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BIFURCATIONS OF BINARY DIFFERENTIAL EQUATIONS

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RESUMO

Apresentamos neste artigo modelos topológicos locais das curvas integrais de uma equação diferencial binária (BDE)

$$a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2 = 0$$

cujos coeficientes a, b, c não são todos nulos na origem, e cujo discriminante possui uma singularidade do tipo Morse. Estudamos também as bifurcações em famílias genéricas a 1-parâmetro. Assim completamos o estudo dos fenômenos de codimensão ≤ 1 que aparecem nas equações diferencias binárias. Os resultados permitem estudar as "conjugate curve congruence", assim como as bifurcações das linhas assintóticas de uma superfície regular nos pontos cúspides de Gauss.

Bifurcations of Binary Differential Equations

J.W. Bruce G.J. Fletcher F.Tari*

Abstract

In this paper we give a local classification of the integral curves of binary differential equations

$$A(x,y)dy^2 + 2B(x,y)dxdy + C(x,y)dx^2 = 0$$

at points where the coefficients A, B, C do not all vanish and where the discriminant $B^2 - AC$ has a Morse singularity. We also give models for generic bifurcations of such equations and apply the results to the differential geometry of smooth surfaces.

AMS Classification: 34Cxx, 58Fxx

1 Introduction

In this paper we study generic 1-parameter families of binary differential equations (BDE's) with non-zero coefficients. A BDE is an implicit differential equation of the form

$$A(x,y)dy^2 + 2B(x,y)dxdy + C(x,y)dx^2 = 0.$$

It defines pairs of directions at points where $\delta = B^2 - AC > 0$. These directions coincide on the discriminant $\delta = 0$. (The BDE has no solution at points where $\delta < 0$.) These type of implicit differential equations have been studied by several authors (eg [4], [5], [7], [8], [9], [10], [12], [13], [14], [15], [19], [20], [21], [22]). They occur in a number of branches of mathematics, and in particular in the differential geometry of surfaces (see [8] for examples).

The bifurcations that occur in generic 1-parameter families of such equations have been dealt with in two cases. In [14] there is given a complete topological classification of elementary singular points in the case where the discriminant is a smooth curve. In [9] we studied the case where the discriminant has a Morse singularity and the coefficients of the BDE all vanish at the origin. That investigation determines, for instance, the way in which the configuration of the asymptotic curves on a smooth surface with a flat umbilic varies as the flat umbilic is destroyed in a generic 1-parameter family of surfaces.

In this paper we deal with the case where the discriminant has a Morse singularity but the coefficients of the BDE do not all vanish at the origin,

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completing the study of generic 1-parameter local bifurcations of BDE's. There are some results related to our work in [21] where Kuz'min studied the behaviour of the integral curves of BDE's with a discriminant of type node. In various sectors of the plane he looks separately at the two line fields making up the BDE. These direction fields are then studied and the resulting integral curves are sketched. The fields are then superimposed. Although there are no proofs the diagrams obtained appear to coincide with those appearing in [13], [20], [5] [7] and here. Applications to gas dynamics are also given in [21].

We shall adopt here, as in [16], the notion of fibre topological equivalence for families of bivalued fields. Two families X_t and Y_s are fibre topologically equivalent if there exist a homeomorphism $s = \psi(t)$ between the parameter space and a family of homeomorphisms of $\mathbb{R}^2$ depending on the parameter t say h_t such that for all t, h_t is a topological equivalence between X_t and $Y_{\psi(t)}$. The map h_t is not required to be continuous in t. This approach may in general exclude bifurcation if the neighbourhood of the phase portrait shrinks to a point as the parameter tends to zero. We can avoid this situation in our case (as in [9]) by fixing the diameter of the neighbourhood for all values of the parameter near zero.

Here we outline the strategy used to classify the given BDE's (see Figure 1). We start, as in [5] and [13], by considering the surface $F(x, y, p) = 0$ and the single valued lift ξ of the double valued field in the (x, y)-plane. The projection to the (x, y)-plane gives rise to an involution on the surface $F(x, y, p) = 0$. Since we have a Morse transition, this surface is singular, in fact a cone (or an isolated point), and the lifted field has an isolated singularity at the origin. We then blow up the cone to obtain a cylinder $\tilde{M}$. The exceptional fibre is a circle (an ellipse in $\mathbb{R}P^2$) and the lifted field has two singular points on that fibre. Both are saddles (in the case of interest) with resonant eigenvalues -1. The involution σ lifts to an involution $\tilde{\sigma}$ on $\tilde{M}$ which fixes the exceptional fibre and interchanges the two saddles. The exceptional fibre is a separatrix for each saddle. So we wish to obtain a model for a pair of coincident saddles sharing a separatrix (Theorem 2.3; In Appendix 1 we give a model for the case when the eigenvalues are not resonant). We do this via a further blow up. Applying a polar blow up we obtain a bivalued field in a neighbourhood of one boundary circle of an annulus. This has 8 singular points on the boundary, all saddles, 2 for each separatrix of the pair of saddles. The pair corresponding to the common separatrices can then be analysed. They each correspond to a pair of saddles with coincident separatrices and eigenvalues now -2. We then apply results of Belitskii [3] and Bruno [11] to normalise one and then bring the other into a standard form (Theorem 2.2). The configuration of the pairs of integral curves around the rest of the boundary is then easily determined and the results patched together (up to homeomorphism). Blowing down we obtain a normal form on the quotient of the cylinder by $\tilde{\sigma}$ and hence in the plane (Theorem 2.6).

In passing, we notice that the proof of the topological reduction of an implicit differential equations to a normal form at a well-folded saddle and node singularities given in [13] appears to have a gap. We present a regorous proof of this result in Appendix 2.

To study the bifurcations, we follow the same procedure in [9]. Note that in

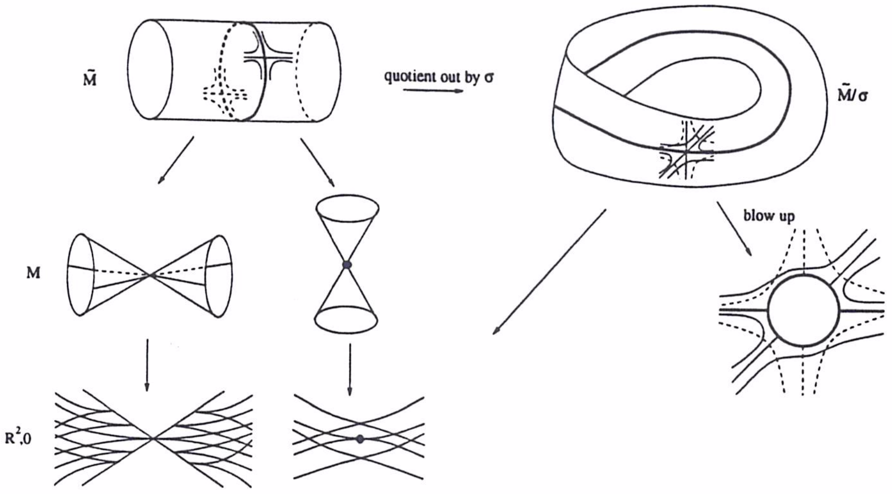

Figure 1: The strategy.

a generic family of BDE's the surface M undergoes a Morse transition, and so is smooth on both sides of the transition. We can then apply the results in [5], [9] and [13] to reduce the integral curves of the perturbation to a normal form.

We give, in Section 4, two applications to the differential geometry of smooth surfaces. The first describes the bifurcations of the asymptotic curves in a generic 1-parameter family of surfaces, and the second is to an apparently new family of BDE's associated to a smooth surface in $\mathbb{R}^3$ called the conjugate curve congruence. This family links the asymptotic and principal curve congruences.

