RT-MAE 9532

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by

Francisco Cribari-Neto, Denise A. Botter
Gauss M. Cordeiro, Silvia L.P. Ferrari

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Classificação AMS: 62F11, 62F12.
(AMS Classification)

- Dezembro de 1995 -

Bias Reduction in One–Parameter Exponential Family Models

Francisco Cribari–Neto

Department of Economics, Southern Illinois University, Carbondale IL, 62901-4515, USA

Denise A. Botter

Dept. de Estatística, Univ. de São Paulo, Caixa Postal 66281, São Paulo SP, 05389-970, Brazil

Gauss M. Cordeiro

Dept. de Estatística, Universidade Federal de Pernambuco, Cidade Universitária, Recife PE, 50740-540, Brazil

Silvia L.P. Ferrari

Dept. de Estatística, Univ. de São Paulo, Caixa Postal 66281, São Paulo SP, 05389-970, Brazil

Abstract. This paper gives closed-form expressions for the second and third order biases of maximum likelihood estimates for a number of distributions in the one-parameter exponential family. Approximations based on asymptotic expansions for some bias-corrections that require the evaluation of unusual functions and long expressions are given. A graphical analysis is also performed to show how such biases and the mean squared errors of both the maximum likelihood estimate and its bias-corrected version vary with the parameter that indexes some distributions. Finally, we present simulation results comparing the performance of the maximum likelihood estimator and its bias-corrected version.

Keywords: Asymptotic expansion; bias correction; exponential family; maximum likelihood estimation.

1. Introduction

Let $Y_1, \ldots, Y_n$ be a set of n independent and identically distributed random variables with probability or density function in the one-parameter exponential family, that is,

$$f(y;\theta) = \frac{1}{\zeta(\theta)} \exp\{-\alpha(\theta)\, d(y) + v(y)\}, \tag{1}$$

where θ is a scalar parameter and $\zeta(\cdot)$, $\alpha(\cdot)$, $d(\cdot)$ and $v(\cdot)$ are known functions and $\theta \in \Theta$. The support of $f(y;\theta)$ is assumed to not depend on θ, α and ζ are assumed to have continuous first five derivatives with respect to θ, with $\zeta(\cdot)$ being positive valued, $\mathrm{d}\alpha(\theta)/\mathrm{d}\theta$ and $\mathrm{d}\beta(\theta)/\mathrm{d}\theta$ being different from zero for all values of θ in the parameter space, where $\beta(\theta)$ is defined as $\beta(\theta) = (\mathrm{d}\zeta(\theta)/\mathrm{d}\theta)(\zeta(\theta)\mathrm{d}\alpha(\theta)/\mathrm{d}\theta)^{-1}$. A number of important distributions can be shown to be special cases of (1). The maximum likelihood estimate $\hat{\theta}$ of θ comes from $n^{-1}\sum_{i=1}^{n} d(y_i) = -\beta(\hat{\theta})$ if the solution to this equation belongs to the parameter space, Θ.

The bias and variance of $\hat{\theta}$ up to order n^{-2} can be written as $B(\theta) = n^{-1}B_1(\theta) + n^{-2}B_2(\theta)$ and $V(\theta) = n^{-1}V_1(\theta) + n^{-2}V_2(\theta)$, respectively. Here, "to order n^{-2}" means that terms of order smaller than n^{-2} are neglected. Three bias-corrected maximum likelihood estimates of θ are

$$\hat{\theta}_1 = \hat{\theta} - \frac{B_1(\hat{\theta})}{n}, \tag{2}$$

$$\hat{\theta}_2 = \hat{\theta} - \frac{B_1(\hat{\theta})}{n} - \frac{B_2(\hat{\theta})}{n^2} \tag{3}$$

and

$$\tilde{\theta}_2 = \hat{\theta} - \frac{B_1(\hat{\theta})}{n} - \frac{B_2^*(\hat{\theta})}{n^2}, \tag{4}$$

where

$$B_2^*(\theta) = B_2(\theta) - B_1(\theta)B_1'(\theta) - \frac{1}{2}B_1''(\theta)V_1(\theta),$$

primes denoting derivatives with respect to θ. Ferrari, Botter, Cordeiro and Cribari–Neto (1996) have shown that these three modified estimates are bias-free to order n^{-1}, but only $\tilde{\theta}_2$ has no bias to order n^{-2}. They have also shown that for one-parameter exponential family models

$$B_1(\theta) = -\frac{\beta''}{2\alpha'\beta'^2}, \tag{5}$$

$$B_2(\theta) = \frac{-12\beta'\alpha''\beta''^2 - 33\alpha'\beta''^3 + 4\beta'^2\alpha''\beta''' + 26\alpha'\beta'\beta''\beta''' - 3\alpha'\beta'^2\beta^{iv}}{24\alpha'^3\beta'^5}, \tag{6}$$

$$\begin{aligned} B_2^*(\theta) = {} & \frac{1}{24\alpha'^4\beta'^5}(12\beta'^2\alpha''^2\beta'' + 18\alpha'\beta'\alpha''\beta''^2 + 15\alpha'^2\beta''^3 - 6\alpha'\beta'^2\beta''\alpha''' - 8\alpha'\beta'^2\alpha''\beta''' \\ & - 16\alpha'^2\beta'\beta''\beta''' + 3\alpha'^2\beta'^2\beta^{iv}), \end{aligned} \tag{7}$$

$$V_1(\theta) = \frac{1}{\alpha'\beta'}, \tag{8}$$

$$V_2(\theta) = \frac{2\beta'\alpha''\beta'' + 5\alpha'\beta''^2 - 2\alpha'\beta'\beta'''}{2\alpha'^3\beta'^4}. \tag{9}$$

Expressions (5)–(9) can then be used in conjunction with (2)–(4) to construct bias-corrected estimates of θ, the parameter that indexes the distribution in use. Bias-corrected maximum likelihood estimates of $\tau = G(\theta)$, a function of θ which does not depend upon n, can also be obtained using the results in their paper. $\hat{\theta}_1$, $\hat{\theta}_2$ and $\tilde{\theta}_2$ have the same mean squared error (MSE) to order n^{-3}, and $\text{MSE}(\hat{\theta}) - \text{MSE}(\tilde{\theta}_2) = \Delta(\theta)/n^2 + O(n^{-3})$, where, in one-parameter exponential family models, $\Delta = \Delta(\theta) = (4\beta'\alpha''\beta'' + 9\alpha'\beta''^2 - 4\alpha'\beta'\beta''')/(4\alpha'^3\beta'^4)$. Δ can be used to decide which estimator should be used in a given application, $\hat{\theta}$ or $\tilde{\theta}_2$, when the mean squared error is the deciding criterion.

In Section 2 we apply the general formulas above to a number of distributions in the one-parameter exponential family, thus giving closed-form expressions for the second and third order biases that can be readily used by practitioners to bias-correct maximum likelihood estimates in studies that involve distributions in the one-parameter exponential family. Section 3 gives approximations based on asymptotic expansions that can be used to avoid the evaluation of unusual functions and long formulas. By making use of our approximations, one can avoid the evaluation of polygamma, Bessel and zeta functions. A graphical analysis that shows the dependence of the corrections on θ is performed is Section 4. This section also examines the difference in the mean squared errors of $\hat{\theta}$ and $\bar{\theta}_2$ to order n^{-2} as a function of θ. Finally, Section 5 concludes the paper with some simulations.

