

RT-MAT 2001-05

The equation $\frac{\partial u}{\partial t} + H\left(t, x_1, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right) = 0$

and the method of characteristics in the
framework of generalized functions

Roseli Fernandez

Março 2001

Esta é uma publicação preliminar (“preprint”).

The equation $\frac{\partial u}{\partial t} + H(t, x_1, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}) = 0$ and the method of characteristics in the framework of generalized functions¹

Roseli Fernandez

Abstract. In this work we study, in the framework of Colombeau's generalized functions, the equation $\frac{\partial u}{\partial t} + H(t, x_1, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}) = 0$ with the initial condition $u|_{\{0\} \times V} = f|_V$, where u , H , and f are generalized functions. We have obtained a theorem on existence of solutions and some partial results on uniqueness. We use a technique which has been based on the classical method of characteristics. This technique, in the framework of generalized mappings, has led us to obtain some results on global invertibility of these mappings and also on ordinary differential equations.

Introduction

Our mains references for Colombeau's theory and notations in general are [1] and [2]. Others relevant references about this theory are [3], [5] and [8].

In Section 2 we have introduced the concept of invertible application and have obtained some results on global invertibility of these applications. Section 3 is devoted to the study of ordinary differential equations. Finally in Section 4 we use the method of characteristics to study the equation $\frac{\partial u}{\partial t} + H(t, x_1, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}) = 0$ with a given initial condition. Others techniques to study generalized solutions to nonlinear partial differential equations can be found in [4] and [6].

1 Preliminaries

To facilitate the reading, we summarize some concepts which will be needed in this work.

¹It is part of the author's Doctoral Thesis written under the guidance of Dr. Jorge Aragona.

If Ω_1 is an open subset of \mathbb{R}^n the notation $K \subset\subset \Omega_1$ means that K is a compact subset of Ω_1 . Moreover

$$\mathcal{E}[\Omega_1] = \{u \in \mathcal{R}^{[0,1] \times \Omega_1} \mid u(\varepsilon, \cdot) \in \mathcal{C}^\infty(\Omega_1; \mathbb{R}) \text{ for all } \varepsilon \in]0, 1[\};$$

$$\mathcal{E}_M[\Omega_1] = \{u \in \mathcal{E}[\Omega_1] \mid \forall K \subset\subset \Omega_1, \forall \alpha \in \mathbb{N}^n, \exists N \in \mathbb{N} \text{ such that} \\ \sup_{x \in K} |\partial^\alpha u(\varepsilon, x)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \downarrow 0 \};$$

$$\mathcal{N}[\Omega_1] = \{u \in \mathcal{E}_M[\Omega_1] \mid \forall K \subset\subset \Omega_1, \forall \alpha \in \mathbb{N}^n \text{ and } \forall q \in \mathbb{N} \text{ one has} \\ \sup_{x \in K} |\partial^\alpha u(\varepsilon, x)| = O(\varepsilon^q) \text{ as } \varepsilon \downarrow 0 \};$$

$$\mathcal{E}_M[\Omega_1; \mathbb{R}^m] = \{(u_1, \dots, u_m) \mid u_i \in \mathcal{E}_M[\Omega_1] \text{ for all } 1 \leq i \leq m\}$$

(the elements of $\mathcal{E}_M[\Omega_1; \mathbb{R}^m]$ are called moderate applications on Ω_1);

$$\mathcal{N}[\Omega_1; \mathbb{R}^m] = \{(u_1, \dots, u_m) \mid u_i \in \mathcal{N}[\Omega_1] \text{ for all } 1 \leq i \leq m\}$$

(the elements of $\mathcal{N}[\Omega_1; \mathbb{R}^m]$ are called null applications on Ω_1);

$$\mathcal{G}(\Omega_1; \mathbb{R}^m) = \frac{\mathcal{E}_M[\Omega_1; \mathbb{R}^m]}{\mathcal{N}[\Omega_1; \mathbb{R}^m]} = \oplus_{i=1}^m E_i, \text{ where } E_i = \mathcal{G}(\Omega_1; \mathbb{R});$$

$$\mathcal{E}_M(\mathbb{R}) = \{u \in \mathcal{R}^{[0,1]} \mid \exists N \in \mathbb{N} \text{ such that } |u(\varepsilon)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \downarrow 0 \};$$

$$\mathcal{N}(\mathbb{R}) = \{u \in \mathcal{E}_M(\mathbb{R}) \mid \forall q \in \mathbb{N} \text{ one has } |u(\varepsilon)| = O(\varepsilon^q) \text{ as } \varepsilon \downarrow 0 \};$$

$$\overline{\mathcal{R}} = \frac{\mathcal{E}_M(\mathbb{R})}{\mathcal{N}(\mathbb{R})} \text{ and } \overline{\mathcal{R}}^n = \oplus_{i=1}^n R_i, \text{ where } R_i = \overline{\mathcal{R}};$$

if Ω_2 is an open subset of \mathbb{R}^m , then $\mathcal{G}_*(\Omega_1; \Omega_2)$ is the set of all the $f \in \mathcal{G}(\Omega_1; \mathbb{R}^m)$ for which there is a representative \hat{f} such that the following holds:

(i) given $K \subset\subset \Omega_1$ there are $K' \subset\subset \Omega_2$ and $\eta \in]0, 1[$ with $\hat{f}([0, \eta[\times K) \subset K'$ (remark that if $f \in \mathcal{G}_*(\Omega_1; \Omega_2)$, then all representative of f satisfy (i));

if $z = (z_1, \dots, z_n)$ denotes points in \mathbb{R}^n , $g \in \mathcal{G}(\Omega_1; \mathbb{R})$, \hat{g} is a representative of g and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ with $|\alpha| = 1$, then

$$\frac{\partial \hat{g}}{\partial z_j} \text{ (resp. } \frac{\partial g}{\partial z_j}) \text{ denotes } \partial^\alpha \hat{g} \text{ (resp. } \partial^\alpha g) \text{ if } \alpha_j = 1 \text{ and } 1 \leq j \leq n;$$

$$\nabla \hat{g} \text{ (resp. } \nabla g) \text{ denotes } \left(\frac{\partial \hat{g}}{\partial z_1}, \dots, \frac{\partial \hat{g}}{\partial z_n} \right) \text{ (resp. } \left(\frac{\partial g}{\partial z_1}, \dots, \frac{\partial g}{\partial z_n} \right));$$

if (l_1, \dots, l_n) denotes points in \mathbb{R}^n , $1 \leq i_s \leq n$ for all $1 \leq s \leq k$ and $l = (l_{i_1}, \dots, l_{i_k})$, then $\frac{\partial g}{\partial l}$ denotes $\left(\frac{\partial g}{\partial l_{i_1}}, \dots, \frac{\partial g}{\partial l_{i_k}} \right)$;

if $\hat{h} = (\hat{h}_1, \dots, \hat{h}_m) \in \mathcal{E}_M[\Omega_1; \mathbb{R}^m]$, then $\partial^\beta \hat{h}$ denotes $(\partial^\beta \hat{h}_1, \dots, \partial^\beta \hat{h}_m)$ for $\beta \in \mathbb{N}^n$;

if J is an open subset of \mathbb{R} and W is an open subset of Ω_1 , then π (resp. π_i , with $1 \leq i \leq n$) denotes the class in $\mathcal{G}(J \times W; \mathbb{R})$ of the moderate function $(\varepsilon, t, y_1, \dots, y_n) \in]0, 1] \times J \times W \mapsto t \in \mathbb{R}$ (resp. $(\varepsilon, t, y_1, \dots, y_n) \in]0, 1] \times J \times W \mapsto y_i \in \mathbb{R}$);

if V is an open subset of \mathbb{R}^n , then 1_V denotes the class in $\mathcal{G}(V; \mathbb{R}^n)$ of the moderate application $(\varepsilon, x) \in]0, 1] \times V \mapsto x \in \mathbb{R}^n$;

if $x = (x_1, \dots, x_n)$ denotes points in \mathbb{R}^n , $f \in \mathcal{G}(\Omega_1; \mathbb{R})$ and \hat{f} is a representative of f , then $J\hat{f}$ denotes the moderate mapping from $]0, 1] \times \Omega_1$ to \mathbb{R} defined by

$$J\hat{f}(\varepsilon, x) = \det(df(\varepsilon, \cdot)_x) = \det\left(\frac{\partial \hat{f}_i}{\partial x_j}(\varepsilon, x)\right)_{1 \leq i, j \leq n}$$

and Jf denotes the class of $J\hat{f}$ in $\mathcal{G}(\Omega_1; \mathbb{R})$;

if I is an open interval of \mathbb{R} , $t_o \in \mathbb{R}$ and $a > 0$, then $I_a(t_o)$ denotes $]t_o - a, t_o + a[$ (sometimes we will write I_a instead of $I_a(0)$);

if $A \subset \mathbb{R}^k$, then \bar{A} denotes the closure of A .

2 Invertible application

Here Ω_1 and Ω_2 will be open subsets of \mathbb{R}^n . The theorem 2.4 is the main result in this section. The proposition 2.5 will be useful in Section 4.

Definition 2.1 *We say that $f \in \mathcal{G}_*(\Omega_1; \Omega_2)$ is an invertible application if, and only if, there is $g \in \mathcal{G}_*(\Omega_2; \Omega_1)$ such that $f \circ g = 1_{\Omega_2}$ and $g \circ f = 1_{\Omega_1}$.*

If $f \in \mathcal{G}_*(\Omega_1; \Omega_2)$ is an invertible application, then there is an unique $g \in \mathcal{G}_*(\Omega_2; \Omega_1)$ that satisfies 2.1 and it is called the inverse application of f .

To obtain 2.4 we will use the characterization of inverse multiplicative in $\mathcal{G}(\Omega_1; \mathbb{R})$ given by F. Villarreal ([9] or [10]). For the benefit of the reader, we write, in 2.2, this characterization.

Proposition 2.2 *Let $f \in \mathcal{G}(\Omega_1; \mathbb{R})$ and \hat{f} a representative of f such that $\hat{f}(]0, 1] \times \Omega_1) \subset \mathbb{R} \setminus \{0\}$. The following are equivalent:*

- (a) f has inverse multiplicative in $\mathcal{G}(\Omega_1; \mathbb{R})$;
- (b) if \hat{f}_1 is a representative of f with $\hat{f}_1(]0, 1[\times \Omega_1) \subset \mathbb{R} \setminus \{0\}$, then $\frac{1}{\hat{f}_1} \in \mathcal{E}_M[\Omega_1; \mathbb{R}]$;
- (c) for all $K \subset\subset \Omega_1$, there are $\eta \in]0, 1[$ and a function μ from $]0, 1[$ to $]0, +\infty[$ such that $\frac{1}{\mu} \in \mathcal{E}_M(\mathbb{R})$ and $\mu(\varepsilon) \leq \inf\{|\hat{f}(\varepsilon, x)| \mid x \in K\}$ for all $\varepsilon \in]0, \eta[$;
- (d) if \hat{f}_1 is a representative of f , then for all $K \subset\subset \Omega_1$, there are $\eta \in]0, 1[$ and a function μ from $]0, 1[$ to $]0, +\infty[$ such that $\frac{1}{\mu} \in \mathcal{E}_M(\mathbb{R})$ and $\mu(\varepsilon) \leq \inf\{|\hat{f}_1(\varepsilon, x)| \mid x \in K\}$ for all $\varepsilon \in]0, \eta[$.
- Moreover, if $\hat{f}(]0, 1[\times \Omega_1) \subset \mathbb{R} \setminus \{0\}$ and (c) holds, then the class $de \frac{1}{\hat{f}}$ in $\mathcal{G}(\Omega_1; \mathbb{R})$ is the inverse multiplicative of f in $\mathcal{G}(\Omega_1; \mathbb{R})$.

The next result is analogous to applications in $C^\infty(\Omega_1; \Omega_2)$.

Proposition 2.3 Let $f \in \mathcal{G}_*(\Omega_1; \Omega_2)$. A necessary condition for f to be an invertible application is that Jf has inverse multiplicative in $\mathcal{G}(\Omega_1; \mathbb{R})$.

Proof. Let $\mathcal{K} = (K_j)_{j \in \mathbb{N}}$ be an exhaustive sequence of compact subsets of Ω_1 , g the inverse application of f and \hat{g} and \hat{f} representatives of g and f respectively.

Since $f \in \mathcal{G}_*(\Omega_1; \Omega_2)$ there is $(\eta_j)_{j \in \mathbb{N}}$ a sequence in $]0, 1[$ with $\eta_j > \eta_{j+1}$ and $\{\hat{f}(\varepsilon, x) \mid (\varepsilon, x) \in]0, \eta_j[\times K_j\} \subset\subset \Omega_2$ for all $j \in \mathbb{N}$.

