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THE QUANTIZED UNIVERSAL ENVELOPING ALGEBRAS $U_q(iso(N))$, $U_q(e(3,1))$ AND $U_q(e(N))$ AND THE REPRESENTATION THEORY FOR $U_q(e(3))$

Sebastian Sachse & Ralf Weixler

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UNIVERSIDADE DE SÃO PAULO
INSTITUTO DE MATEMÁTICA E ESTATÍSTICA
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Sebastian Sachse
Departamento de Matemática Aplicada, Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, Butantã, S.P., Brazil

Ralf Weixler

Sektion Physik der Universität München
Lehrstuhl Professor Wess, Theresienstraße 37
D-80333 München , Germany

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Sebastian Sachse [†]

Departamento de Matemática Aplicada, Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, Bulantã, S.P., Brazil

Ralf Weixler

Sektion Physik der Universität München
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Abstract

We construct the Hopf algebras of $U_q(iso(N))$and $U_q(e(3,1))$ as regular functionals on their function algebras, examine the possible $*$-involutions on these objects, derive the quantized euclidean algebras $U_q(e(N))$ and finally give the irreducible representations of $U_q(e(3))$ depending on two parameters which fix the value of the Casimir operators.

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1 Introduction

In the framework of quantum groups (for conventions used in this article see [1]) there have been several suggestions for a generalization of euclidean symmetry in N dimensions [2, 3] and the Poincaré algebra [4]. For all the deformations with a real deformation parameter two main obstacles persist. The first one consists in a scalar operator which invokes dilatations and seems to be indispensable at least for the representation theory [5]. The second problem lies in the doubling of space directions as soon as a star structure of the Hopf algebra is considered. (For an unimodular parameter the situation seems not to to be any better since even for the homogeneous parts no Hilbert space representation could be found [6].)

Both bugs are connected to the coproduct structure and could be neglected on a purely algebraic level. However this would not allow to consider other than single particle systems. Hence there has been devoted some work of curing the difficulties by modifying the $*$-Hopf algebra structure. This work has just been started [5, 3, 2],but is not yet completely convincing.

We as well have only to suggest our personal point of view which might give an idea how to proceed towards a solution.

In the second part we summarize the Hopf algebra structure of the function algebras A^I of q-euclidean groups $ISO_q(N)$. Then we construct the universal enveloping algebras $U_q(iso(N))$ dual to A^I as its regular functionals. Notice that only with additional conditions, we have the identity $A^{I''} = A^I$ ($'$ designates the dual object). This can be obtained by taking the Pontryagin dual of a C^*-algebra (see as example [7]). Therefore Hilbert space representations of $ISO_q(3)$ have been given in [8]. However we do not proceed here in this direction but rather perform the dualisation on a purely algebraic level by considering left invariant operators on the function algebra. This corresponds to the well known fact that left invariant vector fields on a Lie group generate its Lie algebra.

In part three we discuss the Casimir elements of the universal enveloping algebras.

In the last section we give the irreducible Hilbert space representations of $U_q(e(3))$. As we are dealing with a noncompact algebra we necessarily obtain faithfull representations which are infinite dimensional. Like in conventional space-time, particles existing in such deformed structures should be attached to such representations. In this respect these representations have some significance for the behaviour of quantum systems in q-deformed euclidean space (see for instance [9]).

2 The $*$-Hopf Algebra Structure of Inhomogeneous Quantum Groups

2.1 Function Algebras

To construct $U_q(iso(N))$ and the q-Poincaré algebra we start with the function algebras $SO_q(N)$ and $SO_q(3,1)$. The noncommutative algebra structure is controlled by an $\hat{R}$-matrix fulfilling the Quantum Yang–Baxter equation $\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}$. We refer to $\hat{R}$-matrices for the Series B_n and C_n in their standard form given in [1] or for $SO_q(N)$ in [10]. They are defined by their projector decomposition making use of the antisymmetrizer $\hat{A}_q{}^{ij}{}_{kl}$, the symmetrizer $\hat{S}_q{}^{ij}{}_{kl}$ and the trace projector

$\hat{T}_q^{ij}{}_{kl} \propto C^{ij}C_{kl}$ with the metric C_{ij}.

$$\hat{R}^{ij}{}_{kl} = q\hat{S}_q^{ij}{}_{kl} - q^{-1}\hat{A}_q^{ij}{}_{kl} + q^{1-N}\hat{T}_q^{ij}{}_{kl} \tag{1}$$

