In particular, the Gelfand-Kirillov dimension of U equals n.

Theorem 7.1. *Let U be a PBW algebra generated by the elements $u_1, \dots, u_k$ over Γ, $\operatorname{gr} U$ a polynomial ring in n variables, $\mathcal{M} \subset \operatorname{Aut} L$ a group and $f : U \to L*\mathcal{M}^G$ a homomorphism such that $\cup_i \operatorname{supp} f(u_i)$ contains the generators of $\mathcal{M}$. If $\operatorname{GKdim} \Gamma + \operatorname{growth} \mathcal{M} = n$ then f is an embedding and U is a Galois Γ-algebra.*

Proof. Since $f(U)$ contains the generators of $\mathcal{M}$ then $f(U)$ is a Galois Γ-algebra by Proposition 4.2. Let $I = \operatorname{Ker} f \neq 0$. Then

$$n = \operatorname{GKdim} U = \operatorname{GKdim} \operatorname{gr} U > \operatorname{GKdim} \operatorname{gr} U/\operatorname{gr} I = \operatorname{GKdim} U/I = \operatorname{GKdim} f(U).$$

On the other hand,

$$n = \operatorname{GKdim} U \geq \operatorname{GKdim} f(U) \geq \operatorname{GKdim} \Gamma + \operatorname{growth} \mathcal{M} = n$$

by Theorem 6.1. Therefore $\operatorname{GKdim} f(U) = n$ which is a contradiction. Hence $I = 0$ and f is an injection. We conclude that U is a Galois Γ-algebra.

□

Theorem 7.1 will be applied to construct examples of Galois algebras.

7.3.1. **General linear Lie algebras.** Consider the general lineal Lie algebra gl_n with the standard basis $e_{ij}, i, j = 1, \dots, n$, $[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj}$. Denote by $U_n = U(\mathrm{gl}_n)$ its universal enveloping algebra. Let Z_n be the center of U_n. We identify gl_m for $m \leqslant n$ with a Lie subalgebra of gl_n spanned by $\{e_{ij} \,|\, i, j = 1, \dots, m\}$, so that we have the following chain of inclusions

$$\mathrm{gl}_1 \subset \mathrm{gl}_2 \subset \ldots \subset \mathrm{gl}_n .$$

It induces the inclusions $U_1 \subset U_2 \subset \ldots \subset U_n$. Denote $U = U_n$. The *Gelfand-Tsetlin* subalgebra Γ in U is generated by $\{Z_m \,|\, m = 1, \dots, n\}$. Note that Z_m is a polynomial algebra in m variables $\{c_{mk}\,|\,k=1,\dots,m\}$,

$$c_{mk} \;=\; \sum_{(i_1,\dots,i_k)\in\{1,\dots,m\}^k} e_{i_1i_2}e_{i_2i_3}\cdots e_{i_ki_1}, \tag{37}$$

and Γ is a polynomial algebra in $\dfrac{n(n+1)}{2}$ variables $\{c_{ij}\,|\,1\leqslant j\leqslant i\leqslant n\}$, [Zh]. Let K be the field of fractions of Γ.

Following [DFO2] consider the space $\mathcal{L}=\Bbbk^{\frac{n(n+1)}{2}}$ of double indexed vectors $\ell=(\ell_{ij})$, $\ell_{ij}\in\Bbbk$, $1\leqslant i\leqslant j\leqslant n$, with the standard basis $\{\delta^{ij}\}$, where $(\delta^{ij})_{kl}=1$ if $i=k$, $j=l$ and 0 otherwise. Let $\mathcal{L}_0\subset\mathcal{L}$, $\mathcal{L}_0\simeq\mathbb{Z}^{\frac{n(n-1)}{2}}$ be a lattice generated by δ^{ij}, $1\leqslant j\leqslant i\leqslant n-1$.

The product of symmetric groups $\mathbb{S}_n=\prod_{i=1}^{n}S_i$ acts on $\mathcal{L}$, if $\ell=(\ell_{ij})\in\mathcal{L}$ and $s=(s_i)\in\mathbb{S}_n$, $i=1,\dots,n$ then $(s\cdot\ell)_{ij}=\ell_{s_j(i)j}$. Also the group $\mathcal{L}_0$ acts on $\mathcal{L}$ by the shift $\delta^{ij}\cdot\ell=\ell+\delta^{ij}$, $\delta^{ij}\in\mathcal{L}_0$.

Let Λ be a polynomial algebra in variables $\{\lambda_{ij}\,|\,1\leqslant j\leqslant i\leqslant n\}$ and L be the fraction field of Λ. We will identify $\mathcal{L}$ and $\operatorname{Specm}\Lambda$. Note that Λ is integrally closed in L and coincides with the integral closure of Γ in L.

Let $\imath:\Gamma\longrightarrow\Lambda$ be a $\Bbbk$-algebra monomorphism such that

$$\imath(c_{mk}) \;=\; c_{mk}(\lambda) \;=\; \sum_{i=1}^{m}(\lambda_{mi}+m)^k\prod_{j\neq i}\Big(1-\frac{1}{\lambda_{mi}-\lambda_{mj}}\Big).$$

The image of $\imath$ coincides with the subalgebra of $\mathbb{S}_n$-invariant polynomials $\Lambda^{\mathbb{S}_n}$ in Λ ([Zh]). Choose the generators $\{\gamma_{ij}\}$ of Γ such that $\imath(\gamma_{ij})=\sigma_{ij}(\lambda_{j1},\lambda_{j2},\dots,\lambda_{jj})$, $1\leqslant j\leqslant i\leqslant n$, where σ_{ij} is the i-th elementary symmetric polynomial in j variables. Thus we can identify Γ and $\Lambda^{\mathbb{S}_n}$ by mapping $\gamma\longmapsto\imath(\gamma)$, $\gamma\in\Gamma$. Hence we can view the elements of Γ as polynomial functions on Λ. The homomorphism $\imath$ can be extended to an embedding of the fields $K\longrightarrow L$, $L^{\mathbb{S}_n}=K$ and $G=\mathbb{S}_n$ is the Galois group $G(L/K)$ of the extension $K\subset L$.

Denote by $\pi:\operatorname{Specm}\Lambda\longrightarrow\operatorname{Specm}\Gamma$ the projection induced by $\imath$.

Recall a construction of the Gelfand-Tsetlin basis for finite-dimensional gl_n-modules ([Zh]). Denote $\mathbb{Z}^+\subset\Bbbk$ the set of nonnegative integers and consider $\mathcal{L}^+\subset\mathcal{L}$, consisting of ℓ such that $\ell_{mi}-\ell_{m-1\,i}\in\mathbb{Z}^+$, $\ell_{m-1\,i}-\ell_{m\,i+1}\in\mathbb{Z}^+$ for all possible i,m. Let $\mathfrak{A}\subset\Bbbk^n$ consists of $\alpha=(\alpha_1,\dots,\alpha_n)$ such that $\alpha_i-\alpha_{i+1}\in\mathbb{Z}^+$, $i=1,\dots,n-1$. Set for $\alpha\in\mathfrak{A}$

$$\mathcal{L}_\alpha=\{\ell\in\mathcal{L}\,|\,\ell_{ni}=\alpha_i\ \text{ for }\ i=1,\dots,n\}\ \text{ and }\ \mathcal{L}^+_\alpha=\mathcal{L}_\alpha\cap\mathcal{L}^+.$$

If M is a finite-dimensional irreducible U-module then (for some $\alpha\in\mathfrak{A}$ which is determined by the central character of M) M possesses a $\Bbbk$-base consisting of the elements $[\,\ell\,]$, $\ell\in\mathcal{L}^+_\alpha$. The action of the algebra U is defined by $c_{mk}[\,\ell\,]=c_{mk}(\ell)[\,\ell\,]$, $1\leqslant k\leqslant m\leqslant n$ and

$$E^\pm_m[\,\ell\,]=\sum_{i=1}^{m}a^\pm_{mi}(\ell)[\,\ell\pm\delta^{mi}\,],$$

where $E^+_m=e_{m\,m+1}$, $E^-_m=e_{m+1\,m}$, $m=1,\dots,n-1$ and

$$a^\pm_{mi}(\ell)=\mp\frac{\prod_j(\ell_{m\pm1,j}-\ell_{mi})}{\prod_{j\neq i}(\ell_{mj}-\ell_{mi})}.$$

Set $\mathcal{M} = \mathcal{L}_0$. Let e be the identity element of the group $\mathcal{M}$. Note that all e_{ii}, $i = 1, \dots, n$ are in L.

