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by

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Abstract

In this paper we derive a general closed-form expression for the Bartlett correction for testing a scalar parameter of a two-parameter exponential family model. The correction has the advantage for algebrical and numerical purposes since it involves only trivial operations on suitably defined functions. The formula derived is general enough to cover many important and commonly used distributions. Simulations show that the corrected likelihood ratio tests have empirical sizes closer to the nominal size than the classical uncorrected tests.

Key words: Bartlett correction; chi-squared distribution; exponential family; likelihood ratio statistic; rejection rate.

1 Introduction

The asymptotic chi-squared distribution of the likelihood ratio (LR) statistic ω is frequently used to test hypotheses of interest in regression models. However, as the sample size decreases, the use of such a statistic becomes less justifiable. One way of improving the chi-squared approximation to the LR statistic is by using a Bartlett correction. In fact, under mild regularity conditions, the Bartlett correction c guarantees that all moments of the adjusted LR statistic ω^* are equal to those of the asymptotic χ^2 distribution up to order n^{-1}, where n is the sample size. The Bartlett correction c and the modified statistic ω^* are defined by $c = E(\omega)/p$ and $\omega^* = w/c$, where p is the difference of the dimensions of the parameter spaces under the alternative and the null hypothesis and $E(\omega)$ is obtained up to order n^{-1}. The Bartlett corrections are usually effective in bringing the true sizes of the modified statistic ω^* closer to the nominal levels. A method for obtaining c was developed in full generality by Lawley (1956), who showed by a complicated calculation that all cumulants of ω^* agree to order n^{-1} with those of the reference χ^2_p distribution. The disavantage of this method is that it requires certain joint cumulants of log-likelihood derivatives.

In recent years there has been a renewed interest in Bartlett corrections. Cordeiro (1983, 1987) derived closed-form Bartlett corrections in generalized linear models (Nelder and Wedderburn, 1972) and discussed improved LR tests. Bartlett corrections for models defined by any one-parameter distribution in which the mean is a known function of a linear combination of unknown parameters were obtained by Cordeiro (1985), who generalized his own results of 1983. Several papers have focused on deriving closed-form Bartlett corrections for specific regression problems. For example, Moulton, Weissfeld and St. Laurent (1993) obtained Bartlett corrections for logistic regressions; Attfield (1991) and Cordeiro (1993a) showed how to correct LR tests for heteroskedasticity; Wong (1991) obtained a Bartlett correction for testing several slopes in regression models whose independent variables are subject to error; Wang (1994) derived a Bartlett correction for testing the equality of normal variances against an increasing alternative; Cordeiro, Paula and Botter (1994) derived corrections for the class of dispersion models proposed by Jørgensen (1987); and Chesher and Smith (1997) obtained Bartlett corrections for LR specification tests. A correction to the LR statistic in regression models with Student-t errors was obtained by Ferrari and Arellano-Valle (1996), and similar corrections to heteroskedastic linear models and multivariate regression were obtained by Cribari-Neto and Ferrari (1995) and Cribari-Neto and Zarkos (1995), respectively. Furthermore, Bartlett corrections for general models were discussed by a few authors. An algorithm for computing Bartlett corrections in general statistical models was given by Jensen (1993); see also Andrews and Stafford (1993) and Stafford and Andrews (1993). General matrix formulae for computing Bartlett corrections were developed by Cordeiro (1993b). For a detailed account of the applicability of Bartlett corrections, see Cribari-Neto and Cordeiro (1996).

The purpose of this paper is to obtain simple Bartlett corrections to improve the LR test of a scalar parameter of two-parameter exponential family models where no cumulants are involved. A simple Bartlett correction for one-parameter exponential models that does not depend on cumulants of log-likelihood derivatives was derived by Cordeiro et al. (1995). Then they applied their result to several distributions in the uniparametric exponential family. The present paper can be therefore viewed as an extension of their paper for two-parameter exponential family models.

Consider a set of n independent and identically distributed random variables $y_1, \ldots, y_n$ with density function

$$\pi(y; \mu, \nu) = \exp\{\alpha_1(\mu, \nu)d_1(y) + \alpha_2(\mu, \nu)d_2(y) - \rho(\mu, \nu) + v(y)\}, \tag{1.1}$$

where μ and ν are scalar parameters, $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\rho(\cdot, \cdot)$, $d_1(\cdot)$, $d_2(\cdot)$ and $v(\cdot)$ are known functions. If $y_1, \ldots, y_n$ are continuous, π is assumed to be a density with respect to the Lebesgue measure while, if $y_1, \ldots, y_n$ are discrete, π is assumed to be a density with respect to counting measure. We also assume that the support set of (1.1) is independent of μ and ν and that $\alpha_1(\cdot,\cdot)$, $\alpha_2(\cdot,\cdot)$ and $\rho(\cdot,\cdot)$ have continuous first four derivatives with respect to the parameters μ and ν.

The discussion in this paper proceeds as follows. In the next section we introduce our notation and obtain a simple formula for the Bartlett correction to the LR test of a scalar parameter in the exponential model (1.1) assuming that the parameters μ and ν are globally orthogonal. The formula is simple enough to be used algebrically to obtain closed-form expressions in several special cases since it involves only functions $\alpha_1(\mu,\nu)$, $\alpha_2(\mu,\nu)$ and $\rho(\mu,\nu)$ and their derivatives with respect to the parameters μ and ν. In Section 3, we present a number of distributions of (1.1) with orthogonal parameters in order to show that our result has a wide range of important applications. In Section 4 we show that it is always possible to reparameterize the model (1.1) to achieve the orthogonality between the parameters. We then apply the formula for the Bartlett correction via the reparameterized model to other important tests for which orthogonality does not hold. Finally, in Section 5, some simulation results illustrate the superiority of our Bartlett-corrected statistic ω^* over the usual LR statistic ω with regard to second-order asymptotic theory.

2 Model with orthogonal parameters

Let $l(\mu,\nu)$ be the total log-likelihood for the unknown parameters given the observable data $y = (y_1, \ldots, y_n)$. We have from (1.1)

$$l = l(\mu,\nu) = \sum_{l=1}^{n} t(y_l;\mu,\nu), \tag{2.1}$$

where $t(y_l;\mu,\nu) = \alpha_1 d_1(y_l) + \alpha_2 d_2(y_l) - \rho + v(y_l)$, with $\alpha_1 = \alpha_1(\mu,\nu)$, $\alpha_2 = \alpha_2(\mu,\nu)$ and $\rho = \rho(\mu,\nu)$. We make some assumptions (Cox and Hinkley, 1974, Chapter 9; Jensen, 1993) on the behavior of $l(\mu,\nu)$ as the sample size n approaches infinity such as the regularity of the first four derivatives of $l(\mu,\nu)$ with respect to the parameters μ and ν. We use the following notation: $\alpha_s^{(i,j)} = \partial^{i+j}\alpha_s/\partial\mu^i\partial\nu^j$ and $\rho^{(i,j)} = \partial^{i+j}\rho/\partial\mu^i\partial\nu^i$ for $i,j = 0,1,2,3,4$ and $s = 1,2$. Differentiating (2.1) we can find from $E(\partial l/\partial\mu) = E(\partial l/\partial\nu) = 0$, the following relations:

$$\alpha_1^{(1,0)}\beta_1 + \alpha_2^{(1,0)}\beta_2 = \rho^{(1,0)}, \quad \alpha_1^{(0,1)}\beta_1 + \alpha_2^{(0,1)}\beta_2 = \rho^{(0,1)},$$

where $\beta_r = E\{d_r(y)\}$ for $r = 1,2$. The system of equations for β_1 and β_2 yields

$$\beta_1 = \frac{\rho^{(1,0)}\alpha_2^{(0,1)} - \rho^{(0,1)}\alpha_2^{(1,0)}}{\alpha_1^{(1,0)}\alpha_2^{(0,1)} - \alpha_1^{(0,1)}\alpha_2^{(1,0)}}, \quad \beta_2 = \frac{\rho^{(0,1)}\alpha_1^{(1,0)} - \rho^{(1,0)}\alpha_1^{(0,1)}}{\alpha_1^{(1,0)}\alpha_2^{(0,1)} - \alpha_1^{(0,1)}\alpha_2^{(1,0)}},$$

assuming that

$$\alpha_1^{(1,0)}\alpha_2^{(0,1)} - \alpha_1^{(0,1)}\alpha_2^{(1,0)} \neq 0.$$

The score functions for μ and ν are given by

$$\frac{\partial l}{\partial\mu} = \sum_{j=1}^{n}\sum_{i=1}^{2}\alpha_i^{(1,0)}(d_i(y_j) - \beta_i), \quad \frac{\partial l}{\partial\nu} = \sum_{j=1}^{n}\sum_{i=1}^{2}\alpha_i^{(0,1)}(d_i(y_j) - \beta_i), \qquad (2.2)$$

and the maximum likelihood estimates (MLEs) $\hat{\mu}$ and $\hat{\nu}$ satisfy the equations $\partial l/\partial\mu = 0$ and $\partial l/\partial\nu = 0$. If these equations are nonlinear they can be solved using Fisher's scoring method.