The algebra relations for the generators $M^i{}_j$ of the unital $\mathbb{C}$-algebra $\mathcal{A}$ are expressed by the invariance of these projectors:

$$\hat{R}^{ij}{}_{j'i'}M^{j'}{}_{j''}M^{i'}{}_{i''} = M^i{}_{i'}M^j{}_{j'}\hat{R}^{i'j'}{}_{j''i''} \tag{2}$$

and

$$C_{ij}M^i{}_{i'}M^j{}_{j'} = C_{i'j'}1. \tag{3}$$

In order to obtain inhomogeneous quantum groups $ISO_q(N)$ the set of generators has to be enlarged not only by the coordinate functions x_i obeying the relations $x_ix_j\hat{A}_q^{ij}{}_{kl}$ but by an invertible scaling operator w as well. Its existence is required by consistency of the comultiplication. The additional algebra relations of the extended Hopf algebra $\mathcal{A}^I$ are [11, 12]:

$$\begin{array}{llll} (i) & x_iM^j{}_k = M^j{}_lx_m\hat{R}^{lm}{}_{ik} & (iv) & wM^i{}_j = M^i{}_jw \\ (ii) & \bar{w}w = 1 & (v) & \bar{w}x_i = qx_i\bar{w} \\ (iii) & \bar{w}M^i{}_j = M^i{}_j\bar{w} & (vi) & wx_i = \frac{1}{q}x_iw. \end{array} \tag{4}$$

The comultiplication $\Phi : \mathcal{A}^I \to \mathcal{A}^I \otimes \mathcal{A}^I$, counit $e : \mathcal{A}^I \to \mathbb{C}$ and the antipode $\kappa : \mathcal{A}^I \to \mathcal{A}^I$ are very easily given in matrix notation.
With

$$M^I = \begin{pmatrix} wM & 0 \\ \mathbf{x} & 1 \end{pmatrix} \tag{5}$$

we get

$$\Phi(M^I) = M^I\dot{\otimes}M^I \tag{6}$$

and

$$e(M^I) = \begin{pmatrix} E & 0 \\ 0 & 1 \end{pmatrix} \tag{7}$$

with the unity matrix E.
The antipode is given as:

$$\kappa(M^I) = \begin{pmatrix} \kappa(M)\bar{w} & 0 \\ -\mathbf{x}\kappa(M)\bar{w} & 1 \end{pmatrix}. \tag{8}$$

2.2 Universal Enveloping Algebras

The connection to the enveloping algebra is being made by dualising $\mathcal{A}^I$. We want to summarize the results from [2] which are analogous to those of [13].

The Algebra of generators $\{L^{\pm i}{}_j, p^i, h\}, 1 \le i,j \le N$, is the Hopf algebra of linear functionals of the corresponding inhomogeneous quantum group with comultiplication Δ, counit ε and antipode S. The algebraic relations are:

$$\begin{array}{llll} (i) & \hat{R}^{\pm ij}{}_{i'j'} L^{\pm j'}{}_n L^{\pm i'}{}_m = L^{\pm j}{}_{j'} L^{\pm i}{}_{i'} \hat{R}^{\pm i'j'}{}_{mn}, & (iv) & \hat{A}_q^{ij}{}_{kl} p^l p^k = 0, \\ (ii) & \hat{R}^{\pm ij}{}_{i'j'} L^{+j'}{}_n L^{-i'}{}_m = L^{-j}{}_{j'} L^{+i}{}_{i'} \hat{R}^{\pm i'j'}{}_{mn}, & (v) & h L^{\pm i}{}_j = L^{\pm i}{}_j h, \\ (iii) & L^{\pm i}{}_j p^k = \hat{R}^{\mp ki}{}_{lm} p^m L^{\pm l}{}_j, & (vi) & hp = q^{-1} ph. \end{array} \tag{9}$$

The coproduct is:

$$\begin{aligned} \Delta L^{\pm i}{}_k &= L^{\pm i}{}_j \otimes L^{\pm j}{}_k, \qquad \Delta h = h \otimes h \\ \Delta p^i &= L^{-i}{}_j \otimes p^j + p^i \otimes h. \end{aligned} \tag{10}$$

Counit ε and anitpode S follow from the axioms of a Hopf algebra $\mathcal{H}$. For each $a \epsilon \mathcal{H}$

$$(\varepsilon \otimes \mathrm{id})\Delta a = (\mathrm{id} \otimes \varepsilon)\Delta a = a \tag{11}$$

leads to:

$$\varepsilon(L^{\pm i}{}_j) = \delta^i_j, \quad \varepsilon(p) = 0, \quad \varepsilon(h) = 0. \tag{12}$$

