A NOTE ON MULTISTATE MONOTONE SYSTEM SIGNATURE

by

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Palavras-Chave: System signature, multistate system signature, coherent systems.
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A note on multistate monotone system signature

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Abstract. Using decompositions results for a multistate monotone system (MMS) and through exchangeability properties we obtain a signature representation for any level of a MMS. We extend some ordering applications.

Keywords: System signature; multistate system signature; coherent systems.

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1.Introduction

As in Barlow and Proschan (1981) a complex engineering system is completely characterized by its structure function Φ which relates its lifetime T and its components lifetimes T_i, $1 \leq i \leq n$, defined in a complete probability space $(\Omega, \Im, P)$

$$T = \Phi(\mathrm{T}), \mathrm{T} = (T_1, ..., T_n).$$

A system is said to be coherent if its structure function Φ is increasing in each coordinate and each component is relevant, that is, there exist a time t and a configuration of $\mathbf{T}$ in t such that the system works if, and only if, the component works. The system is of order n if it has n components.

The performance of a coherent system can be measured from this structural relationship and the distribution function of its components lifetimes. The structures functions offer a way of indexing the class of coherent system but such representations make the distribution function of the system lifetime analytically very complicated. An alternative representation for the coherent system distribution function is through the systems signatures, as in Samaniego(1985), that, while narrower in scope than the structure function, is substantially more useful.

Samaniego considers the order statistics of the independent and identically distributed components lifetimes of a coherent system of order n with continuous distribution. Clearly $\{T = T_{(i)}\}$ $1 \leq i \leq n$ is a (P-a.s.) partition of the probability space and

$$P(T \leq t) = \sum_{i=1}^{n} P(T = T_{(i)}) P(T_{(i)} \leq t | T = T_{(i)}) =$$

$$\sum_{i=1}^{n} P(T = T_{(i)}) P(T_{(i)} \leq t) = \sum_{i=1}^{n} \alpha_i P(T_{(i)} \leq t).$$

In the above context Samaniego defines

Definition 1.1 Let T be the lifetime of a coherent system of order n, with components lifetimes $T_1, ..., T_n$ which are independent and identically distributed random variables with continuous distribution F. Then the signature vector α is defined as

$$\alpha = (\alpha_1, ..., \alpha_n)$$

where $\alpha_i = P(T = T_{(i)})$ and the $\{T_{(i)}, 1 \leq i \leq n\}$ are the order statistics of $\{T_i, 1 \leq i \leq n\}$.

The key feature of system signatures that makes them broadly useful in reliability analysis is the fact that, in the context of independent and identically distributed (i.i.d.) continuous components lifetimes, they are distribution-free measures of system quality, depending solely on the design characteristics of the system and independent of the behavior of the systems components .

A detailed treatment of the theory and applications of system signatures may be found in Samaniego (2007). This reference gives detailed justification for the i.i.d. assumption used in the definition of system signatures. By the way there are some applications in which the i.i.d. assumption is appropriate, and in such case, the use of system signatures for comparisons among systems is wholly appropriate; such applications range from batteries in lighting, to wafers or chips in a digital computer to the subsystem of spark plugs in an automobile engine.

Samaniego (2007), Kochar, et al. (1999) and Shaked and Suarez-Llorens (2003) extended the signature concept to the case where the components lifetimes $T_1, ..., T_n$, of a system, are exchangeable, an interesting and practical situation in reliability theory. The random vector $(T_1, T_2, ..., T_n)$ is exchangeable if it has the same jointly distribution as $(T_{\sigma(1)}, T_{\sigma(2)}, ..., T_{\sigma(n)})$, for any permutation σ of the indices $\{1, 2, ..., n\}$.

We pay attention on two results:

The first interesting result is concerning exchangeability:

Theorem 1.2 If $(T_1, T_2, ..., T_n)$ is an exchangeable random vector, $T = \phi(T_1, ..., T_n)$ is the lifetime of a coherent system and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is the signature vector of a system with the same structure function as that of the system with lifetime T, but with independent and identically distributed components having a common absolutely continuous distribution, then

$$P(T > t) = \Sigma_{i=1}^{n} \alpha_i P(T_(i) > t).$$

The second relates signatures of systems with different number of components.