2. Some special cases

In this section we use the expressions for $B_1(\theta)$, $B_2(\theta)$ and $B_2^*(\theta)$ in (5), (6) and (7) to obtain closed-form formulas for the second and third order biases of maximum likelihood estimates for a number of important distributions. Here, we consider 24 special cases and give closed-form expressions for $B_1(\theta)$, $B_2(\theta)$ and $B_2^*(\theta)$ obtained with the help of the computer algebra systems Mathematica (Wolfram, 1991) and Maple V (Abell and Braselton, 1994). These cases cover more than 24 distributions since some of them are indeed families of distributions. For example, the Burr system of distributions covers 10 distributions (all distributions in the Burr system with the exception of the Burr I and the Burr IX). Most of the distributions considered here are well known and have wide range of practical applications in many fields, such as engineering, biology, and economics, among others. Cases (i) through (viii) involve discrete random variables whereas continuous random variables are considered in cases (ix) through (xxiv). For further details on the distributions considered here, see Johnson, Kotz and Balakrishnan (1994a, 1994b) and Johnson, Kotz and Kemp (1992).

The following particular cases are considered:

(i) Binomial ($0 < \theta < 1$, $m \in \mathbb{N}$, m known, $y = 0, 1, 2, \ldots, m$): $\alpha(\theta) = -\log\{\theta/(1-\theta)\}$, $\zeta(\theta) = (1-\theta)^{-m}$, $d(y) = y$, $v(y) = \log\binom{m}{y}$; $\quad B_1(\theta) = B_2(\theta) = B_2^*(\theta) = 0$.

(ii) Negative binomial ($\theta > 0$, $\gamma > 0$, γ known, $y = 0, 1, 2, \ldots$): $\alpha(\theta) = -\log(\theta)$, $\zeta(\theta) = (1-\theta)^{-\gamma}$, $d(y) = y$, $v(y) = \log\binom{\gamma+y-1}{y}$;

$$B_1(\theta) = -\frac{\theta(1-\theta)}{\gamma}, \quad B_2(\theta) = \frac{\theta(1-\theta)(1-2\theta)}{\gamma^2}, \quad B_2^*(\theta) = -\frac{\theta(1-\theta)^2}{\gamma^2}.$$

(iii) Poisson ($\theta > 0$, $y = 0, 1, 2, \ldots$): $\alpha(\theta) = -\log(\theta)$, $\zeta(\theta) = \exp\{\theta\}$, $d(y) = y$, $v(y) = -\log(y!)$; $B_1(\theta) = B_2(\theta) = B_2^*(\theta) = 0$.

(iv) Truncated Poisson ($\theta > 0$, $y = 1,2,\ldots$): $\alpha(\theta) = -\log(\theta)$, $\zeta(\theta) = e^{\theta}(1-e^{-\theta})$, $d(y) = y$, $v(y) = -\log(y!)$;

$$B_1(\theta) = \frac{\theta(e^{-\theta}-1)\{\theta(e^{-2\theta}+e^{-\theta})+2e^{-2\theta}-2e^{-\theta}\}}{2(\theta e^{-\theta}+e^{-\theta}-1)^2},$$

$$\begin{aligned}
B_2(\theta) = &- \{24(\theta e^{-\theta}+e^{-\theta}-1)^5\}^{-1}\theta(e^{-\theta}-1)\{\theta^4(10e^{-6\theta}+2e^{-5\theta}+2e^{-4\theta}+10e^{-3\theta})\\
&+\theta^3(52e^{-6\theta}-44e^{-5\theta}-28e^{-3\theta}+20e^{-2\theta})+\theta^2(97e^{-6\theta}-217e^{-5\theta}+66e^{-4\theta}\\
&+134e^{-3\theta}-83e^{-2\theta}+3e^{-\theta})+\theta(52e^{-6\theta}-272e^{-5\theta}+520e^{-4\theta}-448e^{-3\theta}\\
&+164e^{-2\theta}-16e^{-\theta})-36e^{-6\theta}+156e^{-5\theta}-264e^{-4\theta}+216e^{-3\theta}-84e^{-2\theta}\\
&+12e^{-\theta}\},
\end{aligned}$$

$$\begin{aligned}
B_2^*(\theta) = &- \{24(\theta e^{-\theta}+e^{-\theta}-1)^5\}^{-1}\theta(e^{-\theta}-1)\{\theta^4(-2e^{-6\theta}+2e^{-5\theta}+2e^{-4\theta}-2e^{-3\theta})\\
&+\theta^3(-2e^{-6\theta}+16e^{-5\theta}-12e^{-4\theta}+8e^{-3\theta}-10e^{-2\theta})+\theta^2(19e^{-6\theta}-7e^{-5\theta}-18e^{-4\theta}\\
&-22e^{-3\theta}+31e^{-2\theta}-3e^{-\theta})+\theta(52e^{-6\theta}-116e^{-5\theta}+16e^{-4\theta}+128e^{-3\theta}\\
&-100e^{-2\theta}+20e^{-\theta})+48e^{-6\theta}-216e^{-5\theta}+384e^{-4\theta}-336e^{-3\theta}+144e^{-2\theta}\\
&-24e^{-\theta}\}.
\end{aligned}$$

(v) Logarithmic series ($0 < \theta < 1$, $y = 1,2,\ldots$): $\alpha(\theta) = -\log(\theta)$, $\zeta(\theta) = -\log(1-\theta)$, $d(y) = y$, $v(y) = -\log(y)$;

$$B_1(\theta) = \frac{\theta(1-\theta)\log(1-\theta)[2\{\log(1-\theta)\}^2+(\theta+2)\log(1-\theta)+2\theta]}{2\{\theta+\log(1-\theta)\}^2},$$

$$\begin{aligned}
B_2(\theta) = &- [24\{\theta+\log(1-\theta)\}^5]^{-1}\theta(1-\theta)\log(1-\theta)[(48\theta-24)\{\log(1-\theta)\}^6+(26\theta^2\\
&+164\theta-60)\{\log(1-\theta)\}^5+(14\theta^3+296\theta^2+36\theta-24)\{\log(1-\theta)\}^4+(3\theta^4\\
&+136\theta^3+312\theta^2-48\theta)\{\log(1-\theta)\}^3+(28\theta^4+276\theta^3)\{\log(1-\theta)\}^2+(60\theta^4\\
&+48\theta^3)\log(1-\theta)+24\theta^4],
\end{aligned}$$

$$\begin{aligned}
B_2^*(\theta) = &[24\{\theta+\log(1-\theta)\}^5]^{-1}\theta(1-\theta)\{\log(1-\theta)\}^2[(24\theta-24)\{\log(1-\theta)\}^5+(22\theta^2\\
&+52\theta-72)\{\log(1-\theta)\}^4+(16\theta^3+136\theta^2-120\theta-24)\{\log(1-\theta)\}^3+(3\theta^4\\
&+80\theta^3-72\theta)\{\log(1-\theta)\}^2+(20\theta^4+72\theta^3-72\theta^2)\log(1-\theta)+24\theta^4-24\theta^3].
\end{aligned}$$

(vi) Power series ($\theta > 0$, $a_y \geq 0$, $y = 0,1,2,\ldots$): $\alpha(\theta) = -\log(\theta)$, $\zeta(\theta) = \sum_{y=0}^{\infty} a_y\theta^y$, $d(y) = y$,

