Stochastics An International Journal of Probability and Stochastic Processes
formerly Stochastics and Stochastics Reports
ISSN: 1744-2508 (Print) 1744-2516 (Online) Journal homepage: https://www.tandfonline.com/loi/gssr20

Random walks systems with killing on $\mathbb{Z}$

Elcio Lebensztayn , Fábio Prates Machado & Mauricio Zuluaga Martinez

To cite this article: Elcio Lebensztayn , Fábio Prates Machado & Mauricio Zuluaga Martinez (2008) Random walks systems with killing on $\mathbb{Z}$, Stochastics An International Journal of Probability and Stochastic Processes, 80:5, 451-457, DOI: 10.1080/17442500701748609

To link to this article: https://doi.org/10.1080/17442500701748609

Published online: 10 Oct 2008.

Stochastics: An International Journal of Probability and Stochastics Processes
Vol. 80, No. 5, October 2008, 451–457

Random walks systems with killing on $\mathbb{Z}$†

Elcio Lebensztayn[a1], Fábio Prates Machado[a]* and Mauricio Zuluaga Martinez[b2]

[a]*Department of Statistics, Institute of Mathematics and Statistics, University of São Paulo, Rua do Matão 1010, CEP 05508-090 São Paulo, SP, Brazil;* [b]*Department of Statistics, Federal University of Pernambuco, Cidade Universitária, CEP 50740-540 Recife, PE, Brazil*

(*Received 12 June 2007; final version received 30 August 2007*)

We study random walks systems on $\mathbb{Z}$ whose general description follows. At time zero, there is a number $N \geq 1$ of particles at each vertex of $\mathbb{N}$, all being inactive, except for those placed at the vertex one. Each active particle performs a simple random walk on $\mathbb{Z}$ and, up to the time it dies, it activates all inactive particles that it meets along its way. An active particle dies at the instant it reaches a certain fixed total of jumps ($L \geq 1$) without activating any particle, so that its lifetime depends strongly on the past of the process. We investigate how the probability of survival of the process depends on L and on the jumping probabilities of the active particles.

Keywords: simple random walk; phase transition; epidemic model; contact process; frog model

2000 Mathematics Subject Classification: 60K35; 60G50

1. Introduction

We investigate the asymptotic behavior of three random walks systems on $\mathbb{Z}$. A basic version of the model under study is described next. At time zero, there is a number N of particles at each vertex of $\mathbb{N} = \{1,2,\ldots\}$, all inactive except for those placed at the vertex one. Once activated, each particle moves as a discrete-time independent simple random walk on $\mathbb{Z}$, activating all inactive particles it meets. However, each active particle has a lifetime, which depends on the past of the process, in the sense that its displacement lasts until it reaches a total of L jumps without activating any particle. It is worth mentioning that this dependence rules out the possibility of using ordinary random walk estimates or straightforward comparisons with branching and birth–death processes. We are interested in studying whether the process survives, that is, whether there exists active particles at each instant of time with positive probability.

The many versions of this model (known as *frog model*) studied so far have an essential difference from those we consider here: the lifetime of active (or infected) particles is either infinite or a random variable which is independent of the path. In the first case the main studied subject is a shape theorem for the model on the hypercubic lattice $\mathbb{Z}^d$ and in the second case the purpose is to investigate critical parameters on $\mathbb{Z}^d$ and

*Corresponding author. Email: fmachado@ime.usp.br
†Research supported by FAPESP (05/04001-5), CNPq (305329/2006-5), FAPESP (02/07705-5) and FACEPE (0126-1-02/06).

ISSN 1744-2508 print/ISSN 1744-2516 online

DOI: 10.1080/17442500701748609
http://www.informaworld.com

on homogeneous trees (see [1,2,6] and the references therein). In Ref. [4], a shape theorem is proved for a continuous-time version of the model on $\mathbb{Z}^d$.

Let us give a motivation for the kind of model we work here. Assume that there is an infinite quantity of computers connected in line, and that each computer only communicates with the one on its right and the one on its left. Suppose that at time zero, only one computer is infected by a virus which randomly chooses to jump to the computer on the left or the one on the right, infecting it. When one or more viruses (all viruses jump at discrete times) hit a computer, this computer is infected by a new virus, which starts the same dynamics of jumps. After infected each computer activates an anti-virus, which will kill any virus that jumps in the future on it. Is there a positive probability that an infinite quantity of computers will be infected? What happens if stronger virus is created, which are able to survive a larger number of computers with anti-virus?