The antipode is determined by:

$$\nabla(S \otimes \mathrm{id})\Delta a = \nabla(\mathrm{id} \otimes S)\Delta a = \eta\varepsilon(a), \tag{13}$$

where ∇ designates the algebra multiplication, to be:

$$S(L^{\pm i}{}_l) = C_{lk} L^{\pm k}{}_j C^{ji}, \; S(h) = h^{-1}, \; S(p^i) = -S(L^{-i}{}_j) p^j h^{-1}. \tag{14}$$

The dualisation for the functions on the group can be checked to be:

$$L^{\pm i}{}_j \left(M^k{}_l\right) = \hat{R}^{\pm ik}{}_{lj}, \; p^i(M^k{}_l) = 0, \; h(M^i{}_j) = \delta^i_j, \tag{15}$$

for the 'coordinates' may be chosen

$$L^{\pm i}{}_j(x_m) = 0, \; p^i(x_j) = \delta^i_j, \; h(x_i) = 0, \tag{16}$$

and for the scalar generator it is:

$$L^{\pm i}{}_j(w) = \delta^i_j, \; p^i(w) = 0, \; h(w) = q, \tag{17}$$

obeying relations like for instance:

$$\begin{aligned} hp^i(x_n) &= h(1)p^i(x_n) + h(x_m)p^i(wM^m{}_n) \\ = q^{-1}p^i h(x_n) &= q^{-1}(p^i(1)h(x_n) + p^i(x_m)h(w)h(M^m{}_n)). \end{aligned} \tag{18}$$

We designate these Hopf algebras with $U_q(A')$ with $A' \epsilon \{ISO(N), ISO(3,1)\}$. The necessity of the scalar generator h in the coproduct of the momenta p^i is due to relation (9)(*iv*) which together with $hp^i = q^\alpha p^i h$ gives

$$\begin{aligned} \delta(p^l p^k)\hat{A}_q^{ij}{}_{kl} &= (L^{-l}{}_m \otimes p^m)(L^{-k}{}_n \otimes p^n)\hat{A}_q^{ij}{}_{kl} \\ &+ \left((L^{-l}{}_m \otimes p^m)(p^k \otimes h) + (p^l \otimes h)(L^{-k}{}_n \otimes p^n)\right)\hat{A}_q^{ij}{}_{kl} \\ &+ (p^l \otimes h)(p^k \otimes h)\hat{A}_q^{ij}{}_{kl}. \end{aligned} \quad (19)$$

The last term is vanishing by (9)(*iv*) the first by an $L^{\pm i}{}_j$-invarianz of the projectors. The first part of the remaining expression together with (9)(*iii*) provides

$$\begin{aligned} &(p^{l'} \otimes h)(L^{-k'}{}_m \otimes p^m)\hat{A}_q^{ij}{}_{kl}\hat{R}^{kl}{}_{k'l'}q^{-\alpha} \\ &= -(p^l \otimes h)(L^{-k}{}_m \otimes p^m)q^{-\alpha-1}\hat{A}_q^{ij}{}_{kl}, \end{aligned} \quad (20)$$

and hence $\alpha = -1$.
For the rest of this section let us consider compact quantum groups A only. Following the arguements of [16, 15, 17] we define a vector operator of the contragradient corepresentation to be a set of n operators, n being the dimension of a corepresentation $u_{ij}^{(\alpha)}$ of A, which is carrying a left regular right coaction given by $\tau \circ (\kappa \otimes id) \circ \delta_A$. Let $u = \sum_\alpha u_{ij}^\alpha$ be the direct sum of irreducible matrix representations of A in the sense defined by [14], then:

$$\mathrm{Ad}_u(p^k) = u^*(p^k)u = p^l \otimes \kappa(u_{kl}) \quad (21)$$

and together with (15) we obtain the left A'-module structure

$$L^{\pm i}{}_j(p^k) = L^{\pm i}{}_j(\kappa(M^k{}_l))p^l = \hat{R}^{\mp ki}{}_{jm}p^m. \quad (22)$$