We now derive a simple formula for the Bartlett correction to the LR statistic ω in the two-parameter exponential model (1.1) assuming that the parameters μ and ν are globally orthogonal. There are a number of conceptual and mathematical advantages if the parameters μ and ν are orthogonal. Orthogonality implies, in particular, that the corresponding score components $\partial l/\partial\mu$ and $\partial l/\partial\nu$ are asymptotically uncorrelated.

The standard notation for the log-likelihood derivatives is used (Lawley, 1956; Cordeiro, 1987): $\kappa_{\mu\mu} = E(\partial^2 l/\partial\mu^2)$, $\kappa_{\nu\nu} = E(\partial^2 l/\partial\nu^2)$, $\kappa_{\mu,\mu} = E\{(\partial l/\partial\mu)^2\}$, $\kappa_{\mu,\nu} = E(\partial l/\partial\mu\partial l/\partial\nu)$, $\kappa_{\mu\mu\mu} = E(\partial^3 l/\partial\mu^3)$, $\kappa_{\mu\mu}^{(\nu)} = \partial\kappa_{\mu\mu}/\partial\nu$, $\kappa_{\nu\mu,\nu\mu} = E\{(\partial^2 l/\partial\nu\partial\mu)^2\} - \kappa_{\nu\mu}^2$, etc. Further, we adopt the notation $\beta_r^{(i,j)} = \partial^{i+j}\beta_r/\partial\mu^i\partial\nu^j$ for $i,j = 0,1,2,3,4$ and $r = 1,2$. After some algebra, we can obtain

$$\kappa_{\mu,\mu} = -\kappa_{\mu\mu} = ns_{10,10}, \quad \kappa_{\mu,\nu} = -\kappa_{\mu\nu} = ns_{10,01}, \quad \kappa_{\nu,\mu} = -\kappa_{\nu\mu} = ns_{01,10},$$

$$\kappa_{\nu,\nu} = -\kappa_{\nu\nu} = ns_{01,01}, \quad \kappa_{\mu\mu}^{(\nu)} = -n(s_{11,10} + s_{10,11}), \quad \kappa_{\nu\nu}^{(\nu)} = -n(s_{02,01} + s_{01,02})$$

$$\kappa_{\nu\nu}^{(\mu)} = -n(s_{11,01} + s_{01,11}), \quad \kappa_{\mu\mu\nu} = -n(s_{20,01} + s_{11,10} + s_{10,11}),$$

$$\kappa_{\nu\nu\mu} = -n(s_{02,10} + s_{11,01} + s_{01,11})$$

$$\kappa_{\nu\nu\nu} = -n(2s_{02,01} + s_{01,02}), \quad \kappa_{\mu\mu\mu} = -n(2s_{20,10} + s_{10,20}),$$

$$\kappa_{\mu\mu\nu\nu} = -n(2s_{21,01} + 2s_{11,11} + s_{20,02} + s_{12,10} + s_{10,12}),$$

$$\kappa_{\mu\mu}^{(\nu\nu)} = -n(s_{12,10} + 2s_{11,11} + s_{10,12}),$$

$$\kappa_{\nu\mu\mu}^{(\nu)} = -n(s_{12,10} + s_{11,11}), \kappa_{\mu\nu\nu}^{(\mu)} = -n(s_{21,01} + s_{11,11})$$

$$\kappa_{\nu\mu,\nu\mu} = n\zeta, \quad \text{etc.},$$

where $s_{jk,lm} = \sum_{i=1}^{2} \alpha_i^{(j,k)} \beta_i^{(l,m)}$,

$$\begin{aligned}
\zeta &= (\alpha_1^{(1,1)})^2 V_1 + (\alpha_2^{(1,1)})^2 V_2 + 2\alpha_1^{(1,1)}\alpha_2^{(1,1)} V_{12}, \\
\gamma &= (\alpha_1^{(2,0)})^2 V_1 + (\alpha_2^{(2,0)})^2 V_2 + 2\alpha_1^{(2,0)}\alpha_2^{(2,0)} V_{12}.
\end{aligned}$$

Here, $V_i = \mathrm{Var}(d_i(y))$ for $i = 1,2$ and $V_{12} = \mathrm{Cov}(d_1(y), d_2(y))$. A very simple way to obtain V_1, V_2 and V_{12} is given by Johnson, Ladalla and Liu (1979) who showed that these quantities follow by solving the system of equations

$$\begin{aligned}
s_{01,01} &= (\alpha_1^{(0,1)})^2 V_1 + (\alpha_2^{(0,1)})^2 V_2 + 2\alpha_1^{(0,1)}\alpha_2^{(0,1)} V_{12}, \\
s_{10,10} &= (\alpha_1^{(1,0)})^2 V_1 + (\alpha_2^{(1,0)})^2 V_2 + 2\alpha_1^{(1,0)}\alpha_2^{(1,0)} V_{12}, \\
V_{12} &= -\frac{\sum_{i=1}^{2} \alpha_i^{(1,0)}\alpha_i^{(0,1)} V_i}{(\alpha_1^{(1,0)}\alpha_2^{(0,1)} + \alpha_1^{(0,1)}\alpha_2^{(1,0)})},
\end{aligned}$$

subject to the restrictions

$$s_{10,01} = 0, \quad s_{01,10} = 0. \tag{2.3}$$

Under orthogonality between μ and ν, $\kappa_{\nu,\mu} = 0$ for all (μ, ν) in the parameter space and hence the joint information matrix for the parameters μ and ν is block-diagonal and is given by $K = K(\mu, \nu) = \text{diag}\{-\kappa_{\mu\mu}, -\kappa_{\nu\nu}\}$. Further, the MLEs $\hat{\mu}$ and $\hat{\nu}$ are asymptotically independent and converge in distribution to a bivariate normal distribution $N_2((\mu, \nu)^T, K^{-1})$, where $K^{-1} = K(\mu, \nu)^{-1} = \text{diag}\{-\kappa^{\mu\mu}, -\kappa^{\nu\nu}\}$, with $\kappa^{\mu\mu} = \kappa_{\mu\mu}^{-1}$ and $\kappa^{\nu\nu} = \kappa_{\nu\nu}^{-1}$. Also, the asymptotic variance of $\hat{\mu}(\hat{\nu})$ is the same whether or not ν (μ) is known. A related aspect is that $\hat{\mu}_\nu$, the MLE of μ for specified ν, varies only slowly with ν in the neighbourhood of $\hat{\nu}$, and that there is a corresponding slow variation of $\hat{\nu}_\mu$ with μ. If the parameters μ and ν are not orthogonal, it is possible to determine a new parameterization such that the joint information matrix is block-diagonal. We explore this idea due to Cox and Reid (1987) in Section 4.

Without loss of generality, we assume that μ is the parameter of interest and ν is the nuisance parameter. The composite null hypothesis under test is $H_0 : \mu = \mu^{(0)}$ against a two-sided alternative, where $\mu^{(0)}$ is a given number. Let $\tilde{\nu}$ be the MLE of ν under H_0. Functions evaluated at $(\mu^{(0)}, \tilde{\nu})$ will be denoted by the addition of a tilde. The LR statistic ω for testing H_0 is

$$\omega = 2\{l(\hat{\mu}, \hat{\nu}) - l(\mu^{(0)}, \tilde{\nu})\}. \tag{2.4}$$

Under H_0 and some regularity conditions, (2.4) is asymptotically distributed as χ_1^2. We can obtain $E(\omega)$ from Lawley (1956) and Cordeiro (1987) as

$$E(\omega) = 1 + \epsilon_2 - \epsilon_1, \tag{2.5}$$

where $\epsilon_2 = \sum'_{\mu,\nu}(l_{rstu} - l_{rstuvw})$ and $\epsilon_1 = l_{\nu\nu\nu\nu} - l_{\nu\nu\nu\nu\nu\nu}$ with the l's being obtained from

$$l_{rstu} = \kappa^{rs}\kappa^{tu}\left(\frac{1}{4}\kappa_{rstu} - \kappa_{rst}^{(u)} + \kappa_{rs}^{(tu)}\right) \tag{2.6}$$

and

$$l_{rstuvw} = \kappa^{rs}\kappa^{tu}\kappa^{vu}[\kappa_{rtu}(\frac{1}{6}\kappa_{rsuw} - \kappa_{sw}^{(u)}) + \kappa_{rtu}(\frac{1}{4}\kappa_{svw} - \kappa_{sw}^{(v)}) + \kappa_{rt}^{(v)}\kappa_{sw}^{(u)} + \kappa_{rt}^{(u)}\kappa_{sw}^{(v)}]. \quad (2.7)$$

Here $\sum'_{\mu,\nu}$ denotes the summations over all the combinations of the parameters μ and ν.