Theorem 1.3 If $(T_1, T_2, ..., T_n)$ is an exchangeable random vector and T is the lifetime of a coherent system with components $T_1, ..., T_k$, $k \leq n$, then

$$P(T > t) = \Sigma_{i=1}^{n} \alpha_i^* P(T_(i) > t)$$

for all t where α^* is the signature of order n of the system with the same structure as that the lifetime T but with independent and identically distributed component lifetimes having a common absolutely continuous distribution.

Even so, the widely used concept of signature and its development as in Boland, P.J. and Samaniego, F.,(2004), Bueno ,V.C., (2011a), (2011b), Kochar, S., Mukherjee, H., Samaniego, F.,(1999), Navarro, J., Ruiz, J.M., N. and Sandoval, C.J., (2005), (2007), , Navarro, J., Balakrishnan, N. and Samaniego,F.J.,(2008), (2010), Navarro, J., Balakrishnan, N. and Samaniego,F.J. and Bhattacharya D.(2008), Samaniego, F., (1985), (2006), (2007), Samaniego,F.J., Balakrishnan, N. and Navarro, J., (2009), does not have been explored in the multistate monotone system case.

In this note, in Section 2, we characterizes an monotone multistate system as in Block and Savits (1982) and, in Section 3, we analyse the signature concept in the context of multistate monotone systems. In Section 4 we study ordering for coherent systems.

2.Preliminaries concerning multistate monotone systems and decomposition results

In classical reliability theory, the system state X_t, at time t, is related with its components states, $X_t(i)$, $1 \leq i \leq n$, through its binary structure function ϕ, that is, $X_t = \phi(\mathrm{X}_t)$ where $\mathrm{X}_t = (X_t(1), ..., X_t(n))$. The system (component) state relates the system (component) lifetime through the equality $X_t = 1_{\{T>t\}}(X_t(i) = 1_{\{T_i>t\}})$, in the way that ϕ is defined in $\{0,1\}^n$ and assume values in $\{0,1\}$. $X_t = 1$ if the system is working at time t and its equal to 0, otherwise. Also, the component i is working at time t if $X_t(i) = 1$ and it is failed at t, if $X_t(i) = 0$.

An example of this structural relationship is the parallel series decomposition

$$X_t = \phi(\mathrm{X}_t) = \max_{1 \leq j \leq p} \min_{i \in P_j} X_t(i),$$

where P_j, $1 \leq j \leq p$ is the minimal pat sets of the structure ϕ, that is, the minimal set of components whose functioning insures the functioning of the system.

A multistate monotone system consider several levels of degradation for the system and its components. In this section we use the following structural notation for a multistate monotone system:

(a) $\mathrm{x} = (x_1, ..., x_n)$ denotes an n-dimensional vector with components $x_1, ..., x_n$.

(b) $\mathrm{m} = (m, ..., m)$ and $0 = (0, ..., 0)$.

(c) $\mathrm{x} < \mathrm{y}$ means that $x_i \leq y_i$ for all i and the inequality holds for at least one $i, i = 1, ..., n$.

Consider a system of n components $\mathbf{x} = (x_1, ..., x_n)$ where x_i, $1 \le i \le n$ denotes the performance of component i, in a fixed time,and takes values in the set of states $M = \{0, 1, ..., m\}$. m is the perfect state of functioning and 0 represents its complete failure. We suppose that there exists a nondecreasing function ϕ taking values in the set M such that $\phi(\mathbf{x})$ denotes the performance of the system. Also we assume that $\phi(\mathbf{0}) = 0$ and $\phi(\mathbf{m}) = m$. ϕ is called the structure function of the Multistate Monotone System (MMS).

The following concepts are due to Block and Savits (1982).

Definition 2.1 A vector $\mathbf{x}$ is called an upper vector for level k of an MMS ϕ if, and only if $\phi(\mathbf{x}) \ge k$, $1 \le k \le n$. It is called a critical upper vector for level k if, in addition, $\phi(\mathbf{y}) < k$ for $\mathbf{x} > \mathbf{y}$.

The set of all critical upper vectors for level k is denoted U_k. If $\mathbf{x} \in U_k, 1 \le k \le m$, let $U_k(\mathbf{x}) = \{(i, x_i) : x_i \ne 0\}$.