$v(y) = \log(a_y)$;

$$B_1(\theta) = -\frac{\theta(2g' + \theta g'')}{2(g + \theta g')^2},$$

$$\begin{aligned}
B_2(\theta) = &-\frac{\theta}{24(g + \theta g')^5}(3\theta^2 g^2 g^{iv} + 12g^2 g'' + 16\theta g^2 g''' + 216\theta g'^3 + 33\theta^4 g''^2 - 48gg'^2 \\
&- 26\theta^3 gg''g''' - 26\theta^4 g'g''g''' + 3\theta^4 g'^2 g^{iv} - 180\theta gg'g'' + 204\theta^2 g'^2 g'' + 108\theta^3 g'g''^2 \\
&- 36\theta^3 g'^2 g''' - 90\theta^2 gg''^2 + 6\theta^3 gg'g^{iv} - 20\theta^2 gg'g''' + 16\theta g^2 g''' - 36\theta^3 g'g'''),
\end{aligned}$$

$$\begin{aligned}
B_2^*(\theta) = &\frac{\theta}{24(g + \theta g')^5}(6\theta^3 gg'g^{iv} + 48\theta g'^3 - 72gg' + 8\theta^2 gg'g''' - 16\theta^3 gg''g''' - 16\theta^4 g'g''g''' \\
&+ 24g^2 g'' - 120\theta gg'g'' + 3\theta^4 g'^2 g^{iv} + 3\theta^2 g^2 g^{iv} - 12\theta^3 g'^2 g''' + 20\theta g^2 g''' - 66\theta^2 gg''^2 \\
&+ 24\theta^3 g'g''^2 + 36\theta^2 g'^2 g'' + 15\theta^4 g''^3),
\end{aligned}$$

where $g = g(\theta) = \mathrm{d}\log\zeta(\theta)/\mathrm{d}\theta$.

(vii) Zeta ($\theta > 0$, $y = 1, 2, 3, \ldots$): $\alpha(\theta) = \theta + 1$, $\zeta(\theta) = \mathrm{Zeta}(\theta + 1)$, $d(y) = \log(y)$, $v(y) = 0$;

$$B_1(\theta) = -\frac{g''}{2g'^2}, \quad B_2(\theta) = -\frac{33g''^3 - 26g'g''g''' + 3g'^2 g^{iv}}{24g'^5},$$

$$B_2^*(\theta) = \frac{15g''^3 - 16g'g''g''' + 3g'^2 g^{iv}}{24g'^5},$$

where Zeta$(\cdot)$ is the Riemann zeta-function, *i.e.*, $\mathrm{Zeta}(\theta) = \sum_{i=1}^{\infty} i^{-\theta}$ (see, *e.g.*, Patterson, 1988) and $g = g(\theta) = \mathrm{d}\log \mathrm{Zeta}(\theta + 1)/\mathrm{d}\theta$.

(viii) Non-central hypergeometric ($\theta > 0$, m_1, m_2, r known positive integers, $a = \max\{0, r - m_2\} \leq y \leq \min\{m_1, r\} = b$): $\alpha(\theta) = \theta$, $\zeta(\theta) = D_0(\theta)$, $d(y) = -y$, $v(y) = \log\{\binom{m_1}{y}\binom{m_2}{r-y}\}$;

$$B_1(\theta) = \frac{D_0(-D_0^2 D_3 + 3D_0 D_1 D_2 - 2D_1^3)}{2(D_0 D_2 - D_1)^2},$$

$$\begin{aligned}
B_2(\theta) = &\{24(D_0 D_2 - D_1^2)^5\}^{-1} D_0(-24D_1^9 - 94D_0^4 D_1^3 D_3^2 - 92D_0^2 D_1^6 D_3 - 75D_0^3 D_1^3 D_2^3 \\
&- 120D_0^2 D_1^5 D_2^2 + 108D_0 D_1^7 D_2 - 48D_0^5 D_2^3 D_3 + 144D_0^4 D_1 D_2^4 - 37D_0^3 D_1^5 D_4 \\
&- 3D_0^6 D_2^2 D_5 - 3D_0^4 D_1^4 D_5 + 193D_0^5 D_1 D_2 D_3^2 - 309D_0^4 D_1^2 D_2^2 D_3 + 350D_0^3 D_1^4 D_2 D_3 \\
&+ 26D_0^6 D_2 D_3 D_4 - 63D_0^5 D_1 D_2^2 D_4 + 100D_0^4 D_1^3 D_2 D_4 - 26D_0^5 D_1^2 D_3 D_4 \\
&+ 6D_0^5 D_1^2 D_2 D_5 - 33D_0^6 D_3^3),
\end{aligned}$$

$$\begin{aligned}
B_2^*(\theta) = &-\{24(D_0 D_2 - D_1^2)^5\}^{-1} D_0^3(-15D_0^4 D_3^3 + 71D_0^3 D_1 D_2 D_3^2 - 93D_0^2 D_1^2 D_2^2 D_3 \\
&+ 82D_0 D_1^4 D_2 D_3 + 16D_0^4 D_2 D_3 D_4 - 33D_0^3 D_1 D_2^2 D_4 + 50D_0^2 D_1^3 D_2 D_4 \\
&- 16D_0^3 D_1^2 D_3 D_4 - 3D_0^2 D_1^4 D_5 + 6D_0^3 D_1^2 D_2 D_5 - 26D_0^2 D_1^3 D_3^2 - 16D_1^6 D_3 \\
&- 51D_0 D_1^3 D_2^3 + 12D_1^5 D_2^2 - 18D_0^3 D_2^3 D_3 + 54D_0^2 D_1 D_2^4 - 17D_0 D_1^5 D_4 - 3D_0^4 D_2^2 D_5),
\end{aligned}$$

where $D_p = D_p(\theta) = \sum_{y=a}^{b} y^p \binom{m_1}{y}\binom{m_2}{r-y} \exp\{\theta y\}$, $p = 0, 1, 2, 3, 4$.

(ix) Maxwell ($\theta > 0$, $y > 0$): $\alpha(\theta) = (2\theta^2)^{-1}$, $\zeta(\theta) = \theta^3$, $d(y) = y^2$, $v(y) = \log(y^2\sqrt{2/\pi})$; $B_1(\theta) = -\theta/12$, $B_2(\theta) = \theta/288$, $B_2^*(\theta) = -\theta/288$.

(x) Gamma ($k > 0, \theta > 0$, $y > 0$):

(a) k known: $\alpha(\theta) = \theta$, $\zeta(\theta) = \theta^{-k}$, $d(y) = y$, $v(y) = (k-1)\log(y) - \log\{\Gamma(k)\}$; $B_1(\theta) = \theta k^{-1}$, $B_2(\theta) = \theta k^{-2}$, $B_2^*(\theta) = 0$;

(b) θ known: $\alpha(k) = -(k-1)$, $\zeta(k) = \theta^{-k}\Gamma(k)$, $d(y) = \log(y)$, $v(y) = -\theta y$;

$$B_1(k) = -\frac{\psi''(k)}{2\psi'(k)^2}, \quad B_2(k) = \frac{-33\psi''(k)^3 + 26\psi'(k)\psi''(k)\psi'''(k) - 3\psi'(k)^2\psi^{iv}(k)}{24\psi'(k)^5},$$

$$B_2^*(k) = \frac{15\psi''(k)^3 - 16\psi'(k)\psi''(k)\psi'''(k) + 3\psi'(k)^2\psi^{iv}(k)}{24\psi'(k)^5},$$

where $\Gamma(\cdot)$ and $\psi(\cdot)$ are the gamma and digamma functions, respectively.

(xi) Burr system of distributions ($\theta > 0$, $b > 0$, b known, $y > 0$): $\alpha(\theta) = \theta$, $\zeta(\theta) = c(\theta)/\theta$, $d(y) = -\log G(y)$, $v(y) = \log\{|d\log G(y)/dy|\}$; $B_1(\theta) = B_2(\theta) = \theta$, $B_2^*(\theta) = 0$, where the functions $c(\cdot)$ and $G(\cdot)$ are positive real-valued. Different choices for $c(\theta)$ and $G(y)$ lead to different distributions; see Burr (1942).