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2 Normal form

It was shown in [10] that a BDE with non-zero coefficients at the origin can be transformed by a smooth change of coordinates to the form

$$dy^2 + f(x,y)dx^2 = 0$$

so we assume in this study that the BDE's are in the above form. We suppose that the discriminant function, f in this case, has a Morse singularity. We shall denote the 2-jet of f by $j^2 f(x,y) = ax^2 + 2bxy + cy^2$, so the condition for a Morse singularity is $b^2 \neq ac$. By taking a local chart $dy/dx = p$ in $\mathbb{R}P$ we can view the corresponding set M described in the introduction as a surface in $\mathbb{R}^3$ given by

$$M = \{(x,y,p) \in \mathbb{R}^3 : p^2 + f(x,y) = 0\}.$$

Clearly M is singular; it consists either of a cone or an isolated point. When it is an isolated point the BDE has one integral curve which is the origin, and all such BDE's of Morse type can be considered equivalent. We shall treat the case where M is a cone. The bivalued field in the plane lifts to a single vector field tangent to M given by

$$\xi = p\frac{\partial}{\partial x} + p^2\frac{\partial}{\partial y} - \frac{1}{2}(f_x + pf_y)\frac{\partial}{\partial p}$$

(see [5]), with an isolated zero at the cone point. Equivalently this line field is determined by the restriction of the canonical 1-form $dy - pdx$ to the surface M.

The involution on M that interchanges points which project to the same image in the plane is the reflexion $\sigma(x, y, p) = (x, y, -p)$. This involution has two lines of fixed points on M when f has a saddle singularity (that is when the discriminant is a node) and a single fixed point when f has a maximum/minimum (ie when $f = 0$ is an isolated point).

To study M and the field ξ we blow up the singular point of M and then consider the lift of ξ, say $\tilde{\xi}$, and the lift of the involution $\tilde{\sigma}$ to the blown-up surface $\tilde{M}$ in $\mathbb{R}^2 \times \mathbb{R}P^2$. (Equivalently we blow up M and the canonical 1-form and consider the restriction of the blow-up of that form to $\tilde{M}$.) We start with a general result.

Theorem 2.1 *Let the surface M in $\mathbb{R}^3$ (given by $g(x, y, p) = 0$) pass through the origin, with corresponding vector field ξ. The blow up $\tilde{\xi}$ has singularities at those smooth points of the exceptional fibre of the blow-up $\tilde{M}$ which meet the line corresponding to the p-axis. At a point where they meet transversely this singular point has eigenvalue -1. If there is a non-transverse intersection the eigenvalue is zero.*

Proof We shall blow-up the 1-form rather than the field since the calculations are easier.

Taking a local chart the projection from $\tilde{M}$ to M is given by $\phi(X, Y, P) = (X, XY, XP) = (x, y, p)$, and the surface $\tilde{M}$ is given by

$$G(X, Y, P) = X^{-r}g(X, XY, XP) = 0,$$

where the initial terms in g are of order r. The blow-up of the canonical form $dy - pdx$ is $\alpha = (Y - PX)dX + XdY$. We consider the restriction of this form to $G = 0$, and wish to identify any singular points on the exceptional fibre $X = 0$. The 1-form is tangent to $\tilde{M}$ when $dG \wedge ((Y - PX)dX + XdY) = 0$, so either $G = G_P = XG_X - (Y - PX)G_Y = 0$, or $G = Y - PX = X = 0$. Set $X = 0$ and reduce these to $G(0, Y, P) = G_P(0, Y, P) = G_Y(0, Y, P) = 0$ and $X = Y = G(0, 0, P) = 0$ respectively. The first case corresponds to a singular point of the exceptional fibre; in other words the projective curve $g_r(X, Y, P) = 0$ has a singularity. Suppose we are away from such points but the form has a zero on the exceptional fibre at say $(0, 0, P_0)$. If we can write $G = 0$ locally as the graph of a function $P = P(X, Y)$ then α restricts to $G = 0$ to yield $(Y - P(X, Y)X)dX + XdY$ which has a singular point at $(0, 0, P_0)$ with linear part $(Y - P_0X)dX + XdY$, and the corresponding vector field has eigenvalue -1.

We cannot write $\tilde{M}$ in this form only if $G(0,0,P_0) = G_P(0,0,P_0) = 0$, which corresponds to the failure of transversality in the statement of the theorem. A short calulation shows that in this case the corresponding vector field always has eigenvalue 0.

We start by considering the blow-up of our surface, $p^2 + X^{-2}f(X,XY) = 0$. The Morse condition implies that this is smooth, as can be checked by a straightforward calculation. Indeed the exceptional fibre in $0 \times \mathbb{R}P^2$ is given by $p^2 + ax^2 + 2bxy + cy^2 = 0$, an ellipse in the projective plane.

Using the above we see that for $a \neq 0$, the singularities of $\tilde{\xi}$ on $\tilde{M}$ which lie on the exceptional fibre occur when $Y = 0$ and $P^2 + a = 0$. So the field has two singular points when $a < 0$ and none when $a > 0$. As above we write the surface $\tilde{M}$ locally at the singular point as the graph of a function $(X, Y, P(X,Y))$. In the case where $a < 0$ the matrix of the linear part of the projection of $\tilde{\xi}$ to the (X,Y)-plane at $(0,0,\pm\sqrt{-a})$ is given by

$$\begin{pmatrix} \pm\sqrt{-a} & 0 \\ -a & \mp\sqrt{-a} \end{pmatrix}.$$

As shown in the above theorem the singularities of $\tilde{\xi}$ are therefore both saddles when $a < 0$; indeed the corresponding eigenvalue is -1 in both cases. This is singularly unfortunate since it means that neither can be reduced to linear normal form by Poincarè or Siegel reduction as in [1]. Note that the involution on M lifts to an involution $\tilde{\sigma}(X,Y,P) = (X,Y,-P)$ on $\tilde{M}$. So the image of one singular point by the involution $\tilde{\sigma}$ is the other. The involution also preserves the exceptional fibre. This situation has not arisen in our previous studies of BDE's. We therefore need to determine the normal form for the pairs of direction fields on the quotient space near the corresponding singularity.

Note that, identifying points and their images under $\tilde{\sigma}$, the singular points of $\tilde{\xi}$ yield two germs of vector fields that have a common trajectory along the exceptional fibre and are transverse everywhere else. (Surprisingly this transversality is a stringent condition, and plays a key role in the derivation of the normal form.) Writing $\tilde{M}$ locally, at the singularities of the vector field $\tilde{\xi}$, as the graph of a function $P = \pm P(X,Y)$ and projecting to the (X,Y)-plane the two germs of vector fields are given by $PX\frac{\partial}{\partial X} + (-PY + P^2X)\frac{\partial}{\partial Y}$ and $-PX\frac{\partial}{\partial X} + (PY + P^2X)\frac{\partial}{\partial Y}$. Their direction fields coincide if and only if $XP = 0$, that is if and only if $X = 0$ (since $P(0,0) = \sqrt{-a} \neq 0$). We also see that the zeros of the two vector fields have the same eigenvalues, and a separatrix in common, with the other two separatrices transverse. As remarked above the common eigenvalue is -1; as we have seen above this is essentially due to the fact that we are blowing up the canonical 1-form $dy - pdx$. For setting $x = X,\ y = XY,\ p = XP$ the blow-up of this form is $YdY + (X - PX)dX$.

In order to obtain a local model for a pair of resonant eigenvalue -1 saddles with one separatrix in common, we use a polar blow up and replace the origin with a circle. We obtain local models of the blow up pair at each point of the circle and glue these together to obtain a topological model of the initial pair. We shall see that on this circle we obtain a pair of resonant eigenvalue -2 saddles with the same separatrices. Moreover the *discriminant* of the pair, that is the set of points where the fields are multiples of each other, has a simple form. It is easier to obtain a local model for such pairs than the initial ones. This we do below.