The vector of binary variables $\alpha(\mathbf{x}) = (\alpha_{ij}(\mathbf{x}), 1 \le i \le n, 1 \le j \le m)$, lexicographically ordered, whose components are defined by $\alpha_{ij}(\mathbf{x}) = 1$ if $x_i \ge j$ and 0 otherwise allows us to consider the binary function

$$\phi_k(\alpha(\mathbf{x})) = \max_{\mathbf{y} \in U_k} \min_{(i,j) \in U_k(\mathbf{y})} \alpha_{ij}(\mathbf{x})$$

called the systems decomposition at level k, $1 \le k \le m$.

This binary structure characterizes the level k of the MMS ϕ in the sense that, for $k > 0, \phi(\mathbf{x}) \ge k$ if, and only if, $\phi_k(\alpha(\mathbf{x})) = 1$. Block and Savits (1982) proved that

$$\phi(\mathbf{x}) = \sum_{k=1}^{m} \phi_k(\alpha(\mathbf{x})).$$

Example 2.2

A) Consider the MMS $\phi(\mathbf{x}) = \max\{x_1, x_2\}$ where x_1 and x_2 are the component states, in a fixed time, assuming values in $M = \{0, 1, 2\}$. Clearly

$$U_1 = \{(1,0), (0,1)\}, \;\; U_1((1,0)) = \{(1,1)\}, \;\; U_1((0,1)) = \{(2,1)\},$$

$$U_2 = \{(2,0), (0,2)\}, \;\; U_2((2,0)) = \{(1,2)\}, \;\; U_2((0,2)) = \{(2,2)\}.$$

Therefore

$$\phi_1(\alpha(\mathbf{x})) = \max_{\mathbf{y} \in U_1} \min_{(i,j) \in U_1(\mathbf{y})} \alpha_{ij}(\mathbf{x}) = \max\{\alpha_{11}(\mathbf{x}), \alpha_{21}(\mathbf{x})\}$$

and

$$\phi_2(\alpha(\mathbf{x})) = \max_{\mathbf{y} \in U_2} \min_{(i,j) \in U_2(\mathbf{y})} \alpha_{ij}(\mathbf{x}) = \max\{\alpha_{12}(\mathbf{x}), \alpha_{22}(\mathbf{x})\}.$$

B) For the MMS $\phi(\mathrm{x}) = \min\{x_1, x_2\}$ we have

$$U_1 = \{(1,1)\}, \quad U_1((1,1)) = \{(1,1),(2,1)\},$$

$$U_2 = \{(2,2),\}, \quad U_2((2,2)) = \{(1,2),(2,2)\}.$$

Therefore

$$\phi_1(\alpha(\mathrm{x})) = \max_{\mathrm{y}\in U_1} \min_{(i,j)\in U_1(\mathrm{y})} \alpha_{ij}(\mathrm{x}) = \min\{\alpha_{11}(\mathrm{x}), \alpha_{21}(\mathrm{x})\}$$

and

$$\phi_2(\alpha(\mathrm{x})) = \max_{\mathrm{y}\in U_2} \min_{(i,j)\in U_2(\mathrm{y})} \alpha_{ij}(\mathrm{x}) = \min\{\alpha_{12}(\mathrm{x}), \alpha_{22}(\mathrm{x})\}.$$

Considering the random behavior of the components $X_i, 1 \leq i \leq n$, $X_i \in M$, we can define:

Definition 2.3

The reliability of component i at level l is defined as

$$E[\alpha_{il}(\mathrm{X})] = P(\alpha_{il}(\mathrm{X}) = P(X_i \geq l) = \overline{P}_{il};$$

The reliability of the multistate monotone system at level k is

$$E[\phi_k(\alpha(\mathrm{X}))] = P(\phi(\mathrm{X}) \geq k) = H_k(\overline{\mathrm{P}})$$

where $\overline{\mathrm{P}} = (\overline{P}_{ij}, 1 \leq i \leq n, \;\; 1 \leq j \leq m)$.

The MMS reliability is given by

$$E[\phi(\mathrm{X})] = \sum_{k=1}^{m} P(\phi(\mathrm{X}) \geq k) = \sum_{k=1}^{m} E[\phi_k(\alpha(\mathrm{X}))].$$

3.Signature of a multistate systems signature

In the dynamic case we let $(X_t(i))_{t\geq 0}$, be a right continuous nonincreasing stochastic process with values in $M = \{0, 1, ..., m\}$, representing the stochastic behavior of component $i, 1 \leq i \leq n$. The MMS system is represented by $(\phi(\mathrm{X}_t))_{t\geq 0}$ where $\mathrm{X}_t = (X_t(1), ..., X_t(n))$.