(xii) Rayleigh ($\theta > 0$, $y > 0$): $\alpha(\theta) = \theta^{-2}$, $\zeta(\theta) = \theta^2$, $d(y) = y^2$, $v(y) = \log(2y)$; $B_1(\theta) = -\theta/8$, $B_2(\theta) = \theta/128$, $B_2^*(\theta) = -\theta/128$.

(xiii) Pareto ($\theta > 0$, $k > 0$, k known, $y > k$): $\alpha(\theta) = \theta + 1$, $\zeta(\theta) = (\theta k^\theta)^{-1}$, $d(y) = \log(y)$, $v(y) = 0$; $B_1(\theta) = B_2(\theta) = \theta$, $B_2^*(\theta) = 0$.

(xiv) Weibull ($\theta > 0$, $\phi > 0$, ϕ known, $y > 0$): $\alpha(\theta) = \theta^{-\phi}$, $\zeta(\theta) = \theta^\phi$, $d(y) = y^\phi$, $v(y) = \log(\phi) + (\phi - 1)\log(y)$;

$$B_1(\theta) = \frac{\theta(1-\phi)}{2\phi^2}, \quad B_2(\theta) = -\frac{\theta(2\phi^3 - 9\phi^2 + 10\phi - 3)}{24\phi^4}, \quad B_2^*(\theta) = -\frac{\theta(2\phi^3 - 3\phi^2 - 2\phi + 3)}{24\phi^4}.$$

(xv) Power ($\theta > 0$, $\phi > 0$, ϕ known, $0 < y < \phi$): $\alpha(\theta) = 1 - \theta$, $\zeta(\theta) = \theta^{-1}\phi^\theta$, $d(y) = \log(y)$, $v(y) = 0$; $B_1(\theta) = B_2(\theta) = \theta$, $B_2^*(\theta) = 0$.

(xvi) Laplace ($\theta > 0$, $-\infty < k < \infty$, k known, $y > 0$): $\alpha(\theta) = \theta^{-1}$, $\zeta(\theta) = 2\theta$, $d(y) = |y - k|$, $v(y) = 0$; $B_1(\theta) = B_2(\theta) = B_2^*(\theta) = 0$.

(xvii) Extreme value ($-\infty < \theta < \infty$, $\phi > 0$, ϕ known, $-\infty < y < \infty$): $\alpha(\theta) = \exp\{\theta/\phi\}$, $\zeta(\theta) = \phi\exp\{-\theta/\phi\}$, $d(y) = \exp\{-y/\phi\}$, $v(y) = -y/\phi$; $B_1(\theta) = \phi/2$, $B_2(\theta) = B_2^*(\theta) = \phi/12$.

(xviii) Truncated extreme value ($\theta > 0$, $y > 0$): $\alpha(\theta) = \theta^{-1}$, $\zeta(\theta) = \theta$, $d(y) = \exp\{y\} - 1$, $v(y) = y$; $B_1(\theta) = B_2(\theta) = B_2^*(\theta) = 0$.

(xix) Lognormal ($\theta > 0$, $\mu > 0$, μ known, $y > 0$): $\alpha(\theta) = \theta^{-2}$, $\zeta(\theta) = \theta$, $d(y) = \{\log(y) - \mu\}^2/2$, $v(y) = -\log(y) + \{\log(2\pi)\}/2$; $B_1(\theta) = -\theta/4$, $B_2(\theta) = \theta/32$, $B_2^*(\theta) = -\theta/32$.

(xx) Normal ($\theta > 0$, $-\infty < \mu < \infty$, $-\infty < y < \infty$):

(a) μ known: $\alpha(\theta) = (2\theta)^{-1}$, $\zeta(\theta) = \theta^{1/2}$, $d(y) = (y-\mu)^2$, $v(y) = -\{\log(2\pi)\}/2$; $B_1(\theta) = B_2(\theta) = B_2^*(\theta) = 0$.

(b) θ known: $\alpha(\mu) = -\mu/\theta$, $\zeta(\mu) = \exp\{\mu^2/(2\theta)\}$, $d(y) = y$, $v(y) = -\{y^2 + \log(2\pi\theta)\}/2$; $B_1(\mu) = B_2(\mu) = B_2^*(\mu) = 0$.

(xxi) Inverse Gaussian ($\theta > 0$, $\mu > 0$, $y > 0$):

(a) μ known: $\alpha(\theta) = \theta$, $\zeta(\theta) = \theta^{-1/2}$, $d(y) = (y-\mu)^2/(2\mu^2 y)$, $v(y) = -\{\log(2\pi y^3)\}/2$; $B_1(\theta) = 2\theta$, $B_2(\theta) = 4\theta$, $B_2^*(\theta) = 0$.

(b) θ known: $\alpha(\mu) = \theta/(2\mu^2)$, $\zeta(\mu) = \exp\{-\theta/\mu\}$, $d(y) = y$, $v(y) = -\theta/(2y) + [\log\{\theta/(2\pi y^3)\}]/2$; $B_1(\mu) = B_2(\mu) = B_2^*(\mu) = 0$.

(xxii) McCullagh ($\theta > -1/2$, $-1 \le \mu \le 1$, μ known, $0 < y < 1$): $\alpha(\theta) = -\theta$, $\zeta(\theta) = 4^{-\theta}\mathrm{B}(\theta + 1/2, 1/2)$, $d(y) = \log[y(1-y)/\{(1+\mu)^2 - 4\mu y\}]$, $v(y) = -[\log\{y(1-y)\}]/2$;

$$B_1(\theta) = -\frac{\psi''(\theta+1/2) - \psi''(\theta+1)}{2\{\psi'(\theta+1/2) - \psi'(\theta+1)\}^2},$$

$$\begin{aligned} B_2(\theta) =& [24\{\psi'(\theta+1/2) - \psi'(\theta+1)\}^5]^{-1}[-33\{\psi''(\theta+1/2) - \psi''(\theta+1)\}^3 + 26\{\psi'(\theta \\ &+ 1/2) - \psi'(\theta+1)\}\{\psi''(\theta+1/2) - \psi''(\theta+1)\}\{\psi'''(\theta+1/2) - \psi'''(\theta+1)\} \\ &- 3\{\psi'(\theta+1/2) - \psi'(\theta+1)\}^2\{\psi^{iv}(\theta+1/2) - \psi^{iv}(\theta+1)\}], \end{aligned}$$

$$\begin{aligned} B_2^*(\theta) =& [24\{\psi'(\theta+1/2) - \psi'(\theta+1)\}^5]^{-1}[15\{\psi''(\theta+1/2) - \psi''(\theta+1)\}^3 - 16\{\psi'(\theta \\ &+ 1/2) - \psi'(\theta+1)\}\{\psi''(\theta+1/2) - \psi''(\theta+1)\}\{\psi'''(\theta+1/2) - \psi'''(\theta+1)\} \\ &+ 3\{\psi'(\theta+1/2) - \psi'(\theta+1)\}^2\{\psi^{iv}(\theta+1/2) - \psi^{iv}(\theta+1)\}], \end{aligned}$$

where $\mathrm{B}(\cdot,\cdot)$ is the beta function (see McCullagh, 1989).

(xxiii) Von Mises ($\theta > 0$, $0 < \mu < 2\pi$, μ known, $0 < y < 2\pi$): $\alpha(\theta) = -\theta$, $\zeta(\theta) = 2\pi I_0(\theta)$, $d(y) = \cos(y-\mu)$, $v(y) = 0$;

$$B_1(\theta) = -\frac{r''}{2r'^2}, \quad B_2(\theta) = -\frac{33r''^3 - 26r'r''r''' + 3r'^2r^{iv}}{24r'^5},$$

$$B_2^*(\theta) = \frac{15r''^3 - 16r'r''r''' + 3r'^2r^{iv}}{24r'^5},$$

where $I_\nu(\cdot)$ is the modified Bessel function of first kind and νth order, and $r = r(\theta) = I_0'(\theta)/I_0(\theta)$.