Let $j \in \mathbb{N}$ and \hat{h}_j and \hat{l}_j the applications defined on $]0, 1[\times K_j$ by

$$\hat{h}_j(\varepsilon, x) = \begin{cases} J\hat{g}(\varepsilon, \hat{f}(\varepsilon, x)) & , \text{ if } \varepsilon \in]0, \eta_j[\\ J\hat{g}(\varepsilon, \hat{f}(\frac{\eta_j}{2}, x)) & , \text{ if } \varepsilon \in [\eta_j, 1] \end{cases},$$

$$\hat{l}_j(\varepsilon, x) = \begin{cases} \hat{g}(\varepsilon, \hat{f}(\varepsilon, x)) & , \text{ if } \varepsilon \in]0, \eta_j[\\ \hat{g}(\varepsilon, \hat{f}(\frac{\eta_j}{2}, x)) & , \text{ if } \varepsilon \in [\eta_j, 1] \end{cases}.$$

Then \hat{h}_j and \hat{l}_j are representatives of $(Jg \circ f)|_{K_j}^\circ$ and $(g \circ f)|_{K_j}^\circ$, respectively.

Using that $\hat{h}_j J\hat{f} = J\hat{l}_j$ and that $(J(g \circ f))|_{K_j}^\circ = J(1_{\Omega_1})|_{K_j}^\circ = 1|_{K_j}^\circ$ we have $(Jg \circ f)Jf|_{K_j}^\circ = 1|_{K_j}^\circ$ for all $j \in \mathbb{N}$. Hence $Jg \circ f$ is the inverse multiplicative of Jf in $\mathcal{G}(\Omega_1; \mathbb{R})$. ■

Theorem 2.4 Let $f \in \mathcal{G}_*(\Omega_1; \Omega_2)$ such that there are a representative \hat{f} of f and $\tau \in]0, 1[$ satisfying:

- (i) $\{\hat{f}(\varepsilon, x) \mid x \in \Omega_1\} = \Omega_2$ for all $\varepsilon \in]0, \tau[$;
- (ii) $\hat{f}(\varepsilon, \cdot)$ is an invertible application with inverse g_ε for all $\varepsilon \in]0, \tau[$;
- (iii) for all $K' \subset\subset \Omega_2$, there are $K \subset\subset \Omega_1$ and $\eta \in]0, \tau[$ such that $\{g_\varepsilon(y) \mid (\varepsilon, y) \in]0, \eta[\times K'\} \subset K$;
- (iv) $J\hat{f}(]0, \tau[\times \Omega_1) \subset \mathbb{R} \setminus \{0\}$.

Let \hat{g} the application from $]0, 1[\times \Omega_2$ to Ω_1 such that $\hat{g}(\varepsilon, \cdot) = g_\varepsilon$ for all $\varepsilon \in]0, \tau[$ and $\hat{g}(\varepsilon, \cdot) = g_{\frac{\tau}{2}}$ for all $\varepsilon \in [\tau, 1]$. The following statements are equivalent:

- (a) Jf has inverse multiplicative in $\mathcal{G}(\Omega_1; \mathbb{R})$;
- (b) $\hat{g} \in \mathcal{E}_M[\Omega_2; \mathbb{R}^n]$;
- (c) f is an invertible application and the class of \hat{g} in $\mathcal{G}(\Omega_2; \mathbb{R}^n)$ is the inverse application of f .

(We remark that (iv) and $\hat{g}_\varepsilon \in C^\infty(\Omega_2; \Omega_1)$ for all $\varepsilon \in]0, \tau[$ are equivalent.)

Proof. Let $\hat{f} = (\hat{f}_1, \hat{f}_2, \dots, \hat{f}_n)$ and $\hat{g} = (\hat{g}_1, \hat{g}_2, \dots, \hat{g}_n)$. Suppose that (a) is true and let $K' \subset\subset \Omega_2$ and $\alpha \in \mathbb{N}^n$. We will prove that there is $N \in \mathbb{N}$ such that

$$\sup\{|\partial^\alpha \hat{g}_i(\varepsilon, y)| \mid y \in K' \text{ and } 1 \leq i \leq n\} = O(\varepsilon^{-N}) \text{ as } \varepsilon \downarrow 0, \quad (1)$$

and thus we will have (b).

Take $K \subset\subset \Omega_1$ and $\eta \in]0, \tau[$ as in (iii). If $|\alpha| = 0$ then (1) holds for $N = 0$. Assume that $|\alpha| \geq 1$. We will prove (1), in this case, using induction on $|\alpha|$.

For $\varepsilon \in]0, \eta[$ we know that $\hat{f}(\varepsilon, \hat{g}(\varepsilon, \cdot)) = \hat{1}_{\Omega_2}$, and so $d\hat{g}(\varepsilon, \cdot)$ is the invertible matrix of $d\hat{f}(\varepsilon, \cdot)_{\hat{g}(\varepsilon, \cdot)}$. Therefore

$$\frac{\partial \hat{g}_k}{\partial y_j}(\varepsilon, y) = \frac{1}{J\hat{f}(\varepsilon, \hat{g}(\varepsilon, y))} a_{jk}$$

for all $1 \leq j, k \leq n$, where a_{jk} is sum of products of elements of the set

$$\left\{ \frac{\partial \hat{f}_i}{\partial x_s}(\varepsilon, \hat{g}(\varepsilon, y)) \mid 1 \leq i, s \leq n \right\} \cup \{1, -1\}.$$

Using that Jf has inverse multiplicative in $\mathcal{G}(\Omega_1; \mathbb{R})$, (iv) and 2.2, there are $\eta_2 \in]0, \eta[$ and a function μ from $]0, 1[$ to $]0, +\infty[$ such that $\frac{1}{\mu} \in \mathcal{E}_M(\mathbb{R})$ and

$$0 < \mu(\varepsilon) \leq \inf\{|J\hat{f}(\varepsilon, x)| \mid x \in K\} \text{ for all } \varepsilon \in]0, \eta_2[.$$

Since $\frac{1}{\mu} \in \mathcal{E}_M(\mathbb{R})$ and $\hat{f}_i \in \mathcal{E}_M[\Omega_1; \mathbb{R}]$ for all $1 \leq i \leq n$, there is $N_1 \in \mathbb{N}$ such that

$$\sup\left\{\frac{1}{|J\hat{f}(\varepsilon, \hat{g}(\varepsilon, y))|} \mid y \in K'\right\} = O(\varepsilon^{-N_1}) \text{ as } \varepsilon \downarrow 0;$$

$$\sup_{1 \leq i \leq n} \{|\partial^\gamma \hat{f}_i(\varepsilon, \hat{g}(\varepsilon, y))| \mid y \in K', \gamma \in \mathbb{N}^n \text{ and } |\gamma| \leq |\alpha| + 1\} = O(\varepsilon^{-N_1}) \text{ as } \varepsilon \downarrow 0.$$

If $|\alpha| = 1$ then (1) is true for $N = (n+1)N_1$.

Let $|\alpha| > 1$ and $\nu \in \mathbb{N}^n$ with $|\nu| = |\alpha| + 1$. By hypothesis of induction, there is $N_2 \in \mathbb{N}$ such that

$$\sup_{1 \leq i \leq n} \{|\partial^\beta \hat{g}_i(\varepsilon, y)| \mid y \in K', \beta \in \mathbb{N}^n \text{ and } |\beta| \leq |\alpha|\} = O(\varepsilon^{-N_2}) \text{ as } \varepsilon \downarrow 0.$$

Noting that given $1 \leq k \leq n$ there are $1 \leq j \leq n$ and $\tilde{\gamma} \in \mathbb{N}^n$ with $|\tilde{\gamma}| = |\alpha|$ such that $\partial^\nu \hat{g}_k(\varepsilon, y) = \partial^{\tilde{\gamma}} \frac{\partial \hat{g}_k}{\partial y_j}(\varepsilon, y)$, it is clear that (1) holds for ν .

It is easy to check that (b) implies (c) and, by 2.3, that (c) implies (a). ■

Proposition 2.5 *Let I an open interval of \mathbb{R} with $0 \in I$, $\tau \in]0, 1]$ and $\hat{g} = (\hat{g}_1, \dots, \hat{g}_n) \in \mathcal{E}_M[I \times \mathbb{R}^n; \mathbb{R}^n]$. If f is the class in $\mathcal{G}(I \times \mathbb{R}^n; \mathbb{R}^{n+1})$ of moderate application $\hat{f} : (\varepsilon, t, x) \in]0, 1] \times I \times \mathbb{R}^n \mapsto (t, \hat{g}(\varepsilon, t, x) + x)$ and \hat{g} satisfies*

(i) $\hat{g}_i(\varepsilon, 0, x) = 0$ for all $(\varepsilon, x) \in]0, \tau[\times \mathbb{R}^n$ and $1 \leq i \leq n$;

(ii) there is $M > 0$ such that $\|\partial^\beta \hat{g}(\varepsilon, t, x)\| \leq M$ for all $(\varepsilon, t, x) \in]0, \tau[\times I \times \mathbb{R}^n$ and $\beta = (\beta_0, \beta_1, \dots, \beta_n) \in \mathbb{N}^{n+1}$ with $1 \leq \beta_0 \leq |\beta| \leq 2$,

then there is $a > 0$ with $\overline{I_a} \subset I$ such that

(I) $f|_{I_a \times \mathbb{R}^n} \in \mathcal{G}_*(I_a \times \mathbb{R}^n; I_a \times \mathbb{R}^n)$;

(II) $f|_{I_a \times \mathbb{R}^n}$ is an invertible application;

(III) $\hat{f}(\varepsilon, \cdot)|_{I_a \times \mathbb{R}^n}$ is an invertible application for all $\varepsilon \in]0, \tau[$.

Moreover if $\hat{\Gamma} = (\hat{\Gamma}_0, \hat{\Gamma}_1, \dots, \hat{\Gamma}_n)$ is defined on $]0, 1] \times I_a \times \mathbb{R}^n$ by $\hat{\Gamma}(\varepsilon, \cdot) = (\hat{f}(\varepsilon, \cdot)|_{I_a \times \mathbb{R}^n})^{-1}$ for all $\varepsilon \in]0, \tau[$ and $\hat{\Gamma}(\varepsilon, \cdot) = (\hat{f}(\frac{\tau}{2}, \cdot)|_{I_a \times \mathbb{R}^n})^{-1}$ for all $\varepsilon \in [\tau, 1]$ then

(IV) $\overline{\hat{\Gamma}(]0, \tau[\times J \times K')} \subset I_a \times \mathbb{R}^n$ for all $J \subset \subset I_a$ and $K' \subset \subset \mathbb{R}^n$;

(V) $\hat{\Gamma}$ is a representative of $(f|_{I_a \times \mathbb{R}^n})^{-1}$;

(VI) there is $\overline{M} > 0$ such that $\|\partial^\alpha \widehat{\Gamma}(\varepsilon, t, y)\| \leq \overline{M}$ for all $(\varepsilon, t, y) \in]0, \tau[\times I_a \times \mathbb{R}^n$ and $\alpha \in \mathbb{N}^{n+1}$ with $|\alpha| = 1$.

Proof. Denote by $(t, x) = (t, x_1, \dots, x_n)$ points in $I \times \mathbb{R}^n$. Let M be as in (ii), $a^* > 0$ with $\overline{I_{a^*}} \subset I$ and $a > 0$ such that

$$a < \min\left\{a^*, \frac{1}{nM}, \frac{1}{nn!M^n}, \frac{n^n}{nn!(n+1)^n}\right\}.$$

Using the Mean Value Theorem, (i) and (ii) we have

$$|\widehat{g}_i(\varepsilon, t, x)| \leq M|t| \quad \text{for all } (\varepsilon, t, x) \in]0, \tau[\times I \times \mathbb{R}^n \text{ and } 1 \leq i \leq n; \quad (1)$$

$$\left| \frac{\partial \widehat{g}_i}{\partial x_k}(\varepsilon, t, x) \right| \leq M|t| \quad \text{for all } (\varepsilon, t, x) \in]0, \tau[\times I \times \mathbb{R}^n \text{ and } 1 \leq i, k \leq n. \quad (2)$$

By (1) we have (I) is true. Our next goal is to prove (III) and (IV).

Fix $\varepsilon \in]0, \tau[$. Let $(s, x), (t, y) \in I_a \times \mathbb{R}^n$ with $(s, x) \neq (t, y)$. If $s \neq t$ one has $\widehat{f}(\varepsilon, s, x) \neq \widehat{f}(\varepsilon, t, y)$. Suppose that $s = t$ and let $1 \leq j \leq n$ such that $|x_i - y_i| \leq |x_j - y_j|$ for all $1 \leq i \leq n$. From Mean Value Theorem we know that there is $z \in \mathbb{R}^n$ such that

$$|\widehat{g}_j(\varepsilon, s, x) + x_j - \widehat{g}_j(\varepsilon, s, y) - y_j| = \left| \sum_{i=1}^n \frac{\partial \widehat{g}_j}{\partial x_i}(\varepsilon, s, z)(x_i - y_i) + x_j - y_j \right|$$

and so by (2) we obtain

$$|\widehat{g}_j(\varepsilon, s, x) + x_j - \widehat{g}_j(\varepsilon, s, y) - y_j| \geq |x_j - y_j| - nMa|x_j - y_j| > 0.$$

This implies that $\widehat{f}(\varepsilon, s, x) \neq \widehat{f}(\varepsilon, s, y)$. Hence $\widehat{f}(\varepsilon, \cdot)|_{I_a \times \mathbb{R}^n}$ is one-to-one.