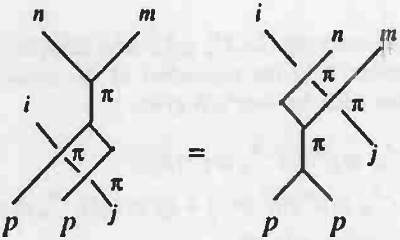
This envokes the mixed commutation relation

$$L^{\pm i}{}_j(p^k a) = \hat{R}^{\mp ki}{}_{nm}p^m L^{\pm n}{}_j(a), \quad a \in M \in {}_{A'}\mathcal{M}. \quad (23)$$

Vector operators of the fundamental corepresentation form comodul algebras by either q-symmetric or q-antisymmetric commutation relations. For this reason the algebraic relations (9) (*iii*) and (*iv*) are compatible. This can be seen from

$$\begin{aligned} L^{\pm i}{}_j(p^k p^l)\hat{A}_q^{nm}{}_{lk} &= p^k p^l \hat{R}^{\mp k'i}{}_{i'k}\hat{R}^{\mp l'i'}{}_{jl}\hat{A}_q^{nm}{}_{l'k'} \\ &= p^k p^l \hat{A}_q^{n'm'}{}_{lk}\hat{R}^{\mp mi}{}_{i'm'}\hat{R}^{\mp ni'}{}_{jn'} = 0. \end{aligned} \quad (24)$$

The "pulling through" of the projector follows if decomposing it into $\hat{R}$-matrix, singlet-projector $\hat{T}_q^{ij}{}_{kl}$ and identity and the relations of the BWM-algebra (60). This may be expressed in the suggestive graphical form:



(25)

We have attached the symbol π to the objects when we want to indicate transposition of the upper and of the lower indices. Without this notation a graphical calculation in A' is akward. The same rules of graphical calculations given in the appendix for untransposed matrices are valid for the "transposed" objects as well. Therefore we will omit the symbol π.
Remarks:

- The generators l of $U_q(iso(N))$ are shown to be left invariant on the corresponding function algebra A^I by the action $D_l : A^I \to A^I$:

$$\mathrm{D}_l(a) = (\mathrm{id} \otimes l)\Phi(a), \qquad a \in A^I,$$

  since it obeys:

$$(\mathrm{id} \otimes \mathrm{D}_l)\Phi(a) = \Phi \mathrm{D}_l(a), \quad \forall l \in U_q(iso(N)).$$

- It is important to notice that for the algebraic part of inhomogeneous enveloping algebras the scalar operator can be omitted. It becomes indispensable as soon as we want to turn $U_q(A^I)$ into a Hopf algebra.

2.3 Complex Conjugation

There are several options in defining a $*$-structure on a Hopfalgebra. The most convenient form is an antilinear, antimultiplicative involution which is in addition a coalgebra homomorphism [11, 12].
For the function algebras of euclidean type it is given as follows:

$$w^* = \bar{w}\,,\ (M^i{}_j)^* = \kappa(M^j{}_i)\,,\ (x_i)^* = \bar{x}^i. \tag{26}$$

The coordinate functions can't be chosen to be real. The $*$-operation on the function algebra A induces naturally a complex structure on its dual space A' which is formed by the elements of the universal envelop via:

$$\mathcal{F}^*(a) = \overline{\mathcal{F}(\kappa^{-1}(a^*))}, \qquad \text{for} \quad a \in A,\ \mathcal{F} \in A^*. \tag{27}$$

(In [14, 18] there has been used κ instead of κ^{-1} which are completetly isomorphic.) Thus we obtain:

$$(L^{\pm i}{}_j)^* = S(L^{\mp j}{}_i) \tag{28}$$

Then from the coproduct of p^i it becomes evident, that as well for the momenta no reality condition is possible in a natural way. We have to admit conjugated momenta

$$\overline{p}_i := (p^i)^*, \tag{29}$$

with coproduct, counit and antipode

$$\begin{aligned} \Delta\overline{p}_i &= S(L^{+j}{}_i) \otimes \overline{p}_j + \overline{p}_i \otimes h, \\ \varepsilon(\overline{p}_i) &= 0 \qquad \text{and} \quad S(\overline{p}_i) = -L^{+j}{}_i \overline{p}_j h^{-1}. \end{aligned} \tag{30}$$