Finally, observe that in this model infection is represented by random walks while individuals are represented by the vertices of the graph. Moreover, the edges represent the possibility of direct contact between pairs of individuals. This representation is clearly different from those used in well-known models as the contact process (see Ref. [7]) and as in mean-field models as SIS or SIR (see Ref. [5]).

2. Definition of the models and results

In this section, we describe the three models and state the results. The corresponding proofs are presented in section 3. The unifying characteristic is L, meaning the number of lives each active particle has. In the first model, the process dies out no matter how strong the virus is (no matter how large L is); in the second model the process survives with positive probability even if the virus is very weak, dying right after jumping to the left ($L = 1$). The third model presents phase transition for the parameter, meaning that it dies out if L is small (the virus is weak) but survives with positive probability if L is large (the virus is strong).

2.1 The uniform process

For the first model, we fix integers $L \geq 1$, $N \geq 1$ and a real number $p \in (0,1)$. Initially there are N particles at each vertex of $\mathbb{N}$. All particles are inactive at time zero, except for those placed at the vertex one. At each time, each active particle, independently, chooses to jump to the right with probability p or to the left with probability $q = 1 - p$, performing a simple random walk on $\mathbb{Z}$. Up to the time it dies, it activates all inactive particles it hits along its way. An active particle dies at the instant it reaches a total of L jumps (consecutive or not) without activating any particle. In other words, each active particle starts with L lives and looses one life whenever its jump does not yield the activation of new particles. When several active particles simultaneously jump on an inactive particle, it is assumed that none of them looses a life. We call this model the *uniform process* on $\mathbb{Z}$.

THEOREM 2.1. The uniform process on $\mathbb{Z}$ dies out a.s. That is, with probability one there exists a time from when there are no active particles.

We emphasize that Theorem 2.1 still holds under the more interesting initial condition where there are N particles at each vertex of $\mathbb{Z}$. For more details, see the remark after its proof.

2.2 The particle process

Now we allow the probability of jumping to the right and left to depend on the initial position of the particle. For this model, we obtain a condition for which the process survives with positive probability.

Fix an integer $L \geq 1$ and a sequence $\{q_n\}_{n\geq 1}$ of real numbers in (0,1). Again, L stands for the number of jumps without activating new particles that an active particle does before dying. At time zero there is one particle at each vertex of $\mathbb{N}$; only the particle at the vertex one being active. The dynamics of the model is the same of the uniform process, except by the fact that the particle initially placed at position n, in the event it is activated, chooses at each step to jump to the right with probability $1 - q_n$ or to the left with probability q_n. We call this model the *particle process* on $\mathbb{Z}$. As a consequence of Theorem 2.1.

Corollary 2.1. If $\inf_n q_n > 0$ then the particle process on $\mathbb{Z}$ dies out a.s.

On the other hand,

Theorem 2.2. If $\sum_{n=1}^{\infty}\prod_{i=0}^{M-1} q_{n+i} < \infty$ for some integer $M \geq 1$, then the particle process on $\mathbb{Z}$ survives with positive probability.

Since, $\sum_{n=1}^{\infty}\prod_{i=0}^{M-1} q_{n+i} \leq M\sum_{n=1}^{\infty}(q_n)^M$, we get the following

Corollary 2.2. If $\sum_{n=1}^{\infty}(q_n)^M < \infty$ for some integer $M \geq 1$, then the particle process on $\mathbb{Z}$ survives with positive probability.

Theorem 2.1 says that no matter how large N and L are and how close p is to 1, the propagation event does not happen with probability one. On the other hand, Theorem 2.2 asserts that the propagation event may occur if the process counts on more efficient individuals. From this we conclude that in our models the way individuals are organized is not as crucial as their behavior. For a model in which the organization of individuals is essential for extinction/survival matters, see Schinazi [7].

2.3 The site process

Finally, we define a family of models exhibiting phase transition. Fixed $\kappa \geq 2$ an even integer, we define a model that dies out a.s. for $L \leq \kappa$ and survives with positive probability for $L > \kappa$. Again L stands for the number of lives that each active particle has. To achieve this objective we let the jumping probabilities to depend on the current position of the active particles.

Suppose that initially there is one particle at each point of $\mathbb{N}$. All of them are inactive except for the one at the vertex one. Assume that an active particle which is at position n jumps to the right with probability $1 - q_n$ or to the left with probability

$$q_n = \begin{cases} (j+1)^{-2/(\kappa n)} & \text{if } n = s^j \text{ for some } j \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $s > \kappa + 1$ is a fixed integer. As usual, an active particle dies at the first time it reaches a total of L jumps without activating any particle. We call this model the *site process* on $\mathbb{Z}$.