Let $(s, y) \in I_a \times \mathbb{R}^n$, $d = \|y\| + nM|s|$, $\overline{B_d(0)} = \{z \in \mathbb{R}^n \mid \|z\| \leq d\}$ and $\Psi : x \in \overline{B_d(0)} \mapsto y - \widehat{g}(\varepsilon, s, x)$. From (1) we have $\psi(\overline{B_d(0)}) \subset \overline{B_d(0)}$ and hence by Brouwer's Theorem we conclude that there is $\overline{x} \in \overline{B_d(0)}$ such that $\psi(\overline{x}) = \overline{x}$. Thus $\widehat{f}(\varepsilon, \cdot)|_{I_a \times \mathbb{R}^n}$ is onto and if $J \subset \subset I_a$ and $K' \subset \subset \mathbb{R}^n$, then $\widehat{\Gamma}(\varepsilon, J \times K') \subset J \times \overline{B_r(0)}$, where $r = \max\{\|z\| + nM|t| \mid (t, z) \in J \times K'\}$. Therefore (III) and (IV) hold.

Let M_1 such that $nn! \max\{M^n, (1 + \frac{1}{n})^n\} < M_1 < \frac{1}{a}$. From (2), (i), (ii) and Mean Value Theorem we have

$$|J\widehat{f}(\varepsilon, t, x)| \geq |J\widehat{f}(\varepsilon, 0, x)| - \sup_{s \in I_a} \left| \frac{\partial J\widehat{f}}{\partial t}(\varepsilon, s, x) \right| |t| \geq 1 - M_1 a > 0 \quad (3)$$

for all $(\varepsilon, t, x) \in]0, \tau[\times I_a \times \mathbb{R}^n$.

As the function $\mu : \varepsilon \in]0, 1[\mapsto 1 - M_1 a$ belongs to $\mathcal{E}_M(\mathbb{R})$ we conclude, by 2.2, that $J\hat{f}$ has inverse multiplicative. This, together with (I), (III), (IV) and 2.4, tells us that (II) and (V) are true.

Finally, assertion (VI) is a consequence of (2), (ii) and (3). ■

More results about invertibility of applications can be found in [7].

3 Existence and uniqueness of solutions to ordinary differential equations

Here Ω_1 will be an open subset of \mathbb{R}^n , Ω_2 an open subset of \mathbb{R}^m , I an open interval of \mathbb{R} and $t_0 \in I$.

To obtain results about existence and uniqueness of solutions to ordinary differential equations we will use the next lemma and definition.

Lemma 3.1 *Let $K \subset \subset \Omega_1$ and $(\hat{g}_i)_{i \in N^*}$ and $(\hat{h}_i)_{i \in N^*}$ sequences of functions in $\mathcal{E}_M[\Omega_1; \mathbb{R}]$ such that given any $i \in N^*$ and $p \in \mathbb{N}$ one has*

$\sup_{x \in K} |\hat{g}_i(\varepsilon, x) - \hat{h}_i(\varepsilon, x)| = O(\varepsilon^p)$ as $\varepsilon \downarrow 0$, then given any $q \in \mathbb{N}$ one has

$\sup_{x \in K} \left| \prod_{i \in A} \hat{g}_i(\varepsilon, x) - \prod_{i \in A} \hat{h}_i(\varepsilon, x) \right| = O(\varepsilon^q)$ as $\varepsilon \downarrow 0$ for all $A \subset N^$ and A finite.*

Proof : It is enough to note that, if $A = \{i_1, \dots, i_k\} \subset N^*$ and $B = \{i_2, \dots, i_k\}$, then $\prod_{j \in A} \hat{g}_j - \prod_{j \in A} \hat{h}_j = (\hat{g}_{i_1} - \hat{h}_{i_1}) \prod_{j \in B} \hat{g}_j + \hat{h}_{i_1} (\prod_{j \in B} \hat{g}_j - \prod_{j \in B} \hat{h}_j)$ and to use the principle of finite induction. ■

Definition 3.2 *We say that*

(i) $\hat{f} \in \mathcal{E}_M[I \times \Omega_1 \times \Omega_2; \mathbb{R}^k]$ has the property (LLL) (locally logarithmically lipschitz) in (I, Ω_1, Ω_2) if given $J \subset \subset I$, $K \subset \subset \Omega_1$, $K' \subset \subset \Omega_2$ and $\alpha \in \mathbb{N}^{n+m+1}$, there are $N \in \mathbb{N}$, $c > 0$ and $\eta \in]0, 1[$ such that for all $(\varepsilon, t, x, z, y) \in]0, \eta[\times J \times K \times K \times K'$ and $1 \leq i \leq k$ one has

$$|\hat{f}_i(\varepsilon, t, x, y) - \hat{f}_i(\varepsilon, t, z, y)| \leq \log(c\varepsilon^{-N}) \|x - z\|;$$

$$|\partial^\alpha \hat{f}_i(\varepsilon, t, x, y) - \partial^\alpha \hat{f}_i(\varepsilon, t, z, y)| \leq c\varepsilon^{-N} \|x - z\|.$$

(ii) $\hat{f} \in \mathcal{E}_M[I \times \Omega_1; \mathbb{R}^k]$ has the property (LLL) (locally logarithmically lipschitz) in (I, Ω_1) if $\hat{g} : (\varepsilon, t, x, y) \in]0, 1] \times I \times \Omega_1 \times \mathbb{R}^m \mapsto \hat{f}(\varepsilon, t, x)$ has de property (LLL) in $(I, \Omega_1, \mathbb{R}^m)$;

(iii) $\hat{f} \in \mathcal{E}_M[U; \mathbb{R}^k]$, where U is an open subset of \mathbb{R}^s , is a bounded mapping on $A \subset U$ if there are $M > 0$ and $\tau \in]0, 1]$ such that $\|\hat{f}(\varepsilon, x)\| \leq M$ for all $(\varepsilon, x) \in]0, \tau[\times A$;

(iv) $\hat{f} \in \mathcal{E}_M[U; \mathbb{R}^k]$, where U is an open subset of \mathbb{R}^s , has the property (LLG) (locally logarithmic growth) if given $K \subset\subset U$, there are $N \in \mathbb{N}$, $c > 0$ and $\eta \in]0, 1]$ such that $\|\hat{f}(\varepsilon, x)\| \leq \log(c\varepsilon^{-N})$ for all $(\varepsilon, x) \in]0, \eta[\times K$.

Clearly if one representative of f has the property (LLG), then each representative of f has it too. Moreover if $f \in \mathcal{G}_*(U; \mathbb{R}^k)$ then each representative of f has the property (LLG).

Remark that if $f \in \mathcal{G}(I \times \Omega_1 \times \Omega_2; \mathbb{R})$, $\partial^\alpha f \in \mathcal{G}_*(I \times \Omega_1 \times \Omega_2; \mathbb{R})$ for all $\alpha = (0, \alpha_1, \dots, \alpha_n, 0, \dots, 0) \in \mathbb{N}^{n+m+1}$ with $|\alpha| = 1$ and Ω_1 is convex, then each representative of f has the property (LLL) in (I, Ω_1, Ω_2) .

Theorem 3.3 (Existence) *Let W be an open subset of Ω_2 , $x_o \in \overline{\mathbb{R}^n}$, $f \in \mathcal{G}(I \times \Omega_1 \times \Omega_2; \mathbb{R}^n)$, \hat{x}_o a representative of x_o and $\hat{f} = (\hat{f}_1, \dots, \hat{f}_n)$ a representative of f . If*

- (i) $\overline{W} \subset\subset \Omega_2$;
- (ii) there are $K_1 \subset\subset \Omega_1$ and $\tau_1 \in]0, 1]$ such that $\hat{x}_o(]0, \tau_1[) \subset K_1$;
- (iii) $f \in \mathcal{G}_*(I \times \Omega_1 \times \Omega_2; \mathbb{R}^n)$;
- (iv) $\partial^\alpha \hat{f}_i$ has the property (LLG), for all $1 \leq i \leq n$ and $\alpha \in \mathbb{N}^{n+m+1}$ with $|\alpha| = 1$ and $\alpha = (0, \alpha_1, \dots, \alpha_n, 0, 0, \dots, 0)$,

then there are $a > 0$ with $\overline{I_a(t_o)} \subset I$, $\tau \in]0, 1]$, $u \in \mathcal{G}(I_a(t_o) \times W; \mathbb{R}^n)$ and a representative \hat{u} of u satisfying:

- (I) there exists $K \subset\subset \Omega_1$ with $\hat{u}(]0, \tau[\times I_a(t_o) \times W) \subset K \subset\subset \Omega_1$;
- (II) $\frac{\partial \hat{u}}{\partial t}(\varepsilon, t, y) = \hat{f}(\varepsilon, t, \hat{u}(\varepsilon, t, y), y)$ for all $(\varepsilon, t, y) \in]0, \tau[\times I_a(t_o) \times W$;
- (III) $\hat{u}(\varepsilon, t_o, \cdot) = \hat{x}_o(\varepsilon)$ for all $\varepsilon \in]0, \tau[$;
- (IV) $u \in \mathcal{G}_*(I_a(t_o) \times W; \Omega_1)$;

$$(V) \frac{\partial u}{\partial t} = f \circ (\pi, u, \pi_1, \dots, \pi_m);$$

$$(VI) u|_{\{t_0\} \times W} = x_0.$$

Moreover

(VII) if $p \in \mathbf{N}$ and $\partial^\alpha f_i \in \mathcal{G}_*(I \times \Omega_1 \times \Omega_2; \mathbf{R})$ for all $1 \leq i \leq n$ and $\alpha \in \mathbf{N}^{n+m+1}$ with $|\alpha| \leq p$, then $\partial^\beta u_i \in \mathcal{G}_*(I_a(t_0) \times W; \mathbf{R})$ for all $\beta \in \mathbf{N}^{m+1}$ with $|\beta| \leq p$ and $1 \leq i \leq n$;

(VIII) if $\partial^\alpha f_i \in \mathcal{G}_*(I \times \Omega_1 \times \Omega_2; \mathbf{R})$ for all $1 \leq i \leq n$ and $\alpha \in \mathbf{N}^{n+m+1}$ with $|\alpha| = 1$ and $\alpha = (0, \alpha_1, \dots, \alpha_{n+m})$, then $\partial^\beta u_i \in \mathcal{G}_*(I_a(t_0) \times W; \mathbf{R})$ for all $1 \leq i \leq n$ and $\beta \in \mathbf{N}^{m+1}$ with $|\beta| = 1$ and $\beta = (0, \beta_1, \dots, \beta_m)$;

(IX) the assertions (VII) and (VIII) are still true if we replace $\partial^\alpha f_i \in \mathcal{G}_*(I \times \Omega_1 \times \Omega_2; \mathbf{R})$ and $\partial^\beta u_i \in \mathcal{G}_*(I_a(t_0) \times W; \mathbf{R})$ by $\partial^\alpha \hat{f}_i$ is a bounded mapping on $J \times \Omega_1 \times \Omega_2$ for all $J \subset\subset I$ and $\partial^\beta \hat{u}_i$ is a bounded mapping on $J_1 \times W$ for all $J_1 \subset\subset I_a(t_0)$, respectively.

Proof. Parts (IV), (V) and (VI) follow from (I), (II) and (III), respectively.

Let $a^* > 0$ with $\overline{I_a(t_0)} \subset I$, K_1 and τ_1 as in (ii), V an open subset of Ω_1 such that $K_1 \subset V \subset \bar{V} \subset\subset \Omega_1$ and $d > 0$ the distance of K_1 to $\Omega_1 \setminus V$.

By (iii) and (i) there are $\tau \in]0, \tau_1[$ and $M > 0$ such that

$$\|\hat{f}(\varepsilon, t, x, z)\| \leq M \text{ for all } (\varepsilon, t, x, z) \in]0, \tau[\times \overline{I_a(t_0)} \times \bar{V} \times \bar{W}.$$

Take $a > 0$ with $a < \min\{a^*, d/M\}$. Fixed any $\varepsilon \in]0, \tau[$ there exists a unique $u_\varepsilon \in \mathcal{C}(\overline{I_a(t_0)} \times \bar{W}; \bar{V}) \cap \mathcal{C}^\infty(I_a(t_0) \times W; \mathbf{R}^n)$ such that

$$u_\varepsilon(t, y) = \hat{x}_0(\varepsilon) + \int_{t_0}^t \hat{f}(\varepsilon, s, u_\varepsilon(s, y), y) ds \text{ for all } (t, y) \in \overline{I_a(t_0)} \times \bar{W}. \quad (1)$$

Remark that $u_\varepsilon(\overline{I_a(t_0)} \times \bar{W}) \subset \{z \in \mathbf{R}^n \mid \|z - \hat{x}_0(\varepsilon)\| \leq d\}$.

Let $\hat{u}:]0, 1] \times I_a(t_0) \times W \rightarrow \mathbf{R}^n$ defined by $\hat{u}(\varepsilon, t, y) = u_\varepsilon(t, y)$, if $\varepsilon \in]0, \tau[$ and $u(\varepsilon, t, y) = u_{\frac{\tau}{\varepsilon}}(t, y)$, if $\varepsilon \in [\tau, 1]$. Then \hat{u} satisfies (I) (for $K = \bar{V}$), (II) and (III).