The vector of binary processes $\alpha(\mathrm{X}_t) = (\alpha_{ij}(\mathrm{X}_t), 1 \leq i \leq n, 1 \leq j \leq m)$, lexicographically ordered, whose components are defined by $\alpha_{ij}(\mathrm{X}_t) = 1$ if $X_t(i) \geq j$ and 0 otherwise allows us to consider, as in Section 2, the binary process

$$\phi_k(\alpha(\mathrm{X}_t)) = \max_{\mathrm{y}\in U_k} \min_{(i,j)\in U_k(\mathrm{y})} \alpha_{ij}(\mathrm{X}_t)$$

called the systems decomposition at level $k, \;\; 1 \leq k \leq m$.

To follows, we define $T_{ij} = \inf\{t \geq 0 : \alpha_{ij}(\mathrm{X}_t) = 0\}$ if $\{\} \neq \emptyset$ and ∞ if $\{\} = \emptyset$, as the first time for which $X_t(i)$ is at any state smaller than j. Also $T_k = \inf\{t \geq o : \phi(\mathrm{X}_t) < k\}$ if $\{\} \neq \emptyset$ and ∞ if $\{\} = \emptyset$ as the first time for which $\phi(\mathrm{X}(t))$ is at any state smaller than k.

In this fashion, the reliability of component i, at level l and time t is

$$P(\alpha_{il}(\mathrm{X}_t) = 1) = P(T_{il} > t).$$

The reliability of the system at level k and time t is

$$P(\phi_k(\alpha(\mathrm{X}_t)) = 1) = P(T_k > t).$$

The reliability of the system at time t is

$$P(T > t) = E[\phi(\mathrm{X}_t)] = \sum_{k=1}^{m} E[\phi_k(\alpha(\mathrm{X}_t))] = \sum_{k=1}^{m} P(T_k > t).$$

Now, we can consider, as in Barlow and Proschan (1981), the equivalent representation

$$T_k = \max_{y \in U_k} \min_{(i,j) \in U_k(y)} T_{ij}$$

where $\alpha_{ij}(\mathrm{X}_t) = 1_{\{T_{ij} > t\}}$.

The first question to be done is what happen with the lifetime T_{ij} under the hypothesis of continuity, identity and independence of the random lifetimes T_i.

Theorem 3.1 If the random lifetimes T_i are independent and identically distributed with continuous distribution, then the random lifetimes T_{ij} are exchangeable.
Proof If we fix the index i we have

$$P(T_{i1} > t_1, T_{i2} > t_2, ..., T_{im} > t_m) = E[\pi_{j=1}^{m} 1_{\{T_{ij} > t_j\}}] =$$

$$E[\pi_{j=1}^{m} \alpha_{ij}(\mathrm{X}_{t_j})] = P(\pi_{j=1}^{m} \alpha_{ij}(\mathrm{X}_{t_j}) = 1) =$$

$$P(X_{t_j}(i) \geq j, 1 \leq j \leq m) = P(X_{\max_{1 \leq j \leq m} t_j}(i) \geq j) =$$

$$P(X_{\max_{1 \leq j \leq m} t_{i_j}}(i) \geq j) = P(T_{i1} > t_{i_1}, T_{i2} > t_{i_2}, ..., T_{im} > t_{i_m})$$

where $t_{i_j}, 1 \leq j \leq n$ is any permutation of $t_j, 1 \leq j \leq n$.

In the case in which we jointly consider more than one component the proof is clear under the hypothesis of independence and identically distributed components.

Now, using the above theorem, we are in conditions to use Theorem 1.2 and Theorem 1.3 to prove that

$$P(T_k > t) = \sum_{y\in U_k} \sum_{(i,j)\in U_k(y)} P(T_k = T_{(ij)})P(T_{(ij)} > t),$$

where $(T(ij),\ \ 1 \le i \le n,\ \ 1 \le j \le m)$ are the ordered $(Tij,\ \ 1 \le i \le n,\ \ 1 \le j \le m)$.