Theorem 2.2 *Suppose given two singular line fields at the origin, both of type saddle with the same eigenvalues -2 and the same separatrices, say the x and y-axis. Suppose further that the discriminant is of the form $\delta = xy^2\tilde{\delta}$, with $\tilde{\delta}(0,0) \neq 0$. Then the line fields can be reduced simultaneously by a diffeomorphism to the pair*

$$\xi_1 = x(1 + x^2y(\epsilon_1 + a_0x^2y))\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}$$
$$\xi_2 = x(1 + y + x^2y(\epsilon_2 + b_0x^2y))\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}$$

where $\epsilon_i \in \{0,1\}$ and $a_0, b_0 \in \mathbb{R}$.

Proof: We shall work with the 1-forms associated to the fields. By reduction of resonant vector fields to normal form we may assume that the 1-form associated with the first field is given by

$$\alpha = x(1 + x^2y(\epsilon_1 + a_0x^2y))dy + 2ydx.$$

(The formal reduction to a normal form is given in [11], and the passage to a smooth reduction is provided in [3].) Since the two forms have the same separatrices, the second form β can be written as $\beta = x(1 + b(x,y))dy + 2ydx$ for some germ of a 1-flat function b. The discriminant of the pair is then given by

$$\delta = 2xy(b(x,y) - x^2y(\epsilon_1 + a_0x^2y)).$$

It follows from the hypothesis of the theorem that $b(x,y) = y(b_0 + c(x,y))$ with $b_0 \neq 0$ and $c(0,0) = 0$. We can set $b_0 = 1$ by a change of scale, so that after a change of notation, we write $\beta = x(1 + y(1 + b(x,y)))dy + 2ydx$ for some new germ of a 1-flat function b.

By [11] and [3], there exist a smooth diffeomorphism conjugating β to

$$\beta_0 = x(1 + y + x^2y(\epsilon_2 + b_0x^2y))dy + 2ydx.$$

Since this diffeomorphism preserves the separatrices, it has the form $\Phi = (x + x\phi, y + y\psi)$, where ϕ and ψ are germs of smooth functions with zero 1-jets.

The claim now is that $\phi = y\tilde{\phi}$ and $\psi = y\tilde{\psi}$, for some smooth germs of $\tilde{\phi}$ and $\tilde{\psi}$. From $\Phi_*(\beta_0) = \beta$, we obtain the following equations:

$$\begin{aligned}
&(x + x\phi)(1 + (y + y\psi)(1 + (x + x\phi)^2(\epsilon_2 + b_0(x + x\phi)^2(y + y\psi)))(1 + \psi + y\psi_y)\\
&\quad + 2x\phi_y(y + y\psi) = x(1 + y(1 + b(x,y)))\\
&(x + x\phi)(1 + (y + y\psi)(1 + (x + x\phi)^2(\epsilon_2 + b_0(x + x\phi)^2(y + y\psi)))y\psi_x\\
&\quad + 2(y + y\psi)(1 + \phi + x\phi_x) = 2y
\end{aligned}$$

for all (x,y) in a neighbourhood of the origin. We want $\phi(x,0)$ and $\psi(x,0)$ to be identically zero.

Dividing by x and setting $y = 0$ in the first equation yield

$$(1 + \phi(x,0))(1 + \psi(x,0)) = 1. \qquad (1)$$

Dividing by y and setting $y = 0$ in the second equation yield

$$x(1+\phi(x,0))\psi_x(x,0) + 2(1+\psi(x,0))(1+\phi(x,0)+x\phi_x(x,0)) = 2. \qquad (2)$$

Using equation (1), equation (2) can be simplified to

$$(1+\phi(x,0))\psi_x(x,0) + 2(1+\psi(x,0))\phi_x(x,0) = 0. \qquad (3)$$

Differentiating equation (1) with respect to x and using equation (3) yield $\phi_x(x,0) = 0$, hence $\phi(x,0) = \psi(x,0) = 0$ for all x in a neighbourhood of the origin.

The diffeomorphism Φ can thus be written on the form $\Phi = (x+xy\phi, y+y^2\psi)$ for some (new) germs of functions ϕ and ψ.

We now follow the proof of Theorem A.II.3 in [22]. Let $\tilde{\Phi}(x,y,t) = (x + txy\phi(x,y), y + ty^2\psi(x,y))$ be a 1-parameter family of germs of diffeomorphisms at the origin. We denote by ω the 1-form $\tilde{\Phi}^*(\beta_0)$ in $\mathbb{R}^2 \times \mathbb{R}$. Write $\omega = A(x,y,t)dy + B(x,y,t)dx + C(x,y,t)dt$ with

$$\begin{aligned} C = \; & 2(y+ty^2\psi)xy\phi + (x+txy\phi)(1+(y+ty^2\psi)(1+(x+txy\phi)^2(\epsilon_2 + \\ & b_0(x+txy\phi)^2(y+ty^2\psi)))y^2\psi \\ = \; & xy^2\tilde{C}(x,y,t) \end{aligned}$$

for some germ of a smooth function $\tilde{C}$. We need to find a vector field of the form

$$Z = U(x,y,t)\frac{\partial}{\partial x} + V(x,y,t)\frac{\partial}{\partial y} + \frac{\partial}{\partial t}$$

which is tangent to the integral surfaces in 3-space determined by ω and $\pi^*(\alpha_1)$ where π is the projection to the parameter t. This holds when $\omega(Z) = 0$ and $\alpha(Z) = 0$, which is equivalent to solving the following linear system for U, V.

$$\begin{aligned} x(1+x^2y(\epsilon_1 + a_0x^2y))U + 2yV = \; & 0 \\ AU + BV + C = \; & 0. \end{aligned}$$

The solution, if it exists, is of the form $U = -2yC/\delta$, $V = x(1 + x^2y(\epsilon_1 + a_0x^2y))C/\delta$, where $\delta = 2yA - x(1+x^2y(\epsilon_1 + a_0x^2y))B$.

Since $\delta = xy^2\tilde{\delta}$ for some smooth function $\tilde{\delta}$, and we can write $C = xy^2\tilde{C}$, the system will have a solution provided that $\tilde{\delta}$ does not vanish on the t-axis. A calculation shows that at $(0,0,t)$ the function $\tilde{\delta}$ is 2, and the result follows. (Note that the condition on the discriminant in the statement of the theorem is essential for the above system to have a solution.)

We now integrate the field Z. It is easy to see that U and V vanish at each point $(0,0,t)$, so the resulting family of diffeomorphisms preserves the t-axis. Since Z is also tangent to the curves formed by the intersections of the integral surfaces of ω and $\pi^*(\alpha)$ it will take the integral curves of β_0 to those of β while preserving those of ω. This establishes the required smooth reduction.

Theorem 2.3 *Suppose given two line fields with a singularity at the origin, both of type saddle with the same eigenvalues -1 and one common separatrix, say the x-axis. Suppose that the two fields are tranverse away from the common* *separatrix. Then the line fields can be reduced simultaneously by a homeomorphism to the pair*

$$\begin{array}{l} \xi_1 = x(1+xy(\epsilon+a_0xy))\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \\ \xi_2 = (x+y+yA_2^5(x,y))\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \end{array}$$

where $\epsilon \in \{0,1\}$, $a_0 \in \mathbb{R}$ *and* A_2^5 *is a polynomial in* (x,y) *of degree* ≥ 2 *and* ≤ 5.

Proof: We shall use, as above, the associated differential 1-forms. Without loss of generality we can set $\alpha = x(1+xy(\epsilon+a_0xy))dy + ydx$ ([11]) and write β in the form $\beta = (x+y+yb(x,y))dy + ydx$, with b a germ of smooth function with zero 1-jets.