(xxiv) Generalized hyperbolic secant ($-\pi/2 \le \theta \le \pi/2$, $0 < y < 1$, $r > 0$, r known): $\alpha(\theta) = \theta$, $\zeta(\theta) = \pi\{\sec(\theta)\}^r$, $d(y) = -\pi^{-1} - \log\{y/(1-y)\}$, $v(y) = -(1/2)\log\{y/(1-y)\}$;

$$B_1(\theta) = -\frac{\sin(\theta)\cos(\theta)}{r}, \quad B_2(\theta) = \frac{\sin(\theta)\cos(\theta)\{10\cos(\theta)^2 - 3\}}{3r^2}, \quad B_2^*(\theta) = -\frac{\sin(\theta)\cos(\theta)^3}{3r^2}.$$

Bias correction for some other distributions can be obtained as special cases of the distributions introduced above. For the Bernoulli distribution, $B_1(\theta) = B_2(\theta) = B_2^*(\theta) = 0$ which follows from the binomial distribution with $m = 1$. The results for the geometric distribution are the same as those for the negative binomial with $\gamma = 1$. Also, the exponential distribution is a special case of the gamma distribution with $k = 1$, so is the chi-squared distributions ($\theta = 1/2$ and $k = \theta/2$). Note, however, that the expressions for the biases given for the gamma distribution must be multiplied by 2 when applied to the chi-squared distribution.

It is interesting to note that for some distributions $B_1(\theta)$, $B_2(\theta)$ and $B_2^*(\theta)$ do not depend on the value of the parameter θ, but this is not always the case. In some cases, the biases vary with θ; see Section 4 for more details. It is also noteworthy that the general expressions for $B_1(\theta)$, $B_2(\theta)$ and $B_2^*(\theta)$ in (5)–(7) are capable of generating both simple and complex expressions for different special cases.

It is possible to verify from first principles or by Taylor series expansion of the expression for the maximum likelihood estimate, that the corresponding formulas for $B_1(\theta)$ and $B_2(\theta)$ given here are correct for the following cases: binomial, Poisson, gamma (a), Pareto, Laplace, extreme value, truncated extreme value, lognormal, normal and inverse Gaussian.

3. Asymptotic expansions

In some cases, the evaluation of bias-corrected estimates requires the evaluation of unusual and long expressions. We shall now obtain approximations based on asymptotic expansions which do not require the computation of functions such as polygamma, Bessel or zeta. The expansions presented below were obtained using Mathematica (Wolfram, 1991) and Maple (Abell and Braselton, 1994).

In order to derive approximations for $B_1(\theta)$, $B_2(\theta)$ and $B_2^*(\theta)$ for the truncated Poisson distribution, we need to make use of the expansion $e^\theta = 1 + \theta + \theta^2/2 + \theta^3/6 + \theta^4/24 + \cdots$ Using this expansion, we obtain, for small θ,

$$B_1(\theta) = -\frac{\theta}{3} + \frac{2\theta^2}{9} - \frac{7\theta^3}{90} + O(\theta^4), \quad B_2(\theta) = \frac{2\theta}{9} - \frac{19\theta^2}{45} + \frac{149\theta^3}{405} + O(\theta^4),$$
$$B_2^*(\theta) = -\frac{\theta}{3} + \frac{56\theta^2}{135} - \frac{112\theta^3}{405} + O(\theta^4).$$

Next, we turn to the logarithmic series distribution. For small values of θ we have that

$$\log(1-\theta) = -\theta - \frac{\theta^2}{2} - \frac{\theta^3}{3} - \frac{\theta^4}{4} - \frac{\theta^5}{5} - \frac{\theta^6}{6} + O(\theta^7).$$

Using this result, we get that for the logarithmic series distribution with small θ

$$B_1(\theta) = -\frac{5\theta}{3} + \frac{19\theta^2}{18} + \frac{11\theta^3}{45} + O(\theta^4), \quad B_2(\theta) = \frac{23\theta}{9} - \frac{407\theta^2}{90} + \frac{59\theta^3}{162} + O(\theta^4),$$
$$B_2^*(\theta) = -\frac{7\theta}{3} + \frac{379\theta^2}{135} - \frac{53\theta^3}{324} + O(\theta^4).$$

Next, we consider the zeta distribution. Let

$$\gamma_j = \lim_{m\to\infty}\left\{\sum_{k=1}^{m}\frac{(\log k)^j}{k} - \frac{(\log m)^{j+1}}{j+1}\right\},$$

$j = 0, 1$, γ_0 being Euler's constant. We then have that

$$B_1(\theta) = \theta + 2(\gamma_0^2 + 2\gamma_1)\theta^3 + O(\theta^4), \quad B_2(\theta) = \theta + 12(\gamma_0^2 + 2\gamma_1)\theta^3 + O(\theta^4),$$
$$B_2^*(\theta) = -2(\gamma_0^2 + 2\gamma_1)\theta^3 + O(\theta^4),$$

for small values of θ. It is possible to simplify these approximations using Maple V (Abell and Braselton, 1994) as

$$B_1(\theta) \approx \theta + 0.375090\theta^3, \quad B_2(\theta) \approx \theta + 2.250560\theta^3, \quad B_2^*(\theta) \approx -0.375090\theta^3.$$

Consider now the gamma distribution (with θ known). For large values of k we have that

$$\psi'(k) = \frac{1}{k} + \frac{1}{2k^2} + \frac{1}{6k^3} - \frac{1}{30k^5} + \frac{1}{42k^7} - \frac{1}{30k^9} + \frac{5}{66k^{11}} - \frac{691}{2730k^{13}} + \cdots$$

Using this expansion we obtain, for large k,

$$B_1(k) = \frac{1}{2} - \frac{1}{24k^2} - \frac{1}{24k^3} + O(k^{-4}), \quad B_2(k) = -\frac{1}{24k} - \frac{1}{48k^2} - \frac{1}{288k^3} + O(k^{-4}),$$
$$B_2^*(k) = -\frac{1}{24k} - \frac{1}{48k^2} + \frac{23}{288k^3} + O(k^{-4}).$$

For small values of k, we obtain

$$B_1(k) = k - \frac{\pi^2}{3}k^3 + 5\,\mathrm{Zeta}(3)k^4 + O(k^5), \quad B_2(k) = k - 2\pi^2k^3 + 44\,\mathrm{Zeta}(3)k^4 + O(k^5),$$
$$B_2^*(k) = \frac{\pi^2}{3}k^3 - 11\,\mathrm{Zeta}(3)k^4 + O(k^5),$$

where $\mathrm{Zeta}(3) \approx 1.2020569$. By making use of the formula

$$\psi'(\theta + 1) - \psi'(\theta + 1/2) = 2\psi'(\theta) - 4\psi'(2\theta) - \frac{1}{\theta^2},$$

we obtain the following asymptotic expansions for the McCullagh distribution, for large θ:

$$B_1(\theta) = 2\theta + \frac{1}{2} + \frac{1}{8\theta^3} + O(\theta^{-4}), \quad B_2(\theta) = 4\theta + 1 + \frac{1}{4\theta} - \frac{1}{16\theta^2} - \frac{9}{32\theta^3} + O(\theta^{-4}),$$
$$B_2^*(\theta) = \frac{1}{4\theta} - \frac{1}{16\theta^2} - \frac{41}{32\theta^3} + O(\theta^{-4}).$$