Definition 3.2 Let $T_1, ..., T_n$ be positive random variables defined in a complete probability space $(\Omega, \Im, P)$, representing the components lifetimes of a multistate monotone system with lifetime T. Let T_k be the systems lifetime at levels greather than or equal to k and let T_{ij} be the components lifetimes at levels greather than or equal to j. Then, the multistate system signature vector at level k, $1 \le k \le m$ is

$$\alpha^k = (\alpha^k_{ij},\ \ 1 \le i \le n,\ \ 1 \le j \le m),$$

where $\alpha^k_{ij} = P(T_k = T_{(ij)})$.

Remark $P(T_k = T_{(ij)})$ is a cumbersome notation. In a fixed time t, each component is in a fixed state $j, 1 \le j \le m$, and T_{ij} are ordered as T_i, independent of j, that is, we can think $T_{(ij)}$ as $T_{(i)}$. To overcome this situation we can use $T_{(ij)} = T_{(i)}, 1 \le i \le n,\ \ 1 \le j \le m$.

Example 3.3

If T_1, T_2, T_3 are independent and identically distributed component's lifetimes of the multistate monotone system lifetime $T = T_1 \wedge (T_2 \vee T_3)$ corresponding to a 3 components system assuming values in $M = \{0, 1, 2\}$.

Clearly

$$U_1 = \{(1,1,0), (1,0,1)\},\ \ U_1((1,1,0)) = \{(1,1), (2,1)\},\ \ U_1((1,0,1)) = \{(1,1), (3,1)\},$$

$$U_2 = \{(2,2,0), (2,0,2)\},\ \ U_2((2,2,0)) = \{(1,2), (2,2)\},\ \ U_2((2,0,2)) = \{(1,2), (3,2)\}.$$

Now

$$\phi_1(\alpha(\mathrm{X}_t)) = (\alpha_{11}(\mathrm{X}_t) \wedge \alpha_{21}(\mathrm{X}_t) \vee (\alpha_{11}(\mathrm{X}_t) \wedge \alpha_{31}(\mathrm{X}_t) =$$
$$\alpha_{11}(\mathrm{X}_t) \wedge (\alpha_{21}(\mathrm{X}_t) \vee \alpha_{31}(\mathrm{X}_t).$$

Therefore $\alpha^1_{11} = P(T_1 = T_{(11)}) = \frac{1}{3}$ and $\alpha^1_{21} = P(T_1 = T_{(21)}) = \frac{2}{3}$, and the signature system survival distribution is

$$P(T_1 > t) = \frac{1}{3}P(T_{(11)} > t) + \frac{2}{3}P(T_{(21)} > t).$$

Also

$$\phi_2(\alpha(\mathrm{X}_t)) = (\alpha_{12}(\mathrm{X}_t) \wedge \alpha_{22}(\mathrm{X}_t) \vee (\alpha_{12}(\mathrm{X}_t) \wedge \alpha_{32}(\mathrm{X}_t) =$$

$$\alpha_{12}(\mathrm{X}_t) \wedge (\alpha_{22}(\mathrm{X}_t) \vee \alpha_{32}(\mathrm{X}_t),$$

$\alpha_{12}^2 = P(T_2 = T_{(12)}) = \frac{1}{3}$ and $\alpha_{22}^2 = P(T_2 = T_{(22)}) = \frac{2}{3}$, following that the signature system survival distribution is

$$P(T_2 > t) = \frac{1}{3}P(T_{(11)} > t) + \frac{2}{3}P(T_{(21)} > t).$$

This representation uses the extension of the signature concept, from the case of independent and identically distributed components to exchangeable components lifetimes. However, this extension is not necessarily true if we take $\alpha_i = P(T_k = T_{(ij)})$. It is true if $(T_{ij}, \quad 1 \leq i \leq n, \quad 1 \leq j \leq m)$ has an exchangeable absolutely continuous jointly distribution (see the proof in Navarro and Rychlik, (2007)). We can observe this, taking the three dimensional Marshall and Olking distribution, obtained from $T_{ij} = S_{ij} \wedge S, 1 \leq i \leq 3, \quad 1 \leq j \leq 2$ where S_{ij} and S are independents and exponentially distributed random variables, S_{ij} with parameter λ and S with parameter θ, we have

$$P(T_{ij} > t_{ij}, 1 \leq i \leq 3, \quad 1 \leq j \leq 2) = e^{-\lambda \Sigma_{i=1}^{3} \Sigma_{j=1}^{2} t_{ij} - \theta \max_{ij} t_{ij}}.$$