Let T be a free associative algebra generated over $\Bbbk$ by E_{ij}, $i, j = 1, \dots, n$. Consider an algebra homomorphism $\mathrm{t} : T \longmapsto L * \mathcal{M}$ such that

$$\text{(38)} \qquad \mathrm{t}(E_{mm}) = e c_{mm}, \ \mathrm{t}(E_{m\,m+1}) = \sum_{i=1}^{m} \delta^{mi} A^{+}_{mi}, \ \mathrm{t}(E_{m+1\,m}) = \sum_{i=1}^{m} (\delta^{mi})^{-1} A^{-}_{mi},$$

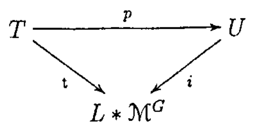
where

$$A^{\pm}_{mi} = \mp \frac{\prod_j (\lambda_{m\pm1,j} - \lambda_{mi})}{\prod_{j \neq i} (\lambda_{mj} - \lambda_{mi})}.$$

Lemma 7.1. $\mathrm{t}(E_{mm+1}) = [\delta^{m1} A^{+}_{m1}]$, $\mathrm{t}(E_{m+1m}) = [(\delta^{m1})^{-1} A^{-}_{m1}]$, *in particular,* t *defines a homomorphism from* T *to* $L * \mathcal{M}^G$.

Proof. To prove it note $H_{\delta^{m1}} = \mathbb{S}_{n-1} \subset G$, consisting of those permutations of G, which fix 1. Also it is easy to see, that $A^{+}_{m1} \in L^{H_{\delta^{m1}}}$. Then for $g \in G$, such that $g(1) = i$ holds $(\delta^{m1})^g = \delta^{mi}$ and $(A^{+}_{m1})^g = A^{+}_{mi}$, which implies the statement. □

Proposition 7.2. *Denote* $p : T \longrightarrow U$ *the projection defined by* $E_{ij} \longmapsto e_{ij}$. *Then there exists an embedding of algebras* $i : U \longrightarrow L * \mathcal{M}^G$, *such that the diagram*



commutes. The embedding i *turns* U *into a Galois algebra with respect to* Γ.

Proof. Let $z \in T$ and $\mathrm{t}(z) = \sum_{i=1}^{s} [m_i a_i]$, $m_i \in \mathcal{M}$, $a_i \in L$. Then there exists a dense subset $\Omega(z)$ of $[\ell] \in \mathfrak{A}$, such that $[\ell]$ is a basic vector of some finite dimensional U-module M and $p(z) \cdot [\ell] = \sum_{i=1}^{s} a_i(\ell)[m_i + \ell]$.

Let $z \in T$ be a Jacobson-Serre relation, [D]. Then it turns 0 in all finite dimensional representations of U. If $[\ell] \in \Omega(z)$ then $a_i(\ell) = 0$ for all i. Since each a_i is a rational function on $\operatorname{Specm} \mathrm{L}$ it implies that $a_i = 0$, and hence $z \in \operatorname{Ker} \mathrm{t}$. Therefore, there exists a homomorphism $i : U \longrightarrow L * \mathcal{M}^G$, such that the diagram commutes. It remains to show that i is an embedding. Since U is a PBW algebra and

$$n^2 = \operatorname{GKdim} U(\mathrm{gl}_n) = \operatorname{GKdim} \Gamma + \operatorname{growth} \mathcal{M} = \frac{n(n+1)}{2} + \frac{n(n-1)}{2},$$

we conclude that i is an embedding. Moreover, by Proposition 4.2 and by Theorem 7.1 U is a Galois Γ-algebra . □

Corollary 7.2. *The universal enveloping algebra* $U(\mathrm{gl}_n)$ *is a Galois subalgebra of* $L * \mathcal{M}$.

Remark 7.2. *The fact that the homomorphism $i : U \longrightarrow L * \mathcal{M}^G$ is an embedding follows also from the generalized Harish-Chandra theorem ([Ov]).*

Remark 7.3. *Realization of $U(\mathrm{gl}_n)$ as a Galois algebra is equivalent to the embedding of $U(\mathrm{gl}_n)$ into a product of localized Weyl algebras constructed in [Kh].*

7.3.2. **Restricted Yangians for gl_2.** Let p be a positive integer. The *level p Yangian* $\mathrm{Y}_p(\mathrm{gl}_2)$ for the Lie algebra gl_2 can be defined as the algebra over $\Bbbk$ with generators $t_{ij}^{(1)}, \ldots, t_{ij}^{(p)}$, $i, j = 1, 2$, subject to the relations

$$[T_{ij}(u), T_{kl}(v)] = \frac{1}{u - v}(T_{kj}(u)\ T_{il}(v) - T_{kj}(v)\ T_{il}(u)), \tag{39}$$

where u, v are formal variables and

$$T_{ij}(u) = \delta_{ij}\ u^p + \sum_{k=1}^{p} t_{ij}^{(k)}\ u^{p-k} \in \mathrm{Y}_p(\mathrm{gl}_2)[u].$$

It means

$$[t_{ij}^{(r)}, t_{kl}^{(s)}] = \sum_{a=1}^{\min(r,s)} \left(t_{kj}^{(a-1)} t_{il}^{(r+s-a)} - t_{kj}^{(r+s-a)} t_{il}^{(a-1)}\right),$$

where $t_{ij}^{(0)} = \delta_{ij}$ and $t_{ij}^{(r)} = 0$ for $r \geq p + 1$. Note that the level 1 Yangian $\mathrm{Y}_1(\mathrm{gl}_2)$ coincides with the universal enveloping algebra $\mathrm{U}(\mathrm{gl}_2)$.

Denote by $D(u)$ the *quantum determinant*

$$D(u) = T_{11}(u)\ T_{22}(u - 1) - T_{21}(u)\ T_{12}(u - 1),$$

which is a polynomial in u of degree $2p$,

$$D(u) = u^{2p} + d_1\ u^{2p-1} + \cdots + d_{2p}, \qquad d_i \in \mathrm{Y}_p(\mathrm{gl}_2).$$

The coefficients $d_1, \ldots, d_{2p}$ are algebraically independent generators of the center of the algebra $\mathrm{Y}_p(\mathrm{gl}_2)$. Denote by Γ the subalgebra of $\mathrm{Y}_p(\mathrm{gl}_2)$ generated by the coefficients of $D(u)$ and by the elements $t_{22}^{(k)}$, $k = 1, \ldots, p$. This algebra is commutative Harish-Chandra subalgebra in $\mathrm{Y}_p(\mathrm{gl}_2)$.