The scalar generator turns out to be hermitic:

$$h^*(w) = \overline{h(w)} = q. \tag{31}$$

The commutation relations with the second set of momenta are:

$$\begin{array}{llll} (i) & L^{\pm i}{}_j \overline{p}_n = \overline{p}_m L^{\pm k}{}_j \hat{R}^{\mp im}{}_{nk} & (iii) & h\overline{p}_i = q\overline{p}_i h \\ (ii) & \overline{p}_i \overline{p}_j \hat{A}_q^{ij}{}_{mn} = 0. & (iv) & p^i \overline{p}_j = q\overline{p}_k p^l \hat{R}^{ik}{}_{jl}. \end{array} \tag{32}$$

Now the pairing is completed by:

$$p^i(\bar{x}^n) = -q^{(N-1)} C^{ni}, \qquad \overline{p}_i(x_n) = C_{ni}, \qquad \overline{p}_i(\bar{x}^n) = -q^{(N-1)} C_{im} C^{nm}. \tag{33}$$

All other pairing brackets vanish. Together with the $*$-operation we obtain the $*$-Hopf algebras $U_q^*(A^I)$. We have mentioned that they can not be considered to be good deformations of classical euclidean Lie algebras. If however we are interested in the representation theory of single particles we are free to take into account only a subalgebra without coproduct which we will call $U_q(e(N))$ or $U_q(e(3,1))$ respectively. This subalgebra shall be generated by the set $\{L^{\pm i}{}_j, p^i\}$ and allows to impose the reality condition

$$(p^i)^* = C_{ij}\, p^j \tag{34}$$

The scalar operator did show up on the coproduct level only and can be omitted here as well.

2.4 Comment on an Unitary Scalar Operator

It is equally possible to find $U_q^*(A^I)$s in which the scalar operator h is unitary. Then the scalar operator of the corresponding function algebra has to be chosen hermitic. All relations in that case are the same as above except for the coproduct (30) and relations (32) (iii) and (iv):

$$\begin{aligned} \Delta\overline{p}_i &= S(L^{+j}{}_i) \otimes \overline{p}_j + \overline{p}_i \otimes h^{-1}, \\ (iii) h\overline{p}_i &= q^{-1}\overline{p}_i h, \qquad (iv) p^i \overline{p}_j = q^{-1} \overline{p}_k p^l \hat{R}^{ik}{}_{jl}. \end{aligned} \tag{35}$$

The new aspect for this setting is that in case of $U_q^*(ISO_q(3,1))$ it is possible to impose (34) on the representations of light cone momenta (i.e. $C_{ij}p^jp^i = 0$) since:

$$p^ip^mC_{nm} = q^{-1}\hat{R}^{ni}{}_{n'i'}p^{i'}p^{n'}C_{jn} - \frac{q^{1-N}-1}{Q_N}C_{kl}p^lp^k\delta^i_n. \tag{36}$$

In spite of the problems described above it should be possible to find representations for massless particles (even for systems of more then one massless particles) in which momenta have real expectation values.

3 Casimir Operators

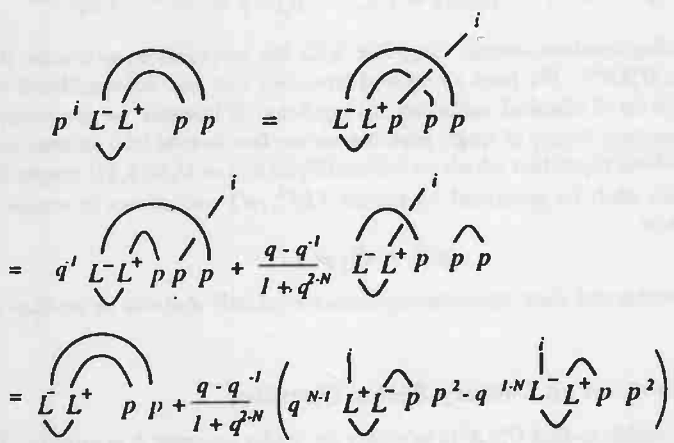
We find two Casimir operators for $U_q(e(N))$ and $U_q(e(3,1))$ and one for the $*$-Hopf algebras $U_q^*(A^I)$. For the first group they are:

$$\begin{aligned} p^2 &= C_{ij}p^jp^i, \\ w^2 &= C_{lk}C_{nm}L^{-k}{}_iL^{+m}{}_jC^{ji}p^np^l - \frac{q}{[2]_q}L^2p^2, \end{aligned} \tag{37}$$

with q-numbers:

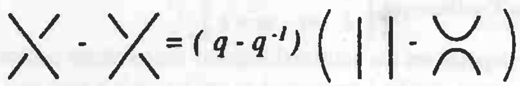
$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$

and L^2 designates the Casimir of the corresponding homogeneous subalgebra. For p^2 this follows directly from the commutation relations (9) whereas for the second Casimir we find for its commuting with the momenta the diagram (4):