Let $J \subset\subset I_a(t_0)$ and $K' \subset\subset W$. We will prove, by induction on $|\beta|$, where $\beta \in \mathbf{N}^{m+1}$, that there exists $N_2 \in \mathbf{N}$ such that

$$\sup_{(t, y) \in J \times K'} |\partial^\beta \hat{u}_i(\varepsilon, t, y)| = O(\varepsilon^{-N_2}) \text{ as } \varepsilon \downarrow 0 \text{ for all } 1 \leq i \leq n. \quad (2)$$

Thus $\hat{u} = (\hat{u}_1, \dots, \hat{u}_n) \in \mathcal{E}_M[I_a(t_0) \times W; \mathbf{R}^n]$.

It is clear that (2) is true for $|\beta| = 0$. Let $p \in \mathbf{N}$ and suppose that $|\beta| = p + 1$ and (2) holds for all $\gamma \in \mathbf{N}^{m+1}$ with $|\gamma| \leq p$.

Denote by $(t, x_1, \dots, x_n, y_1, \dots, y_m)$ points in $I \times \Omega_1 \times \Omega_2$.

Using that $\overline{I_a(t_0)} \times \overline{V} \times \overline{W} \subset \subset I \times \Omega_1 \times \Omega_2$, $\hat{f} \in \mathcal{E}_M[I \times \Omega_1 \times \Omega_2; \mathbb{R}^n]$ and (iv), there are $c > 0$, $\eta \in]0, \tau[$ and $N \in \mathbb{N}$ such that

$$\left| \frac{\partial \hat{f}_i}{\partial x_j}(\varepsilon, s, \hat{u}(\varepsilon, s, y), y) \right| \leq \log(c\varepsilon^{-N}) \text{ and } |\partial^\alpha \hat{f}_i(\varepsilon, s, \hat{u}(\varepsilon, s, y), y)| \leq c\varepsilon^{-N} \quad (3)$$

for all $(\varepsilon, s, y) \in]0, \eta[\times I_a(t_0) \times W$, $1 \leq i, j \leq n$ and $|\alpha| \leq p + 1$.

Assume that $\beta = (\beta_0, \beta_1, \dots, \beta_m)$ and let

$$A_{\varepsilon, t, y} = \{ \partial^\mu \hat{f}_k(\varepsilon, t, \hat{u}(\varepsilon, t, y), y) \mid \mu \in \mathbb{N}^{n+m+1}, |\mu| \leq p + 1 \text{ and } 1 \leq k \leq n \};$$

$$B_{\varepsilon, t, y} = \{1\} \cup \{ \partial^\gamma \hat{u}_k(\varepsilon, t, y) \mid \gamma \in \mathbb{N}^{m+1}, |\gamma| \leq p \text{ and } 1 \leq k \leq n \}.$$

If $\beta_0 \neq 0$ then $\partial^\beta \hat{u}_i(\varepsilon, t, y)$ is sum of products of elements of $A_{\varepsilon, t, y} \cup B_{\varepsilon, t, y}$. Applying (3) and the hypothesis of induction it is clear that (2) is true.

If $\beta_0 = 0$, then

$$\partial^\beta \hat{u}_i(\varepsilon, t, y) = \int_{t_0}^t (l_1(\varepsilon, s, y) + \sum_{k=1}^n \frac{\partial \hat{f}_i}{\partial x_k}(\varepsilon, s, \hat{u}(\varepsilon, s, y), y) \partial^\beta \hat{u}_k(\varepsilon, s, y)) ds,$$

where $l_1(\varepsilon, s, y)$ is sum of products of elements of $A_{\varepsilon, s, y} \cup B_{\varepsilon, s, y}$.

Take $a_1, b_1 \in \mathbb{R}$ such that $J \cup \{t_0\} \subset [a_1, b_1] \subset I_a(t_0)$. From the hypothesis of induction and (3), there are $c_1 > 0$, $N_1 \in \mathbb{N}$ and $\eta_1 \in]0, \eta[$ such that, on $]0, \eta_1[\times [a_1, b_1] \times K'$ one has

$$\| \partial^\beta \hat{u}(\varepsilon, t, y) \| \leq \sum_{i=1}^n | \partial^\beta \hat{u}_i(\varepsilon, t, y) | \leq \left| \int_{t_0}^t (c_1 \varepsilon^{-N_1} + n^2 \log(c\varepsilon^{-N})) \| \partial^\beta \hat{u}(\varepsilon, s, y) \| ds \right|.$$

This inequality, together with Gronwall's Lemma, proves (2).

The remaining statements are established by similar arguments, replacing $\log(c\varepsilon^{-N})$ and $c\varepsilon^{-N}$ in (3) by the appropriate constants in each case. ■

Theorem 3.4 Let $x_0 \in \overline{\mathbb{R}^n}$, $f \in \mathcal{G}(I \times \Omega_1 \times \Omega_2; \mathbb{R}^n)$, \hat{x}_0 a representative of x_0 and $\hat{f} = (\hat{f}_1, \dots, \hat{f}_n)$ a representative of f . If \hat{f} is a bounded mapping on $I \times \Omega_1 \times \Omega_2$ and the assertions 3.3.ii and 3.3.iv are true, then the assertions from 3.3.I to 3.3.IX hold, replacing W by Ω_2 .

Proof. Let V as in the proof of 3.3. Using similar arguments there are $a > 0$ with $\overline{I_a(t_0)} \subset I$ and $\tau \in]0, 1]$ such that, if $\varepsilon \in]0, \tau[$ and $y \in \Omega_2$, then there is a unique $u_{r_y}^\varepsilon \in \mathcal{C}(\overline{I_a(t_0)} \times \overline{B_{r_y}(y)}; \overline{V})$ satisfying $u_{r_y}^\varepsilon(t, z) = \hat{x}_0(\varepsilon) + \int_{t_0}^t \hat{f}(\varepsilon, s, u_{r_y}^\varepsilon(s, z), z) ds$, where $r_y > 0$ and $\overline{B_{r_y}(y)} \subset \Omega_2$ is the closed ball

of center y and radius r_y . As in the proof of 3.3, the mapping \hat{u} from $]0, 1[\times I_a(t_0) \times \Omega_2$ defined by $\hat{u}(\varepsilon, \cdot)|_{I_a(t_0) \times B_{r_y}(y)} = \hat{u}_{r_y}^\varepsilon$ if $\varepsilon \in]0, \tau[$ and \hat{u} constant in $[\tau, 1] \times I_a(t_0) \times \Omega_2$ is a moderate application and its class satisfies the requirements of our theorem. ■

In the next theorem we will replace x_0 by a generalized application.

Theorem 3.5 (Existence) *Let W be an open subset of Ω_2 , $g \in \mathcal{G}(W; \mathbf{R}^n)$, \hat{g} a representative of $g = (g_1, \dots, g_n)$, $f \in \mathcal{G}(I \times \Omega_1 \times \Omega_2; \mathbf{R}^n)$ and $\hat{f} = (\hat{f}_1, \dots, \hat{f}_n)$ a representative of f . If*

- (i) $\overline{W} \subset \subset \Omega_2$;
- (ii) there are U an open subset of Ω_1 and $\tau_1 \in]0, 1[$ such that $\overline{U} \subset \subset \Omega_1$ and $\hat{g}(]0, \tau_1[\times W) \subset U$;
- (iii) $g \in \mathcal{G}_*(W; U)$, where U is as in (ii);
- (iv) $f \in \mathcal{G}_*(I \times \Omega_1 \times \Omega_2; \mathbf{R}^n)$;
- (v) $\partial^\alpha \hat{f}_i$ has the property (LLG) for all $1 \leq i \leq n$ and $\alpha \in \mathbf{N}^{n+m+1}$ with $|\alpha| = 1$ and $\alpha = (0, \alpha_1, \dots, \alpha_n, 0, 0, \dots, 0)$,

then there are $a > 0$ with $\overline{I_a(t_0)} \subset I$, $\tau \in]0, 1[$, $u \in \mathcal{G}(I_a(t_0) \times W; \mathbf{R}^n)$ and a representative \hat{u} of u satisfying:

- (I) there is V an open subset of Ω_1 with $\hat{u}(]0, \tau[\times I_a(t_0) \times W) \subset V \subset \overline{V} \subset \subset \Omega_1$;
- (II) $\frac{\partial \hat{u}}{\partial t}(\varepsilon, t, y) = \hat{f}(\varepsilon, t, \hat{u}(\varepsilon, t, y), y)$ for all $(\varepsilon, t, y) \in]0, \tau[\times I_a(t_0) \times W$;
- (III) $\hat{u}(\varepsilon, t_0, \cdot) = \hat{g}(\varepsilon, \cdot)$ for all $\varepsilon \in]0, \tau[$;
- (IV) $u \in \mathcal{G}_*(I_a(t_0) \times W; \Omega_1)$;
- (V) $\frac{\partial u}{\partial t} = f \circ (\pi, u, \pi_1, \dots, \pi_m)$;
- (VI) $u|_{\{t_0\} \times W} = g$.

Moreover

(VII) if $p \in \mathbf{N}$ and for all $1 \leq i \leq n$ one has $\partial^\alpha f_i \in \mathcal{G}_*(I \times \Omega_1 \times \Omega_2; \mathbf{R})$ and $\partial^\gamma g_i \in \mathcal{G}_*(W; \mathbf{R})$ for all $\alpha \in \mathbf{N}^{n+m+1}$ with $|\alpha| \leq p$ and $\gamma \in \mathbf{N}^m$ with $|\gamma| \leq p$, then $\partial^\beta u_i \in \mathcal{G}_*(I_a(t_0) \times W; \mathbf{R})$ for all $\beta \in \mathbf{N}^{m+1}$ with $|\beta| \leq p$ and $1 \leq i \leq n$;

(VIII) if for all $1 \leq i \leq n$ one has $\partial^\alpha f_i \in \mathcal{G}_*(I \times \Omega_1 \times \Omega_2; \mathbf{R})$ and $\partial^\gamma g_i \in \mathcal{G}_*(W; \mathbf{R})$ for all $\alpha = (0, \alpha_1, \dots, \alpha_{n+m}) \in \mathbf{N}^{n+m+1}$ and $\gamma \in \mathbf{N}^m$

with $|\alpha| = |\gamma| = 1$, then $\partial^\beta u_i \in \mathcal{G}_*(I_a(t_o) \times W; \mathbb{R})$ for all $1 \leq i \leq n$ and $\beta = (0, \beta_1, \dots, \beta_m) \in \mathbb{N}^{m+1}$ with $|\beta| = 1$;

(IX) the assertions (VII) and (VIII) are still true if we replace $\partial^\alpha f_i \in \mathcal{G}_*(I \times \Omega_1 \times \Omega_2; \mathbb{R})$, $\partial^\gamma g_i \in \mathcal{G}_*(W; \mathbb{R})$ and $\partial^\beta u_i \in \mathcal{G}_*(I_a(t_o) \times W; \mathbb{R})$ by $\partial^\alpha \hat{f}_i$ is a bounded mapping on $J \times \Omega_1 \times \Omega_2$ for all $J \subset\subset I$, $\partial^\gamma \hat{g}_i$ is a bounded mapping on W and $\partial^\beta \hat{u}_i$ is a bounded mapping on $J_1 \times W$ for all $J_1 \subset\subset I_a(t_o)$, respectively.

Proof. Let U and τ_1 as in (ii), take U_1 and U_2 open subsets of Ω_1 such that $\overline{U} \subset U_1 \subset \overline{U_1} \subset U_2 \subset \overline{U_2} \subset\subset \Omega_1$ and let $4d$ the distance of $\overline{U_1}$ to $\Omega_1 \setminus U_2$.

Note that, if $x \in B_d(0) = \{y \in \mathbb{R}^n \mid \|y\| < d\}$, then $x + z \in \Omega_1$ for all $z \in \overline{U_1}$. Thus we can define the moderate application $\hat{h} = (\hat{h}_1, \dots, \hat{h}_n)$ on $]0, 1] \times I \times B_d(0) \times U_1 \times \Omega_2$ by $\hat{h}(\varepsilon, s, x, z, y) = \hat{f}(\varepsilon, s, x + z, y)$.

We will denote by $(s, x, z, y) = (s, x_1, \dots, x_n, z_1, \dots, z_n, y_1, \dots, y_m)$ points in $I \times B_d(0) \times (U_1 \times \Omega_2)$.

Let $J \subset\subset I$, $K \subset\subset B_d(0)$ and $L \subset\subset U_1 \times \Omega_2$. Take $K_2 \subset\subset U_1$ and $K' \subset\subset \Omega_2$ such that $L \subset K_2 \times K'$.