THEOREM 2.3. The site process on $\mathbb{Z}$ dies out a.s. if and only if

$$L \le \kappa.$$

3. Proofs

The following result (see e.g. Bremaud [3]) is useful for what comes next.

LEMMA 3.1. Let $\{a_n\}_{n\ge 1}$ be a sequence of real numbers in (0,1). Then,

$$\prod_{n=1}^{\infty}(1-a_n)=0 \Longleftrightarrow \sum_{n=1}^{\infty} a_n=\infty.$$

Proof of Theorem 2.1. Instead of considering the uniform process as defined, let us assume that each active particle looses one life whenever it jumps towards the origin. We call this process the *independent uniform process* because each active particle has a life, which depends only on its own random walk. Clearly, the result is proved once we show that the independent uniform process dies out a.s.

For $i \in \mathbb{N}$, let $\mathcal{R}_i$ be the set of all vertices visited by particles placed originally at i during their virtual lives. We use the word "virtual" because one does not know whether vertex i is visited by some particle; the set $\mathcal{R}_i$ becomes "real" if i is actually visited. For $i,j \in \mathbb{N}$, we define the following events: $A_i :=$ the particles at vertex i are activated, $\{i \to j\} := \{j \in \mathcal{R}_i\}$, and $\{i \nrightarrow j\} := \{i \to j\}^c$. Finally, we consider

$$B_i = \bigcup_{j=1}^{i-1} \{j \to 2i\} \quad \text{and} \quad C_i = \bigcup_{j=i}^{2i-1} \{j \to 2i\}.$$

Since, $\{\mathcal{R}_i; i \in \mathbb{N}\}$ is an independent family (for the independent uniform process),

$$\mathbb{P}(A_{2i}) \le \mathbb{P}(B_i) + \mathbb{P}(A_i)\mathbb{P}(C_i). \tag{3.1}$$

Observe now that for $j < 2i$,

$$\mathbb{P}(j \to 2i) = \mathbb{P}(j \to j+1) \prod_{k=j+1}^{2i-1} \mathbb{P}(j \to k+1 | j \to k) \le \mathbb{P}(1 \to 2)^{2i-j}.$$

Therefore, for a positive constant c,

$$\mathbb{P}(B_i) \le c\,\mathbb{P}(1 \to 2)^i, \tag{3.2}$$

where $\mathbb{P}(1 \to 2) \le 1-(pq)^{LN} < 1$. Also, for all i,

$$\mathbb{P}(C_i) = 1 - \prod_{j=i}^{2i-1} \mathbb{P}(j \nrightarrow 2i) = 1 - \prod_{j=2}^{i+1} \mathbb{P}(1 \nrightarrow j) \le 1 - \prod_{j=2}^{\infty} \mathbb{P}(1 \nrightarrow j) =: r, \tag{3.3}$$

where $r < 1$ by Lemma 3.1.

Since $\mathbb{P}(1 \longrightarrow 2) \leq r$, we conclude from (3.1), (3.2) and (3.3) that

$$\mathbb{P}(A_{2i}) \leq cr^{i} + r\mathbb{P}(A_i).$$

Thus, for any $j \geq 1$, we have that

$$\mathbb{P}(A_{2^j}) \leq cr^{2^{j-1}} + r\mathbb{P}(A_{2^{j-1}}),$$

which implies that $\lim_{j\to\infty}\mathbb{P}(A_{2^j}) = 0$. Consequently,

$$\lim_{i\to\infty}\mathbb{P}(A_i) = 0, \tag{3.4}$$

and from this it follows that the process dies out a.s. □

Remark. A simple modification in the previous proof allows us to prove Theorem 2.1 in the case that there are initially N particles at each vertex of $\mathbb{Z}$. For this, we redefine $B_i = \bigcup_{j=-\infty}^{i-1}\{j \longrightarrow 2i\}$ and note that (3.2) still is valid, so that the same reasoning leads us to (3.4). Analogously, we obtain (3.4) with A_{-i} instead of A_i, and the almost sure extinction follows.

Proof of Theorem 2.2. It is enough to consider $L = 1$. In this case, an active particle dies as soon as it jumps to the left. Consider the events A_i, $\{i \longrightarrow j\}$ and $\{i \nrightarrow j\}$ as defined in the proof of Theorem 2.1. Recall that $\{i \longrightarrow k\}$ and $\{j \longrightarrow \ell\}$ are independent for $i \neq j$ and observe that for $n \geq M$,

$$A_{n+1} \supset (A_n \cap \{n \longrightarrow n+1\}) \cup \bigcup_{i=n-M+1}^{n-1}\left(A_i \cap \{i \longrightarrow n+1\} \cap \bigcap_{j=i+1}^{n}\{j \nrightarrow n+1\}\right).$$