We now consider the polar blow-up $x = r\cos(\theta)$, $y = r\sin(\theta)$. This results, after dividing by r, in the following two 1-forms on $S^1 \times \mathbb{R}^+$

$$\begin{array}{rl} \tilde{\alpha} = & r(\cos^2\theta - \sin^2\theta + r^2\cos^3\theta\sin\theta(\epsilon + a_0r^2\cos\theta\sin\theta))d\theta + \\ & \sin\theta(2\cos\theta + r^2\cos^2\theta\sin\theta(\epsilon + a_0r^2\cos\theta\sin\theta))dr \\ \tilde{\beta} = & r(\cos^2\theta + \cos\theta\sin\theta - \sin^2\theta + \cos\theta\sin\theta b(r\cos\theta, r\sin\theta))d\theta + \\ & \sin\theta(2\cos\theta + \sin\theta + \sin\theta b(r\cos\theta, r\sin\theta))dr. \end{array}$$

The 1-form $\tilde{\alpha}$ has 4 zeros of type saddle on the circle S^1 ($r = 0$) at $\theta = 0, \pi, \pm\pi/2$. The zeros of $\tilde{\beta}$ on S^1 occur at $\theta = 0, \pi$ and at the solutions of $2\cos\theta + \sin\theta = 0$. So $\tilde{\alpha}$ and $\tilde{\beta}$ have two zeros (saddles) in common. At these zeros, the forms have common separatrices and have the same linear part $rd\theta + 2\theta dr$. Furthermore, the discriminant of the two forms can be written in the form $\delta = r\theta^2\delta_1$ at $\theta = 0$ and $\delta = r(\theta-\pi)^2\delta_2$ at $\theta = \pi$, with the δ_i not vanishing at the points in consideration. So at the zeros the hypothesis of Theorem 2.2 are satisfied, and can reduce $\tilde{\beta}$ to its 7-jet which depends only on the the 5-jet of b.

At $\theta = \pm\pi/2$, $\tilde{\alpha}$ has a singularity of type saddle and $\tilde{\beta}$ is regular. Applying Proposition 4.1 in [5] we can reduce the pair by a homeomorphism to the model $(dy, xdy + ydx)$. Note that the result in [5] is valid even when the eigenvalues of $\tilde{\alpha}$ are resonant, which is the case here.

This is also the case at the zeros of $\tilde{\beta}$ given by $2\cos\theta + \sin\theta = 0$. At such points $\tilde{\alpha}$ is regular so we apply again Proposition 4.1 in [5] to obtain local models.

Away from the singular points the two fields are regular and have a common integral curve along S^1. Their difference is not degenerate, so a model is given by Lemma 3.1 in [5].

We can glue the local models together by sliding along integral curves to construct a homeomorphism in a neighbourhood of S^1 (see [5] for details). Projecting down to $\mathbb{R}^2, 0$ yields the required homeomorphism.

For the case of the pair of line fields resulting from the second blowing up of the BDE $dy^2 + f(x,y)dx^2 = 0$, we can refine the result in Theorem 2.3. We observe that the terms in A_5^2 in the theorem result from the fact that we have two moduli in the normal form in Theorem 2.2.

Lemma 2.4 *Let* $\xi = x(1 + ya(x,y))\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}$ *be a smooth vector field, and denote by* $\sum_{i=0}^{k} a_{ki}x^{k-i}y^i$ *the homogeneous part of degree* k *in the Taylor expansion of* a*. Then* ξ *is equivalent to* $x(1 + Ax^2y + Bx^4y^2)\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}$ *with*

$$
\begin{aligned}
A &= a_{31}, \\
B &= a_{62} - 2a_{51}a_{11} - 2a_{41}a_{21}.
\end{aligned}
$$

Proof: Since ξ is determined by its 7-jet (it is smoothly equivalent to $x(1 + x^2y(\epsilon + a_0x^2y))\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}$, $\epsilon = 0$ or 1, [11]) it is enough to reduce j^6a to the required form. This is done inductively on the jet level using the computer algebra package Maple.

Proposition 2.5 *The pair of germs, at their common singularity, of the direction fields resulting from the second blowing up of the BDE* $dy^2 + f(x,y)dx^2 = 0$ *is completely determined by* j^3f.

Proof: Using the 1-forms again, the pair is given by $\alpha = XdY + (Y + PX)dX$ and $\beta = XdY + (Y - PX)dX$, where $P(X,Y) = (-X^{-2}f(X,XY))^{\frac{1}{2}}$ (see discussion before Theorem 2.2). Changing variables, $X = y$ and $Y = x$ we can write $\alpha = (x + yP(y,x))dy + ydx$. To simplify the calculations, we shall use the directional blowing up. The blowing up $x = X, y = XY$ applied to α yields

$$\tilde{\alpha} = X(1 + YP(XY,Y))dY + 2Y(1 + YP(XY,Y)/2)dX.$$

Dividing by the coefficient of $2Y$ in dX we obtain an equivalent 1-form of the form

$$\tilde{\alpha} = X(1 + Ya(X,Y))dY + 2YdX$$

as in Lemma 2.4. Since $P(XY,Y) = (-(XY)^{-2}f(XY,X^2Y))^{\frac{1}{2}}$ the coefficients A and B in Lemma 2.4, applied to this situation, depend only on j^3f. Suppose that $j^3f = ax^2 + 2bxy + cy^2 + d_0x^3 + d_1x^2y + d_2xy^2 + d_3y^3$, then these coefficients are as follow:

$$
\begin{aligned}
A_\alpha &= \frac{ac-b^2}{4(-a)^{\frac{3}{2}}}, \\
B_\alpha &= \frac{1}{8(-a)^{\frac{7}{2}}}[b(3ac - 5b^2)d_0 + a(ac - 3b^2)d_1 - 2a^2bd_2 + 2a^3d_3] + A_\alpha^2.
\end{aligned}
$$

As for the 1-form $\beta = XdY + (Y - PX)dX$, the coefficients in Lemma 2.4 associated to the blow up 1-form $\tilde{\beta}$ (obtained in the same way as $\tilde{\alpha}$) are given by

$$
\begin{aligned}
A_\beta &= -\frac{ac-b^2}{4(-a)^{\frac{3}{2}}}, \\
B_\beta &= -\frac{1}{8(-a)^{\frac{7}{2}}}[b(3ac - 5b^2)d_0 + a(ac - 3b^2)d_1 - 2a^2bd_2 + 2a^3d_3] + A_\beta^2.
\end{aligned}
$$

In Lemma 2.4 one can set the coefficient of x^3y to 1 by a change of scale, if it is non-zero. The resulting coefficient of x^4y^2 is given by B/A^2. This is in fact the modulus that appears in the normal form $x(1 + x^2y(\epsilon + a_0x^4y^2)\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}$.

For the 1-forms $\tilde{\alpha}$ and $\tilde{\beta}$ above, since we are assuming that the discriminant has a Morse singularity the coefficients A_α and A_β are non-zero so that the value of ϵ in the model is 1. Furthermore, one can see that $B_\alpha/A_\alpha^2 = \mu + 1$ and $B_\beta/A_\beta^2 = -\mu + 1$, and μ depends only on the coefficients of $j^3 f$. So, in particular, we have only one modulus in the normal form of the pair $(\tilde{\alpha}, \tilde{\beta})$ in Theorem 2.2.

Theorem 2.6 *Suppose the BDE $dy^2 + f(x,y)dx^2 = 0$ has a discriminant with a Morse singularity with branches transverse to the unique direction defined by the equation at the origin (this is equivalent to $f_{xx}(0,0) \neq 0$). Then there exists a germ of a homeomorphism $h : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ taking the integral curves of the BDE to one of the following normal forms.*

(i) $f_{xx}(0,0) > 0$:

$$dy^2 + (x^2 \pm y^2)dx^2 = 0.$$

(ii) $f_{xx}(0,0) < 0$:

$$dy^2 + (-x^2 \pm y^2 + \lambda y^3)dx^2 = 0,$$

for some $\lambda \in \mathbb{R}$. The value of λ is completely determined by $j^3 f$.