The inconvenience for the practical use of formulae (2.5) – (2.7) is that they require joint cumulants of log-likelihood derivatives with respect to the parameters μ and ν. We now give expressions for computing $E(\omega)$ that involves only simple differential operations. By inserting the cumulants κ's given before into equations (2.5) – (2.7), $c = E(\omega)$ can be decomposed, due to the orthogonality between μ and ν, into the sum of three components as

$$c = 1 + A_\mu + C_{1_{\mu,\nu}} - C_{2_{\mu,\nu}}, \quad (2.8)$$

where

$$A_\mu = \frac{1}{4n\left(\sum_{r=1}^2 \alpha_r^{(1,0)}\beta_r^{(1,0)}\right)^2}\sum_{r=1}^2(\alpha_r^{(3,0)}\beta_r^{(1,0)} + \alpha_r^{(2,0)}\beta_r^{(2,0)} - \alpha_r^{(1,0)}\beta_r^{(3,0)}) - \frac{1}{3n\left(\sum_{r=1}^2 \alpha_r^{(1,0)}\beta_r^{(1,0)}\right)^3}\sum_{r=1}^2\sum_{s=1}^2(a_r a_s + a_r b_s - \frac{5}{4}b_r b_s), \quad (2.9)$$

$$C_{1_{\mu,\nu}} = \frac{1}{2n\left(\sum_{r=1}^2 \alpha_r^{(1,0)}\beta_r^{(1,0)}\right)\left(\sum_{r=1}^2 \alpha_r^{(0,1)}\beta_r^{(0,1)}\right)}\sum_{r=1}^2(\alpha_r^{(1,2)}\beta_r^{(1,0)} + \alpha_r^{(2,1)}\beta_r^{(0,1)} + 3\alpha_r^{(1,1)}\beta_r^{(1,1)}) \quad (2.10)$$

and

$$C_{2_{\mu,\nu}} = \frac{1}{4n\left(\sum_{r=1}^2 \alpha_r^{(1,0)}\beta_r^{(1,0)}\right)\left(\sum_{r=1}^2 \alpha_r^{(0,1)}\beta_r^{(0,1)}\right)}\Big\{\frac{1}{\sum_{r=1}^2 \alpha_r^{(1,0)}\beta_r^{(1,0)}} \Big[\left(\sum_{r=1}^2 \alpha_r^{(1,1)}\beta_r^{(1,0)}\right)^2 + 4\left(\sum_{r=1}^2 \alpha_r^{(1,1)}\beta_r^{(1,0)}\right)\left(\sum_{r=1}^2 \alpha_r^{(1,0)}\beta_r^{(1,1)}\right)$$

$$+2\left(\sum_{r=1}^{2}\alpha_r^{(1,1)}\beta_r^{(0,1)}\right)\left(\sum_{r=1}^{2}(2\alpha_r^{(2,0)}\beta_r^{(1,0)}+\alpha_r^{(1,0)}\beta_r^{(2,0)}\right)$$
$$-4\left(\sum_{r=1}^{2}\alpha_r^{(1,1)}\beta_r^{(0,1)}\right)\left(\sum_{r=1}^{2}\alpha_r^{(2,0)}\beta_r^{(1,0)}\right)\Big] + \frac{1}{\sum_{r=1}^{2}\alpha_r^{(0,1)}\beta_r^{(0,1)}}\Big[\left(\sum_{r=1}^{2}\alpha_r^{(1,1)}\beta_r^{(0,1)}\right)^2$$
$$+4\left(\sum_{r=1}^{2}\alpha_r^{(1,1)}\beta_r^{(0,1)}\right)\left(\sum_{r=1}^{2}\alpha_r^{(0,1)}\beta_r^{(1,1)}\right)+2\left(\sum_{r=1}^{2}\alpha_r^{(1,1)}\beta_r^{(1,0)}\right)\left(\sum_{r=1}^{2}\alpha_r^{(0,2)}\beta_r^{(0,1)}+\alpha_r^{(0,1)}\beta_r^{(0,2)}\right)$$
$$-4\left(\sum_{r=1}^{2}\alpha_r^{(1,1)}\beta_r^{(1,0)}\right)\left(\sum_{r=1}^{2}\alpha_r^{(0,2)}\beta_r^{(0,1)}\right)\Big]\Big\}, \tag{2.11}$$

with $a_r = \alpha_r^{(2,0)}\beta_r^{(1,0)}$ and $b_r = \alpha_r^{(1,0)}\beta_r^{(2,0)}$ for $r = 1, 2$.

Expressions (2.8) – (2.11) are suitable for routine computation since they depend only on simple derivatives of the functions $\alpha_1, \alpha_2, \beta_1$ and β_2. The improved LR test of $H_0 : \mu = \mu^{(0)}$ can be computed from (2.4) and (2.8) – (2.11) using $\omega^* = \omega/c$ and comparing the transformed statistic ω^* with the χ_1^2 distribution. The distribution of ω is generally order n^{-1} away from χ_1^2, but the Bartlett correction (2.8) renders the n^{-1} term equal to zero and the distribution of ω^* becomes much closer to χ^2 than is the distribution of ω.

We now give Bartlett corrections for testing $H_0 : \mu = \mu^{(0)}$ under two special cases of (1.1). First, consider a two-parameter exponential family with $\alpha_1(\mu,\nu) = \alpha_1(\mu)$ and $\alpha_2(\mu,\nu) = \alpha_2(\nu)$. In this case, we find

$$\alpha_1^{(0,j)} = \partial^j\alpha_1(\mu,\nu)/\partial\nu^j = 0, \quad \alpha_2^{(i,0)} = \partial^i\alpha_2(\mu,\nu)/\partial\mu^i = 0,$$

and

$$\alpha_r^{(i,j)} = \partial^{i+j}\alpha_r(\mu,\nu)/\partial\mu^i\partial\nu^j = 0 \text{ for } r = 1,2 \text{ and } i,j = 1,2,\ldots$$

which implies

$$\beta_1 = \frac{\rho^{(1,0)}}{\alpha_1^{(1,0)}} \quad \text{and} \quad \beta_2 = \frac{\rho^{(0,1)}}{\alpha_2^{(0,1)}}.$$

Finally, the orthogonality between μ and ν gives $\beta_1^{(0,j)} = \beta_2^{(i,0)} = 0$ and $\beta_r^{(i,j)} = 0$ for $r = 1, 2$ and $i, j = 1, 2, \ldots$ and therefore $C_{1_{\mu,\nu}} = C_{2_{\mu,\nu}} = 0$ and the Bartlett correction in

(2.8) reduces to $c = 1 + A_\mu$, with

$$A_\mu = \frac{1}{4n(\alpha_1^{(1,0)}\beta_1^{(1,0)})^2}\left[\alpha_1^{(3,0)}\beta_1^{(1,0)} + \alpha_1^{(2,0)}\beta_1^{(2,0)} - \alpha_1^{(1,0)}\beta_1^{(3,0)}\right] - \frac{1}{3n(\alpha_1^{(1,0)}\beta_1^{(1,0)})^3}\left[\alpha_1^{(2,0)}\beta_1^{(1,0)} - \frac{5}{4}(\alpha_1^{(1,0)}\beta_1^{(2,0)})^2 + \alpha_1^{(2,0)}\beta_1^{(1,0)}\alpha_1^{(1,0)}\beta_1^{(2,0)}\right]. \quad (2.12)$$

We now assume that the function $\alpha_2(\mu,\nu)$ depends only on the nuisance parameter ν, namely $\alpha_2(\mu,\nu) = \alpha_2(\nu)$, implying

$$\beta_1 = \frac{\rho^{(1,0)}}{\alpha_1^{(1,0)}}, \quad \beta_2 = \frac{\rho^{(1,0)}\alpha_1^{(1,0)} - \rho^{(1,0)}\alpha_1^{(0,1)}}{\alpha_2^{(0,1)}\alpha_1^{(1,0)}}.$$