Using that $K + K_2 \subset\subset \Omega_1$, (iv) and (v), there are $M > 0$, $c > 0$, $\eta \in]0, \tau_1[$ and $N \in \mathbb{N}$ such that on $]0, \eta[\times J \times K \times (K_2 \times K')$ one has

$$\|h(\varepsilon, s, x, z, y)\| \leq M \text{ and } \left| \frac{\partial \hat{h}_i}{\partial x_j}(\varepsilon, s, x, z, y) \right| \leq \log(c\varepsilon^{-N}) \text{ for all } 1 \leq i, j \leq n.$$

Applying 3.3 (replacing \hat{f} , Ω_1 , Ω_2 and W by \hat{h} , $B_d(0)$, $U_1 \times \Omega_2$ and $U \times W$, respectively) there are $a > 0$ with $I_a(t_o) \subset I$, $\tau \in]0, \eta[$, V_1 open subset of $B_d(0)$ and an application $\hat{v} \in \mathcal{E}_M[I_a(t_o) \times U \times W; \mathbb{R}^n]$ such that

$$\hat{v}(]0, \tau[\times I_a(t_o) \times U \times W) \subset V_1 \subset \overline{V_1} \subset\subset B_d(0);$$

$$\hat{v}(\varepsilon, t_o, \cdot) = 0 \text{ for all } \varepsilon \in]0, \tau[;$$

$$\frac{\partial \hat{v}}{\partial s}(\varepsilon, s, z, y) = \hat{h}(\varepsilon, s, \hat{v}(\varepsilon, s, z, y), z, y) \text{ on }]0, \tau[\times I_a(t_o) \times U \times W.$$

By (ii) we may define $\hat{u} = (\hat{u}_1, \dots, \hat{u}_n)$ on $]0, 1] \times I_a(t_o) \times W$ by $\hat{u}(\varepsilon, t, y) = \hat{g}(\varepsilon, y) + \hat{v}(\varepsilon, t, \hat{g}(\varepsilon, y), y)$, if $\varepsilon \in]0, \tau[$ and $\hat{u}(\varepsilon, t, y) = \hat{g}(\frac{\tau}{2}, y) + \hat{v}(\frac{\tau}{2}, t, \hat{g}(\frac{\tau}{2}, y), y)$, if $\varepsilon \in [\tau, 1]$.

Since $\hat{g} \in \mathcal{E}_M[W; \mathbb{R}^n]$, $\hat{v} \in \mathcal{E}_M[I_a(t_o) \times U \times W; \mathbb{R}^n]$ and (iii) holds is easy to prove that $\hat{u} \in \mathcal{E}_M[I_a(t_o) \times W; \mathbb{R}^n]$. Let u the class of \hat{u} .

To establish from (I) to (IX) it is enough remark that $\hat{u}(\varepsilon, t_0, \cdot) = \hat{g}(\varepsilon, \cdot)$, $\frac{\partial \hat{u}}{\partial t}(\varepsilon, t, y) = \hat{f}(\varepsilon, t, \hat{u}(\varepsilon, t, y), y)$ on $]0, \tau[\times I_a(t_0) \times W$, if $V = U + V_1$ then $\hat{u}(]0, \tau[\times I_a(t_0) \times W) \subset V \subset \bar{V} \subset \bar{U}_1 + B_d(0)$ and \hat{v} and its class v satisfy from 3.3.VII to 3.3.IX (replacing \hat{f} , Ω_1 , Ω_2 , W , \hat{u} and u by \hat{h} , $B_d(0)$, $U_1 \times \Omega_2$, $U \times W$, \hat{v} and v , respectively). ■

In the next theorem we will replace W by Ω_2 .

Theorem 3.6 Let $g \in \mathcal{G}(\Omega_2; \mathbb{R}^n)$, \hat{g} a representative of $g = (g_1, \dots, g_n)$, $f \in \mathcal{G}(I \times \Omega_1 \times \Omega_2; \mathbb{R}^n)$ and $\hat{f} = (\hat{f}_1, \dots, \hat{f}_n)$ a representative of f . If \hat{f} is a bounded mapping on $I \times \Omega_1 \times \Omega_2$ and the assertions 3.5.ii, 3.5.iii and 3.5.v are true (replacing W by Ω_2) or if $\Omega_1 = \mathbb{R}^n$, \hat{f} is a bounded mapping on $I \times \Omega_1 \times \Omega_2$, $g \in \mathcal{G}_*(\Omega_2; \mathbb{R}^n)$ and 3.5.v holds, then the assertions from 3.5.II to 3.5.IX are true (replacing W by Ω_2).

Proof. Similar to the proof of 3.5, replacing W and 3.3 by Ω_2 and 3.4, respectively. Remark that if $\Omega_1 = \mathbb{R}^n$, then \hat{h} may be defined on $]0, 1] \times I \times \mathbb{R}^n \times \mathbb{R}^n \times \Omega_2$. ■

Now we present a theorem about uniqueness of solutions.

Theorem 3.7 (Uniqueness) Let $f \in \mathcal{G}(I \times \Omega_1 \times \Omega_2; \mathbb{R}^n)$ and $\hat{f} = (\hat{f}_1, \dots, \hat{f}_n)$ a representative of f such that

- (i) \hat{f} has the property (LLL) in (I, Ω_1, Ω_2) ;
- (ii) $\partial^\alpha \hat{f}_i$ has the property (LLG) for all $\alpha = (0, \alpha_1, \dots, \alpha_n, 0, 0, \dots, 0) \in \mathbb{N}^{n+m+1}$ with $|\alpha| = 1$ and $1 \leq i \leq n$.

If u and v belong to $\mathcal{G}_*(I \times \Omega_2; \Omega_1)$, $u|_{\{t_0\} \times \Omega_2} = v|_{\{t_0\} \times \Omega_2}$ and they are solutions of $\frac{\partial \omega}{\partial t} = f \circ (\pi, \omega, \pi_1, \dots, \pi_m)$, then $u = v$.

Proof. Let $(J_j)_{j \in \mathbb{N}}$ be an exhaustive sequence of compact subsets of I with $t_0 \in \bigcap_{j \in \mathbb{N}} J_j$ and J_j closed interval for all $j \in \mathbb{N}$, $(K'_j)_{j \in \mathbb{N}}$ an exhaustive sequence of compact subsets of Ω_2 , \hat{u} a representative of u and \hat{v} a representative of v . It suffices to prove that, if $j \in \mathbb{N}$, $J \subset \subset J'_j$, $K' \subset \subset K'_j$ and $\beta \in \mathbb{N}^{m+1}$, then given $q \in \mathbb{N}$ one has

$$\sup_{(t,y) \in J \times K'} |(\partial^\beta \hat{v}_i - \partial^\beta \hat{u}_i)(\varepsilon, t, y)| = O(\varepsilon^q) \text{ as } \varepsilon \downarrow 0 \text{ for all } 1 \leq i \leq n. \quad (1)$$

Fix $j \in N$ and let $q \in N$. Take $a, b \in \mathbb{R}$ such that $J \cup \{t_o\} \subset [a, b] \subset \subset \mathring{J}_j$. Since $u, v \in \mathcal{G}_*(I \times \Omega_2; \Omega_1)$, there are $K_j \subset \subset \Omega_1$ and $\eta_j \in]0, 1[$ such that $\hat{u}(]0, \eta_j[\times J_j \times K'_j) \cup \hat{v}(]0, \eta_j[\times J_j \times K'_j) \subset K_j \subset \subset \Omega_1$.

Denote by $(t, x_1, \dots, x_n, y_1, \dots, y_m)$ points in $I \times \Omega_1 \times \Omega_2$.

Using that $\hat{f} \in \mathcal{E}_M[I \times \Omega_1 \times \Omega_2; \mathbb{R}^n]$, (i) and (ii) there are $N \in N$, $c > 0$ and $\tau \in]0, \eta_j[$ such that on $]0, \tau[\times [a, b] \times K'$ one has

$$\left| \frac{\partial \hat{f}_i}{\partial x_k}(\varepsilon, t, \hat{u}(\varepsilon, t, y), y) \right| \leq \log(c\varepsilon^{-N}); \quad \left| \frac{\partial \hat{f}_i}{\partial x_k}(\varepsilon, t, \hat{v}(\varepsilon, t, y), y) \right| \leq \log(c\varepsilon^{-N}); \quad (2)$$

$$|\hat{f}_i(\varepsilon, t, \hat{u}(\varepsilon, t, y), y) - \hat{f}_i(\varepsilon, t, \hat{v}(\varepsilon, t, y), y)| \leq \log(c\varepsilon^{-N}) \|(\hat{u} - \hat{v})(\varepsilon, t, y)\|; \quad (3)$$

$$|\partial^\alpha \hat{f}_i(\varepsilon, t, \hat{u}(\varepsilon, t, y), y)| \leq c\varepsilon^{-N}; \quad |\partial^\alpha \hat{f}_i(\varepsilon, t, \hat{v}(\varepsilon, t, y), y)| \leq c\varepsilon^{-N}; \quad (4)$$

$$|\partial^\alpha \hat{f}_i(\varepsilon, t, \hat{u}(\varepsilon, t, y), y) - \partial^\alpha \hat{f}_i(\varepsilon, t, \hat{v}(\varepsilon, t, y), y)| \leq c\varepsilon^{-N} \|(\hat{u} - \hat{v})(\varepsilon, t, y)\| \quad (5)$$

for all $|\alpha| \leq |\beta|$ and $1 \leq i, k \leq n$.

Let $\hat{g} = (\hat{g}_1, \dots, \hat{g}_n) \in \mathcal{N}[\mathring{J}_j \times K'_j; \mathbb{R}^n]$ and $\hat{h} = (\hat{h}_1, \dots, \hat{h}_n) \in \mathcal{N}[K'_j; \mathbb{R}^n]$ such that, if $l = (l_1, \dots, l_n) :]0, \eta_j[\times \mathring{J}_j \times K'_j \rightarrow \mathbb{R}^n$ is defined by

$$l(\varepsilon, t, y) = (\hat{v} - \hat{u})(\varepsilon, t, y) - \hat{h}(\varepsilon, y),$$

then

$$l(\varepsilon, t, y) = \int_{t_o}^t (\hat{f}(\varepsilon, s, \hat{v}(\varepsilon, s, y), y) - \hat{f}(\varepsilon, s, \hat{u}(\varepsilon, s, y), y) + \hat{g}(\varepsilon, s, y)) ds.$$

As \hat{u} and \hat{v} are moderate mappings and \hat{h} and \hat{g} are null there are $c_1 > 0$, $\tau_1 \in]0, \tau[$ and $N_1 \in N$ with $N_1 > \max\{N, Nn^2(b - t_o), Nn^2(t_o - a)\}$ such that for all $\varepsilon \in]0, \tau_1[$, $t \in [a, b]$ and $y \in K'$ one has

$$\max\{ \prod_{(\lambda, j) \in \Lambda \times A} |\partial^\lambda \hat{u}_j(\varepsilon, t, y)|, \prod_{(\lambda, j) \in \Lambda \times A} |\partial^\lambda \hat{v}_j(\varepsilon, t, y)| \} \leq c_1 \varepsilon^{-N_1} \quad (6)$$

for all $\Lambda \times A \subset \{\theta \in N^{m+1} \mid |\theta| \leq |\beta|\} \times \{1, \dots, n\}$;

$$\max\{ |\partial^{\tilde{\mu}} \hat{h}_i(\varepsilon, y)|, |\partial^\gamma \hat{g}_i(\varepsilon, t, y)| \} \leq c_1 \varepsilon^{q+2N_1} \quad (7)$$

for all $\tilde{\mu} \in N^m$, $\gamma \in N^{m+1}$ with $|\tilde{\mu}| \leq |\beta|$, $|\gamma| \leq |\beta|$ and $1 \leq i \leq n$.

By (3) and (7) we have

$$\|l(\varepsilon, t, y)\| \leq \left| \int_{t_o}^t (n \log(c\varepsilon^{-N}) \|l(\varepsilon, s, y)\| + (nc + 1)nc_1 \varepsilon^{q+N_1}) ds \right|$$

on $]0, \tau_1[\times [a, b] \times K'$. This, together Gronwall's Lemma, implies that

$$\sup_{(t, y) \in [a, b] \times K'} |(\hat{v} - \hat{u})(\varepsilon, t, y) - \hat{h}(\varepsilon, y)| = O(\varepsilon^q) \quad \text{as } \varepsilon \downarrow 0.$$

Hence (1) is true for $|\beta| = 0$.

Let $|\beta| = |(\beta_0, \beta_1, \dots, \beta_m)| > 0$. Using the induction hypothesis and 3.1 given $3N_1 + q$ there are $c_2 > 0$ and $\tau_2 \in]0, \tau_1[$ such that

$$\begin{aligned} & |\Pi_{(\lambda, j) \in \Lambda \times A} \partial^\lambda \hat{u}_j(\varepsilon, t, y) - \Pi_{(\lambda, j) \in \Lambda \times A} \partial^\lambda \hat{v}_j(\varepsilon, t, y)| \leq c_2 \varepsilon^{3N_1 + q} \quad (8) \\ & \text{on }]0, \tau_1[\times [a, b] \times K' \text{ for all } \Lambda \times A \subset \{\theta \in \mathbf{N}^{m+1} \mid |\theta| < |\beta|\} \times \{1, \dots, n\}. \end{aligned}$$

Let $1 \leq i \leq n$. If $\bar{\beta} = (\beta_0 - 1, \beta_1, \dots, \beta_m)$, $(\varepsilon, t, y) \in]0, \tau_2[\times \overset{\circ}{J}_j \times \overset{\circ}{K}'_j$, $w_1(\varepsilon, t, y) = \hat{f}_i(\varepsilon, t, \hat{v}(\varepsilon, t, y), y)$ and $w_2(\varepsilon, t, y) = \hat{f}_i(\varepsilon, t, \hat{u}(\varepsilon, t, y), y)$ we have

$$\partial^\beta l_i(\varepsilon, t, y) = \int_{t_0}^t \partial^\beta (w_1 - w_2 + \hat{g}_i)(\varepsilon, s, y) ds, \text{ if } \beta_0 = 0; \quad (9)$$

$$\partial^\beta l_i(\varepsilon, t, y) = \partial^{\bar{\beta}} (w_1 - w_2 + \hat{g}_i)(\varepsilon, t, y), \text{ if } \beta_0 > 0. \quad (10)$$

Now we will calculate $\partial^\mu (w_1 - w_2)$, where $\mu \in \mathbf{N}^{m+1}$ with $|\mu| \leq |\beta|$.