Proof: As before we shall write $j^2 f = ax^2 + 2bxy + cy^2$ and distinguish the cases when the discriminant δ is an isolated point or a crossing. In the case where δ is an isolated point, the involution $\tilde{\sigma}$ on the cylinder $\tilde{M}$ has no fixed points. We need to identify points on $\tilde{M}$ that project to the same image in the plane. So we are reduced to classifying a certain bivalued vector field on the quotient space which is a Mobius band. The central circle of the Mobius band is an integral curve for both fields. These fields are transverse elsewhere and away from their common singular points their difference has a non-degenerate singularity. By Lemma 3.1 in [5] the topological configuration of the integral curves is given locally at non-singular points on the central circle by the pair $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.

We have seen that $\tilde{\xi}$ has two singularities on the exceptional fibre if $a < 0$ and none if $a > 0$. (These singularities are both of type resonant saddles.) The two singularities project to the same point on the Mobius band and the two fields satisfy the hypothesis of Theorem 2.3. Therefore the topological configuration of the integral curves at the common singular point of the bivalued field is given by the normal form in Theorem 2.3. But it follows from the proof of Proposition 2.5 that the two moduli in Theorem 2.3 are dependent for our pair of direction fields on the Mobius band. It is also clear from the proof of Proposition 2.5, that given a $j^3 f$, one can choose an appropriate value of λ so that the modulus associated to $dy^2 + (-x^2 \pm y^2 + \lambda y^3)dx^2 = 0$ coincides with that associated to the initial BDE.

We can now glue together the local normal forms to obtain a homeomorphism in a neighbourhood of the central circle by sliding along integral curves (see [5] for details).

When δ is a crossing the involution $\tilde{\sigma}$ has two lines of fixed points. When blowing-up in the x-direction these lines cross the exceptional fibre at $P = 0$

and $Y_i = (-b \pm \sqrt{b^2 - ac})/c$, where a, b, c are the coefficients of the quadratic part of f. These points are generically (for $a \neq 0$) distinct from the singular points of the vector field. So identifying points q and $\tilde{\sigma}(q)$ we are studying a bivalued field on the quotient space. Away from the images of $q_i = (0, Y_i, 0)$ the situation is exactly as for the case where the discriminant is an isolated point. At q_i the normal form is given by Theorem 4.3 in [7].

We observe here that the configurations of the integral curves of the BDE depends on the nature of the Morse singularity of δ (i.e., an isolated point or a crossing) and on the sign of a. The modulus λ in the normal form results from our use of the smooth model provided by Theorem 2.3. It would be interesting to see if it is possible to produce a topological model for the pair in Theorem 2.3 without moduli.

Remark 2.7 The condition $a \neq 0$ ensures that the branches of the discriminant $f = 0$ (when this has an A_1^{-1} singularity) are transverse to the unique direction defined by the BDE at the origin. In this case a result in [10] states that the multiplicity of the BDE, which is the maximum number of zeros of the bivalued field that can occur in a generic perturbation, is equal to 2. When $a = 0$, a branch of $f = 0$ is tangent to the unique direction at the origin determined by our BDE (namely $y = 0$). Then the multiplicity of the BDE is one plus the order of tangency between $f = 0$ and $y = 0$ (see [10]).

3 Generic unfoldings

Consider a 1-parameter family of BDE's with non-zero coefficients at the origin. The same proof as in [10] shows that such a family can be taken in the form

$$dy^2 + f(x, y, t)dx^2 = 0$$

with $f_0(x, y) = f(x, y, 0)$ having a Morse singularity at $(0, 0)$. (As before we shall write $j^2 f_0(x, y) = ax^2 + 2bxy + cy^2$.) We lift the family of bivalued fields defined by the above equation to a vector field on the surface M in $\mathbb{R}^3 \times \mathbb{R}P^1$ defined by

$$M = \{(x, y, t, [\alpha, \beta]) : \alpha^2 + f(x, y, t)\beta^2 = 0\}.$$

By choosing a local chart in $\mathbb{R}P^1$ we can write M as a surface in $\mathbb{R}^4$ given by the zero set of the function

$$F(x, y, t, p) = p^2 + f(x, y, t).$$

It is clear that M is a smooth surface if and only if $\frac{\partial f}{\partial t}(0, 0, 0) \neq 0$, that is if and only if the resulting deformation of the discriminant $\Delta(x, y, t) = f(x, y, t)$ of the BDE is a versal unfolding of the Morse singularity at $t = 0$. In this case the projection to the parameter t yields Morse functions on both M and Δ. We shall call such a family of BDE's a *versal family* of the BDE at $t = 0$.

A lift of the family of bivalued field on the surface M is given by

$$\xi = p\frac{\partial}{\partial x} + p^2\frac{\partial}{\partial y} - \frac{1}{2}(f_x(x, y, t) + pf_y(x, y, t))\frac{\partial}{\partial p}$$

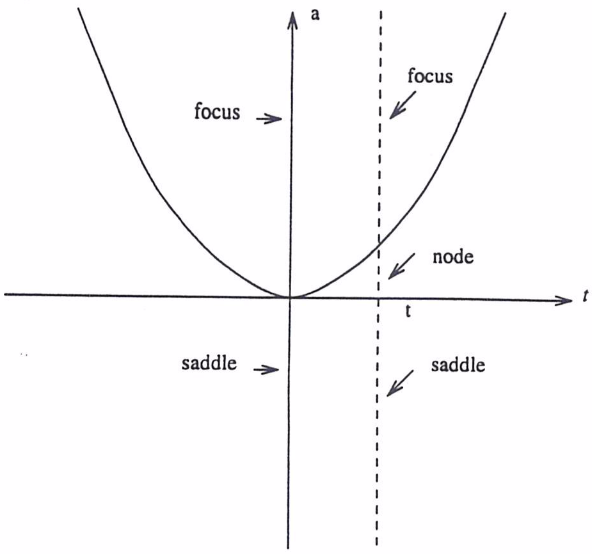

Figure 2: Birth/annihilation of two singularities of type saddles or foci.

Proposition 3.1 *Given a versal family of BDE's, the zeros of the lifted field form a smooth curve on the discriminant $\Delta = 0$. The perturbed BDE has two zeros on one side of the transition and none on the other. The zeros are both of type well-folded saddles or foci.*

Proof The zeros of ξ are given by $p = f(x, y, t) = f_x(x, y, t) = 0$. Since we assume that $\frac{\partial^2 f}{\partial x^2}(0, 0, 0) = 2a \neq 0$ and $\frac{\partial f}{\partial t}(0, 0, 0) \neq 0$, it follows from the implicit function theorem that the singular points of ξ form a smooth curve on Δ. A short calculation shows that the curve of singular points has exactly 2-point contact with the hyperplane $t = 0$, so we get two singular points on one side of the transition and none on the other.

The surface M_t, which is the t-constant fibre on M, is smooth for $t \neq 0$ (versality condition). The normal to this surface at the singular points is parallel to $(0, 1, 0)$. Let (x_0, y_0, t) be the singular point, so in particular $f_y(x_0, y_0, t) \neq 0$. We can parametrise M_t locally by (x, p) and projecting ξ to the tangent plane (x, p) we find that the linear part of the projected field at the zero point $(x_0, 0)$ has the form

$$\begin{pmatrix} 0 & 1 \\ -f_{xx}(x_0, y_0, t) & -f_y(x_0, y_0, t) \end{pmatrix}.$$

For t small the type of singular point is determined by the sign of $f_{xx}(0, 0, 0)$, that is by the sign of a. So the singular points are both saddles if $a < 0$ and both foci if $a > 0$. That these saddles/foci are well-folded follows from the fact that $F_{pp} = 2 \neq 0$ (see Proposition 3.2 in [9]).