Therefore

$$P(T_{(1)}(T_{11}, T_{21}.T_{31}) > t) = e^{-(3\lambda+\theta)t};$$

$$P(T_{(2)}(T_{11}, T_{21}.T_{31}) > t) = 3e^{-(2\lambda+\theta)t} - 2e^{-(3\lambda+\theta)t};$$

and

$$P(T_1 = T_{11} \wedge (T_{21} \vee T_{31}) > t) = 2e^{-(2\lambda+\theta)t} - e^{-(3\lambda+\theta)t},$$

performing

$$P(T_1 > t) = \frac{1}{3}P(T_{(11)} > t) + \frac{2}{3}P(T_{(21)} > t).$$

However,

$$P(T_1 = T_{(3)}(T_{11}, T_{21}.T_{31}) = P(T_{11} = T_{21} \vee T_{31}) = \frac{\theta(\theta + 4\lambda)}{(\theta + 2\lambda)(\theta + 3\lambda)} > 0$$

and $P(T_1 = T_{(1)}(T_{11}, T_{21}.T_{31}) \neq \frac{1}{3}$ or $P(T_1 = T_{(2)}(T_{11}, T_{21}.T_{31}) \neq \frac{2}{3}$

4.Ordering multistate monotone systems

In this section we assume that the system components processes $(X_t(i))_{t \geq 0}, \ 1 \leq i \leq n$ are independent and identically distributed processes taking values in $M = \{0, 1, ..., m\}$.

As in Section 3, the component level processes $(\alpha_{ij}(\mathrm{X}_t), \quad 1 \leq i \leq n, \quad 1 \leq j \leq m)$, is exchangeable.

We analyse results for stochastic, hazard rate and likelihood ratio orders.

A random vector $\mathbf{T}$, or its jointly distribution function, is stochastically lower than a random vector S, or its jointly distribution function, denoted by $\mathrm{T} \leq^{st} \mathrm{S}$ if, and only if

$$E[H(\mathrm{T})] \leq E[H(\mathrm{S})]$$

holds for all increasing Borel measurable function such that the expected values exists.

A random variable T, or its distribution function, is lower than the random variable S, or its distribution function, denoted by $T \leq^{hr} S$ if, and only if,

$$P(T > t)P(S > s) \geq P(T > s)P(S > t), \quad \forall t \leq s.$$

If random variables T and S are absolutely continuous, we say that T, or its distribution function, is lower than S, or its distribution function, in likelihood ratio order, denoted by $T \leq^{lr} S$ if, and only if

$$\frac{f_T(t)}{f_S(t)}, \downarrow t, \quad \forall t \geq 0,$$

where $f_T(t)$ and $f_S(t)$ are the probability densities functions of T and S, respectively.

To follows we are going to consider the well know results proved in Shaked and Shanthikumar (2007):

Lemma 4.1 If $P(T > t|\theta)$ and $P(S > t|\theta)$ are two families of reliability functions, for $\theta \in \Theta$, such that

$$P(T > t|\theta) \leq P(S > t|\theta), \quad \forall t \in \Re^+, \quad \forall \theta \in \Theta,$$

then

$$P(T > t) = \int_\Theta P(T > t|\theta)dG_1(\theta) \leq \int_\Theta P(S > t|\theta)dG_2(\theta) = P(S > t)$$

for all distribution function G_1 and G_2 such that $G_1 \leq^{st} G_2$.

Lemma 4.2 If $P(T > t|\theta)$ and $P(S > t|\theta)$ are two families of reliability functions, for $\theta \in \Theta$, such that

$$P(T > t|\theta) \leq^{hr} (\leq^{lr}) P(S > t|\theta^*)$$

whenever $\theta \leq \theta^*$, then

$$P(T > t) = \int_\Theta P(T > t|\theta)dG_1(\theta) \leq \int_\Theta P(S > t|\theta)dG_2(\theta) = P(S > t)$$

for all distribution function G_1 and G_2 such that $G_1 \leq^{hr} (\leq^{lr}) G_2$.