Let $\mu = (\mu_0, \mu_1, \dots, \mu_m)$ with $|\mu| \leq |\beta|$ and $\tilde{\gamma} = (\mu_0, 0, \dots, 0, \mu_1, \dots, \mu_m)$, then $\partial^\mu (w_1 - w_2)(\varepsilon, t, y) = \bar{a}_1 + \bar{a}_2 + \bar{a}_3 + \bar{a}_4$, where

$$\bar{a}_1 = \partial^{\tilde{\gamma}} \hat{f}_i(\varepsilon, t, \hat{v}(\varepsilon, t, y), y) - \partial^{\tilde{\gamma}} \hat{f}_i(\varepsilon, t, \hat{u}(\varepsilon, t, y), y);$$

$$\bar{a}_2 = \sum_{k=1}^n \left(\frac{\partial \hat{f}_i}{\partial x_k}(\varepsilon, t, \hat{v}(\varepsilon, t, y), y) - \frac{\partial \hat{f}_i}{\partial x_k}(\varepsilon, t, \hat{u}(\varepsilon, t, y), y) \right) \partial^\mu \hat{v}_k(\varepsilon, t, y);$$

$$\bar{a}_3 = \sum_{k=1}^n \frac{\partial \hat{f}_i}{\partial x_k}(\varepsilon, t, \hat{u}(\varepsilon, t, y), y) (\partial^\mu \hat{v}_k(\varepsilon, t, y) - \partial^\mu \hat{u}_k(\varepsilon, t, y));$$

\bar{a}_4 is sum of elements of the kind $b_{\Lambda A}^\gamma + c_{\Lambda A}^\gamma$, where

$$b_{\Lambda A}^\gamma = [\partial^{\tilde{\gamma}} \hat{f}_i(\varepsilon, t, \hat{v}(\varepsilon, t, y), y) - \partial^{\tilde{\gamma}} \hat{f}_i(\varepsilon, t, \hat{u}(\varepsilon, t, y), y)] \Pi_{(\lambda, k) \in \Lambda \times A} \partial^\lambda \hat{v}_k(\varepsilon, t, y);$$

$$c_{\Lambda A}^\gamma = \partial^{\tilde{\gamma}} \hat{f}_i(\varepsilon, t, \hat{u}(\varepsilon, t, y), y) [\Pi_{(\lambda, k) \in \Lambda \times A} \partial^\lambda \hat{v}_k(\varepsilon, t, y) - \Pi_{(\lambda, k) \in \Lambda \times A} \partial^\lambda \hat{u}_k(\varepsilon, t, y)]$$

with $|\gamma| \leq |\mu|$ and $\Lambda \times A \subset \{\delta \in \mathbf{N}^{m+1} \mid |\delta| < |\mu|\} \times \{1, \dots, n\}$.

From (2), (4), (5), (6), (7) and (8) there is $c_3 > 0$ such that for all $(\varepsilon, t, y) \in]0, \tau_1[\times [a, b] \times K'$ one has

$$|\partial^\mu (w_1 - w_2)(\varepsilon, t, y)| \leq n \log(c\varepsilon^{-N}) \|\partial^\mu l(\varepsilon, t, y)\| + c_3 \varepsilon^{q+N_1}. \quad (11)$$

Suppose that $\beta_0 = 0$. Using (7), (9) and (11) (taking $\mu = \beta$) we have

$$\|\partial^\beta l(\varepsilon, t, y)\| \leq \left| \int_{t_0}^t (n^2 \log(c\varepsilon^{-N}) \|\partial^\beta l(\varepsilon, s, y)\| + n(c_3 + c_1) \varepsilon^{q+N_1}) ds \right|.$$

Then, by Gronwall's Lemma, we obtain

$$\sup_{(t, y) \in [a, b] \times K'} |\partial^\beta l(\varepsilon, t, y)| = O(\varepsilon^q) \text{ as } \varepsilon \downarrow 0.$$

Thus we have proved (1) for $\beta_0 = 0$.

Applying (7), (10) and (11) (replacing μ by $\bar{\beta} = (\beta_0 - 1, \beta_1, \dots, \beta_m)$) we conclude that (1) holds for $\beta_0 \neq 0$. \blacksquare

Remark 3.8 Let J be an open interval of I , $\hat{\pi}_* :]0, 1] \times J \rightarrow \mathcal{R}$ defined by $\hat{\pi}_*(\varepsilon, t) = t$ and π_* its class in $\mathcal{G}(J; \mathcal{R})$. With similar arguments to proof 3.3 and 3.7 it is easy to verify that:

If $f \in \mathcal{G}_*(I \times \Omega_1; \mathcal{R}^n)$, $x_o \in \overline{\mathcal{R}^n}$ and there are \hat{x}_o a representative of x_o and \hat{f} a representative of f such that $\hat{x}_o(]0, \tau[) \subset \subset \Omega_1$ for some $\tau \in]0, 1]$ and \hat{f} has the property (LLL) in (I, Ω_1) , then there are $a > 0$ with $\overline{I_a(t_o)} \subset I$ and an unique $u \in \mathcal{G}_*(I_a(t_o); \Omega_1)$ such that $u' = f \circ (\pi_*, u)$ and $u(t_o) = x_o$. Moreover, from $\mathcal{G}(\cdot; \mathcal{R}^n)$ be a sheaf of vector spaces on \mathcal{R} , there is an unique maximal solution of $\omega' = f \circ (\pi_*, \omega)$ satisfying $\omega(t_o) = x_o$ (a maximal solution is a solution that can not be extended).

4 Method of characteristics

In this Section Ω and Ω' will be open subsets of \mathcal{R}^n , I an open interval of \mathcal{R} with $0 \in I$, I' an open interval of \mathcal{R} and we will denote by $x = (x_1, \dots, x_n)$, $(t, x) = (t, x_1, \dots, x_n)$ (or $(s, \tau) = (s, \tau_1, \dots, \tau_n)$) and $(t, x, y, p) = (t, x_1, \dots, x_n, y, p_1, \dots, p_n)$ points in Ω , $I \times \Omega$ and $I \times \Omega \times I' \times \Omega'$, respectively. Moreover $f \in \mathcal{G}(\Omega; \mathcal{R})$ and $H \in \mathcal{G}(I \times \Omega \times I' \times \Omega'; \mathcal{R})$.

The aim here is to study the following problem:

Problem 4.1 Given H and f , there are W open subset of $I \times \Omega$ with $V = \{z \in \mathcal{R}^n \mid (0, z) \in W\} \neq \emptyset$ and $u \in \mathcal{G}(W; \mathcal{R})$ such that

- (i) $(u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}) \in \mathcal{G}_*(W; I' \times \Omega')$;
- (ii) $\frac{\partial u}{\partial t} + H \circ (\pi, \pi_1, \dots, \pi_n, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}) = 0$;
- (iii) $u|_{\{0\} \times V} = f|_V$.

We say that $u \in \mathcal{G}(W; \mathcal{R})$ is a solution for 4.1 in $\mathcal{G}(W; \mathcal{R})$ if and only if u satisfies (i), (ii) and (iii).

Definition 4.2 If J is an open interval of \mathcal{R} with $0 \in J \subset I$ and W is an open subset of $I \times \Omega$ with $V = \{z \in \mathcal{R}^n \mid (0, z) \in W\} \neq \emptyset$, then we will denote by $\mathcal{S}(I, \Omega, I', \Omega', H, f, J, W)$ the set of (X, U, P) for which one has:

- (i) $X \in \mathcal{G}_*(J \times V; \Omega)$, $U \in \mathcal{G}_*(J \times V; I')$ and $P \in \mathcal{G}_*(J \times V; \Omega')$;

(ii) (X, U, P) is a solution of sistem:

$$\frac{\partial X}{\partial s} = \frac{\partial H}{\partial p} \circ (\pi, X, U, P)$$

$$\frac{\partial U}{\partial s} = -H \circ (\pi, X, U, P) + \sum_{j=1}^n \frac{\partial H}{\partial p_j} \circ (\pi, X, U, P) P_j,$$

where $P = (P_1, \dots, P_n)$

$$\frac{\partial P}{\partial s} = -\frac{\partial H}{\partial x} \circ (\pi, X, U, P) - P \frac{\partial H}{\partial y} \circ (\pi, X, U, P);$$

(iii) $(X, U, P)|_{\{0\} \times V} = (1_V, f|_V, \nabla f|_V)$;

(iv) $Y = (\pi, X) \in \mathcal{G}_*(J \times V; W)$ and Y is an invertible application.

Theorem 4.3 (Existence) *If there is a representative \widehat{H} of H such that $\frac{\partial \widehat{H}}{\partial y}$ has the property (LLG) and if $(X, U, P) \in \mathcal{S}(I, \Omega, I', \Omega', H, f, J, W)$ and $u = U \circ (\pi, X)^{-1}$, then*

(I) u is a solution for 4.1 in $\mathcal{G}(W; \mathbb{R})$;

(II) $(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}) = P \circ (\pi, X)^{-1}$.

Proof. Let $Y = (\pi, X)$ and $h = (h_0, h_1, \dots, h_n) = Y^{-1}$. Then

$$\begin{aligned} (\pi|_{\{0\} \times V}, \pi_1|_{\{0\} \times V}, \dots, \pi_n|_{\{0\} \times V}) &= (Y \circ h)|_{\{0\} \times V} \\ &= (h_0|_{\{0\} \times V}, X \circ (h|_{\{0\} \times V})) \\ &= (h_0|_{\{0\} \times V}, h_1|_{\{0\} \times V}, \dots, h_n|_{\{0\} \times V}) \end{aligned}$$

and thus $u|_{\{0\} \times V} = U \circ (h|_{\{0\} \times V}) = f|_V$.

As $U = u \circ Y$ and 4.2.ii holds we have

$$\left(\frac{\partial u}{\partial t} \circ Y \right) + H \circ (\pi, X, U, P) + \sum_{j=1}^n \left(\frac{\partial u}{\partial x_j} \circ Y - P_j \right) \frac{\partial X_j}{\partial s} = 0; \quad (1)$$

$$\frac{\partial U}{\partial r_i} = \sum_{j=1}^n \left(\frac{\partial u}{\partial x_j} \circ Y \right) \frac{\partial X_j}{\partial r_i} \quad \text{for all } 1 \leq i \leq n. \quad (2)$$

Fix $1 \leq i \leq n$ and let $\varphi = \frac{\partial U}{\partial r_i} - \sum_{j=1}^n P_j \frac{\partial X_j}{\partial r_i}$.

Suppose proved that $\varphi = 0$. Then by (1) and (2) we have

$$\begin{aligned}
\left[\left(\frac{\partial u}{\partial t} \circ Y \right) + H \circ (\pi, X, U, P) \right] + \sum_{j=1}^n \left[\left(\frac{\partial u}{\partial x_j} \circ Y \right) - P_j \right] \frac{\partial X_j}{\partial s} &= 0 \\
\sum_{j=1}^n \left[\left(\frac{\partial u}{\partial x_j} \circ Y \right) - P_j \right] \frac{\partial X_j}{\partial r_1} &= 0 \\
&\vdots \\
\sum_{j=1}^n \left[\left(\frac{\partial u}{\partial x_j} \circ Y \right) - P_j \right] \frac{\partial X_j}{\partial r_n} &= 0
\end{aligned}$$

Applying that Y is an invertible application we conclude, by 2.3, that JY has inverse multiplicative in $\mathcal{G}(J \times V; \mathbf{R})$. This implies that the system above has only the trivial solution. Hence

$$\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right) = P \circ Y^{-1} \in \mathcal{G}_*(W; \Omega');$$

$$\frac{\partial u}{\partial t} = -H \circ (\pi, X, U, P) \circ Y^{-1} = -H \circ (\pi, \pi_1, \dots, \pi_n, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}).$$

To conclude this proof we will prove that $\varphi = 0$.

Let $(J_j)_{j \in \mathbf{N}}$ be an exhaustive sequence of compact subsets of J with $0 \in \bigcap_{j \in \mathbf{N}} J_j$ and J_j closed interval for all $j \in \mathbf{N}$, $(K'_j)_{j \in \mathbf{N}}$ an exhaustive sequence of compact subsets of V , \widehat{X} , \widehat{U} , \widehat{P} and $\widehat{\varphi}$ representatives of X , U , P and φ , respectively.