It is noteworthy that these expansions are in agreement with the quasi-linear behavior of $B_1(\theta)$ and $B_2(\theta)$ shown in Figure 4 in the next section. It is also clear that $B_2^*(\theta)$ should be close to zero for large values of θ. Indeed, Figure 4 shows that the latter holds for all values (small and large) of θ. It is also possible to obtain expansions for $B_1(\theta)$, $B_2(\theta)$ and $B_2^*(\theta)$ for the McCullagh distribution when θ is small. However, the coefficients of such expansions involve long expressions. To save space, we replaced these espression by approximated values, thus obtaining

$$B_1(\theta) \approx 0.66638 + 1.64355\theta + 0.43163\theta^2 - 0.35751\theta^3,$$
$$B_2(\theta) \approx 1.20140 + 3.51507\theta + 1.18735\theta^2 - 1.66760\theta^3,$$
$$B_2^*(\theta) \approx -0.02503 - 0.01070\theta + 0.15110\theta^2 - 0.05311\theta^3.$$

For the von Mises distribution, $B_1(\theta)$, $B_2(\theta)$ and $B_2^*(\theta)$ are functions of modified Bessel functions of first kind. However, it is possible to design approximations which do not involve such functions. For large values of θ, it is possible to write an expansion for $r(\theta)$ by making use of the expansions in Abramowitz and Stegun (1970, 9.7.1), and then to use this result to obtain, for large θ,

$$B_1(\theta) = 2\theta - \frac{1}{2} + \frac{15}{16\theta^2} + \frac{39}{8\theta^3} + O(\theta^{-4}), \quad B_2(\theta) = 4\theta - 1 - \frac{3}{4\theta} - \frac{3}{\theta^2} - \frac{171}{16\theta^3} + O(\theta^{-4}),$$
$$B_2^*(\theta) = -\frac{3}{4\theta} - \frac{27}{4\theta^2} - \frac{765}{16\theta^3} + O(\theta^{-4}).$$

Using equation (3.4.46) in Mardia (1972) we obtain that, for small θ,

$$r(\theta) = \frac{\theta}{2}\left\{1 - \frac{\theta^2}{8} + \frac{\theta^4}{48} + \frac{11\theta^6}{3072} + \frac{19\theta^8}{30720} - \cdots\right\}.$$

Therefore, we obtain the following expansions for $B_1(\theta)$, $B_2(\theta)$ and $B_2^*(\theta)$

$$B_1(\theta) = \frac{3\theta}{4} + \frac{7\theta^3}{48} + O(\theta^5), \quad B_2(\theta) = \frac{19\theta}{16} + \frac{505\theta^3}{768} + O(\theta^5), \quad B_2^*(\theta) = -\frac{\theta}{4} - \frac{53\theta^3}{768} + O(\theta^5).$$

Notice that these approximations do not require the evaluation of Bessel functions.

4. Graphical analysis

It was shown in Section 2 that for many special cases $B_1(\theta)$, $B_2(\theta)$ and $B_2^*(\theta)$ vary with the unknown parameter θ. In this section we give plots of $B_1(\theta)$, $B_2(\theta)$ and $B_2^*(\theta)$ against θ in order to examine how these three quantities vary with θ. The following distributions are considered: truncated Poisson, logarithmic series, gamma (with known θ), McCullagh, von Mises and generalized hyperbolic secant (with $r = 1$). A plot of $B_1(\theta)$, $B_2(\theta)$ and $B_2^*(\theta)$ against θ is given for each of these distributions. These plots are given in Figures 1 through 6, respectively.

[Figures 1 through 6 near here]

It is clear from Figures 1 to 6 that: (i) the second and third order biases and $B_2^*(\theta)$ can vary substantially depending on the true value of the parameter θ; (ii) except for the gamma distribution, the behavior of $B_2(\theta)$ and that of $B_2^*(\theta)$ are quite different; for the McCullagh distribution, for example, $B_2^*(\theta)$ is nearly zero for all values of θ whereas $B_2(\theta)$ grows almost linearly with θ; (iii) in some cases, $B_1(\theta)$, $B_2(\theta)$ and $B_2^*(\theta)$ display rather interesting behavior as θ changes, as for instance in the case of the generalized hyperbolic secant distribution, where all three quantities show periodic fluctuation patterns.

It is common practice to choose among competing estimators using the mean squared error as a deciding criterion. It is then important to look at Δ, which gives the difference between the mean squared errors of $\hat{\theta}$ and $\tilde{\theta}_2$ to order n^{-2}. That is, to order n^{-2}, $\Delta > 0$ when $\text{MSE}(\hat{\theta}) > \text{MSE}(\tilde{\theta}_2)$, and vice-versa. For some distributions, the expression for Δ is quite simple. For example, for the geometric distribution, $\Delta = -\theta(1-\theta^2)(2-5\theta)$, and hence $\Delta < 0$ if $\theta < 0.4$, $\Delta = 0$ if $\theta = 0.4$ and $\Delta > 0$ if $\theta > 0.4$. Also, for the inverse Gaussian (with known mean μ), $\Delta = 12\theta^2$ which is positive for all values of θ; for the Pareto and power distributions, $\Delta = 3\theta^2$ which again is positive for all values of θ; for the log-normal distribution (with known μ), $\Delta = -3\theta^2/16$, thus being negative for all values of θ. The expression for Δ, however, can be quite complex, and in such cases a graphical analysis can be helpful. The idea is to plot Δ against θ in order to examine which regions of the parameter space correspond to positive, zero and negative values of Δ. This is done in Figures 7 through 10. Here we plot Δ against θ for the following distributions: truncated Poisson, gamma (with θ known), McCullagh and generalized hyperbolic secant.

[Figures 7 through 10 near here]

Figure 7 shows that for the truncated Poisson distribution, the mean squared error of the bias-corrected estimate is smaller than the mean squared error of the maximum likelihood estimate, up to order n^{-2}, for (approximately) $\theta > 1.4$, and larger for (appproximately) $\theta < 1.4$. Figure 8 shows that for the gamma distribution Δ is negative only for small values of k. For the McCullagh distribution, Figure 9 shows that, to third order, $\tilde{\theta}_2$ dominates $\hat{\theta}$ in terms of mean sequared error. Also, Δ displays an interesting periodic behavior as a function of θ for the generalized hyperbolic secant distribution, being negative when θ is close to 0, and postive when θ is close to $-\pi/2$ or $\pi/2$.

5. Simulation results

This section gives some Monte Carlo simulation results comparing the finite-sample performance of the maximum likelihood estimator $\hat{\theta}$ and its bias-corrected version $\tilde{\theta}_2$. We consider an inverse Gaussian distribution with known mean μ, and focus on the estimation of the precision parameter, θ. The experiment was performed by setting $\mu = 1$, $\theta = 1, 5, 10$, and using 10,000 replications. Random numbers were generated using the algorithm outlined in Devroye (1986, pp.148-149). The estimated means and mean squared errors of $\hat{\theta}$ and $\tilde{\theta}_2 = \hat{\theta} - 2\hat{\theta}/n$ are given in Table 1.

The figures in Table 1 show that $\tilde{\theta}_2$ outperforms the maximum likelhood estimator $\hat{\theta}$ both in terms of smaller biases and smaller mean squared errors.

The next simulation experiment involves the estimation of the parameter θ in a Weibull distribution when the parameter ϕ is known. In Tables 2 and 3 we give results for $\theta = 5, 10$ and $\phi = 0.5, 3, 7$ obtained from 10,000 replications.

The simulation results for the Weibull case show that the bias of $\hat{\theta}$ tend to be larger when ϕ is small. In all cases, the bias of $\tilde{\theta}_2$ was nearly zero, and its mean squared error was no larger than the mean squared of the maximum likelihood estimate.