In view of the global orthogonality between μ and ν, we obtain $\kappa_{\mu\nu} = -n\sum_{r=1}^{2}\alpha_r^{(1,0)}\beta_r^{(0,1)} = -n\alpha_1^{(1,0)}\beta_1^{(0,1)} = 0$ and $\kappa_{\nu\mu} = -n(\alpha_1^{(0,1)}\beta_1^{(1,0)} + \alpha_2^{(0,1)}\beta_2^{(1,0)}) = 0$ and equations (2.9) - (2.11) become

$$A_\mu = \frac{1}{4n(\alpha_1^{(1,0)}\beta_1^{(1,0)})^2}\left[\alpha_1^{(3,0)}\beta_1^{(1,0)} + \alpha_1^{(2,0)}\beta_1^{(2,0)} - \alpha_1^{(1,0)}\beta_1^{(3,0)}\right] - \frac{1}{3n(\alpha_1^{(1,0)}\beta_1^{(1,0)})^3}\left[(\alpha_1^{(2,0)}\beta_1^{(1,0)})^2 - \frac{5}{4}(\alpha_1^{(1,0)}\beta_1^{(2,0)})^2 + \alpha_1^{(2,0)}\beta_1^{(1,0)}\alpha_1^{(1,0)}\beta_1^{(2,0)}\right], \quad (2.13)$$

$$C_{1_{\mu,\nu}} = \frac{1}{2n\alpha_1^{(1,0)}\beta_1^{(1,0)}\sum_{r=1}^{2}\alpha_r^{(0,1)}\beta_r^{(0,1)}}\left[\alpha_1^{(1,2)}\beta_1^{(1,0)} + \alpha_1^{(2,1)}\beta_1^{(0,1)} + 3\alpha_1^{(1,1)}\beta_1^{(1,1)}\right] \quad (2.14)$$

and

$$\begin{aligned}C_{2_{\mu,\nu}} = \frac{1}{4n\alpha_1^{(1,0)}\beta_1^{(1,0)}\sum_{r=1}^{2}\alpha_r^{(0,1)}\beta_r^{(0,1)}}\Bigg\{&\frac{1}{\alpha_1^{(1,0)}\beta_1^{(1,0)}}\Big[(\alpha_1^{(1,1)}\beta_1^{(1,0)})^2 + 4\alpha_1^{(1,1)}\beta_1^{(1,0)}\alpha_1^{(1,0)}\beta_1^{(1,1)} \\ &+2\alpha_1^{(1,1)}\beta_1^{(0,1)}\left(2\alpha_1^{(2,0)}\beta_1^{(1,0)} + \alpha_1^{(1,0)}\beta_1^{(2,0)}\right) - 4\alpha_1^{(1,1)}\beta_1^{(0,1)}\alpha_1^{(2,0)}\beta_1^{(1,0)}\Big] \\ &+\frac{1}{\sum_{r=1}^{2}\alpha_r^{(0,1)}\beta_r^{(0,1)}}\Big[(\alpha_1^{(1,1)}\beta_1^{(0,1)})^2 + 4\alpha_1^{(1,1)}\beta_1^{(0,1)}\sum_{r=1}^{2}\alpha_r^{(0,1)}\beta_r^{(1,1)} \\ &+2\alpha_1^{(1,1)}\beta_1^{(1,0)}\sum_{r=1}^{2}\left(2\alpha_r^{(0,2)}\beta_r^{(0,1)} + \alpha_r^{(0,1)}\beta_r^{(0,2)}\right) - 4\alpha_1^{(1,1)}\beta_1^{(1,0)}\sum_{r=1}^{2}\alpha_r^{(0,2)}\beta_r^{(0,1)}\Big]\Bigg\}. \quad (2.15)\end{aligned}$$

3 Special cases

This section develops Bartlett corrections to improve LR tests for a number of important distributions in the exponential model (1.1). We assume that these distributions are parameterized in such a way that the parameter of interest and the nuisance parameter are orthogonal. Section 4 is devoted to distributions for which orthogonality is not available. The distributions considered, namely normal, inverse Gaussian, gamma, log-gamma and inverse gamma distributions, are well known and have a wide range of practical applications in fields such as engineering, economics, biology and medicine, among others. For each distribution, we give closed-form expressions for Bartlett corrections to improve LR tests for both parameters. For the gamma distribution, the Bartlett corrections are quite complex and require the evaluation of polygamma functions and we give some simple approximations based on asymptotic expansions.

The distributions considered here are the following:

(i) Normal $N(\theta, \phi)$ distribution with mean θ and variance ϕ ($\phi > 0$, $-\infty < \theta < \infty$, $-\infty < y < \infty$): $\alpha_1(\theta,\phi) = \theta/\phi$, $\alpha_2(\theta,\phi) = -1/(2\phi)$, $\rho(\theta,\phi) = \theta^2/(2\phi) + \log\phi/2$, $d_1(y) = y$, $d_2(y) = y^2$, $v(y) = -\log(2\pi)/2$, $\beta_1 = \theta$, $\beta_2 = (\phi + \theta^2)$, $V_1 = \phi$, $V_2 = 2\phi^2 + 4\phi\theta^2$ and $V_{12} = 2\theta\phi$. We begin with the test of the mean $H_0 : \theta = \theta^{(0)}$ against $H : \theta \neq \theta^{(0)}$, where $\theta^{(0)}$ is a given number and the variance ϕ is the nuisance parameter. The LR statistic for testing $H_0 : \theta = \theta^{(0)}$ is given by

$$\omega = n \log \left\{ \frac{\sum_{i=1}^{n}(y_i - \theta^{(0)})^2}{\sum_{i=1}^{n}(y_i - \overline{y})^2} \right\}.$$

Thus, we apply equations (2.13) – (2.15), on moving from (μ,ν) to (θ,ϕ) and noting that α_2 depends only on the nuisance parameter ϕ. We have $A_\theta = 0$, $C_{1_{\theta,\phi}} = 2/n$ and $C_{2_{\theta,\phi}} = 1/2n$ and the Bartlett correction (2.8) reduces to $C = 1 + 3/2n$ which is a widely known result. This provides a partial check on these equations.

We now consider the test of the variance $H_0 : \phi = \phi^{(0)}$ against $H : \phi \neq \phi^{(0)}$, where $\phi^{(0)}$ is a specified value and the mean θ is now the nuisance parameter. The LR statistic is given by

$$\omega = n\left\{\log\left(\frac{\phi^{(0)}}{\hat{\phi}}\right) + \frac{\hat{\phi}-\phi^{(0)}}{\phi^{(0)}}\right\},$$

where $\hat{\phi} = \sum_{i=1}^{n}(y_i-\bar{y})^2/n$. Equations (2.8) – (2.11) yield the Bartlett correction $c = 1+\frac{11}{6n}$ which is also a known result.

(ii) Inverse Gaussian $N^{-}(\theta,\phi)$ distribution with mean $\theta > 0$ and precision parameter $\phi > 0$ ($y > 0$): $\alpha_1(\theta,\phi) = -\phi/(2\theta^2)$, $\alpha_2(\theta,\phi) = -\phi/2$, $\rho(\theta,\phi) = -(\phi/\theta + \log\phi/2)$, $d_1(y) = y$, $d_2(y) = y^{-1}$, $v(y) = -\log(2\pi y^3)/2$, $\beta_1 = \theta$, $\beta_2 = (\theta+\phi)/(\theta\phi)$, $V_1 = \theta^3/\phi$, $V_2 = 2/\phi^2 + 1/(\phi\theta)$ and $V_{12} = (\phi-\theta)/\phi$. The variance of y is $\mathrm{Var}(y) = \theta^3/\phi$. For testing the mean $H_0 : \theta = \theta^{(0)}$ against $H : \theta \neq \theta^{(0)}$, the LR statistic is $\omega = n\log(\hat{\phi}/\tilde{\phi})$, where $\hat{\phi} = (\tilde{y}^{-1} - \bar{y}^{-1})^{-1}$ and $\tilde{\phi} = n\theta^{(0)2}\left[\sum_{i=1}^{n}\frac{(y_i-\theta^{(0)})^2}{y_i}\right]^{-1}$ are the unrestricted and restricted MLEs of ϕ, respectively, and $\tilde{y} = \left[\frac{1}{n}\sum_{i=1}^{n}y_i^{-1}\right]^{-1}$ is the harmonic mean of the y's. From (2.13) – (2.15) we obtain $A_\theta = C_{1_{\theta,\phi}} = 0$ and $C_{2_{\theta,\phi}} = 3/2n$ which leads to $c = 1 + 3/2n$.

We now consider the LR statistic for testing the precision parameter $H_0 : \phi = \phi^{(0)}$ against $H : \phi \neq \phi^{(0)}$, where the mean θ is the nuisance parameter. It is given by

$$w = n\log\left(\frac{\hat{\phi}}{\phi^{(0)}}\right) - \frac{(\hat{\phi}-\phi^{(0)})}{\bar{y}^2}\sum_{i=1}^{n}\frac{(y_i-\bar{y})^2}{y_i}.$$

From (2.9) – (2.11) we obtain $A_\phi = 1/3n$, $C_{1_{\phi,\theta}} = 0$ and $C_{2_{\phi,\theta}} = -3/2n$ and then by (2.8) we obtain the Bartlett correction $c = 1 + 11/6n$.

(iii) Gamma $G(\theta,\phi)$ distribution with mean $\theta > 0$ and shape parameter $\phi > 0$ ($y > 0$): $\alpha_1(\theta,\phi) = -\phi/\theta$, $\alpha_2(\theta,\phi) = \phi$, $\rho(\theta,\phi) = \phi\log\theta - \phi\log\phi + \log\Gamma(\phi)$, $d_1(y) = y$, $d_2(y) = \log y$, $v(y) = -\log y$, $\beta_1 = \theta$, $\beta_2 = \log(\theta/\phi) + \psi(\phi)$, $V_1 = \theta^2/\phi$, $V_2 = \psi'(\phi)$ and $V_{12} = \theta/\phi$, where $\Gamma(\phi)$ is the gamma function and $\psi(\phi) = \mathrm{d}\log\Gamma(\phi)/\mathrm{d}\phi$ is the digamma function. The variance of y is $\mathrm{Var}(y) = \theta^2/\phi$. Here, we have used a parameterization such that the orthogonality between θ and ϕ is achieved. Evidently, the MLE of θ is $\hat{\theta} = \bar{y}$. The MLE of ϕ is obtained as a solution of the nonlinear equation

$$\log \hat{\phi} = \psi(\hat{\phi}) = \log\left(\frac{\overline{y}}{\overline{y}_g}\right), \tag{3.1}$$

where $\overline{y}_g = \left(\prod_{i=1}^{n} y_i\right)^{1/n}$ is the geometric mean of the y's.