Let $j \in \mathbf{N}$, $[a, b] \subset \mathring{J}_j$, $K' \subset \mathring{K}'_j$ and $\beta \in \mathbf{N}^{n+1}$. We will prove, using induction on $|\beta|$, that

$$\text{given } q \in \mathbf{N} \text{ one has } \sup_{(t,r) \in [a,b] \times K'} |\partial^\beta \widehat{\varphi}(\varepsilon, t, r)| = 0(\varepsilon^q) \text{ as } \varepsilon \downarrow 0. \quad (3)$$

By 4.2 we have $\frac{\partial \varphi}{\partial s} = -\varphi \frac{\partial H}{\partial y} \circ (\pi, X, U, P)$ and $\varphi|_{\{0\} \times V} = 0$. Thus there are $\widehat{g} \in \mathcal{N}[\mathring{J}_j \times \mathring{K}'_j; \mathbf{R}]$, $\widehat{h} \in \mathcal{N}[\mathring{K}'_j; \mathbf{R}]$ and $\eta_j \in]0, 1]$ such that

$$\widehat{\varphi}(\varepsilon, t, r) - \widehat{h}(\varepsilon, r) = \int_0^t (\widehat{\psi}(\varepsilon, s, \widehat{\varphi}(\varepsilon, s, r), r) + \widehat{g}(\varepsilon, s, r)) ds,$$

where

$$\widehat{\psi}(\varepsilon, s, t, r) = -t \frac{\partial \widehat{H}}{\partial y}(\varepsilon, s, \widehat{X}(\varepsilon, s, r), \widehat{U}(\varepsilon, s, r), \widehat{P}(\varepsilon, s, r)) \text{ on }]0, \eta_j[\times \mathring{J}_j \times \mathbf{R} \times \mathring{K}'_j.$$

Using that \widehat{H} is a moderate function, the Mean Value Theorem and that $\frac{\partial \widehat{H}}{\partial y}$ has the property (LLG) we can find $N \in \mathbb{N}$, $c > 0$ and $\tau \in]0, \eta_j[$ such that for all $\alpha \in \mathbb{N}^{n+1}$ with $|\alpha| \leq |\beta|$ and $(\varepsilon, s, r) \in]0, \tau[\times [a, b] \times K'$ one has

$$|\partial^\alpha \widehat{\psi}(\varepsilon, s, \widehat{\varphi}(\varepsilon, s, r), r)| \leq c\varepsilon^{-N}; \quad (4)$$

$$|\partial^\alpha \widehat{\psi}(\varepsilon, s, \widehat{\varphi}(\varepsilon, s, r), r)| \leq c\varepsilon^{-N} |\widehat{\varphi}(\varepsilon, s, r)|, \quad (5)$$

$$\left| \frac{\partial \widehat{\psi}}{\partial t}(\varepsilon, s, \widehat{\varphi}(\varepsilon, s, r), r) \right| \leq \log(c\varepsilon^{-N}); \quad (6)$$

$$|\widehat{\psi}(\varepsilon, s, \widehat{\varphi}(\varepsilon, s, r), r)| \leq \log(c\varepsilon^{-N}) |\widehat{\varphi}(\varepsilon, s, r)|, \quad (7)$$

and thus

$$|\widehat{\varphi}(\varepsilon, t, r) - \widehat{h}(\varepsilon, r)| \leq \left| \int_0^t (\log(c\varepsilon^{-N}) |\widehat{\varphi}(\varepsilon, s, r) - \widehat{h}(\varepsilon, s)| + \log(c\varepsilon^{-N}) |\widehat{h}(\varepsilon, s)| + |\widehat{g}(\varepsilon, s, r)|) ds \right|.$$

This, together with Gronwall's Lemma, gives us

$$\sup_{(t,r) \in [a,b] \times K'} |\widehat{\varphi}(\varepsilon, t, r) - \widehat{h}(\varepsilon, r)| = O(\varepsilon^q) \quad \text{as } \varepsilon \downarrow 0.$$

Hence we have proved (3) for $|\beta| = 0$.

Let $l :]0, \eta_j[\times \overset{\circ}{J}_j \times \overset{\circ}{K}'_j \rightarrow \mathbb{R}$ defined by $l(\varepsilon, t, r) = \widehat{\varphi}(\varepsilon, t, r) - \widehat{h}(\varepsilon, r)$ and $\beta = (\beta_0, \beta_1, \dots, \beta_n)$. Using (4), (5), (6), (7), the induction hypothesis and Gronwall's Lemma for $\beta_0 = 0$ we have $\sup_{(t,r) \in [a,b] \times K'} |\partial^\beta l(\varepsilon, t, r)| = O(\varepsilon^q)$ as $\varepsilon \downarrow 0$. Thus (3) holds for $|\beta| > 0$. \blacksquare

Now we will present some results about uniqueness.

Theorem 4.4 *Let \widehat{H} a representative of H such that*

(i) $\partial^\alpha \widehat{H}$ has the property (LLG) for all $\alpha = (0, \alpha_1, \dots, \alpha_{2n+1}) \in \mathbb{N}^{2n+2}$ with $1 \leq |\alpha| \leq 2$;

(ii) \widehat{H} , $\frac{\partial \widehat{H}}{\partial x}$, $\frac{\partial \widehat{H}}{\partial y}$ and $\frac{\partial \widehat{H}}{\partial p}$ have the property (LLL) in $(I, \Omega \times I' \times \Omega')$.

If (X, U, P) and $(\overline{X}, \overline{U}, \overline{P})$ belong to $\mathcal{S}(I, \Omega, I', \Omega', H, f, J, W)$, then $X = \overline{X}$, $U = \overline{U}$ and $P = \overline{P}$.

Proof. Let $V = \{z \in \mathbb{R}^n \mid (0, z) \in W\}$, $\widehat{\varphi} = (\widehat{\varphi}_1, \dots, \widehat{\varphi}_{2n+1})$ the moderate application from $]0, 1[\times J \times (\Omega \times I' \times \Omega') \times V$ to \mathbb{R}^{2n+1} defined by

$$(\widehat{\varphi}_1, \dots, \widehat{\varphi}_n)(\varepsilon, t, x, y, p, z) = \frac{\partial \widehat{H}}{\partial p}(\varepsilon, t, x, y, p);$$

$$\widehat{\varphi}_{n+1}(\varepsilon, t, x, y, p, z) = -\widehat{H}(\varepsilon, t, x, y, p) + \sum_{j=1}^n \frac{\partial \widehat{H}}{\partial p_j}(\varepsilon, t, x, y, p) p_j;$$

$$(\widehat{\varphi}_{n+2}, \dots, \widehat{\varphi}_{2n+1})(\varepsilon, t, x, y, p, z) = -\frac{\partial \widehat{H}}{\partial x}(\varepsilon, t, x, y, p) - p \frac{\partial \widehat{H}}{\partial y}(\varepsilon, t, x, y, p)$$

and φ the class of $\widehat{\varphi}$ in $\mathcal{G}(J \times (\Omega \times I' \times \Omega') \times V, \mathbb{R}^{2n+1})$.

As the elements of $\mathcal{S}(I, \Omega, I', \Omega', H, f, J, W)$ are solution of

$$\left(\frac{\partial \bar{X}}{\partial s}, \frac{\partial \bar{U}}{\partial s}, \frac{\partial \bar{P}}{\partial s} \right) = \varphi \circ (\pi, \bar{X}, \bar{U}, \bar{P}, \pi_1, \dots, \pi_n)$$

$$(\bar{X}, \bar{U}, \bar{P})|_{\{0\} \times V} = (1_V, f|_V, \nabla f|_V)$$

we conclude, by 3.7, that $(X, U, P) = (\bar{X}, \bar{U}, \bar{P})$. ■

Theorem 4.5 *If $\Omega = \mathbb{R}^n$, $I' = \mathbb{R}$, $\mathcal{S}(I, \mathbb{R}^n, \mathbb{R}, \Omega', H, f, J, J \times \mathbb{R}^n) \neq \emptyset$ and there is a representative \widehat{H} of H as in 4.4 and such that $\frac{\partial \widehat{H}}{\partial p}$ is a bounded mapping on $I \times \mathbb{R}^n \times \mathbb{R} \times \Omega'$, then if there is $u \in \mathcal{G}(J \times \mathbb{R}^n; \mathbb{R})$ such that*

(i) *u is a solution for 4.1 in $\mathcal{G}(J \times \mathbb{R}^n; \mathbb{R})$;*

(ii) *there are a representative \widehat{u} of u and $\tau \in]0, 1]$ such that*

$$\left(\frac{\partial \widehat{u}}{\partial x_1}, \dots, \frac{\partial \widehat{u}}{\partial x_n} \right)(]0, \tau[\times J \times \mathbb{R}^n) \subset \Omega',$$

(iii) *$\partial^\gamma u \in \mathcal{G}_*(J \times \mathbb{R}^n; \mathbb{R})$ for all $\gamma \in \mathbb{N}^{n+1}$ with $\gamma = (0, \gamma_1, \dots, \gamma_n)$ and $1 \leq |\gamma| \leq 2$,*

then u is as in 4.3. Moreover if $v \in \mathcal{G}(J \times \mathbb{R}^n; \mathbb{R})$ satisfies from (i) to (iii), then $v = u$.

Proof. Let $(X, U, P) \in \mathcal{S}(I, \mathbb{R}^n, \mathbb{R}, \Omega', H, f, J, J \times \mathbb{R}^n)$, $\widehat{\psi}$ the moderate application defined on $]0, 1] \times J \times \mathbb{R}^n \times \mathbb{R}^n$ by the relation

$$\widehat{\psi}(\varepsilon, t, x, r) = \frac{\partial \widehat{H}}{\partial p}(\varepsilon, t, x, \widehat{u}(\varepsilon, t, x), \frac{\partial \widehat{u}}{\partial x_1}(\varepsilon, t, x), \dots, \frac{\partial \widehat{u}}{\partial x_n}(\varepsilon, t, x))$$

and ψ the class of $\widehat{\psi}$ in $\mathcal{G}(J \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$.

Applying 3.6 we know that for $t \in J$ there are $a_t > 0$ with $\overline{I_{a_t}(t)} \subset J$ and $\tilde{X}_t \in \mathcal{G}_*(I_{a_t}(t) \times \mathbb{R}^n; \mathbb{R}^n)$ such that $\frac{\partial \tilde{X}_t}{\partial s} = \psi \circ (\pi, \tilde{X}_t, \pi_1, \dots, \pi_n)$ and $\tilde{X}_t|_{\{t\} \times \mathbb{R}^n} = X|_{\{t\} \times \mathbb{R}^n}$.

Since $J \times \mathbb{R}^n = \cup_{t \in J} J_{a_t}(t) \times \mathbb{R}^n$ and $\mathcal{G}_*(\cdot, \mathbb{R}^n)$ is a sheaf of vector spaces on \mathbb{R} , there is $\tilde{X} \in \mathcal{G}_*(J \times \mathbb{R}^n; \mathbb{R}^n)$ such that $\tilde{X}|_{I_{a_t}(t) \times \mathbb{R}^n} = \tilde{X}_t$ for all $t \in J$. Thus $\tilde{X}|_{\{0\} \times \mathbb{R}^n} = \tilde{X}_0|_{\{0\} \times \mathbb{R}^n} = X|_{\{0\} \times \mathbb{R}^n} = 1_{\mathbb{R}^n}$ and

$$\frac{\partial \tilde{X}}{\partial s} = \frac{\partial H}{\partial p} \circ (\pi, \tilde{X}, u \circ (\pi, \tilde{X}), \frac{\partial u}{\partial x_1} \circ (\pi, \tilde{X}), \dots, \frac{\partial u}{\partial x_n} \circ (\pi, \tilde{X})).$$

Let $\tilde{U} = u \circ (\pi, \tilde{X})$ and $\tilde{P} = (\frac{\partial u}{\partial x_1} \circ (\pi, \tilde{X}), \dots, \frac{\partial u}{\partial x_n} \circ (\pi, \tilde{X}))$. If $\tilde{\varphi}$ and φ are as in the proof of 4.4, then, by (i), we have $(\tilde{X}, \tilde{U}, \tilde{P})|_{\{0\} \times \mathbb{R}^n} = (1_{\mathbb{R}^n}, f|_{\mathbb{R}^n}, \nabla f|_{\mathbb{R}^n})$ and

$$\left(\frac{\partial \tilde{X}}{\partial s}, \frac{\partial \tilde{U}}{\partial s}, \frac{\partial \tilde{P}}{\partial s} \right) = \varphi \circ (\pi, \tilde{X}, \tilde{U}, \tilde{P}, \pi_1, \dots, \pi_n).$$

This, together with 3.7, gives us $(\tilde{X}, \tilde{U}, \tilde{P}) = (X, U, P)$. Thus we have $u = U \circ (\pi, X)^{-1}$.

A similar computation establishes $v = U \circ (\pi, X)^{-1}$. ■

Proposition 4.6 *Let $I' = \mathbb{R}$ and $\Omega = \Omega' = \mathbb{R}^n$. If $(f, \nabla f) \in \mathcal{G}_*(\mathbb{R}^n; \mathbb{R}^{n+1})$, $H \in \mathcal{G}_*(I \times \mathbb{R}^{2n+1}; \mathbb{R})$ and if there are a representative \hat{f} of f and a representative \hat{H} of H such that*

(i) $\partial^\alpha \hat{f}$ is a bounded mapping on \mathbb{R}^n for all $\alpha \in \mathbb{N}^n$ with $|\alpha| = 2$;

(ii) $\frac{\partial \hat{H}}{\partial y} = 0$ on $]0, 1] \times I \times \mathbb{R}^{2n+1}$;

(iii) $\partial^\gamma \hat{H}$ is a bounded mapping on $I \times \mathbb{R}^{2n+1}$ for all $\gamma \in \Lambda$, where $\Lambda = \{\beta = (\beta_0, \beta_1, \dots, \beta_{2n+1}) \in \mathbb{N}^{2n+2} \mid \beta_0 = 0 < |\beta| \leq 2 \text{ or } \beta_j = 0 < \beta_0 = 1 < |\beta| = 2 \text{ for all } 1 \leq j \leq n+1\}$,

then there are $a > 0$ with $\overline{I_a} \subset I$ and an unique solution u for 4.1 in $\mathcal{G}(I_a \times \mathbb{R}^n; \mathbb{R})$ such that $\partial^\gamma u \in \mathcal{G}_*(I_a \times \mathbb{R}^n; \mathbb{R})$ for all $\gamma \in \Lambda_1$, where $\Lambda_1 = \{\mu = (\mu_0, \mu_1, \dots, \mu_n) \in \mathbb{N}^{n+1} \mid |\mu| \leq 1 \text{ or } \mu_0 = 0 < |\mu| = 2\}$.