In the final simulation experiment, we consider a logarithmic series distribution with parameter

Table 1. Simulation results I

inverse Gaussian distribution, $\mu = 1$					
		$\hat{\theta}$		$\bar{\theta}_2$	
θ	n	mean	MSE	mean	MSE
1	5	1.68	5.28	1.01	1.74
	10	1.24	0.54	1.00	0.31
	15	1.15	0.28	1.00	0.19
	20	1.11	0.16	1.00	0.12
	25	1.08	0.12	1.00	0.10
	30	1.07	0.09	1.00	0.08
5	5	8.54	381.77	5.13	137.45
	10	6.24	14.97	5.00	8.59
	15	5.73	6.38	4.97	4.39
	20	5.54	3.99	4.99	2.99
	25	5.44	3.01	5.00	2.39
	30	5.37	2.38	5.01	1.95
10	5	16.68	442.72	10.01	143.33
	10	12.60	59.17	10.08	33.55
	15	11.48	25.53	9.95	17.52
	20	11.17	17.50	10.06	13.06
	25	10.85	11.92	9.99	9.47
	30	10.71	9.21	9.99	7.59

Table 2. Simulation results II

Weibull distribution, $\theta = 5$					
		$\hat{\theta}$		$\bar{\theta}_2$	
ϕ	n	mean	MSE	mean	MSE
0.5	5	6.00	31.90	5.20	30.94
	10	5.48	12.27	5.03	12.04
	15	5.34	7.84	5.03	7.73
	20	5.23	5.61	4.99	5.55
	25	5.19	4.39	5.00	4.35
	30	5.14	3.61	4.98	3.59
3.0	5	4.89	0.57	5.00	0.56
	10	4.94	0.28	4.99	0.28
	15	4.96	0.18	5.00	0.18
	20	4.97	0.14	5.00	0.14
	25	4.98	0.11	5.00	0.11
	30	4.99	0.09	5.00	0.09
7.0	5	4.94	0.11	5.00	0.11
	10	4.97	0.05	5.00	0.05
	15	4.98	0.04	5.00	0.03
	20	4.98	0.03	5.00	0.03
	25	4.99	0.02	5.00	0.02
	30	4.99	0.02	5.00	0.02

Table 3. Simulation results III

Weibull distribution, $\theta = 10$					
		$\hat{\theta}$		$\dot{\theta}_2$	
ϕ	n	mean	MSE	mean	MSE
0.5	5	11.84	124.31	10.24	120.97
	10	11.00	49.80	10.10	48.80
	15	10.69	32.09	10.07	31.62
	20	10.48	22.96	10.00	22.73
	25	10.45	18.13	10.07	17.93
	30	10.35	14.59	10.03	14.46
3.0	5	9.78	2.27	10.00	2.22
	10	9.89	1.13	10.00	1.12
	15	9.94	0.76	10.01	0.76
	20	9.93	0.56	9.98	0.55
	25	9.95	0.45	9.99	0.45
	30	9.97	0.36	10.00	0.36
7.0	5	9.88	0.45	10.00	0.43
	10	9.94	0.22	10.01	0.21
	15	9.96	0.14	10.00	0.14
	20	9.97	0.11	10.00	0.10
	25	9.98	0.08	10.00	0.08
	30	9.98	0.07	10.00	0.07

θ. We used the following approximation for the maximum likelihood estimate of θ (Birch, 1963):

$$\hat{\theta} \approx 1 - \left[1 + \left\{\left(\frac{5}{3} - \frac{1}{16}\log(\bar{y})\right)(\bar{y} - 1) + 2\right\}\log(\bar{y})\right]^{-1}.$$

Table 4 presents the resuts for $\theta = 0.2, 0.5, 0.7, 0.9$ obtained from 10,000 replications.

Again, the third-order bias-corrected maximum likelihood estimator $\tilde{\theta}_2$ outperformed the uncorrected MLE in the sense that it displayed smaller bias. The mean squared error of both estimators were nearly the same in all cases, being undistinguishable up to the second decimal most of the time.

Acknowledgements

The partial financial support of CNPq/Brazil is gratefully acknowledged.

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Table 4. Simulation results IV

logarithmic series distribution, $\theta = 0.2, 0.5, 0.7, 0.9$					
		$\hat{\theta}$		$\bar{\theta}_2$	
θ	n	mean	MSE	mean	MSE
0.2	5	0.16	0.04	0.23	0.04
	10	0.18	0.02	0.21	0.02
	15	0.18	0.02	0.20	0.02
	20	0.19	0.01	0.20	0.01
	25	0.19	0.01	0.20	0.01
	30	0.19	0.01	0.20	0.01
0.5	5	0.41	0.07	0.53	0.06
	10	0.45	0.04	0.51	0.03
	15	0.47	0.02	0.50	0.02
	20	0.48	0.02	0.50	0.02
	25	0.48	0.01	0.50	0.01
	30	0.48	0.01	0.50	0.01
0.7	5	0.60	0.06	0.72	0.06
	10	0.65	0.03	0.70	0.02
	15	0.67	0.02	0.70	0.01
	20	0.67	0.01	0.70	0.01
	25	0.68	0.01	0.70	0.01
	30	0.68	0.01	0.70	0.01
0.9	5	0.83	0.03	0.89	0.03
	10	0.86	0.01	0.90	0.01
	15	0.88	0.00	0.90	0.00
	20	0.88	0.00	0.90	0.00
	25	0.89	0.00	0.90	0.00
	30	0.89	0.00	0.90	0.00