If $\hat{\phi}$ is large enough, the approximantion $\psi(\phi) = \log(\phi - \frac{1}{2})$ may be used to yield $\hat{\phi} = \overline{y}/\{2(\overline{y} - \overline{y}_g)\}$. For a better approximation 1/12 should be subtracted from the right-hand side of this formula. Another approximation to solve (3.1) (Thom, 1968) is given by

$$\hat{\phi} = \frac{1}{4g}\left\{1 + (1 + \frac{4g}{3})^2\right\},$$

where $g = \log(\overline{y}/\overline{y}_g)$.

For testing the mean $H_0 : \theta = \theta^{(0)}$ against $H : \theta \neq \theta^{(0)}$, the LR statistic is given by

$$\begin{aligned} \omega &= 2n\Big\{(\hat{\phi} - \tilde{\phi}) \log \overline{y}_g - (\hat{\phi} - \frac{\tilde{\phi}\overline{y}}{\theta^{(0)}}) \\ &\quad + \log\left[\left(\frac{\hat{\phi}}{\overline{y}}\right)^{\hat{\phi}} \frac{1}{\Gamma(\hat{\phi})}\right] - \log\left[\left(\frac{\tilde{\phi}}{\theta^{(0)}}\right)^{\tilde{\phi}} \frac{1}{\Gamma(\tilde{\phi})}\right]\Big\}, \end{aligned} \tag{3.2}$$

where $\tilde{\phi}$ is the MLE of ϕ under H_0 which comes from

$$\log \tilde{\phi} - \psi(\tilde{\phi}) = \log\left(\frac{\theta^{(0)}}{\overline{y}_g}\right) + \frac{\overline{y}}{\theta^{(0)}} - 1. \tag{3.3}$$

Equation (3.3) can be solved for $\tilde{\phi}$ analogously to $\hat{\phi}$ in (3.1).

Since α_2 depends only on the nuisance parameter ϕ, equations (2.13) – (2.15) yield the values of A_θ, $C_{1_{\theta,\phi}}$ and $C_{2_{\theta,\phi}}$. Then, we obtain

$$c = 1 + \frac{1}{12n\phi}\left[\frac{-3(2\phi^2\psi'' + \phi\psi' + 1)}{(\phi\psi' - 1)^2} + 2\right], \tag{3.4}$$

where ψ' and ψ'' are the first two derivatives of the $\psi(\phi)$ function. The improved LR statistic can easily be obtained as $\omega^* = \tilde{c}^{-1}\omega$ from (3.2) and (3.4), where $\tilde{c}$ is the value of (3.4) at $\tilde{\phi}$, and has under H_0 a χ_1^2 distribution to order n^{-1}.

We now consider the test of the shape parameter $H_0 : \phi = \phi^{(0)}$ against $H : \phi \neq \phi^{(0)}$, where $\phi^{(0)}$ is a specified value and the mean θ is now the nuisance parameter. The LR statistic for testing $H_0 : \phi = \phi^{(0)}$ reduces to

$$\omega = 2n\left\{(\hat{\phi} - \phi^{(0)})[\log(\bar{y}_g/\bar{y}) - 1] + \log\left(\frac{\hat{\phi}^{\hat{\phi}}}{\Gamma(\hat{\phi})}\right) - \log\left(\frac{\phi^{(0)\phi^{(0)}}}{\Gamma(\phi^{(0)})}\right)\right\}, \qquad (3.5)$$

where $\hat{\phi}$ is obtained from (3.1). Using the results (2.9) – (2.11) we can obtain the Bartlett correction to improve the test of $H_0 : \phi = \phi^{(0)}$ as

$$c = 1 + \frac{1}{12n\phi^{(0)}}\left[\frac{5(\phi^{(0)^2}\psi_0'' + 1)^2}{(\phi^{(0)}\psi_0' - 1)^3} - \frac{3\phi^{(0)^2}(\phi^{(0)}\psi_0''' + 2\psi_0'')}{(\phi^{(0)}\psi_0' - 1)^2} - \frac{3}{\phi^{(0)}\psi_0' - 1}\right], \qquad (3.6)$$

where ψ_0', ψ_0'' and ψ_0''' are the first three derivatives of $\psi(\phi)$ evaluated at $\phi^{(0)}$.

For both tests in the gamma distribution, the evaluation of the corrected statistics requires the computation of polygamma functions. We now obtain approximations based on asymptotic expansions which do not involve such functions. We use the expansion $\psi'(\phi) = 1/\phi + 1/(2\phi^2) + 1/(6\phi^3) - 1/(30\phi^5) + O(\phi^{-7})$ to obtain for the test of $H_0 : \theta = \theta^{(0)}$ when ϕ is large

$$c = 1 + \frac{1}{6n}\left(9 - \frac{1}{3\phi^2} - \frac{4}{15\phi^3}\right) + O(\phi^{-4}).$$

For the test of $H_0 : \phi = \phi^{(0)}$ we also obtain from (3.6) for large $\phi^{(0)}$

$$c = 1 + \frac{1}{6n}\left(11 - \frac{1}{\phi^{(0)}} + \frac{7}{27\phi^{(0)}}\right) + O(\phi^{(0)^{-4}}).$$

(iv) Log-gamma $LG(\theta, \phi)$ distribution ($-\infty < \theta < \infty$, $-\infty < y < \infty$) defined as the distribution of $y = \log x$ where x has a gamma $G(\theta, \phi)$ distribution with mean θ and shape parameter ϕ (see (iii) above). The log-gamma distribution belongs to the exponential family model (1.1) with $\alpha_1(\theta, \phi) = -\phi e^{-\theta}$, $\alpha_2(\theta, \phi) = \phi$, $\rho(\theta, \phi) = \phi\theta - \phi \log \phi + \log\{\Gamma(\phi)\}$, $d_1(y) = \exp(y)$, $d_2(y) = y$, $v(y) = 0$, $\beta_1 = \exp(\theta)$, $\beta_2 = \theta + \psi(\phi) - \log \phi$, $V_1 = \exp(2\theta)/\phi$, $V_2 = \psi'(\phi)$ and $V_{12} = \phi^{-1}\exp(\theta)$.

For testing $H_0 : \theta = \theta^{(0)}$ against $H : \theta \neq \theta^{(0)}$, the LR statistic and its modified version have the same expressions of the corresponding statistics for the test of the mean of the gamma distribution described in (iii). The derivation of this result relies on the fact that the LR statistic and its Bartlett correction are invariant under a transformation of the data in the situation where the null hypothesis remains unaltered. Analogously, for the test of $H_0 : \phi = \phi^{(0)}$ against $H : \phi \neq \phi^{(0)}$, the LR statistic and the Bartlett correction coincide with the corresponding expressions for the test of the gamma distribution discussed in case (iii).

(v) Inverse gamma $G^{-}(\theta, \phi)$ distribution ($\theta > 0$, $\phi > 0$, $y > 0$) defined as the distribution of $y = x^{-1}$ where x has a gamma $G(\theta, \phi)$ distribution. The distribution belongs to the exponential family model (1.1) where $\alpha_1(\theta, \phi) = -\theta\phi$, $\alpha_2(\theta, \phi) = -\phi$, $\rho(\theta, \phi) = -\phi \log(\theta\phi) + \log\Gamma(\phi)$, $d_1(y) = y^{-1}$, $d_2(y) = \log y$, $v(y) = -\log y$, $\beta_1 = \theta^{-1}$, $\beta_2 = \log(\theta\phi) - \psi(\phi)$, $V_1 = 1/(\phi\theta^2)$, $V_2 = \psi'(\phi)$ and $V_{12} = -1/(\theta\phi)$. The LR statistics and their Bartlett corrections for both tests $H_0 : \theta = \theta^{(0)}$ against $H : \theta \neq \theta^{(0)}$ and $H_0 : \phi = \phi^{(0)}$ against $H : \phi \neq \phi^{(0)}$ are identical to the corresponding ones for the tests of the gamma $G(\theta, \phi)$ distribution described in (iii).