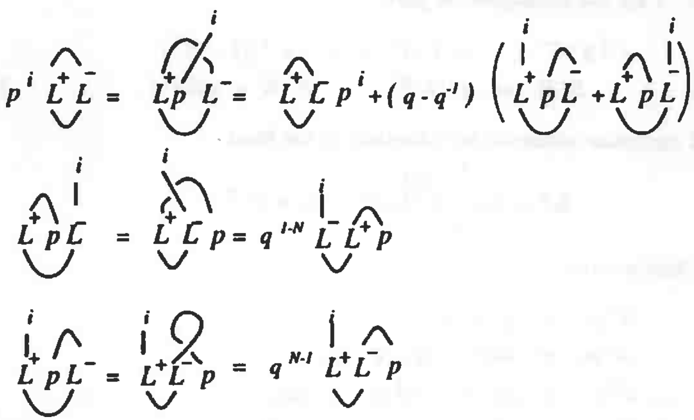


(38)

and with



(39)



(40)

These diagrams are translated into:

$$[p^a, C_{lk}C_{nm}L^{-k}{}_iL^{+m}{}_jC^{ji}p^np^l] = \frac{q-q^{-1}}{[1+q^{2N}]_q}\left(q^{N-1}C_{lk}L^{-a}{}_iL^{+k}{}_jC^{ji}p^l - q^{1-N}C_{lk}L^{+a}{}_iL^{-k}{}_jC^{ji}p^l\right)p^2 \quad (41)$$

and

$$[p^a, L^2] = (q-q^{-1})\left(q^{N-1}C_{lk}L^{-a}{}_iL^{+k}{}_jC^{ji}p^l - q^{1-N}C_{lk}L^{+a}{}_iL^{-k}{}_jC^{ji}p^l\right). \quad (42)$$

They are called the q-length or mass operators and the 'q-Pauli Lubanski' operator. For $U_q^*(A^I)$ the Casimir operator has the form:

$$p^4 = h^{-2}C_{ij}p^jp^i\,C^{nm}\bar{p}_m\bar{p}_n \quad (43)$$

A deformed 'q-Pauli Lubanski' operator is still missing in this case.

4 The Irreducible Representations of $U_q(e(3))$

As discussed previously the N-dimensional momentum space p^i provides a modul algebra of each corresponding $U_q(e(N))$. It forms a vector operator for the universal enveloping algebra and its matrix elements are given (by the q-version of Wigner

Ekkard theorem) as the product of reduced matrix elements $\langle l'|p|l\rangle$, where $l' = l \pm 1, l$ and deformed Clebsch Gordan Coefficients $\begin{bmatrix} 1 & l & l' \\ i & m & m+i \end{bmatrix}_q$.

The reduced matrix elements depend on the minimal angular momentum occurring in the representation which as usual can be interpreted as the particle's spin and the value of p^2.

The algebraic relations for $U_q(e(N))$ in explicit form are with $S^+ = {L^{+1}}_2, S^- = {L^{-3}}_2, K = {L^{+1}}_1$ for the homogeneous part:

$$\begin{aligned} q^{-\frac{1}{2}} S^+ S^- &- q^{\frac{1}{2}} S^- S^+ = (q - q^{-1})(1 - K^2) \\ S^+ K &= q^{-1} K S^+, \qquad S^- K = q K S^-. \end{aligned} \tag{44}$$

The classical amngular momenta are obtained in the limit

$$\mathrm{L}^{\pm} = \lim_{q \to 1} \frac{\sqrt{[2]}}{q - q^{-1}} S^{\pm}, \quad K = q^{-H}.$$

The mixed relations are:

$$\begin{aligned} S^+ p_+ &= q^{-1} p_+ S^+ \\ S^+ p_0 &= p_0 S^+ - (q - q^{-1}) p_+ \\ S^+ p_- &= q p_- S^+ + q^{\frac{1}{2}} (q - q^{-1}) p_+ \\ S^- p_+ &= q^{-1} p_+ S^- - q^{-\frac{1}{2}} (q - q^{-1}) p_0 \\ S^- p_0 &= p_0 S^- + (q - q^{-1}) p_- \\ S^- p_- &= q p_- S^- \\ K p_+ &= q^{-1} p_+ K, \ K p_0 = p_0 K, \ K p_- = p_- K, \end{aligned} \tag{45}$$

and for the covariant momenta:

$$\begin{aligned} p_+ p_- &= p_- p_+ + (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) {p_0}^2 \\ p_+ p_0 &= q^{-1} p_0 p_- \ , \ p_- p_0 = q p_0 p_- \end{aligned} \tag{46}$$