Let Λ be a polynomial algebra in the variables $b_1, \ldots, b_p, g_1, \ldots g_{2p}$. Define a $\Bbbk$-homomorphism $\imath : \Gamma \to \Lambda$ by

$$\imath(t_{22}^{(k)}) = \sigma_{k,p}(b_1, \ldots, b_p), \qquad \imath(d_i) = \sigma_{i,2p}(g_1, \ldots, g_{2p}), \tag{40}$$

where $\sigma_{i,j}$ is the i-th elementary symmetric polynomial in j variables. We will identify the elements of Γ with their images in Λ and treat them as polynomials in the variables $b_1, \ldots, b_p, g_1, \ldots g_{2p}$ invariant under the action of the group $S_p \times S_{2p}$. Set $\mathcal{L} = \mathrm{Specm}\,\Lambda$. We will identify $\mathcal{L}$ with $\Bbbk^{3p}$. If

$$\beta = (\beta_1, \ldots, \beta_p), \qquad \gamma = (\gamma_1, \ldots, \gamma_{2p}) \qquad \text{and} \qquad \ell = (\beta_1, \ldots, \beta_p, \gamma_1, \ldots, \gamma_{2p})$$

then we shall write $\ell = (\beta, \gamma)$. The monomorphism $\imath$ induces the epimorphism

$$\imath^* : \mathcal{L} \to \mathrm{Specm}\,\Gamma. \tag{41}$$

If $\ell \in \mathcal{L}$ and $\mathrm{m} = \imath^*(\ell)$ then $D(\ell)$ will denote the equivalence class of m in $\Delta(\mathrm{Y}_p(\mathrm{gl}_2), \Gamma)$.

Let $\mathcal{P}_0 \subseteq \mathcal{L}$, $\mathcal{P}_0 \simeq \mathbb{Z}^p$, be the lattice generated by the elements $\delta_i \in \mathbb{k}^{3p}$ for $i = 1, \dots, p$, where

$$\delta_i = (\delta_i^1, \dots, \delta_i^{3p}), \qquad \delta_i^j = \delta_{ij}.$$

Then $\mathcal{P}_0$ acts on $\mathcal{L}$ by shifts $\delta_i(\ell) := \ell + \delta_i$. Furthermore, the group $S_p \times S_{2p}$ acts on $\mathcal{L}$ by permutations. Denote by S a multiplicative set in Λ generated by the elements $b_i - b_j - m$ for all $i \neq j$ and all $m \in \mathbb{Z}$ and by $\mathbb{L}$ the localization of Λ by S.

For arbitrary $3p$-tuple $\ell = (\beta, \gamma) \in \mathcal{L}$ set

$$\beta(u) = (u + \beta_1) \cdots (u + \beta_p), \qquad \gamma(u) = (u + \gamma_1) \cdots (u + \gamma_{2p}).$$

Let I_ℓ be the left ideal of $\mathrm{Y}_p(\mathrm{gl}_2)$ generated by the coefficients of the polynomials $T_{22}(u) - \beta(u)$ and $D(u) - \gamma(u)$. Define the corresponding quotient module over $\mathrm{Y}_p(\mathrm{gl}_2)$ by

$$M(\ell) = \mathrm{Y}_p(\mathrm{gl}_2)/I_\ell. \tag{42}$$

We shall call it the *universal module*. It was shown in [FMO], that I_ℓ is a proper ideal of $\mathrm{Y}_p(\mathrm{gl}_2)$ and so $M(\ell)$ is a non-trivial module.

Set $\mathcal{P}_1 = \operatorname{Specm} \mathbb{L} \subseteq \mathcal{L}$, i.e. $\mathcal{P}_1$ consists of *generic* $3p$-tuples $\ell = (\beta, \gamma)$ such that

$$\beta_i - \beta_j \notin \mathbb{Z} \qquad \text{for all} \quad i \neq j. \tag{43}$$

If $\ell \in \mathcal{P}_1$ then the modules from the category $\mathbb{H}(\mathrm{Y}_p(\mathrm{gl}_2), \Gamma, D(\ell))$ are called *generic* Harish-Chandra modules.

Theorem 7.2. ([FMO]) *There exist vectors $\xi^{(k)}, (k) \in \mathbb{Z}^p$, which form a basis of $M(\ell)$. Moreover, we have the formulas*

$$T_{22}(u)\, \xi^{(k)} = \prod_{i=1}^{p} (u + \beta_i + k_i)\, \xi^{(k)}, \tag{44}$$

$$\begin{aligned} T_{21}(u)\, \xi^{(k)} &= \sum_{i=1}^{p} A_i(k)\, \frac{(u + \beta_1 + k_1) \cdots \wedge_i \cdots (u + \beta_p + k_p)}{(\beta_1 - \beta_i + k_1 - k_i) \cdots \wedge_i \cdots (\beta_p - \beta_i + k_p - k_i)}\, \xi^{(k+\delta_i)}, \\ T_{12}(u)\, \xi^{(k)} &= \sum_{i=1}^{p} B_i(k)\, \frac{(u + \beta_1 + k_1) \cdots \wedge_i \cdots (u + \beta_p + k_p)}{(\beta_1 - \beta_i + k_1 - k_i) \cdots \wedge_i \cdots (\beta_p - \beta_i + k_p - k_i)}\, \xi^{(k-\delta_i)}, \end{aligned} \tag{45}$$

where

$$A_i(k) = \begin{cases} 1 & \text{if } k_i \geq 0 \\ -\gamma(-\beta_i - k_i) & \text{if } k_i < 0 \end{cases}$$

and

$$B_i(k) = \begin{cases} -\gamma(-\beta_i - k_i + 1) & \text{if } k_i > 0 \\ 1 & \text{if } k_i \leq 0. \end{cases}$$

The action of $T_{11}(u)$ is found from the relation

$$\Big(T_{11}(u)\, T_{22}(u-1) - T_{21}(u)\, T_{12}(u-1)\Big)\, \xi^{(k)} = \gamma(u)\, \xi^{(k)}. \tag{46}$$

Now quite analogous to 7.3.1 we can prove that $Y_p(\mathrm{gl}_2)$ is a Galois Γ-subalgebra in $L * \mathcal{M}^G$, where L is the field of fractions of Λ, $\mathcal{M} = \mathcal{P}_0$ and $G = S_p \times S_{2p}$. The formulae from Theorem 7.2 define the homomorphism $i : Y_p(\mathrm{gl}_2) \to L * \mathcal{M}^G$. Note that $Y_p(\mathrm{gl}_2)$ is a PBW algebra and its GK-dimension equals $4p$. On the other hand $\mathrm{GKdim}\,\Lambda = 3p$ and growth $\mathcal{P}_0 = p$. Theorem 6.1 shows that i is an embedding and Proposition 4.2 shows $Y_p(\mathrm{gl}_2)$ is Γ-Galois.

8. Gelfand-Kirillov conjecture

If $\mathcal{G}$ is a finite dimensional Lie algebra then its universal enveloping algebra $U(\mathcal{G})$ is a noetherian domain, and thus it admits a skew field of fractions. The celebrated Gelfand-Kirillov conjecture ([GK], [D], Problems, I, 3, and 4.9.21) asserts that this skew field of fractions is isomorphic to the skew fraction field of a certain Weyl algebra over a purely transcendental field extension of $\Bbbk$. This conjecture is known to be true in the case of $\mathcal{G} = \mathrm{gl}_n$ (or $\mathcal{G} = \mathrm{sl}_n$). For other known cases, counterexamples and generalization for quantized algebras see [BG] and references therein. Using the technique of Galois algebras we reprove the Gelfand-Kirillov conjecture for gl_n and show it for the Yangians of gl_2.