4 Orthogonalized parameters

Consider the exponential family model (1.1) and assume that μ is the parameter of interest and ν is the nuisance parameter. If the parameters μ and ν are not orthogonal in the expected information sense, i.e., the conditions (2.3) are not satisfied, it is of relevance to know whether there exists a complementary parameter δ such that (μ, δ) parameterizes the model and μ and δ are orthogonal. When the dimension of the parameter of interest is one, it is in general possible to find such a orthogonal parameterization which comes as a solution of the differential equation

$$\kappa_{\nu\nu}\frac{\partial\nu}{\partial\mu} = -\kappa_{\nu\mu}; \tag{4.1}$$

see Cox and Reid (1987). From $\kappa_{\nu\nu} = -ns_{01,01}$ and $\kappa_{\nu\mu} = -ns_{10,01}$, equation (4.1) becomes

$$\frac{\partial\nu}{\partial\mu} = -\frac{s_{10,01}}{s_{01,01}}.$$

It is possible to solve this equation in generality because the solution depends on the forms of the quantities $s_{01,01}$ and $s_{10,01}$. We now apply (4.1) to some exponential models whose parameters are not orthogonal.

(i) Normal $N(\theta, \theta^2\phi^2)$ distribution with mean θ and coefficient of variation ϕ ($\theta > 0$, $\phi > 0$, $-\infty < y < \infty$): $\alpha_1(\theta,\phi) = -1/(2\theta^2\phi^2)$, $\alpha_2(\theta,\phi) = 1/(\theta\phi^2)$, $\rho(\theta,\phi) = 1/(2\phi^2) + \log(\theta\phi)$, $d_1(y) = y^2$, $d_2(y) = y$, $v(y) = 0$, $\beta_1 = \theta^2(\phi^2+1)$, $\beta_2 = \theta$, $V_1 = 2\theta^4\phi^2(\phi^2+2)$, $V_2 = \theta^2\phi^2$ and $V_{12} = 2\phi^2\theta^3$. We obtain

$$\kappa_{\theta\theta} = -\frac{n(2\phi^2+1)}{\theta^2\phi^2}, \quad \kappa_{\phi\phi} = -\frac{2n}{\phi^2} \quad \text{and} \quad \kappa_{\theta\phi} = -\frac{2n}{\theta\phi} \neq 0.$$

First, we consider the test of the coefficient of variation $H_0 : \phi = \phi^{(0)}$ against $H : \phi \neq \phi^{(0)}$, where the mean θ is the nuisance parameter. The parameters ϕ and θ are not orthogonal and equation (4.1) takes the form

$$\frac{\partial\theta}{\partial\phi} = -\frac{2\theta\phi}{2\phi^2+1},$$

with a possible solution $\delta = \theta(2\phi^2+1)^{1/2}$.

The density function in the new parameterization (ϕ, δ) is

$$\pi^*(y;\delta,\phi) = \frac{(2\phi^2+1)^{1/2}}{\sqrt{2\pi}\phi\delta}\exp\left\{-\frac{(2\phi^2+1)}{\delta^2\phi^2}\left(y - \frac{\delta}{(2\phi^2+1)^{1/2}}\right)^2\right\}, \tag{4.2}$$

where $\alpha_1(\phi,\delta) = -(2\phi^2+1)/(2\delta^2\phi^2)$, $\alpha_2(\phi,\delta) = (2\phi^2+1)^{1/2}/(\delta\phi^2)$, $\rho(\phi,\delta) = 1/(2\phi^2) + \log(\delta\phi/(2\phi^2+1)^{1/2})$, $d_1(y) = y^2$, $d_2(y) = y$, $\beta_1 = \delta^2(\phi^2+1)/(2\phi^2+1)$, $\beta_2 = \delta/((2\phi^2+1)^{1/2})$,

$V_1 = 2\phi^2(\phi^2+2)\delta^4/(2\phi^2+1)^2$, $V_2 = \delta^2\phi^2/(2\phi^2+1)$ and $V_{12} = 2\phi^2\delta^3/(2\phi^2+1)^{3/2}$. It is now easy to check that $\kappa_{\delta,\phi} = \kappa_{\phi,\delta} = 0$, i.e., δ is orthogonal to ϕ. Thus, equations (2.8) – (2.11) apply with μ and ν replaced by ϕ and δ, respectively.

Let $l(\phi,\delta) = \sum_{i=1}^{n} \log \pi^*(y_i;\delta,\phi)$ be the log-likelihood for the parameterized model which comes from (4.2). The unrestricted estimates of ϕ and δ are $\hat{\phi} = s/\overline{y}$ and $\hat{\delta} = (2s^2+\overline{y}^2)^{1/2}$, where $s^2 = \sum_{i=1}^{n}(y_i-\overline{y})^2/n$. The restricted estimate of δ is $\tilde{\delta} = \tilde{\theta}(2\phi^{(0)^2}+1)^{1/2}$, where

$$\tilde{\theta} = \frac{(\overline{y}^2 + 4\phi^{(0)2}m_2)^{1/2} - \overline{y}}{2\phi^{(0)2}}, \quad \text{if } \overline{y} > 0$$

and

$$\tilde{\theta} = \frac{-(\overline{y}^2 + 4\phi^{(0)2}m_2)^{1/2} - \overline{y}}{2\phi^{(0)2}}, \quad \text{if } \overline{y} < 0,$$

where $m_2 = \sum_{l=1}^{m} y_l^2/n$.

The LR statistic is simply given by $\omega = 2\{l(\hat{\delta},\hat{\phi}) - l(\tilde{\delta},\phi^{(0)})\}$. We obtain $\zeta = 4(\phi^2+1)/\{\delta^2\phi^4(2\phi^2+1)^{1/2}\}$ and $\gamma = 4(3+9\phi^2+4\phi^4)^2/\{\phi^6(2\phi^2+1)^4\}$. The Bartlett correction can be calculated from (2.8) – (2.11). Using MATHEMATICA (Wolfram, 1996) we find

$$c = 1 + \frac{1}{6n}\left[\frac{3(3+8\phi^2+7\phi^4)}{(1+2\phi^2)^2} + \frac{2(1+6\phi^2+18\phi^4+15\phi^6)}{(1+2\phi^2)^3}\right], \tag{4.3}$$

which does not involve the nuisance parameter δ.

Now consider the test of $H_0 : \theta = \theta^{(0)}$ against $H : \theta \neq \theta^{(0)}$, where the coefficient of variation ϕ is the nuisance parameter. In this case, the new parameter δ orthogonal to θ comes from the differential equation

$$\frac{\partial\phi}{\partial\theta} = \frac{\phi}{\theta},$$

which yields $\delta = \theta/\phi$. For testing the mean of the normal $N(\theta,\theta^2\phi^2)$ distribution, the Bartlett corrected statistic has the same form of that one for testing the mean in the normal $N(\theta, \phi)$ distribution with variance ϕ (Section 3, case (i)), since it is invariant under transformation of the nuisance parameter.

(ii) Inverse Gaussian $N^{-}(\theta, \theta\phi^2)$ distribution ($\theta > 0$, $\phi > 0$, $y > 0$) with mean θ and coefficient of variation ϕ^{-1}. The density function in this parameterization becomes

$$\pi(y; \theta, \phi) = \exp\left\{-\frac{(y-\theta)^2}{2y\theta\phi^2} + \frac{1}{2}\log\left(\frac{\theta}{\phi^2}\right) - \frac{1}{2}\log(2\pi y^3)\right\},$$

where $\alpha_1(\theta, \phi) = -1/(2\theta\phi^2)$, $\alpha_2(\theta, \phi) = -\theta/(2\phi^2)$, $\rho(\theta, \phi) = \log(\phi^2)/2 - \log\theta/2 - 1/\phi^2$, $d_1(y) = y$, $d_2(y) = y^{-1}$, $v(y) = -\log(2\pi y^3)/2$, $\beta_1 = \theta$, $\beta_2 = (\phi^2+1)/\theta$, $V_1 = \theta^2\phi^2$, $V_2 = \phi^2(2\phi^2+1)/\theta^2$ and $V_{12} = -\phi$. We obtain

$$\kappa_{\theta\theta} = -\frac{n(\phi^2+2)}{2\theta^2\phi^2}, \quad \kappa_{\phi\phi} = -\frac{2n}{\phi^2} \text{ and } \kappa_{\theta\phi} = \frac{n}{\theta\phi} \neq 0.$$

The unrestricted MLEs of θ and ϕ are $\hat{\theta} = \bar{y}$ and $\hat{\phi} = \frac{\bar{y}}{\tilde{y}}$, where $\tilde{y}$ is the harmonic mean of the y's.