Remark 3.2 That nodes do not occur in the above transition is illustrated by looking at the following example. In [13], Davydov showed that the normal forms for well-folded singularities of BDE's with smooth discriminants are given by the equation $dy^2 + (-ax^2 + 2y)dx^2 = 0$, where the singularity is of type well-folded saddle if $a < 0$, node if $0 < a < 1/4$ and focus if $a > 1/4$. Consider now the 1-parameter family of BDE's

$$dy^2 - (ax^2 + y^2 + 2ty)dx^2 = 0.$$

At $t = 0$ the discriminant has a Morse singularity. For $t \neq 0$ the vector field ξ_t has a singularity at the origin. At this point, the characteristic equation of the linear part of the projection of the ξ_t to the (x, p)-plane is given by $\lambda^2 - \lambda t + a = 0$. So the singularity is of type saddle if $a < 0$, node if $0 < a < t^2/4$ and focus if $a > t^2/4$. In Figure 2 we draw the parabola $a = t^2/4$, and from that figure it becomes clear that when $t = 0$ we are either in the saddle region, or in the foci region. So for a small perturbation of the equation the resulting singularities are saddles if $a < 0$ and foci if $a > 0$.

Theorem 3.3 *Suppose that the family of BDE's at $t = 0$ has non-zero constant coefficients and a discriminant of Morse type with branches transverse to the unique direction defined by the BDE at the origin. Suppose further that the family is versal. Then it is (fibre) topologically equivalent to*

$$dy^2 + (x^2 \pm y^2 + t)dx^2 = 0,$$

or to

$$dy^2 + (-x^2 \pm y^2 + \lambda y^3 + t)dx^2 = 0.$$

See Figure 3.

Proof We have seen that as the parameter varies we obtain Morse sections on the surface M and the discriminant Δ. We have two cases to consider.

(i) Δ_0 is an isolated point: the discriminant Δ_t is a smooth closed curve on one side of the transition and empty on the other. There are two cases to consider here depending on the sign $\pm$ of x^2 in the function f. When the sign is $-$, the solutions of the BDE lie outside the closed curve Δ_t. There are two singularities of the BDE on Δ_t and they are well-folded saddles (Proposition 3.1). Applying Davydov's results [13], we can find a homeomorphism taking the integral curves of ξ_t at these zeros to those of our model. We slide along the integral curves ([5]) to extend this homeomorphism to a neighbourhood of Δ_t. As the vector field ξ_0 has no singular points away from the origin, we can fix the diameter of the neighbourhood of Δ_t where the homeomorphism is defined.

When the sign is $+$, we have to be a bit more careful (essentially because we cannot push problems off to infinity). Now for t small and say positive the set $p^2 + f_t = 0$ is a sphere S projecting down to a disk D. On S there are just two zeros, which project down to two well-folded foci on D at say q_1 and q_2. The boundary of D, Δ_t, consists of two intervals I and J having endpoints q_1 and q_2 in common. Think of say q_1 as a source and q_2 as a sink; we shall establish that the picture for the integral curves is as in Figure 4. We claim that each curve leaving an interior point of I must next meet the boundary at a point of J. Otherwise it meets I next (at neither q_1 or q_2, as we know from the normal forms for well-folded foci). If the offending curve lies, say, on the upper half of the sphere all the integral curves in the half disk bounded by the critical set

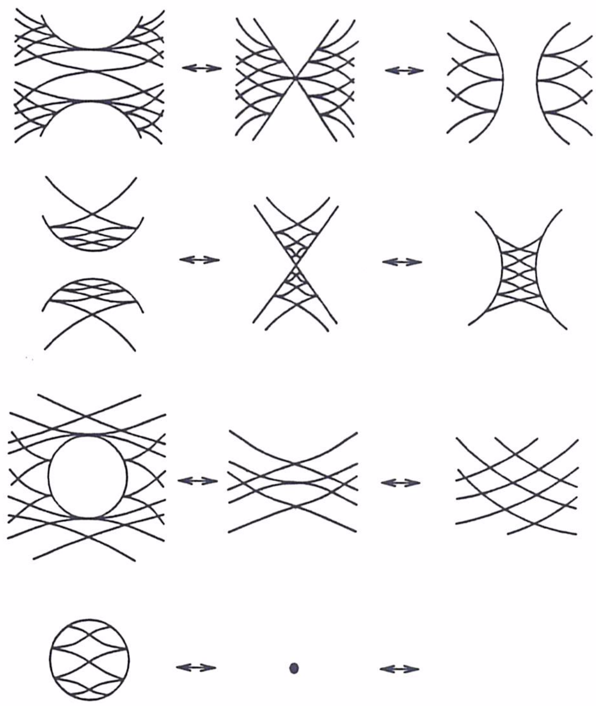

Figure 3: Bifurcations of the integral curves top to bottom: $dy^2 + (-x^2 + y^2 + t)dx^2 = 0$, $dy^2 + (x^2 - y^2 + t)dx^2 = 0$, $dy^2 + (-x^2 - y^2 + t)dx^2 = 0$, $dy^2 + (x^2 + y^2 + t)dx^2 = 0$.

and this curve also start and end on I, but this clearly implies a singular point on the boundary other than q_1 or q_2, a contradiction.

So we have a map $\alpha : I \to J$ which is continuous, fixes q_1 and q_2 and is smooth on the interior (it comes from the projection of a smooth flow on say the upper half sphere). In the same way we obtain a map $\beta : J \to I$ by flowing along the integral curves lying in the lower half of the sphere. So we obtain self-maps $R(I) : I \to I$ and $R(J) : J \to J$ by composing α and β. With respect to the given orientations of I and J (from q_1 to q_2) these are increasing (and smooth) on the interiors, continuous and fix q_1 and q_2. Using the normal forms for well-folded foci the structure of these maps is clear at q_1 and q_2.

Now suppose given another such deformation of our BDE with corresponding intervals labelled I_1 and J_1 and self-maps $R(I_1)$ and $R(J_1)$. Then the self maps $R(I)$ and $R(I_1)$ are conjugate via a map $h(I) : I \to I_1$ which is smooth in the interior of I and a homeomorphism. If we then define $h(J) : J \to J_1$ by $h(J) = \alpha_1 \circ h \circ \alpha^{-1}$ (or $\beta_1^{-1} \circ h \circ \beta$) then $\alpha_1 \circ h(I) = h(J) \circ \alpha$ and $\beta_1 \circ h(J) = h(I) \circ \beta$. If the two disks in which the integral curves lie are labelled D and D_1 then we define a map $H : D \to D_1$ as follows. Given any point $v \in D$ it lies on two upward integral curves, one emanating from some $x \in I$ the other from some $y \in J$. Define $H(v)$ to be the intersection point of the upward integral curves from $h(I)(x)$ and $h(J)(y)$ (it is not hard to see that there is such a point from our construction of $h(I)$ and $h(J)$). Since the integral curves meet transversally the map is smooth on the interior, and it is smooth on the boundary too away

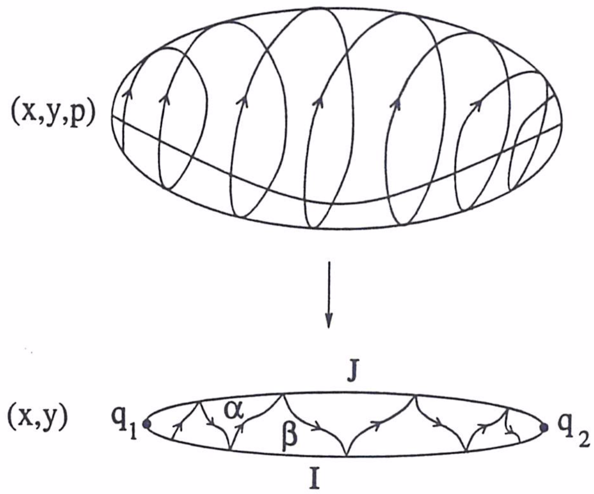

Figure 4: Configuration of the integral curves inside Δ.

from q_1 and q_2. At q_1 and q_2 it is continous. Finally it has an inverse map defined in exactly the same way. The result now follows.