The metric C_{ij} which enters the reality condition (34) is in the basis (p_+, p_0, p_-):

$$C_{ij} = \begin{pmatrix} 0 & 0 & q^{-\frac{1}{2}} \\ 0 & 1 & 0 \\ q^{\frac{1}{2}} & 0 & 0 \end{pmatrix}. \tag{47}$$

The subalgebra of angular momenta has for each (half) integer positive value l the $(2l+1)$-dimensional representation on $\mathcal{H}^l$:

$$\begin{aligned} S^{\pm}|l, m\rangle &= \frac{q - q^{-1}}{\sqrt{[2]}} q^{-\frac{m \pm 1}{2}} \sqrt{[l \pm m][l \pm m + 1]} |l, m \pm 1\rangle \\ K|l, m\rangle &= q^{-m} |l, m\rangle \end{aligned} \tag{48}$$

The noncompact generators have to be represented on the direct sums $\sum_{l \geq l_0}^{\oplus} \mathcal{H}^l$. The matrix elements in the orthonormal basis are:

$$
\begin{aligned}
\langle l+1,m+1|p_+|l,m\rangle &= -p_{l,l+1}\,q^{\frac{m-l-1}{2}}\sqrt{[l+m+1][l+m+2]}\\
\langle l,m+1|p_+|l,m\rangle &= p_{l,l}\,q^{\frac{m}{2}}\sqrt{[l-m][l+m+1]}\\
\langle l-1,m+1|p_+|l,m\rangle &= p_{l,l-1}\,q^{\frac{l+m}{2}}\sqrt{[l-m][l-m-1]}\\
\langle l+1,m|p_0|l,m\rangle &= p_{l,l+1}\,q^{\frac{m}{2}}\sqrt{[2]}\sqrt{[l-m+1][l+m+1]}\\
\langle l,m|p_0|l,m\rangle &= \frac{p_{l,l}}{q^{\frac{1}{2}}}\sqrt{[2]}^{-1}\left([2]q^m-\left(q^{l+\frac{1}{2}}+q^{-l-\frac{1}{2}}\right)\right) \qquad (49)\\
\langle l-1,m|p_0|l,m\rangle &= p_{l,l-1}q^{\frac{m}{2}}\sqrt{[2]}\sqrt{[l-m][l+m]}\\
\langle l+1,m-1|p_-|l,m\rangle &= -p_{l,l+1}\,q^{\frac{l+m+1}{2}}\sqrt{[l-m+1][l+m-2]}\\
\langle l,m-1|p_-|l,m\rangle &= p_{l,l}\,q^{\frac{m}{2}}\sqrt{[l+m][l-m+1]}\\
\langle l-1,m-1|p_-|l,m\rangle &= -p_{l,l-1}\,q^{\frac{m-l}{2}}\sqrt{[l+m][l+m-1]}
\end{aligned}
$$

Here we have omitted each $\sqrt{q}$-number index so that one should identify: $[n]\equiv[n]_{\sqrt{q}}$ The reduced matrix elements are:

$$
p_{l,l} = \frac{C}{[l]_q[l+1]_q}, \qquad (50)
$$

$$
p_{l,l-1} = \left(C_0{}^2-\frac{C^2}{[l]_q^2}\right)\frac{1}{[2l]^2-1}. \qquad (51)
$$

The value of p^2 is related to $C_0{}^2$ by $p^2=[2]\,C_0{}^2$ and C to the minimal angular momentum by $C^2=[l_0]_q^2C_0{}^2$.
The dependence on the Clebsch Gordan coefficients is calculated from inserting the well known action of the compact generators first into $S^+p_+=q^{-1}p_+S^+$ and then into $S^-p_+=q^{-1}p_+S^- - q^{-\frac{1}{2}}(q-q^{-1})p_0$.