We start with the null hypothesis $H_0 : \phi = \phi^{(0)}$ to be tested against the alternative hypothesis $H : \phi \neq \phi^{(0)}$, where now the mean θ is the nuisance parameter. The parameters θ and ϕ are not orthogonal. By comparison with (4.1) we obtain the differential equation

$$\frac{\partial\theta}{\partial\phi} = \frac{2\theta\phi}{\phi^2+2}.$$

Consequently, choosing $\delta = \theta/(\phi^2+2)$, we have that δ is orthogonal to ϕ. Then, the associated density in the new parameterization takes the form

$$\pi^*(y; \delta, \phi) = \exp\left\{-\frac{(y-\delta(\phi^2+2))^2}{2y\phi\delta(\phi^2+2)} + \frac{1}{2}\log\left(\frac{\delta(\phi^2+2)}{\phi^2}\right) - \frac{1}{2}\log(2\pi y^3)\right\},$$

where $\alpha_1(\delta, \phi) = -1/\{2\delta\phi^2(\phi^2+2)\}$, $\alpha_2(\delta, \phi) = -\delta(\phi^2+2)/(2\phi^2)$, $\rho(\delta, \phi) = \log(\phi^2)/2 - \log\{\delta(\phi^2+2)\}/2 - 1/\phi^2$, $\beta_1 = \delta(\phi^2+2)$, $\beta_2 = (\phi^2+1)/\{\delta(\phi^2+2)\}$, $V_1 = \delta^2\phi^2(\phi^2+2)^2$,

$V_2 = \phi^2(2\phi^2+1)/\{\delta^2(\phi^2+2)^2\}$ and $V_{12} = -\phi^2$. The restricted MLE of δ is easily shown to be

$$\tilde{\delta} = \frac{\tilde{y}}{2(2+\phi^{(0)^2})}\{\phi^{(0)^2} + \sqrt{\phi^{(0)^4} + 4\overline{y}/\tilde{y}}\},$$

and the LR statistic for testing $H_0 : \phi = \phi^{(0)}$ reduces to

$$\begin{aligned} \omega &= \frac{1}{(\phi^{(0)^2}+2)\tilde{\delta}\phi^{(0)^2}} \sum_{i=1}^{n} y_i^{-1} y_i - (\phi^{(0)^2}+2)\tilde{\delta}\}^2 \\ &\quad + n\left\{\log\left[\frac{\overline{y}\tilde{y}\phi^{(0)^2}}{(\phi^{(0)^2}+2)\tilde{\delta}(\overline{y}-\tilde{y})}\right] - 1\right\}. \end{aligned} \tag{4.4}$$

We can show that $\zeta = 4(\phi^4 + 6\phi^2 + 4)/\{\delta^2\phi^4 + (\phi^2+2)^2\}$ and $\gamma = 4(25\phi^6 + 78\phi^4 + 108\phi^2 + 72)/\{\phi^4(\phi^2+2)^4\}$. The Bartlett correction to improve (4.4) can be calculated directly from (2.8) – (2.11) using MATHEMATICA, for example. We find

$$c = 1 + \frac{1}{6n}\left[\frac{2(8+12\phi^2-3\phi^4-6\phi^6)}{(2+\phi^2)^3} + \frac{3(12+14\phi^2+7\phi^4)}{(2+\phi^2)^2}\right]. \tag{4.5}$$

We note that (4.5) does not depend on the nuisance parameter δ.

We now move to the null hypothesis which specifies the value of the mean θ, namely $H_0 : \theta = \theta^{(0)}$ with alternative hypothesis $H : \theta \neq \theta^{(0)}$, where now the coefficient of variation ϕ^{-1} is the nuisance parameter. The parameters ϕ and θ are not orthogonal. To find a transformation to orthogonalized parameters (θ, δ), the differential equation

$$\frac{\partial\phi}{\partial\theta} = \frac{\phi}{2\theta},$$

should be solved. A possible solution is $\delta = \phi\theta^{-1/2}$ which yields the density function in the orthogonalized parameterization

$$\pi^*(y;\theta,\delta) = \exp\left\{-\frac{y}{2\theta^2\delta^2} - \frac{1}{2y\delta^2} + \frac{1}{\theta\delta^2} - \frac{1}{2}\log(\delta^2) - \frac{1}{2}\log(2\pi y^3)\right\},$$

where $\alpha_1(\theta,\delta) = -1/(2\theta^2\delta^2)$, $\alpha_2(\theta,\delta) = -1/(2\delta^2)$, $\rho(\theta,\delta) = \log(\delta^2)/2 - 1/(\theta\delta^2)$, $d_1(y) = y$, $d_2(y) = y^{-1}$, $v(y) = -\frac{1}{2}\log(2\pi y^3)$, $\beta_1 = \theta$, $\beta_2 = (\delta^2\theta+1)/\theta$, $V_1 = \delta^2\theta^3$, $V_2 = \delta^2(2\delta^2\theta+1)/\theta$

and $V_{12} = -\delta^2\theta$. The LR statistics and their Bartlett corrected versions for testing the mean of the inverse Gaussian distribution, parameterized as $N^-(\theta, \phi)$ or $N(\theta, \theta^2\delta^2)$, are identical because they are invariant under transformation of nuisance parameter. Thus, the Bartlett correction for testing $H_0 : \theta = \theta^{(0)}$ in the inverse Gaussian $N^-(\theta, \theta\phi^2)$ distribution is $c = 1 + 3/2n$ as obtained in Section 4 (case (ii)).

(iii) Gamma $G(\theta\phi, \phi)$ distribution ($\theta > 0$, $\phi > 0$, $y > 0$). In this parameterization, $\theta\phi$ is the mean and ϕ is the inverse of the square of the coefficient of variation. The corresponding density function is

$$\pi(y; \theta, \phi) = \frac{1}{\Gamma(\phi)} \theta^{-\phi} y^{\phi-1} \exp(-y/\theta).$$

Here, $\alpha_1(\theta, \phi) = -\theta^{-1}$, $\alpha_2(\theta, \phi) = \phi$, $\rho(\theta, \phi) = \phi \log \theta + \Gamma(\phi)$, $d_1(y) = y$, $d_2(y) = \log y$, $v(y) = -\log y$, $\beta_1 = \theta\phi$, $\beta_2 = \log \theta + \psi(\phi)$, $V_1 = \phi\theta^2$, $V_2 = \psi'(\phi)$ and $V_{12} = \theta$. We find

$$\kappa_{\theta\theta} = -\frac{n\phi}{\theta^2}, \quad \kappa_{\phi\phi} = -n\psi'(\phi) \quad \text{and} \quad \kappa_{\theta\phi} = -\frac{n}{\theta} \neq 0.$$

First, consider testing the null hypothesis $H_0 : \phi = \phi^{(0)}$ with two-sided alternative hypotesis $H : \phi \neq \phi^{(0)}$. A new parameterization (ϕ, δ) such that ϕ and δ are orthogonal parameters follows from (4.1) as

$$\left(-\frac{n\phi}{\theta^2}\right) \frac{\partial \theta}{\partial \phi} = \frac{n}{\theta},$$

which yields $\delta = \theta\phi$. In the new parameterization the density function takes the form

$$\pi^*(y; \delta, \phi) = \exp\left\{-\frac{y\phi}{\delta} + \phi \log y - \log y + \phi \log\left(\frac{\phi}{\delta}\right) - \log \Gamma(\phi)\right\},$$

where $\alpha_1(\delta, \phi) = -\phi/\delta$, $\alpha_2(\delta, \phi) = \phi$, $\rho(\delta, \phi) = -\phi \log(\phi/\delta) + \log \Gamma(\phi)$, $v(y) = -\log y$, $d_1(y) = y$, $d_2(y) = \log y$, $\beta_1 = \delta$, $\beta_2 = \log(\delta/\phi) + \psi(\phi)$, $V_1 = \delta^2/\phi$, $V_2 = \psi'(\phi)$ and $V_{12} = \delta/\phi$. Clearly, we obtained the same parameterization for the gamma $G(\delta, \phi)$ distribution adopted in Section 3 (case (iii)). The hypothesis testing is invariant under transformation of the nuisance parameter. Hence, the Bartlett correction given by (3.6) holds here.

Suppose now that we wish to test the null hypothesis $H_0 : \theta = \theta^{(0)}$ with two sided alternative $H : \theta \neq \theta^{(0)}$, where ϕ is the nuisance parameter. Equation (4.1) yields

$$\psi'(\phi)\frac{\partial\phi}{\partial\theta} = -\frac{1}{\theta},$$

with solution $\delta = \psi(\phi)+\log\theta$. Thus, δ is orthogonal to θ. We have $\psi'(\phi) = \int_0^\infty \frac{te^{-\phi t}}{1-e^{-t}}dt > 0$ for $\phi > 0$ (Abramowitz and Stegun, 1979, p. 260). Since $\psi(\phi)$ is strictly increasing as ϕ increases, it has a strictly increasing inverse function, say, $\psi^{-1}(\cdot)$. Let $r(u) = \psi^{-1}(u)$, with $u = u(\theta,\delta) = \delta - \log\theta$. Under (δ,θ) parameterization, we obtain

$$\pi^*(y;\theta,\delta) = \exp\left\{-\frac{y}{\theta} + r(u)\log y - r(u)\log\theta - \log\Gamma(r(u)) - \log y\right\}, \qquad (4.6)$$

with $\alpha_1(\theta,\delta) = -\frac{1}{\theta}$, $\alpha_2(\theta,\delta) = r(u)$, $\rho(\theta,\delta) = r(u)\log\theta + \log\Gamma(r(u))$, $d_1(y) = y$, $d_2(y) = \log y$, $v(y) = -\log y$, $\beta_1 = \theta r(u)$, $\beta_2 = \delta$, $V_1 = \theta^2 r(u)$, $V_2 = (r'(u))^{-1}$ and $V_{12} = \theta$, where $r'(u) = \frac{dr(u)}{du} = \frac{1}{\psi'(r(u))}$. We can easily obtain $\alpha_1^{(1,0)} = \frac{1}{\theta^2}$, $\alpha_2^{(1,0)} = -\frac{r'(u)}{\theta}$, $\beta_1^{(0,1)} = \theta r'(u)$, $\beta_2^{(0,1)} = 1$, etc. We also have

$$\zeta = r''(u)^2/\{\theta^2 r'(u)\}$$

and

$$\gamma = \frac{\{(4r(u)r'(u) - 3r'(u)^2 - 2r'(u)r''(u) + r''(u)^2)\}}{\theta^4 r'(u)}.$$