Proof. Let $\hat{g} : (\varepsilon, z) \in]0, 1] \times \mathbb{R}^n \mapsto (z, \nabla \hat{f}(\varepsilon, z))$, g the class of \hat{g} in $\mathcal{G}(\mathbb{R}^n; \mathbb{R}^{2n})$, $\hat{\varphi}(\varepsilon, s, x, p, r) = \left(\frac{\partial \hat{H}}{\partial p}(\varepsilon, s, x, 0, p), -\frac{\partial \hat{H}}{\partial x}(\varepsilon, s, x, 0, p) \right)$ on $]0, 1] \times I \times \mathbb{R}^{3n}$ and φ the class of $\hat{\varphi}$ in $\mathcal{G}(I \times (\mathbb{R}^n \times \mathbb{R}^n) \times \mathbb{R}^n; \mathbb{R}^{2n})$.

From 3.6 there are $a^* > 0$ with $\overline{I_{a^*}} \subset I$ and $(X, P) \in \mathcal{G}_*(I_{a^*} \times \mathbb{R}^n; \mathbb{R}^{2n})$ such that there are $\widehat{X} = (\widehat{X}_1, \dots, \widehat{X}_n)$, $\widehat{P} = (\widehat{P}_1, \dots, \widehat{P}_n)$ representatives of X and P , respectively, and $\tau \in]0, 1]$ satisfying

$$\left(\frac{\partial \widehat{X}}{\partial s}, \frac{\partial \widehat{P}}{\partial s} \right) = \hat{\varphi} \circ (\hat{\pi}, \widehat{X}, \widehat{P}, \hat{\pi}_1, \dots, \hat{\pi}_n) \text{ on }]0, \tau[\times I_{a^*} \times \mathbb{R}^n;$$

$$(\widehat{X}, \widehat{P})(\varepsilon, 0, r) = \hat{g}(\varepsilon, r) \text{ for all } (\varepsilon, r) \in]0, \tau[\times \mathbb{R}^n;$$

$\partial^\alpha \widehat{X}_i$ and $\partial^\alpha \widehat{P}_i$ are bounded mappings on $\overline{I_b} \times \mathbb{R}^n$ for all $b \in]0, a^*[$, $\alpha = (0, \alpha_1, \dots, \alpha_n) \in \mathbb{N}^{n+1}$ with $|\alpha| = 1$ and $1 \leq i \leq n$.

Let \widehat{U} be the moderate mapping defined on $]0, 1] \times I_{a^*} \times \mathbb{R}^n$ by

$$\widehat{U}(\varepsilon, s, r) = \hat{f}(\varepsilon, r) + \int_0^s (-\widehat{H} \circ (\hat{\pi}, \widehat{X}, \widehat{0}, \widehat{P})) + \sum_{j=1}^n \frac{\partial \widehat{H}}{\partial p_j} \circ (\hat{\pi}, \widehat{X}, \widehat{0}, \widehat{P}) \widehat{P}_j(\varepsilon, t, r) dt$$

and U the class of \widehat{U} in $\mathcal{G}(I_{a^*} \times \mathbb{R}^n; \mathbb{R})$. Then $U \in \mathcal{G}_*(I_{a^*} \times \mathbb{R}^n; \mathbb{R})$, $U|_{\{0\} \times \mathbb{R}^n} = f$ and, by (ii), (X, U, P) satisfies 4.2.ii.

Let $b \in]0, a^*[$, $\widehat{Y} = (\hat{\pi}, \widehat{X})|_{]0, 1] \times I_b \times \mathbb{R}^n}$, \widehat{l} the moderate mapping from $]0, 1] \times I_b \times \mathbb{R}^n$ to \mathbb{R}^n defined by $\widehat{l}(\varepsilon, s, r) = \widehat{X}(\varepsilon, s, r) - r$ and Y the class of \widehat{Y} in $\mathcal{G}(I_b \times \mathbb{R}^n; \mathbb{R}^{n+1})$. Thus $\widehat{Y}(\varepsilon, s, r) = (s, \widehat{l}(\varepsilon, s, r) + r)$ and, by 2.5, there is $a > 0$ with $\overline{I_a} \subset I_b$ such that

$Y|_{I_a \times \mathbb{R}^n} \in \mathcal{G}_*(I_a \times \mathbb{R}^n; I_a \times \mathbb{R}^n)$ and it is an invertible application;

there is a representative $\widehat{\Gamma}$ of $(Y|_{I_a \times \mathbb{R}^n})^{-1}$ such that

$$\widehat{\Gamma}(\varepsilon, \cdot) = (\widehat{Y}(\varepsilon, \cdot)|_{I_a \times \mathbb{R}^n})^{-1} \text{ for all } \varepsilon \in]0, \tau[;$$

$\partial^\alpha \widehat{\Gamma}$ is a bounded mapping on $I_a \times \mathbb{R}^n$ for all $\alpha \in \mathbb{N}^{n+1}$ with $|\alpha| = 1$.

Hence $(X, U, P)|_{I_a \times \mathbb{R}^n} \in \mathcal{S}(I, \mathbb{R}^n, \mathbb{R}, \mathbb{R}^n, H, f, I_a, I_a \times \mathbb{R}^n)$. Thus, by (ii) and 4.3, we have $u = U \circ (Y|_{I_a \times \mathbb{R}^n})^{-1}$ is a solution for 4.1 in $\mathcal{G}(I_a \times \mathbb{R}^n; \mathbb{R})$

and $\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right) = P \circ (Y|_{I_a \times \mathbb{R}^n})^{-1}$. Clearly u satisfies the requirements of our proposition. The uniqueness is a consequence of 4.5. \blacksquare

References

- [1] Aragona J., Biagioni H.A., Intrinsic definition of the Colombeau algebra of generalized functions, *Anal. Math.*, 17(1991), 75-132.
- [2] Aragona J., Villarreal F., Colombeau's theory and shock waves in a problem of hydrodynamics, *J. Anal. Math.*, 61(1993), 113-144.
- [3] Biagioni H.A., *A nonlinear theory of generalized functions*, Lect. Notes Math. 1421, Springer-Verlag, Berlin, 1990.
- [4] Biagioni H.A., Generalized solutions to nonlinear first-order systems, *Mh. Math.*, 118(1994), 7-20.
- [5] Colombeau J.F., *Elementary introduction to new generalized functions*, North Holland, Amsterdam 1985.
- [6] Colombeau J.F., Heibig A., Oberguggenberger M., Le problème de Cauchy dans un espace de fonctions généralisées I, *C. R. Acad. Sci. Paris Sér. I Math.*, 317(1993), 851-855.
- [7] Fernandez R., *A equação de Hamilton-Jacobi no contexto das funções generalizadas*, Doctoral Thesis, Universidade de São Paulo, 1996.
- [8] Oberguggenberger M., *Multiplication of distributions and applications to partial differential equations*, Pitman Res. Notes 259 New York, 1992.
- [9] Villarreal F., *Sobre soluções na forma de onda de choque de certos sistemas de equações diferenciais parciais na hidrodinâmica*, Doctoral Thesis, Universidade de São Paulo, 1990.
- [10] Villarreal F., Composition and invertibility for a class of generalized functions in the Colombeau's Theory, *Integr. Trans. and Spec. Func.*, 6(1998), 339-345

Roseli Fernandez
Universidade de São Paulo
Instituto de Matemática e Estatística
Caixa Postal 66281 - CEP 05315-970 - São Paulo, Brazil
E-mail: roselif@ime.usp.br

TRABALHOS DO DEPARTAMENTO DE MATEMÁTICA

TÍTULOS PUBLICADOS

- 2000-01 BARROS, S.R.M., PEREIRA, A.L., POSSANI, C. and SIMONIS, A. Spatially periodic equilibria for a non local evolution equation. 11p.
- 2000-02 GOODAIRE, E.G. and POLCINO MILIES, C. Moufang unit loops torsion over the centre. 10p.
- 2000-03 COSTA, R. and MURAKAMI, L.S.I. On idempotents and isomorphisms of multiplication algebras of Bernstein algebras. 12p.
- 2000-04 KOSZMIDER, P. On strong chains of uncountable functions. 24p.
- 2000-05 ANGELERI-HÜGEL, L and COELHO, F.U. Infinitely generated complements to partial tilting modules. 11p.
- 2000-06 GIANNONI, F., MASIELLO, A., PICCIONE, P. and TAUSK, D. V. A Generalized Index Theorem for Morse-Sturm Systems and applications to Semi-Riemannian Geometry. 45p.
- 2000-07 PICCIONE, P. and TAUSK, D.V. Lagrangian and Hamiltonian formalism for constrained variational problems. 29p.
- 2000-08 PICCIONE, P. and TAUSK, D.V. An index theorem for non periodic solutions of Hamiltonian Systems. 67p.
- 2000-09 GIAMBÒ R., GIANNONI, F. and PICCIONE, P. Existence multiplicity and regularity for sub-Riemannian geodesics by variational methods. 25p.
- 2000-10 PICCIONE, P. and TAUSK, D.V. Variational Aspects of the Geodesic Problem in sub-Riemannian Geometry. 30p.
- 2000-11 GIANNONE, F. and PICCIONE, P. The Arrival Time Brachistochrones in General Relativity. 58p.
- 2000-12 GIANNONE, F., PICCIONE, P. and TAUSK, V. D. Morse Theory for the Travel Time Brachistochrones in stationary spacetimes. 70p.
- 2000-13 PICCIONE, P. Time minimizing trajectories in Lorentzian Geometry. The General-Relativistic Brachistochrone Problem. 23p.
- 2000-14 GIANNONI, F., PICCIONE, P. and SAMPALMIERI, R. On the geodesical connectedness for a class of semi-Riemannian manifolds. 31p.
- 2000-15 PICCIONE, P. and TAUSK, D. V. The Maslov Index and a generalized Morse Index Theorem for non positive definite metrics. 7p.
- 2000-16 PICCIONE, P. and TAUSK, D. V. On the Banach Differential Structure for sets of Maps on Non-Compact Domains. 26p.
- 2000-17 GONZÁLEZ, S., GUZZO JR, H. and VICENTE, P. Special classes of n^{th} -order Bernstein algebras. 16p.

- 2000-18 VICENTE, P. and GUZZO, H. Classification of the 5-dimensional power-associative 2^{nd} -order Bernstein algebras. 23p.
- 2000-19 KOSZMIDER, P. On Banach Spaces of Large Density but Few Operators. 18p.
- 2000-20 GONÇALVES, D. L. and GUASCHI, J. About the structure of surface pure braid groups. 12p.
- 2000-21 ARAGONA, J. and SOARES, M. An existence theorem for an analytic first order PDE in the framework of Colombeau's theory. 11p.
- 2000-22 MARTIN, P. A. The structure of residue class fields of the Colombeau ring of generalized numbers. 21p.
- 2000-23 BEKKERT, V. and MERKLEN, H.A. Indecomposables in Derived Categories of Gentle Algebras. 18p.
- 2000-24 COSTA, R. and MURAKAMI, L.S.I. Two Numerical Invariants for Bernstein Algebras. 17p.
- 2000-25 COELHO, F.U., ASSEM, I. and TREPODE, S. Simply connected tame quasi-tilted algebras. 26p.
- 2000-26 CARDONA, F. S. P. and WONG, P. N. S. Addition formulas for Relative Reidemeister Numbers. 25p.
- 2000-27 MELO, SEVERINO T. and MERKLEN, M.I. On a Conjectured Noncommutative Beals-Cordes-type Characterization. 7p.
- 2000-28 DOKUCHAEV, M. A., JURIAANS, O. S., POLCINO MILIES, F. C. and SINGER, M. L. S. Finite Conjugacy in Algebras and Orders. 17p.
- 2001-01 KOSZMIDER, P. Universal Matrices and Strongly Unbounded Functions. 18p.
- 2001-02 JUNQUEIRA, L. and KOSZMIDER, P. On Families of Lindelöf and Related Subspaces of 2^{th} . 30p.
- 2001-03 KOSZMIDER, P. and TALL, FRANKLIN D. A Lindelöf Space with no Lindelöf Subspace of Size \aleph_1 . 11p.
- 2001-04 COELHO, F. U. and VARGAS, R. R. S. Mesh Algebras. 20p.
- 2001-05 FERNANDEZ, R. The equation $\frac{\partial u}{\partial t} + H\left(t, x_1, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right) = 0$ and the method of characteristics in the framework of generalized functions. 24p.