Next one computes the reduced matrix elements like in the classical case from the relation $p_+p_0=q^{-1}p_0p_+$. For the transition $l\to l+1$ we obtain:

$$
\frac{p_{l,l}}{p_{l+1,l+1}} = \frac{\left\{([l+2]_q-[l+1]_q)q^{-\frac{l+2}{2}}-q^{\frac{l+2}{2}}\right\}}{\left\{([l+1]_q-[l]_q)q^{-\frac{1}{2}}-q^{\frac{1}{2}}\right\}}, \qquad (52)
$$

and this implies (50) whereas the transition $l\to l$ yields:

$$
[2]\left([2l+3]p_{l,l+1}^2-[2l-1]p_{l,l-1}^2\right)-p_{l,l}^2\left(q^{l+\frac{1}{2}}+q^{-(l+\frac{1}{2})}\right)=0. \qquad (53)
$$

We put $\phi(l)=:p_{l,l+1}^2[2l+3][2l+1]$ such that:

$$
\phi(l)-\phi(l-1)=C^2\frac{[l+\frac{1}{2}]_q}{[l]_q^2[l+1]_q^2}=\frac{C^2}{[l]_q^2}-\frac{C^2}{[l+1]_q^2}. \qquad (54)
$$

This verifies (51). In order to inhibit negative values for l, the constant C_0 has to be put equal to $\frac{C^2}{[l_0]_q^2}$, which shows that l_0 is the smallest occurring angular momentum in a representation and can be identified with the spin of the represented object. At last we check the dependence of p^2 on C_0.

$$\langle l,m|p^2|l,m\rangle = \frac{[2]}{[2l+1]}\left\{\left(C_0^2 - \frac{C^2}{[l+1]_q^2}\right)\frac{q^{l+\frac{1}{2}} - q^{m-\frac{1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\right. \tag{55}$$
$$\left. + \left(C_0^2 - \frac{C^2}{[l]_q^2}\right)\frac{q^{m-\frac{1}{2}} - q^{-l-\frac{1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\right\} + \frac{C^2}{[l]_q^2[l+1]_q^2}\left(\frac{q^{l+m} + q^{m-l-1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} + D(l)\right)$$

with

$$D(l) = \frac{q^{-1}\left(q^{l+\frac{1}{2}} + q^{-l-\frac{1}{2}}\right)^2 - \left(q^{\frac{1}{2}} + q^{-\frac{1}{2}}\right)^2}{\left(q - q^{-1}\right)\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)^3}. \tag{56}$$

A few steps later we obtain:

$$p^2 = [2]C_0^2.$$

The in the classical limit unusual factor 2 which is showing up in these results is due to the differing normalization for the operators of the commutator $[L^+, L^-] = H$.

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A graphical Calculus

An important property of the $\hat{R}$-matrices is that they leave the metric C_{ij} invariant. This is mediated by the equations:

$$\begin{aligned} C_{ij}\hat{R}^{\pm ki}{}_{rm}\hat{R}^{\pm mj}{}_{sl} &= C_{rs}\delta_l^k \\ C_{ij}\hat{R}^{\pm jk}{}_{ms}\hat{R}^{\pm im}{}_{lr} &= C_{rs}\delta_l^k \\ C^{ij}\hat{R}^{\pm kr}{}_{im}\hat{R}^{\pm ms}{}_{jl} &= C^{rs}\delta_l^k \\ C^{ij}\hat{R}^{\pm sk}{}_{mj}\hat{R}^{\pm rm}{}_{li} &= C^{rs}\delta_l^k. \end{aligned} \tag{57}$$

These equations have simple graphical representations. With

$\hat{R}^{ij}{}_{kl}$ = 

$\hat{R}^{-1\,ij}{}_{kl}$ = 

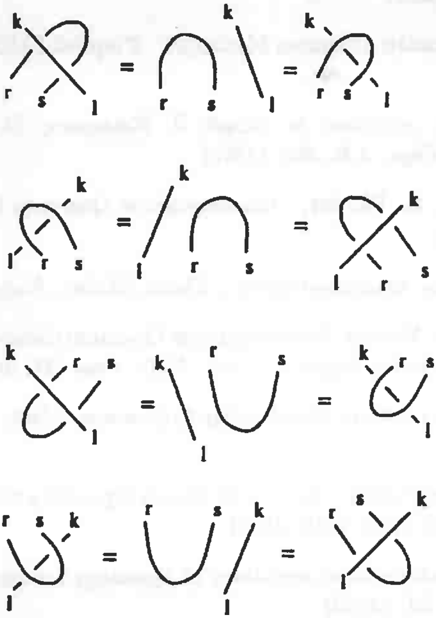
(58)

C^{ij} = 

C_{kl} = 

(59)

the equations (57) are put into the form:



(60)

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