8.1. **Symmetric differential operators.** Fix $k, 1 \leq k \leq n-1$ and denote by A_k the k-th Weyl algebra generated over $\Bbbk$ by $x_1, \dots, x_k$ and $\partial_1, \dots, \partial_k$ subject to the relations

$$x_i x_j = x_j x_i,\ \partial_i \partial_j = \partial_j \partial_i,\ \partial_i x_j - x_j \partial_i = \delta_{ij}. \tag{47}$$

The symmetric group S_k has a natural action on A_k by permutation of variables x_i's and ∂_j's simultaneously.

Let $\Lambda = \Bbbk[x_1, \dots, x_n]$. Consider the algebra $D(\Lambda)$ of differential operators on Λ, $D(\Lambda)$ contains Λ as a subalgebra of the operators of multiplication by the elements of Λ. One can identify $D(\Lambda)$ with the Weyl algebra A_n by the following isomorphism v: we identify $v(x_i)$ with the operator of multiplication by x_i and identify $v(\partial_i)$, $i = 1, \dots, n$ with the operator of partial derivation by x_i. Note also, that if A is a localizations of Λ then $D(A)$ is generated over A by $\partial_1, \dots, \partial_n$ subject to the same relations (47). In this case $\sum_{i=1}^{n} A\partial_i \subset D(A)$ is just the Lie algebra of all $\Bbbk$-derivations of A.

The action of S_n on A_n induces the action of S_n on $D(\Lambda)$ by conjugations via the isomorphism v. Indeed for $\pi \in S_n$, $i, j = 1, \dots, n$, $f \in \Lambda$

$$\begin{aligned} &(\pi v(x_i)\pi^{-1})(f) = \pi(x_i \pi^{-1}(f)) = x_{\pi(i)} f \\ &(\pi \partial_i \pi^{-1})(x_j) = \pi \partial_i (x_{\pi^{-1}(j)}) = \begin{cases} 1, & j = \pi(i) \\ 0, & j \neq \pi(i) \end{cases},\ \text{hence } \pi \partial_i \pi^{-1} = \partial_{\pi(i)}. \end{aligned} \tag{48}$$

Let σ_i be the i-th symmetric polynomial in $x_1, \dots, x_n$, $i = 1, \dots, n$, $\Lambda^{S_n} = \Bbbk[\sigma_1, \dots, \sigma_n] \subset \Lambda$, $\delta = \prod_{1 \leq i < j \leq n} (x_i - x_j)$, $\Delta = \delta^2 \in \Lambda^{S_n}$ the discriminant and Λ_Δ and $\Lambda_\Delta^{S_n}$ the localizations of corresponding algebras by the multiplicative set generated by Δ.

The canonical embedding $i : \Lambda_\Delta^{S_n} \to \Lambda_\Delta$ induces a homomorphism of algebras i_D : $D(\Lambda_\Delta)^{S_n} \to D(\Lambda_\Delta^{S_n})$.

The key result of this section is

Theorem 8.1. *i_D is an isomorphism.*

8.1.1. **i_D is an epimorphism.** Since $\Lambda_\Delta^{S_n}$ is a localization of the polynomial ring $\Bbbk[\sigma_1, \dots, \sigma_n]$, the ring $D(\Lambda_\Delta)^{S_n}$ is generated over $\Lambda_\Delta^{S_n}$ by differentiations $\partial'_1, \dots, \partial'_n$ such that $\partial'_i(\sigma_j) = \delta_{ij}$, $i, j = 1, \dots, n$. Hence it is enough to construct S_n-invariant differentiations $d_1, \dots, d_n$: $\Lambda_\Delta \to \Lambda_\Delta$ which in restriction on $\Lambda_\Delta^{S_n}$ coincide with $\partial'_1, \dots, \partial'_n$.

Let $d = \sum_{i=1}^{n} f_i \partial_i$, $f_i \in \Lambda_\Delta$, be a S_n-invariant differential operator. Then any f_i should be invariant with respect to the stabilizer of i in S_n, $i = 1, \dots, n$. Denote by σ_i^j the i-th symmetrical polynomial in the variables $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n$, $i = 0, 1, \dots, n-1$, $j = 1, \dots, n$ and consider the $n \times n$ matrix $X = (\partial_j(\sigma_i))_{i,j=1,\dots,n}$. It is easy to see that

$$X = (\sigma_i^j)_{\substack{i=0,\dots,n-1, \\ j=1,\dots,n}}.$$

If e_i is the standard i-th basic vector and $f_i = (f_{1i}, \dots, f_{ni})$ is a vector of solutions of the system $Xf = e_i$, then the differential operator $d_i = \sum_{k=1}^{n} f_{ki} \partial_k$, with coefficients from $\Bbbk(x_1, \dots, x_n)$, satisfies the relation $d_i(\sigma_j) = \delta_{ij}$. It remains to prove that $d_i \in D(\Lambda_\Delta)^{S_n}$.

Lemma 8.1. $\det X = \delta$.

Proof.

$$X = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \sigma_1^1 & \sigma_1^2 & \dots & \sigma_1^n \\ \sigma_2^1 & \sigma_2^2 & \dots & \sigma_2^n \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n-1}^1 & \sigma_{n-1}^2 & \dots & \sigma_{n-1}^n \end{pmatrix} \tag{49}$$

□

Then $\det X$ belongs to the space of homogeneous polynomials of degree $\frac{n(n-1)}{2}$, if we set in this polynomial x_i to be equal x_j for $i \neq j, i, j = 1, \dots, n$ it turns in 0, hence δ is a divisor of $\det X$, moreover (due to the equality of degrees) $\det X = \lambda\delta, \lambda \in \Bbbk$. Since in both of them the monomial $x_1^{n-1} x_2^{n-1} \dots x_{n-1}^{n-1}$ enters with the coefficient 1, we conclude $\det X = \delta$.

Applying the Kramer rule we obtain

Corollary 8.1. *$f_{ij} \in \Lambda_\Delta$, $d_i \in D(\Lambda_\Delta)^{S_n}, i, j = 1, \dots, n$.*

Proof. The first statement follows from the lemma above. It remains to prove that for a fixed i the rational functions $f_{i1}, \dots, f_{in}$ form an orbit of the action of S_n. Denote σ^i the

i-th column of the matrix X. Then for $\pi \in S_n$ there holds $\pi \cdot \sigma^i = \sigma^{\pi(i)}$, $i = 1, \dots, n$ and hence

$$\pi f_{ij} = \text{(applying } \pi \text{ element wise) } \pi \frac{\det(\sigma^1, \dots, \sigma^{j-1}, \overbrace{e_i}^{j}, \sigma^{j+1}, \dots, \sigma^n)}{\det(\sigma^1, \dots, \sigma^{j-1}, \sigma^j, \sigma^{j+1}, \dots, \sigma^n)} = \tag{50}$$

$$\frac{\det(\sigma^{\pi(1)}, \dots, \sigma^{\pi(j-1)}, \overbrace{e_i}^{j}, \sigma^{\pi(j+1)}, \dots, \sigma^{\pi(n)})}{\det(\sigma^{\pi(1)}, \dots, \sigma^{\pi(j-1)}, \sigma^{\pi(j)}, \sigma^{\pi(j+1)}, \dots, \sigma^{\pi(n)})} = \text{(permuting columns with } \pi) \tag{51}$$

$$= \frac{\operatorname{sign} \pi \det(\sigma^1, \dots, \sigma^{\pi(j)-1}, \overbrace{e_i}^{\pi(j)}, \sigma^{\pi(j)+1}, \dots, \sigma^n)}{\operatorname{sign} \pi \det(\sigma^1, \dots, \sigma^{\pi(j)-1}, \sigma^{\pi(j)}, \sigma^{\pi(j)+1}, \dots, \sigma^n)} = f_{i\pi(j)}, \tag{52}$$

which completes the proof. □

8.1.2. i_d **is a monomorphism** . We need to show that for every $D \in D(\Lambda)^{S_n}$ there exists $f \in \Lambda^{S_n}$ such that $D(f) \neq 0$.