Theorem 4.3 Let T be the lifetime of a multistate monotone system with components dynamically represented by the processes $(X_t(i))_{t\geq 0}$, $1 \leq i \leq n$, independent and identically distributed, taking values in $M = \{0, 1, ..., m\}$ and let S be the lifetime of other multistate monotone systems with components representation $(Y_t(i))_{t\geq 0}$, $1 \leq i \leq n$, which are independent and identically distributed processes. Let α^k and β^k, be the signature vector, at level $k, 1 \leq k \leq m$, of the system with lifetimes T and S, respectively. If $\alpha^k \leq^{st} \beta^k$ and $P(X_i(t) \geq j) \leq P(Y_i(t) \geq j)$, $\forall t > 0$, $j \in M$, then, $T_k \leq^{st} S_k, 1 \leq k \leq m$. Furthermore, $T \leq^{st} S$.

Proof

Let, as before, $\alpha_{ij}(\mathrm{X}_t) = 1_{\{T_{ij}>t\}}$ and $\beta_{ij}(\mathrm{X}_t) = 1_{\{S_{ij}>t\}}$.
Follows that

$$P(T_{ij} > t) = E[1_{\{T_{ij}>t\}}] = E[\alpha_{ij}(\mathrm{X}_t)] = P(\alpha_{ij}(\mathrm{X}_t) = 1) = P(X_t(i) \geq j) \leq$$

$$P(Y_i(t) \geq j) = P(\beta_{ij}(\mathrm{X}_t) = 1) = E[\beta_{ij}(\mathrm{X}_t)] = E[1_{\{S_{ij}>t\}}] = P(S_{ij} > t).$$

Therefore $T_{ij} \leq^{st} S_{ij}$. As $T_{(ij)}$ is an increasing function, say Ψ_T, of $(T_{ij}, 1 \leq i \leq n,\ 1 \leq j \leq m)$ and $S_{(ij)}$ is an increasing function, say Ψ_S, of $(S_{ij}, 1 \leq i \leq n,\ 1 \leq j \leq m)$, for any increasing and Borel measurable function Φ we have

$$E[\Phi(T_{(ij)})] = E[\Phi(\Psi_T(T_{ij}, 1 \leq i \leq n,\ 1 \leq j \leq m) \leq$$

$$E[\Phi(\Psi_S(S_{ij}, 1 \leq i \leq n,\ 1 \leq j \leq m) = E[\Phi(S_{(ij)})].$$

So, we also have $T_{(ij)} \leq^{st} S_{(ij)}$. From Theorem 3.1 we have

$$P(T_k > t) = \sum_{y\in U_k} \sum_{(i,j)\in U_k(y)} P(T_k = T_{(ij)}) P(T_{(ij)} > t).$$

From Lemma 4.1 and the hypothesis $\alpha^k \leq^{st} \beta^k$, we have $T_k \leq^{st} S_k, 1 \leq k \leq m$. Furthermore

$$P(T > t) = E[\phi(\mathrm{X}_t)] = \Sigma_{k=1}^m E[\phi_k(\alpha(\mathrm{X}_t))] = \Sigma_{k=1}^m P(T_k > t) \leq$$

$$\Sigma_{k=1}^m P(S_k > t) = \Sigma_{k=1}^m E[\phi_k(\beta(\mathrm{Y}_t))] = E[\phi(\mathrm{Y}_t)] = P(S > t),$$

and $T \leq^{st} S$.

Theorem 4.4 Let T be the lifetime of a multistate monotone system with components dynamically represented by the processes $(X_t(i))_{t\geq 0}$, $1 \leq i \leq n$,independent and identically distributed , taking values in $M = \{0, 1, ..., m\}$ and let S be the lifetimes of other multistate monotone systems with components representation $(Y_t(i))_{t\geq 0}$, $1 \leq i \leq n$, which are independent and identically distributed processes. Let α^k and β^k, be the signature vector, at level $k, 1 \leq k \leq m$, of the system with lifetimes T and S, respectively. If $\alpha^k \leq^{hr} (\leq^{lr}) \beta^k$ and $T_{(ij)} \leq^{hr} (\leq^{lr}) S_{(ij)}$ then, $T_k \leq^{hr} (\leq^{lr}) S_k, 1 \leq k \leq m$.

This Theorem is only an application of Lemma 4.2.

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