(ii) Δ_0 *is a crossing*: two singularities appear on one side of the transition and none on the other (this follows from the versality condition). There is one singularity on each branch of Δ_t. These are also well-folded saddles or foci and we can proceed as for the case when Δ_0 is an isolated point. On one side of the transition (Figure 3 left) we produce local models at each point on the discriminant and then glue them to obtain the required homeomorphism. On the other side of the transition (Figure 3 right) we also obtain models in a neighbourhood of the discriminant curves as before. For the intervening strip the model is obtained by sliding along integral curves (see [5]).

4 Applications

4.1 1-parameter families of surfaces

We studied in [6] the way in which the geometry associated with the height function varies in a 1-parameter family. The transitions in such families occur generically in three ways at the following singularities of the height function: (i) non versal A_3, (ii) A_4, (iii) D_4 (flat umbilic).

The stable configurations of the asymptotic curves are well-known (see for example [2]) and follow from the results of Dara [12] and Davydov [13]. We wish to determine here the way in which these configurations vary when the family of height functions on the surface passes through the transitions listed above. The case of the flat umbilic is already dealt with in [7] and [9]. We study the case (i) and (ii).

(i) *The* $A_3^{\pm}$ *transitions*: Since the contact of a surface with planes is affine invariant [6], we can write the surface locally, away from flat umbilics, in Monge form

$$f(x,y) = a_0x^2 + \Sigma_{i=0}^3 b_i x^{3-i} y^i + \Sigma_{i=0}^4 c_i x^{4-i} y^i + \Sigma_{i=0}^5 d_i x^{5-i} y^i + \ldots$$

The $A_3^{\pm}$ transitions occur when the family of height functions given by $F(x,y,u,v) = f(x,y) + ux + vy$ fails to versally unfold the A_3 singularity. Geometrically this occurs when the the parabolic set is singular, or equivalently when the Monge-Taylor map fails to be transverse to the A_3 stratum in the jet space [6]. In terms of the coefficients of f these transitions happen when

$$b_2 = b_3 = 0, \quad c_4 \neq 0.$$

The BDE that determines the asymptotic direction is given by

$$f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2 = 0,$$

where the discriminant $\delta = f_{xy}^2 - f_{xx}f_{yy}$ is precisely the parabolic set. Note that as we are away from a flat umbilic, the coefficients of the above equation do not all vanish at the origin, so the equation is equivalent to $dx^2 + g(x,y)dy^2 = 0$ for some g. The 2-jet of δ at the $A_3^{\pm}$ transitions is

$$j^2\delta = 4[(b_1^2 - c_2)x^2 - 3c_3xy - 6c_4y^2].$$

This has generically a Morse singularity, and the branches of δ are transverse to the unique direction $dx = 0$ defined by the BDE at the origin if and only if $c_4 \neq 0$. This is the case at the $A_3^{\pm}$ transitions. Calculations show that we can reduce the 2-jet of the BDE by smooth changes of coordinates to

$$dx^2 + [(-b_1^2 + c_2)x^2 + 3c_3xy + 6c_4y^2]dy^2.$$

So Theorem 2.6 and 3.3 apply here, and in particular the family of surfaces in [6] given by

$$x^2 \pm x^2y^2 \pm y^4 + ty^2$$

exhibits all the different transitions.

Proposition 4.1 *At the $A_3^{\pm}$-transitions the configurations of the asymptotic curves generically undergo the transitions described in Theorem 3.3 (see Figure 3). More precisely, at an A_3^- two hyperbolic cusps of Gauss (ie well-folded saddles) are generically created/destroyed, and at an A_3^+ two elliptic cusps Gauss of the second type (ie well-folded foci) appear/disappear (on one side of the transition).*

(ii) *The A_4 transition*: This transition occur on a smooth parabolic set when

$$b_3 = 0, \quad b_2^2 - 4c_4 = 0, \; d_5 - \frac{1}{2}c_3b_2 - \frac{1}{4}b_1b_2^2 \neq 0.$$

Here a calculation shows that the 4-jet can be transformed to

$$dx^2 + (x + g(x,y))dy^2$$

with $g(0,y) = Ay^3 + ..$ where $A = d_5 - \frac{1}{2}c_3b_2 - \frac{1}{4}b_1b_2^2$. So at an A_4 we can apply results in [14] and deduce the following.

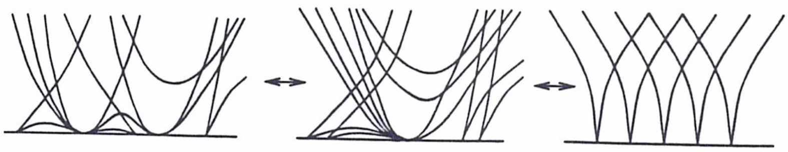

Figure 5: Well-folded saddle-node bifurcations.

Proposition 4.2 *At an A_4 singularity of the height function the configurations of the asymptotic curves generically undergo a well-folded saddle-node bifurcation. See Figure 5*

4.2 Conjugate Curve Congruence

We now describe an example from differential geometry of a 1-parameter family of BDE's that exhibit the umbilic type bifurcation described in [9], all codimension 1 phenomena described here and those in [13] and [14]. This family is constructed as follows ([17], [18]). Take the set of all directions in all tangent planes making a fixed (signed) angle $\alpha \in [-\pi/2, \pi/2]$ with their conjugate direction with respect to the second fundamental form of the surface. Note that when $\alpha = 0$ this gives the asymptotic directions, since these are the self conjugate directions, and when $\alpha = \pm\pi/2$ it yields the principal directions. For a fixed angle α this set is called C_α; *the conjugate curve congruence.*

For a surface parametrised by a map $(x, y) \to \mathbf{r}(x, y) \in \mathbb{R}^3$ with shape operator S, as usual we define $E = \mathbf{r}_x.\mathbf{r}_x$, $F = \mathbf{r}_x.\mathbf{r}_y$, $G = \mathbf{r}_y.\mathbf{r}_y$, $l = S(\mathbf{r}_x).\mathbf{r}_x$, $m = S(\mathbf{r}_x).\mathbf{r}_y$, $n = S(\mathbf{r}_y).\mathbf{r}_y$, and can write C_α [17, Chapter 7] as

$$\begin{aligned} &dy^2(\sin\alpha(Gm - Fn) - n\cos\alpha\sqrt{EG - F^2}) + \\ &dydx(\sin\alpha(Gl - En) - 2m\cos\alpha\sqrt{EG - F^2}) + \\ &dx^2(\sin\alpha(Fl - Em) - l\cos\alpha\sqrt{EG - F^2}) = 0. \end{aligned}$$

In fact if we are away from umbilics we can adopt the special parametrisation where x-constant and y-constant curves are principal curves. Then writing $\kappa_1(x, y)$ and $\kappa_2(x, y)$ for the principal curvatures, C_α can be simplified considerably to the form

$$\kappa_2 \cos\alpha dy^2 + (\kappa_2 - \kappa_1)\sin\alpha dydx + \kappa_1 \cos\alpha dx^2 = 0.$$

It is then straightforward to calculate the discriminant. Writing K for the Gauss curvature and H for the mean curvature the discriminant is

$$H^2 \sin^2\alpha - K = 0.$$

We observe that the discriminant for $\alpha = 0$ (corresponding to the asymptotic directions) is the parabolic curve, and for $\alpha = \pm\pi/2$ (principal directions) the discriminant consists of the umbilics.