Figure 1: $B_1(\theta)$, $B_2(\theta)$ and $B_2^*(\theta)$, Truncated Poisson

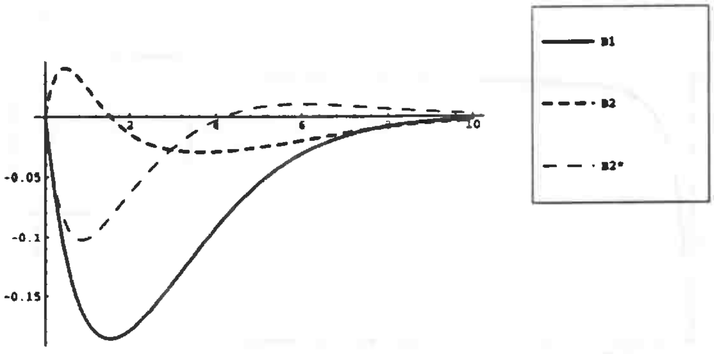


Figure 2: $B_1(\theta)$, $B_2(\theta)$ and $B_2^*(\theta)$, Logarithmic Series

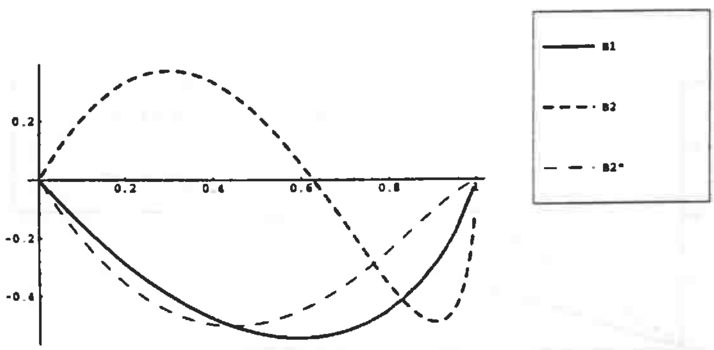


Figure 3: $B_1(\theta)$, $B_2(\theta)$ and $B_2^*(\theta)$, Gamma

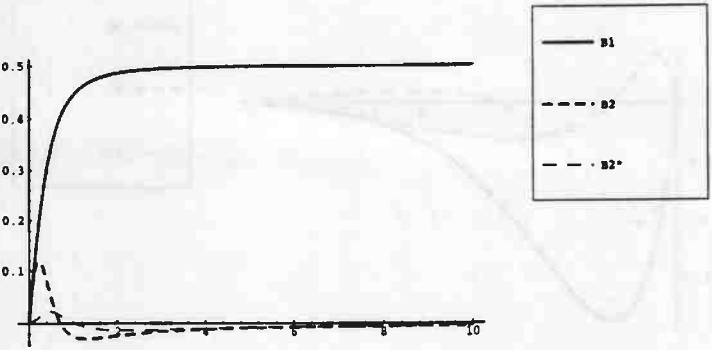


Figure 4: $B_1(\theta)$, $B_2(\theta)$ and $B_2^*(\theta)$, McCullagh

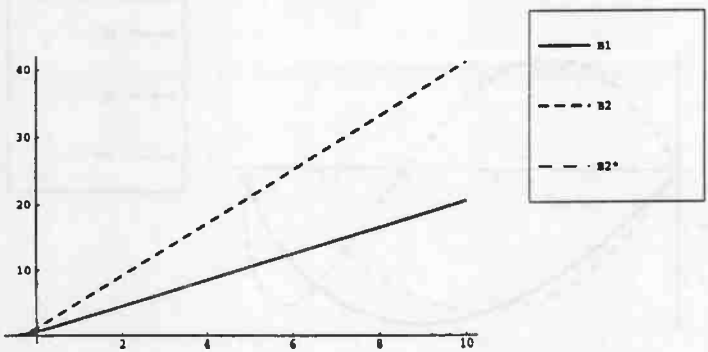


Figure 5: $B_1(\theta)$, $B_2(\theta)$ and $B_2^*(\theta)$, Von Mises

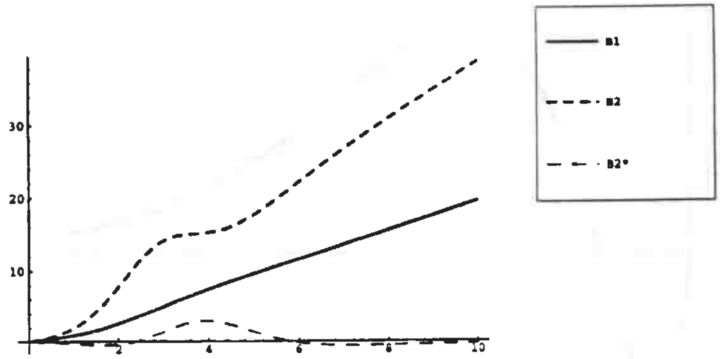


Figure 6: $B_1(\theta)$, $B_2(\theta)$ and $B_2^*(\theta)$, Generalized Hyperbolic Secant

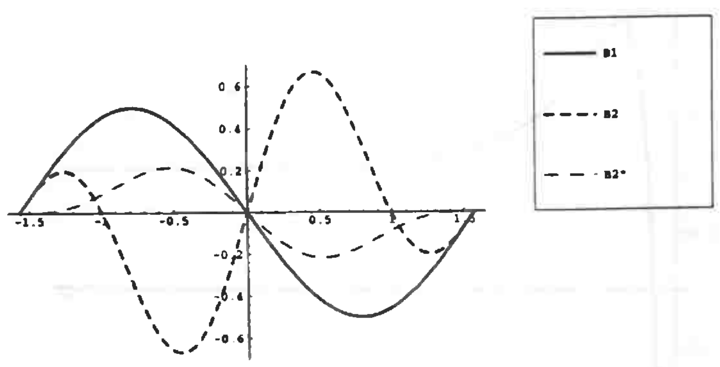


Figure 7: Δ, Truncated Poisson

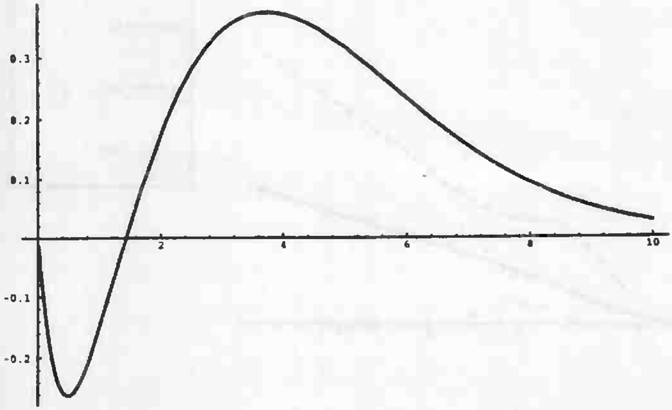


Figure 8: Δ, Gamma

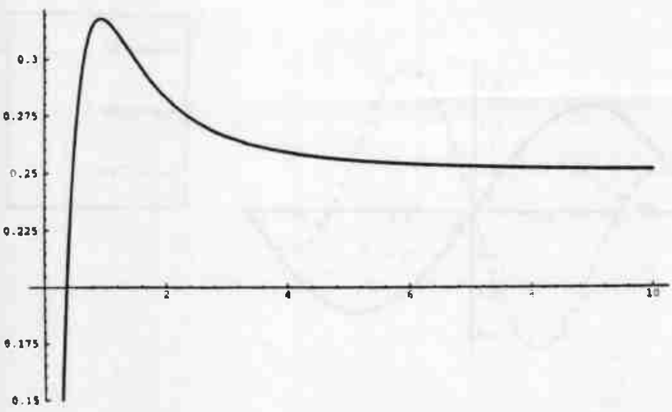


Figure 9: Δ, McCullagh

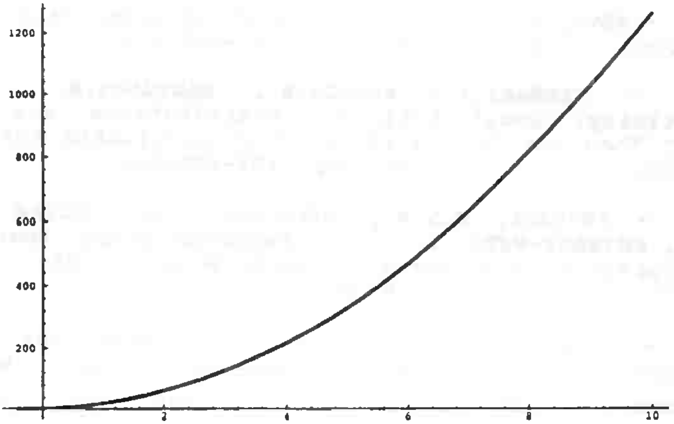
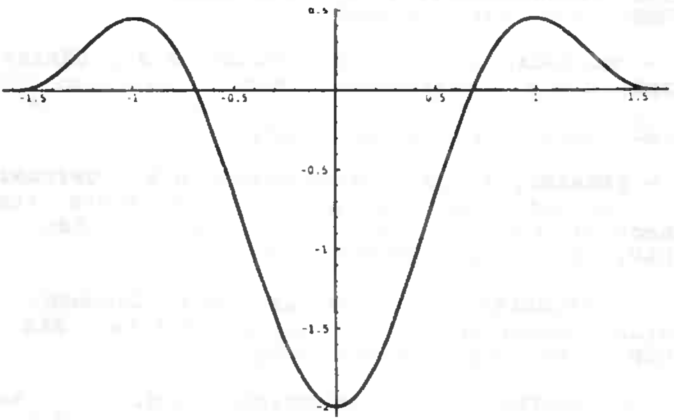


Figure 10: Δ, Generalized Hyperbolic Secant



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Departamento de Estatística
IME-USP
Caixa Postal 66.281
05389-970 - São Paulo, Brasil