The unrestricted MLE of δ is $\hat{\delta} = \log\overline{y}_g$ and the unrestricted MLE of θ is obtained as a solution of the nonlinear equation

$$\psi^{-1}\left(\log\left(\frac{\overline{y}_g}{\hat{\theta}}\right)\right) = \frac{\overline{y}}{\hat{\theta}},$$

where $\overline{y}_g$ is the geometric mean of the y's. The restricted MLE of δ is $\tilde{\delta} = \log\overline{y}_g$. Then, the LR statistic for testing $H_0 : \theta = \theta^{(0)}$ is given by

$$w = 2n\left[\overline{y}\left(\frac{1}{\theta^{(0)}} - \frac{1}{\hat{\theta}}\right) + \psi^{-1}\left(\log\left(\frac{\overline{y}_g}{\hat{\theta}}\right)\right)\log\left(\frac{\overline{y}_g}{\hat{\theta}}\right)\right.$$

$$-\psi^{-1}\left(\log\left(\frac{\overline{y}_g}{\theta^{(0)}}\right)\right)\log\left(\frac{\overline{y}_g}{\theta^{(0)}}\right)-\log\frac{\Gamma\left\{\psi^{-1}\left(\log\left(\frac{\overline{y}_g}{\hat{\theta}}\right)\right)\right\}}{\Gamma\left\{\psi^{-1}\left(\log\left(\frac{\overline{y}_g}{\theta^{(0)}}\right)\right)\right\}}\Bigg].$$

Using (2.12) due to the orthogonality between θ and δ, we obtain the Bartlett correction to improve the test of $H_0 : \theta = \theta^{(0)}$ as

$$\begin{aligned} c &= 1+\frac{1}{12n(r-r')^3}\{2(9r+4\theta^3)r+(28r'-41r-8\theta^3)r' \\ &\quad +(2r-12r'+5r'')r''+3(r-r')r'''\}, \end{aligned}$$

where r', r'' and r''' are the derivatives of $r(u)$ evaluated at $\theta = \theta^{(0)}$, $\delta = \tilde{\delta}$.

5 Simulation results

We now perform two Monte Carlo simulations studies to check the adequacy of the asymptotic chi-squared distribution as an approximation to the finite-sample null distribution of the LR statistic ω and its corrected version ω^*. We consider a gamma $G(\theta, \phi)$ distribution with mean $\theta > 0$ and precision parameter $\phi > 0$ as presented in case (iii) of Section 3. In the first study our interest is to test $H_0 : \theta = \theta^{(0)}$ against $H : \theta \neq \theta^{(0)}$, assuming that the precision parameter ϕ is unknown. The LR statistic ω and its Bartlett correction c follow from (3.2) and (3.4), respectively. The unrestricted $\tilde{\phi}$ and restricted $\hat{\phi}$ estimates of ϕ are given by (3.1) and (3.3), respectively. Without loss of generality, the experiment was performed by setting $\theta^{(0)} = 4$ with the values of the nuisance parameter ϕ taken as 0.5, 1, 2 and 5. The sample sizes considered were $n = 5, 10, 15$ and 20. We carried out size simulations based on 10,000 replications. In each simulation the uncorrected statistic ω and the corrected one $\omega^* = \tilde{c}^{-1}\omega$ were computed, where $\tilde{c}$ is the value of (3.4) estimated at $\tilde{\phi}$. Rejection rates under the null hypothesis, i.e., the percentage of times that both statistics ω and ω^* exceed the appropriate upper points of the χ^2_1 distribution, are given in Table 1 for $\alpha = 5\%$ and in Table 2 for $\alpha = 10\%$. The program for the computation of the rejection rates was written using NAG FORTRAN subroutines.

The figures in Tables 1 and 2 show the tendency of the unmodified LR statistic ω to reject the null hypothesis $H_0 : \theta = 4$ more often than is expected for the selected nominal levels. Moreover, the empirical sizes of the adjusted statistic ω^* are closer to the nominal levels than the empirical sizes of the unmodified statistic ω. Thus, the Bartlett correction is very effective in bringing the rejection rates of the modified statistic toward to the nominal levels. Clearly, the asymptotic χ_1^2 approximation for both statistics ω and ω^* works better for large values of n as expected. For fixed n, however, the rejection rates of both statistics are not very sensitive to the value of the nuisance parameter, at least for the values of ϕ considered here.

Table 1 Simulated rejection rates of the Bartlett corrected (ω^*) and LR (ω) statistics for the hypothesis $H : \theta = 4$ (5% nominal level)

ϕ	0.5		1.0		2.0		5.0	
n	ω	ω^*	ω	ω^*	ω	ω^*	ω	ω^*
5	8.5	7.1	8.4	7.2	8.6	7.1	8.5	6.9
10	6.6	5.7	6.5	5.7	6.6	5.7	6.5	5.4
15	6.0	5.5	6.0	5.4	6.2	5.4	6.2	5.3
20	5.7	5.1	5.6	5.1	5.8	5.2	5.7	5.1

Table 2 Simulated rejection rates of the Bartlett corrected (ω^*) and LR (ω) statistics for the hypothesis $H : \theta = 4$ (10% nominal level)

ϕ	0.5		1.0		2.0		5.0	
n	ω	ω^*	ω	ω^*	ω	ω^*	ω	ω^*
5	14.4	12.1	14.1	12.2	14.4	12.0	14.3	11.8
10	12.3	11.3	12.1	11.3	12.4	11.0	12.2	10.7
15	11.6	11.0	11.4	10.8	11.9	10.7	11.6	10.5
20	11.1	10.7	10.8	10.4	10.8	10.4	10.7	10.2

We move to the test of the shape parameter, namely $H_0 : \phi = \phi^{(0)}$ versus $H : \phi \neq \phi^{(0)}$. The LR statistic and its Bartlett correction are given by equations (3.5) and (3.6), respectively. All simulations are based on 10,000 replications and are performed for $\phi^{(0)} = 0.5, 1, 2$ and 5 and for the following sample sizes $n = 5, 10, 15$ and 20, assuming that $\theta = 5$ in all cases. The rejection rates of ω and ω^* are reported for the nominal sizes of 5% and 10% in Tables 3 and 4, respectively.

Table 3 Simulated rejection rates of the Bartlett corrected (ω^*) and LR (ω) statistics for the hypothesis $H : \phi = \phi^{(0)}$ (5% nominal level)

$\phi^{(0)}$	0.5		1.0		2.0		5.0	
n	ω	ω^*	ω	ω^*	ω	ω^*	ω	ω^*
5	8.1	7.3	9.0	7.6	9.3	7.7	9.4	7.8
10	6.9	6.0	7.0	6.0	7.1	6.2	7.4	6.5
15	6.3	5.7	6.4	5.9	6.5	5.9	6.8	6.4
20	5.9	5.3	5.9	5.3	6.1	5.6	6.2	5.8

Table 4 Simulated rejection rates of the Bartlett corrected (ω^*) and LR (ω) statistics for the hypothesis $H : \phi = \phi^{(0)}$ (10% nominal level)

$\phi^{(0)}$	0.5		1.0		2.0		5.0	
n	ω	ω^*	ω	ω^*	ω	ω^*	ω	ω^*
5	14.9	12.4	15.4	12.8	15.9	12.9	15.8	12.8
10	12.7	11.4	12.9	11.7	12.9	11.8	13.0	11.6
15	11.7	10.6	11.9	10.7	11.9	10.5	12.0	10.5
20	10.8	10.2	10.9	10.4	11.2	10.4	11.2	10.2

From these tables it is clear the unmodified LR test is oversized and that the Bartlett corrected statistic ω^* outperforms the original statistic ω in all cases, thus delivering estimated sizes closer to their nominal levels. Finally, the size distortion of the LR statistic and the Bartlett corrected statistic decreases when n increases.

In view of the discussion above tests based on ω^* can be viewed as meaningful improvements of tests based on ω since the first-order asymptotics usually employed with LR statistics can deliver inaccurate inferences with samples of small to moderate sizes. Bartlett corrections based on second-order asymptotic theory can then be used to make inference with more reliable finite-sample behaviour. In that sense, if μ is the parameter of interest the set $\{\mu; |\omega^*| \leq x\}$, where x is the appropriate upper point of a χ_1^2 distribution, is an improved confidence interval for μ relative to the usual confidence interval $\{\mu; |\omega| \leq x\}$.

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