For $I = \{i_1, \dots, i_n\} \in \mathbb{N}^n$ denote $x^I = x_1^{i_1} \dots x_n^{i_n}$ and $\partial^I = \partial_1^{i_1} \dots \partial_n^{i_n}$. We use the following degree deg on $D(\Lambda)$: $\deg x^I \partial^J = |J|$, where $|J| = j_1 + \dots + j_n$ for $J = (j_1, \dots, j_n)$. For $D = \sum_{J \in \mathbb{N}^n} F_J \partial^J$, $F_J \in \Lambda$ we set

$$\operatorname{supp} D = \{J \in \mathbb{N}^n \mid F_J \neq 0\}.$$

Remark 8.1. *If for $I, J \in \mathbb{N}^n$ holds $I > J$ then $\partial^J(x^I) = P_J(I) x^{I-J}$, where*

$$P_J(I) = \prod_{k=1}^{n} p_{J,k}(i_k), \ p_{J,k}(z) = z(z-1)\dots(z-j_k+1), \ \textit{so } \deg p_{J,k} = j_k, \ k = 1, \dots, n. \tag{53}$$

If $J \in \mathbb{N}^n$ satisfies $j_1 > j_2 > \dots > j_n$ then such vector is called *senior*. Denote by $\mathbb{S}$ the set of all senior vectors. For some $F_J \in \Lambda$ and $J \in \mathbb{S}$ denote $F_J \partial^J + \dots = \sum_{\sigma \in S_n} F_J^\sigma \partial^{\sigma J} \in D(\Lambda)^{S_n}$. Let $D \in D(\Lambda)^{S_n}$ and assume that no $J \in \operatorname{supp} D$ contains equal coordinates. Then $D = \sum_{J \in O} (F_J \partial^J + \dots)$, where O consists of the senior representatives of the orbits of the action of S_n on $\operatorname{supp} D$. Analogously for a senior I we introduce $x^I + \dots = \sum_{\sigma \in S_n} x^{\sigma I} \in \Lambda^{S_n}$.

Let $D = \sum_{J \in \operatorname{supp} D} F_J \partial^J \in \operatorname{Ker} i_d$, $D \neq 0$. Possibly multiplying D from the left on $\partial_1^{k_1} \dots \partial_n^{k_n} + \dots$ for some $k_1 \gg k_2 \gg \dots \gg k_n \gg 0$ we can assume that D satisfies the condition above. In other words, for $J \in \operatorname{supp} D$ and $\sigma \in S_n$, $\sigma J = J$ if and only if σ the trivial permutation and $D = \sum_{J \in O} (F_J \partial^J + \dots)$.

Lemma 8.2. *Let* $D = \sum_{J\in O}(F_J\partial^J + \dots)$ *and* $I > J$ *(lexicographical) for any* $J \in \operatorname{supp} D$. *Then*

$$D(x^I + \dots) = \sum_{J\in O}\sum_{\sigma,\tau\in S_n} P_{\tau J}(\sigma I)F_{\tau J}x^{\sigma I-\tau J}, \tag{54}$$

where $F_{\tau J} = F_J^\tau$.

Proof.

$$D(x^I + \dots) = D\Big(\sum_{\sigma\in S_n} x^{\sigma I}\Big) = \Big(\sum_{J\in O}\sum_{\tau\in S_n} F_{\tau J}\partial^{\tau J}\Big)\Big(\sum_{\sigma\in S_n} x^{\sigma I}\Big) =$$
$$\sum_{J\in O}\sum_{\sigma,\tau\in S_n} F_{\tau J}\partial^{\tau J}(x^{\sigma I}) = \sum_{J\in O}\sum_{\sigma,\tau\in S_n} P_{\tau J}(\sigma I)F_{\tau J}x^{\sigma I-\tau J}.$$

□

We say that $I \in \mathbb{S}$ is *segregating* for a set $\mathcal{K} = \{K_1, \dots, K_t\}$ provided that for any $s = 1, \dots, t$ holds $I > K_s$ and all the vectors $\{\sigma I + \tau K \mid \sigma, \tau \in S_n\}$ are distinct.

Let $\sigma_1, \dots, \sigma_{n!}$ be all elements of S_n. The sequence of elements $I_1 < I_2 < \dots < I_{n!} \in \mathbb{N}^n$ is called *segregating* for $\mathcal{K}$, provided that every I_k is segregating for $\mathcal{K}$ and for any $\mathsf{s} \in S_{n!}$, $\mathsf{s} \neq e$, holds $\sum_{i=1}^{n!}\sigma_i I_i \neq \sum_{i=1}^{n!}\sigma_{\mathsf{s}(i)}I_i$ (equivalently $\sum_{i=1}^{n!}(\sigma_i I_i - \sigma_{\mathsf{s}(i)}I_i) \neq 0$).

For $D \in D(\Lambda)$ denote by $\mathcal{K}(D) \subset \mathbb{Z}^n$ the set of all differences $M - J$, where J runs $\operatorname{supp} D$ and for a fixed J, M runs the degrees of monomials in F_J. We say that I is segregating for D provided I is segregating for $\mathcal{K}(D)$.

Lemma 8.3. (1) *Let* $\mathcal{K} \subset \mathbb{Z}^n$ *and let* $I_1, \dots, I_{n!}$ *be a* $\mathcal{K}$*-segregating sequence. Then the matrix* $X = (x^{\sigma_j I_i})_{i,j=1,\dots,n!}$ *is non-degenerate.*

(2) *Let* $\mathcal{K} \subset \mathbb{Z}^n$, $s_1 > s_2 > \dots > s_n > 1$ *a sequence of integers,* $I(t) = (t^{s_1}, \dots, t^{s_n})$, $t \in \mathbb{N}$ *and* $J \in \mathbb{N}^n$. *Then for any* $N \geq 0$ *there exist integers* $l_1 > \dots > l_{n!} > N$ *such that the sequence* $I_k = I(l_k) - J, k = 1, \dots, n!$ *is* $\mathcal{K}$*-segregating.*

Proof. $\det X = \sum_{\mathsf{s}\in S_{n!}} \operatorname{sign}(\mathsf{s})x^{\sigma_{\mathsf{s}(1)}I_1}\dots x^{\sigma_{\mathsf{s}(n!)}I_{n!}}$. By the definition of a segregating sequence all the monomials in this sum are different implying (1).

To prove (2) consider α (respectively β), the maximal by absolute value coordinate in vectors from $\mathcal{K}$ (respectively J). Then $I(t)$ will be $\mathcal{K}$-segregating if $t^{s_n+1} - t^{s_n} > 2|\alpha| + |\beta|$, since in this case $|t^{s_i} - t^{s_j}| > 2|\alpha| + |\beta|$ for all $i \neq j$.