Thus, using that $\mathbb{P}(A_n)$ is non-increasing in n,

$$\begin{aligned}
\mathbb{P}(A_{n+1}) &\geq \mathbb{P}(A_n)\left[\mathbb{P}(n \longrightarrow n+1) + \sum_{i=n-M+1}^{n-1}\mathbb{P}(i \longrightarrow n+1)\prod_{j=i+1}^{n}\mathbb{P}(j \nrightarrow n+1)\right] \\
&= \mathbb{P}(A_n)\left[1 - \prod_{i=n-M+1}^{n}\mathbb{P}(i \nrightarrow n+1)\right] \\
&= \mathbb{P}(A_n)\left[1 - \prod_{i=n-M+1}^{n} 1 - (1-q_i)^{n+1-i}\right] \\
&\geq \mathbb{P}(A_n)\left[1 - \prod_{i=n-M+1}^{n}(n+1-i)q_i\right] \\
&= \mathbb{P}(A_n)\left[1 - M!\prod_{i=n-M+1}^{n} q_i\right].
\end{aligned}$$

So, using Lemma 3.1, we have that for t large enough

$$\mathbb{P}(A_{n+t}) \geq C\mathbb{P}(A_t)$$

where $C > 0$ does not depend on n. Letting $n \to \infty$ the result follows. □

Proof of Theorem 2.3. First recall that κ is an even integer. We have to prove two claims:

(i) The process dies out a.s. for $L = \kappa$.
(ii) The process survives with positive probability for $L = \kappa + 1$.

For $j \geq 1$, let D_j be the event that the particle at the vertex s^j is activated. Observe that at the instant that D_j occurs, all active particles are at the vertex s^j and the number of active particles is between $s^j - s^{j-1} + 1$ and s^j. For $L = \kappa$, we work by considering the best case for survival in terms of active particles: s^j active particles with L lives each. We obtain that

$$\mathbb{P}(D_j) = \mathbb{P}(D_1)\prod_{i=1}^{j-1} \mathbb{P}(D_{i+1}|D_i) \leq \prod_{i=1}^{j-1} 1 - ((q_{s^i})^{\kappa/2})^{s^i} = \prod_{i=1}^{j-1} 1 - (i+1)^{-1}.$$

Observe the importance of assuming κ even: if the process dies out with L odd, then by construction it also dies out with $L + 1$.

On the other hand, if $L = \kappa + 1$, then, by considering the worst case ($s^j - s^{j-1} + 1$ active particles with L lives each),

$$\mathbb{P}(D_j) \geq \prod_{i=1}^{j-1} 1 - ((q_{s^i})^{1+\kappa/2})^{s^i - s^{i-1}+1} \geq \prod_{i=1}^{j-1} 1 - (i+1)^{-\alpha},$$

where $\alpha = (1 - s^{-1})(1 + 2\kappa^{-1}) > 1$. As $\mathbb{P}(\text{survival}) = \lim_{j\to\infty}\mathbb{P}(D_j)$, both claims follow from Lemma 3.1. □

Acknowledgements

We wish to thank two anonymous referees for helpful suggestions. Also to Valdivino Vargas Júnior for the careful reading of the first draft of the paper.

Notes

1. Email: elcio@ime.usp.br
2. Email: zuluaga@ime.usp.br

References

[1] O.S.M. Alves, F.P. Machado, and S. Yu. Popov, *Phase transition for the frog model*, Electron. J. Probab. 7(16) (2002), pp. 1–25.
[2] O.S.M. Alves, F.P. Machado, and S. Yu. Popov, *The shape theorem for the frog model*, Ann. Appl. Probab. 12(2) (2002), pp. 534–547.
[3] P. Bremaud, *Markov chains. Gibbs fields, Monte Carlo simulation and queues*, in *Texts in Applied Mathematics*, Vol. 31, Springer-Verlag, New York, 1999.
[4] H. Kesten, and V. Sidoravicius, *The spread of a rumor or infection in a moving population*, Ann. Probab. 33(6) (2005), pp. 2402–2462.

[5] S. Lalley, *Spatial epidemics: Critical behavior in one dimension*. Avaliable at http://www.stat.uchicago.edu/lalley/Papers/spatial.pdf (2007).

[6] E. Lebensztayn, F.P. Machado, and S. Popov, *An improved upper bound for the critical probability of the frog model on homogeneous trees*, J. Stat. Phys. 119(1–2) (2005), pp. 331–345.

[7] R.B. Schinazi, *Mass extinctions: An alternative to the Allee effect*, Ann. Appl. Probab. 15(1B) (2005), pp. 984–991.