Further, set $l_{n!} = N + 1$. Assume $l_{i+1}, \dots, l_{n!}$ are constructed. Then l_i should satisfy the condition

$$l_i^{s_n+1} - l_i^{s_n} > 2(n! - i)(2|\alpha| + |\beta|) + \sum_{t=i+1}^{n!} l_t^{s_1}$$

and hence

$$|l_i^{s_j} - l_i^{s_k}| > 2(n! - i)(2|\alpha| + |\beta|) + \sum_{t=i+1}^{n!} l_t^{s_1}$$

for every $j \neq k$. Assume that for some $\mathsf{s} \in S_{n!}$, $\mathsf{s} \neq e$, holds $\sum_{i=1}^{n!}(\sigma_i I(l_i) - \sigma_{\mathsf{s}(i)} I(l_i)) = 0$. Assume also that $\sigma_i(k) \neq \sigma_{\mathsf{s}(i)}(k)$ for some $k, 1 \leq k \leq n$. Taking the $\sigma_i(k)$-th coordinate in the sum above we obtain

$$\sum_{i=1}^{n!}(\sigma_i I(l_i) - \sigma_{\mathsf{s}(i)} I(l_i))_k = 0. \tag{55}$$

Let j be the index of the first nonzero summand here. Then

$$|\sigma_j I(l_j)_k - \sigma_{\mathsf{s}(j)} I(l_j)_k| > \sum_{t=j+1}^{n!} l_t^{s_1} >$$

$$\sum_{i=k+1}^{n!} |\sigma_i I(l_i)_k - \sigma_{\mathsf{s}(i)} I(l_i))_k| \geq 0,$$

which is a contradiction. Thus the sum (55) is not zero. □

Let $I \in \mathbb{N}^n$ be such that $I > J$ for all $J \in \operatorname{supp} D$ and assume that I is segregating for D. Then all the monomials which appear in the summands in (54), have different degrees. Indeed, if M is a degree of a monomial in some F_J, then in (54) we have the monomials with degrees of the form $\tau M + (\sigma I - \tau J) = \sigma I + \tau(M - J) = \sigma I + \tau K$ for some $K \in \mathcal{K}(D)$, $\sigma, \tau \in S_n$, which are different for different pairs σ, τ.

Hence the monomials $x^{L(m,\sigma,\tau)}$ in $D(x^I + \dots)$ are parameterized by a monomial $m \in \Lambda$, that appears in some F_J, and by a pair $\sigma, \tau \in S_n$. The coefficient $a_{L(m,\sigma,\tau)} \in \Bbbk$ by $x^{L(m,\sigma,\tau)}$ is a polynomial in $i_1, \dots, i_n$. [1]

We choose $s_1, \dots, s_n \in \mathbb{Z}$ such that $s(J) = s_1 j_1 + \dots + s_n j_n$ are different for all $J \in \operatorname{supp} D$ and $s_1 > s_2 > \dots > s_n > 1$. Fix a senior $J \in \operatorname{supp} D$ with the maximal $s(J)$ and denote $I(t) = (t^{s_1}, \dots, t^{s_n})$. Recall (Lemma 8.3) that for large t the vector $I(t)$ is D-segregating. Then from (54) we obtain that the term of the highest degree of t equals

$$t^{s_1 j_1 + \dots + s_n j_n} \sum_{\sigma \in S_n} F_{\sigma J} x^{\sigma(I(t) - J)}. \tag{56}$$

Hence for large enough t holds $\sum_{\sigma \in S_n} F_{\sigma J} x^{\sigma(I(t)-J)} = 0$. Note that $F_{\sigma J}$ does not depend on t. But by Lemma 8.3, (2) we can construct a segregating sequence $I_1, \dots, I_{n!}$, $I_j = I(l_j) - J$

[1] Since those I's that segregate D, form a Zariski dense set in $\Bbbk^n$, we obtain that these polynomials are zero.

such that $\sum_{\sigma \in S_n} F_{\sigma J} x^{\sigma(I_k)} = 0$ for all $k = 1, \ldots, n!$. On the other hand (Lemma 8.3, (1)) the matrix $\left(x^{\sigma(I_i)}\right)_{i=1,\ldots,N;\sigma\in S_n}$ is non-degenerated. Hence $F_J = 0$ which contradicts the assumption $J \in \operatorname{supp} D$.

8.2. **Case of $U(\mathrm{gl}_n)$.** Using a realization of $U(\mathrm{gl}_n)$ as a Galois algebra (cf. section 7.3.1) we obtain an embedding

$$U(\mathrm{gl}_n) \subset (B * \mathbb{Z}^m)^G,$$

where $m = n(n-1)/2$, $G = S_1 \times S_2 \times \ldots \times S_n$. Here B is a certain localization of the polynomial ring in $n(n+1)/2$ variables. Therefore $U(\mathrm{gl}_n)$ has a natural embedding into

$$Q = \mathcal{A}_1^{S_1} \otimes \mathcal{A}_2^{S_2} \otimes \ldots \otimes \mathcal{A}_{n-1}^{S_{n-1}} \otimes \Bbbk[t_1, \ldots, t_n]^{S_n},$$

where $\mathcal{A}_k$ is a certain localization of the k-th Weyl algebra A_k. The algebra A_k is a simple noetherian noncommutative domain. Denote by $\mathcal{L}_k$ the skew field of fractions of A_k. By Proposition 5.1 the skew field of fractions of $A_k^{S_k}$ equals $\mathcal{L}_k^{S_k}$, $k = 1, \ldots, n-1$. Hence

Corollary 8.2. *$U(\mathrm{gl}_n)$ and Q are rationally equivalent, i.e. the skew field of fractions of $U(\mathrm{gl}_n)$ is isomorphic to*

$$\mathcal{L}_1^{S_1} \otimes \ldots \otimes \mathcal{L}_{n-1}^{S_{n-1}} \otimes \Bbbk(t_1, \ldots, t_n),$$

where $\Bbbk(t_1, \ldots, t_n)$ is the field of fractions of $\Bbbk[t_1, \ldots, t_n]$.

The Gelfand-Kirillov conjecture is true for gl_n and states that the skew field of $U(\mathrm{gl}_n)$ is isomorphic to the skew field of A_m over the field of fractions of the center $Z(U(\mathrm{gl}_n))$. It is also known that if $A_k^{S_k}$ is rationally equivalent to $A_t \otimes \Bbbk[z_1, \ldots z_s]$ (noncommutative Noether's problem) then $t = k$, $s = 0$, and $\mathcal{L}_k^{S_k} \simeq \mathcal{L}_k$ [AD]. Therefore we conclude

Corollary 8.3. *The Gelfand-Kirillov conjecture for gl_n follows from the noncommutative Noether's problem for $A_k^{S_k}$, $k = 1, \ldots, n-1$.*

Our goal now is to prove that $\mathcal{L}_k^{S_k} \simeq \mathcal{L}_k$.

Remark 8.2. *As it was pointed to us by T.Levasseur the validity of the Gelfand-Kirillov conjecture for gl_n implies that $\mathcal{L}_k^W \simeq \mathcal{L}_k$, where $W \simeq S_k$ is the Weyl group of gl_n. But the problem here is with the identification of the explicit action of W on $\mathcal{L}_k$. Our approach is based on the application of the symmetric differential operators.*

Corollary 8.4. *Let $\mathcal{L}_n$ be the fraction field of A_n endowed with the induced action of S_n. Then $\mathcal{L}_n^{S_n} \simeq \mathcal{L}_n$.*

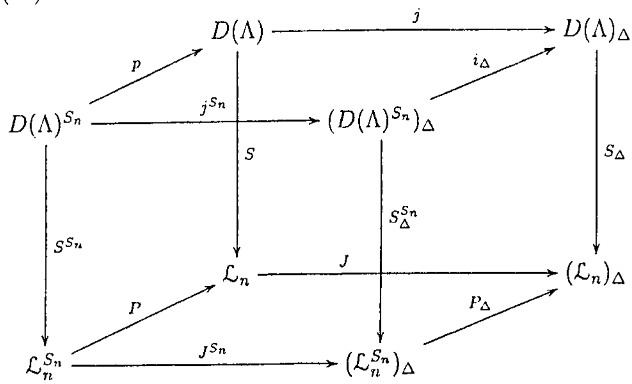
Proof. We use here the following facts

(1) $D(\Lambda)_S \overset{\varphi}{\simeq} D(\Lambda_S)$ for a multiplicative set S.
(2) If $\Delta \in \Lambda$ is a S_n-invariant element then $D(\Lambda_\Delta)^{S_n} \simeq (D(\Lambda)^{S_n})_\Delta$.
(3) $(\Lambda^{S_n})_\Delta \simeq (\Lambda_\Delta)^{S_n}$.
(4) $D(\Lambda_\Delta)^{S_n} \simeq D((\Lambda^{S_n}))_\Delta$.

The first statement can be found in [MCR], Theorem 15.1.25. If $D \in D(\Lambda_\Delta)^{S_n}$ then (as any differential operator after localization) $D_1 = \Delta^k D \in D(\Lambda)$ for some $k \geq 0$. Hence D_1 is S_n-invariant, since both Δ and D are, i.e. $D_1 \in D(\Lambda)^{S_n}$ implying (2). The third statement is obvious and (4) follows from the previous statements and Theorem 8.1.

Consider a commutative diagram

(57)



All the horizontal arrows in the diagram are just embeddings in the localization by Δ (the horizontal arrows on the front face are induced by the corresponding horizontal arrows on the rare face on S_n-invariants). The vertical arrow $S : D(\Lambda) \longrightarrow \mathcal{L}_n$ is just an embedding into the skew field of fractions. Other vertical arrows are induced by localizations and taking S_n-invariants. All other arrows are just embeddings of the S_n-invariants (the arrows on the right face are the localizations by Δ of the arrows on the left face).

By Proposition 5.1 the arrow $S^{S_n} : D(\Lambda)^{S_n} \longrightarrow \mathcal{L}_n^{S_n}$ is just the embedding into the fraction field. On the other side $D(\Lambda)^{S_n}$ and $(D(\Lambda)^{S_n})_\Delta$ have the same skew fraction field. Both J and J_{S_n} are isomorphisms, since they are embeddings into the localization by an invertible element Δ. Hence the skew field of fractions of $(D(\Lambda)^{S_n})_\Delta$ is isomorphic to $\mathcal{L}_n^{S_n}$.

Then

$$(D(\Lambda)^{S_n})_\Delta \simeq (D(\Lambda)_\Delta)^{S_n} \simeq D(\Lambda_\Delta)^{S_n} \simeq D((\Lambda_\Delta)^{S_n}) \simeq \tag{58}$$

$$D(\mathbb{k}[\sigma_1, \ldots, \sigma_n]_\Delta) \simeq D(\mathbb{k}[\sigma_1, \ldots, \sigma_n])_\Delta. \tag{59}$$

It implies that $(D(\Lambda)^{S_n})_\Delta$ is just a localization of the Weyl algebra A_n, and thus its skew fraction field is isomorphic to $\mathcal{L}_n$. It implies $\mathcal{L}_n^{S_n} \simeq \mathcal{L}_n$.

□

Remark 8.3. *Corollaries 8.3 and 8.4 give a new proof of the Gelfand-Kirillov conjecture for* gl_n.

8.3. **Yangian.** Suppose now $U = Y_p(\mathrm{gl}_2)$, the restricted Yangian of level p for gl_2 (cf. section 7.3.2). Then U is a Galois subalgebra in $L * (\mathbb{Z}^p)^G$, where L is the field of fractions of a polynomial algebra Λ in $3p$ variables and $G = S_p \times S_{2p}$. Therefore

Corollary 8.5. (1) *The restricted Yangian* $Y_p(\mathrm{gl}_2)$ *is rationally equivalent to*

$$A_p^{S_p} \otimes \Bbbk[x_1, \ldots, x_{2p}].$$

(2) *The Gelfand-Kirillov conjecture holds for* $Y_p(\mathrm{gl}_2)$ *and its skew field of fractions is isomorphic to* $\mathcal{L}_p \otimes \Bbbk(x_1, \ldots, x_{2p})$.

9. Some remarks on the quantum case

9.1. **Quantum algebras as subalgebras in skew group algebras.** In this subsection we discuss the possibility to realize quantizations of $U(\mathrm{gl}_n)$. As in the classical case the existence of Gelfand-Tsetlin formulas is the main hint for the existence of the structure of a Galois algebra. We write down a conjectural presentation of some quantum algebras as Galois algebras and discuss possible consequences. The case $n = 2$ is considered in details.

Denote by $\breve{U}(\mathrm{gl}_n)$ ([KS], 7.3) the algebra, generated over $\Bbbk(q^{1/2})$ by elements

$$E_i, F_i, K_j, K_j^{-1}, i = 1, \ldots, n-1, j = 1, \ldots, n \tag{60}$$

subject to the relations

$$\begin{aligned}
&K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,\\
&K_iE_jK_i^{-1} = q^{\delta_{ij}/2}q^{-\delta_{ij+1}/2}E_j, \quad K_iF_jK_i^{-1} = q^{-\delta_{ij}/2}q^{\delta_{ij+1}/2}F_j,\\
&[E_i, F_r] = \delta_{ir}\frac{K_i^2K_{i+1}^{-2} - K_i^{-2}K_{i+1}^{-2}}{q - q^{-1}}, \quad [E_i, E_j] = [F_i, F_j] = 0, \quad |i-j| \geq 2,\\
&E_i^2E_{i\pm1} - (q + q^{-1})E_iE_{i\pm1}E_i + E_{i\pm1}E_i^2 = 0,\\
&F_i^2F_{i\pm1} - (q + q^{-1})F_iF_{i\pm1}F_i + F_{i\pm1}F_i^2 = 0.
\end{aligned} \tag{61}$$

The Gelfand-Tsetlin bases of irreducible finite dimensional representations of gl_n are parameterized by the families of $|M\rangle = (m_{ij})_{1\leq i\leq j\leq n} \in \mathbb{Z}^{n(n-1)/2}$ with some conditions of integrality, positivity and betweenness conditions ([KS], 7.3). Denote by $e_{ij}, 1 \leq i \leq j \leq n$ the standard basic in $\mathbb{Z}^{n(n-1)/2}$.

The canonical embedding of the sets of generators induces the canonical embedding of the algebras $i_k : \breve{U}_q(\mathrm{gl}_k) \hookrightarrow \breve{U}_q(\mathrm{gl}_{k+1}), k \geq 1$, so we will assume $\breve{U}_q(\mathrm{gl}_k) \subset \breve{U}_q(\mathrm{gl}_l)$ for $k \leq l$. Denote by Z_k the center of $\breve{U}_q(\mathrm{gl}_k)$ and by $\Gamma = \Gamma_n$ the Gelfand-Tsetlin subalgebra in $\breve{U}_q(\mathrm{gl}_n)$ generated by $Z_1, \ldots, Z_n$.

The Gelfand-Tsetlin formulae for the action of the generators (60) are defined as

(62)

$$K_k|M\rangle = q^{a_k/2}|M\rangle, \quad a_k = \sum_{i=1}^{k} m_{ki} - \sum_{i=1}^{k-1} m_{k-1i}$$

$$E_k|M\rangle = \sum_{j=1}^{k} -\frac{\prod_{i=1}^{k+1}[l_{i\,k+1} - l_{j\,k}]}{\prod_{i\neq j}[l_{i\,k} - l_{j\,k}]}\,|M + e_{kj}\rangle, \quad F_k|M\rangle = \sum_{j=1}^{k} \frac{\prod_{i=1}^{k-1}[l_{i\,k-1} - l_{j\,k}]}{\prod_{i\neq j}[l_{i\,k} - l_{j\,k}]}\,|M - e_{kj}\rangle,$$

where $l_{ij} = m_{ij} - j$ and $[x] = [x]_q = \dfrac{q^x - q^{-x}}{q - q^{-1}}$.

We rewrite these formulas without using the square bracket notations. We apply the following dictionary, where the first one implies the rest

(63)
$$\begin{array}{rcl} q^{l_{ij}} & \longleftrightarrow & X_{ij}, \\ [l_{i\,k\pm1} - l_{i\,k}] & \longleftrightarrow & (q - q^{-1})^{-1}(X_{i\,k\pm1}/X_{i\,k} - X_{i\,k}/X_{i\,k\pm1}), \\ [l_{i\,k\pm1} - l_{i\,k} + 1] & \longleftrightarrow & (q - q^{-1})^{-1}(qX_{i\,k\pm1}/X_{i\,k} - X_{i\,k}/qX_{i\,k\pm1}), \\ [l_{i\,k\pm1} - l_{i\,k} - 1] & \longleftrightarrow & (q - q^{-1})^{-1}(X_{i\,k\pm1}/qX_{i\,k} - qX_{i\,k}/X_{i\,k\pm1}). \end{array}$$

It allows to rewrite the algebra $\check{U}_q(\mathrm{gl}_n)$ as follows. Let

$$\Lambda = \Bbbk(q^{1/2})[X_{ij}^{\pm1/2}; 1 \leq i \leq j \leq n]$$

and L denotes the field of fractions of Λ. Consider a free abelian group $\mathcal{M} = \mathbb{Z}^{\frac{n(n-1)}{2}}$ with a basis $\delta_{ij}, 1 \leq i \leq j \leq n$. Endow Λ with the action of $\mathcal{M}$ as follows:

$$\delta_{ij} \cdot X_{kl}^{1/2} = \begin{cases} q^{1/2}X_{ij}^{1/2} & \text{if } i = k, j = l, \\ X_{kl}^{1/2} & \text{otherwise.} \end{cases}$$

Let $\mathbb{A}$ be a free associative algebra over $\Bbbk$ generated by the generators (60) and Φ a $\Bbbk$-algebras homomorphism $\Phi : \mathbb{A} \longrightarrow L * \mathcal{M}$ defined by

(64)
$$\Phi(K_k) = (q^{k/2}\prod_{i=1}^{k} X_{ik}^{1/2} \prod_{i=1}^{k-1} X_{i\,k-1}^{-1/2})\,e, \quad 1 \leq k \leq n,$$

$$\Phi(E_k) = (q - q^{-1})^{-1}\sum_{j=1}^{k} -\frac{\prod_{i=1}^{k+1}\left(X_{i\,k+1}/X_{j\,k} - X_{j\,k}/X_{i\,k+1}\right)}{\prod_{i\neq j}\left(X_{i\,k}/X_{j\,k} - X_{j\,k}/X_{i\,k}\right)}\delta_{j\,k},$$

$$\Phi(F_k) = \sum_{j=1}^{k} \frac{\prod_{i=1}^{k-1}(X_{i\,k-1}/X_{j\,k} - X_{j\,k}/X_{i\,k-1})}{\prod_{i\neq j}(X_{i\,k}/X_{j\,k} - X_{j\,k}/X_{i\,k})}\delta_{j\,k}^{-1},$$

The proof of the next proposition is analogous to the proof Proposition 7.2.

Proposition 9.1. *The mapping Φ defines an algebra embedding $i : \check{U}_q(\mathrm{gl}_n) \longrightarrow L * \mathcal{M}$. Besides $L \cdot \mathrm{Im}(i) = L * \mathcal{M}$.*

On the other hand, the group $G = \tilde{S}_{n-1} \times \tilde{S}_{n-2} \times \cdots \times \tilde{S}_1$ acts on Λ and on its field of fractions L, where $\tilde{S}_{2k+1} = S_{2k+1}$, $\tilde{S}_{2k} = S_{2k} \times C_2^k$, C_2^k is a cyclic group of order 2, S_i acts on $X_{1i}, \ldots, X_{ii}$ by permutations of the second indices and C_2^k acts by the change of sign $X_{ik} \mapsto -X_{ik}$ for all $i = 1, \ldots, k$. Denote by K the field of invariants L^G, $\Gamma = \Lambda^G$. By the construction $\operatorname{Im} i \subset L * \mathcal{M}^G$.

Conjecture 1. *The algebra $\check{U}_q(\mathrm{gl}_n) \subset L * \mathcal{M}^G$ is a Galois algebra with respect to the Gelfand-Tsetlin subalgebra Γ with the field of fractions K.*

9.2. **Example of $\check{U}_q(\mathrm{gl}_3)$.** As an evidence of the conjecture above we consider the case of the quantized algebra $\check{U}_q(\mathrm{gl}_2)$. Following (61) this algebra is defined by generators and relations

$$\begin{aligned}
&K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1, i = 1, 2,\\
&K_1E_1 = E_1(q^{1/2}K_1), \quad K_1F_1 = F_1(q^{-1/2}K_1),\\
&K_2E_1 = E_1(K_2q^{-1/2}), \quad K_2F_1 = F_1(K_2q^{1/2}),\\
&[E_1, F_1] = \frac{K_1^2K_2^{-2} - K_1^{-2}K_2^{-2}}{q - q^{-1}}.
\end{aligned} \tag{65}$$

Its easy to see that the quadratic Casimir element equals

$$C = E_1F_1 + \frac{q^{-1}(K_1/K_2)^2 + q(K_2/K_1)^2}{(q - q^{-1})^2} = F_1E_1 + \frac{q(K_1/K_2)^2 + q^{-1}(K_2/K_1)^2}{(q - q^{-1})^2}. \tag{66}$$

Let us write the formulas for the mapping Φ

(67)

$$\begin{aligned}
&\Phi(K_1) = q^{1/2}X_{11}^{1/2}\, e, \quad \Phi(K_2) = q(X_{21}X_{22})^{1/2}X_{11}^{-1/2}\, e,\\
&\Phi(E_1) = -(q - q^{-1})^{-2}(X_{21}/X_{11} - X_{11}/X_{21})(X_{22}/X_{11} - X_{11}/X_{22})\, \delta_{11}, \quad \Phi(F_2) = \delta_{11}^{-1}
\end{aligned}$$

and calculate $\Phi(C)$.

$$\begin{aligned}
\Phi(C) = &-(q - q^{-1})^{-2}(X_{21}/X_{11} - X_{11}/X_{21})(X_{22}/X_{11} - X_{11}/X_{22}) +\\
&(q - q^{-1})^{-2}(X_{11}^2/(X_{21}X_{22}) + (X_{21}X_{22})/X_{11}^2) = (q - q^{-1})^{-2}(X_{21}/X_{22} + X_{22}/X_{21}).
\end{aligned} \tag{68}$$

Hence, the image of the Gelfand-Tsetlin subalgebra is generated by

$$X_{11}^{1/2}, X_{21}^{1/2}X_{22}^{1/2} \text{ and } X_{21} + X_{22}. \tag{69}$$

It means that the Galois group involved contains four elements and is generated by the transformations

$$X_{2i}^{1/2} \longleftrightarrow X_{2\,2-i}^{1/2}, \text{ and } X_{2i}^{1/2} \longleftrightarrow -X_{2i}^{1/2}, i = 1, 2.$$

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