We now establish the generic bifurcations of C_α using the methods developed here. For a geometric interpretation of the following conditions and other related geometry concerning this family, see [17] and [18]. Working locally we write our surface in Monge form $(x, y, h(x, y))$ with

$$f(x,y) = \frac{1}{2}\Sigma_{i=0}^{2}\binom{i}{2}a_i x^{2-i}y^i + \frac{1}{6}\Sigma_{i=0}^{3}\binom{i}{3}b_i x^{3-i}y^i + \frac{1}{24}\Sigma_{i=0}^{4}\binom{i}{4}c_i x^{4-i}y^i + \ldots$$

By taking $\sin\alpha_0 = \frac{a_0}{\sqrt{a_0^2+a_1^2}}$, $\cos\alpha_0 = \frac{-a_1}{\sqrt{a_0^2+a_1^2}}$, we fix the x-axis as a member of C_{α_0} at the origin. Then the origin is a smooth point of the discriminant if and only if

$$a_0^2 - a_0a_2 + 2a_1^2 = 0 \text{ and } b_0a_1 - b_1a_0 \neq 0.$$

The 1-jet of the equation of C_{α_0} is equivalent in this case to $dy^2 + xdx^2 = 0$, and hence by a result of Dara [12], the equation can be reduced locally by a diffeomorphism to this linear form. The configuration of the conjugate curve congruence is a family of cusps, with the cusp points forming the discriminant. This configuration is stable.

The origin is a singularity of C_α lying on the discriminant if and only if

$$a_0^2 - a_0a_2 + 2a_1^2 = b_0a_1 - b_1a_0 = 0.$$

Then the 2-jet of the equation of C_{α_0} is equivalent to $dy^2 + (-y + \lambda x^2)dx^2 = 0$ with

$$\lambda = -\tfrac{1}{8a_1^3(a_0^2+a_1^2)(a_0b_2-a_1b_1)}\left(4a_1^3(a_0^2+a_1^2)(a_0(a_1^2+a_0^2)+(a_1c_0-a_0c_1))+\right.$$
$$\left.a_0((2a_1^2+a_0^2)b_1 - a_0a_1b_2)((3a_1^2+a_0^2)b_1 - 2a_0a_1b_2)\right) \neq 0.$$

If $\lambda \neq 0$ and $\lambda \neq 1/16$, this 2-jet is sufficient [13], i.e. determines the local normal form. The zero of the bivalued field is of type well-folded saddle if $\lambda < 0$, well-folded node if $0 < \lambda < 1/16$ and well-folded focus if $\lambda > 1/16$. These elementary singularities are also topologically stable, so appear in nearby C_α for α near α_0.

When $\lambda = 0$ the equation generically undergoes a well-folded saddle-node bifurcation [14]. The equation of C_{α_0} is equivalent to $dy^2+(-y+x^3+Ax^4)dx^2 = 0$, and the family C_α unfolds this singularity (i.e. it is generic). See Figure 5.

We now compute the conditions for the discriminant to have a Morse singularity at the origin in the case where not all the coefficients of C_{α_0} vanish. The equation can then be written as in section 2. With the above set up this occurs when

$$a_0^2 - a_0a_2 + 2a_1^2 = 0, \quad b_0a_1 - b_1a_0 = 0, \quad b_1a_1 - b_2a_0 = 0.$$

Generically these singularities are isolated points in the elliptic region of the surface. The coordinate changes detailed in [8] can be performed explicitly with the computer algebra package MAPLE, reducing the 2-jet of C_{α_0} to the form $dy^2 + (l_1x^2 + l_2xy + l_3y^2)dx^2 = 0$, where

$$l_1 = \frac{-a_0}{16(a_0^2+a_1^2)a_1^4}\left(a_0(a_0^2+a_1^2)b_1^2 - 4a_0a_1^4(a_0^2+a_1^2) + 4a_1^3(a_0c_1 - a_1c_0)\right),$$

$$l_2 = \frac{-a_0}{8a_1^3(a_0^2+a_1^2)}\left((a_0^2+2a_1^2)b_1^2 - a_0^2b_3b_1 + 4a_1^2(c_2a_0 - a_1c_1)-\right.$$

$$4a_2(a_0 + 2a_1^2)(a_0^2 + a_1^2))\,,$$

$$l_3 = \frac{-1}{16a_1^2(a_0^2 + a_1^2)^2}\left(a_0^2 b_3 - (2a_1^2 + a_0^2)b_1^2 + 4a_1(a_0^2 + a_1^2)(a_0(a_0c_3 - a_1c_2) - (a_1(a_0^2 + a_1^2)(3a_2^2 + 4a_1^2))\right).$$

For a generic surface we expect $l_2^2 - 4l_1l_3 \neq 0$ and $l_1 \neq 0$, and therefore Theorem 2.6 applies here. By fixing the coefficients a_i and b_i and varying the c_i's we can obtain all the four types in Theorem 2.6. A short calculation shows that the family C_α satisfies the hypotheses of Theorem 3.3, so the bifurcations are as in Figure 3.

We now study the case where all the coefficients of the equation vanish. Then the discriminant generically has a Morse singularity. Calculations detailed in [17, Chapter 7] show that the only Morse singularity is the isolated point type and this occurs at the umbilics when $\alpha_0 = \pm\pi/2$. It is also shown in [17] that provided the umbilic is ordinary (versally unfolded D_4 of the distance squared function) then C_α is versal in the sense of [9]. In this case [9] provides the following normal forms for C_α,

Lemon:	$(y+\alpha)dy^2 + 2xdydx - ydx^2 = 0$
Star:	$(y+\alpha)dy^2 - 2xdydx - ydx^2 = 0$
Monstar:	$(y+\alpha)dy^2 + (1/2)xdydx - ydx^2 = 0.$

(See also [9] for figures.)

Proposition 4.3 *For a generic surface the family C_α exhibits the well-folded elementary singularities and the well-folded saddle-node bifurcation. It can also undergo the transitions described in Theorem 3.3. At an umbilic C_α versally unfolds the lemon, star and monstar singularities.*

5 Appendix 1

In this appendix we give a model for pairs of singular direction fields both of type non-resonant saddle/node and having a common separatrix. The model is simpler than that of Theorem 2.3. The proof follows the same pattern as that of Theorem 2.2. We ask that both fields are analytic.

Theorem 5.1 *Suppose given two analytic line fields with a singularity at the origin, both of type non-resonant saddle or node having a common separatrix. Suppose further that the zeros have the same eigenvalues, and that away from the common separatrix the fields are transverse (even at complex points). Then the line fields can be reduced simultaneously by a diffeomorphism to one of the following pair*

$$\xi_1 = x\frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial y}$$
$$\xi_2 = (x+y)\frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial y}$$

for generic values of λ (indeed we need $\lambda \neq -1$ and k for any natural number $k \geq 2$).

Proof We shall work with the 1-forms associated to the fields. When the singularities are not resonant we can assume that the 1-form associated with the first field is given by

$$\alpha_1 = xdy - \lambda ydx.$$

(See ([1])). From the hypothesis the 1-jet of the second form β can be written $\beta_1 = (x+y)dy - \lambda ydx$. Away from $y=0$ the two forms are transverse. We claim that β may consequently be written in the form

$$\beta = (x + y + y\phi_1)dy - (\lambda y + y^2\phi_2)dx.$$

To see this we write

$$\beta = (x + y + y\psi_1(x,y) + \psi_2(x))dy - (\lambda y + y^2\psi_3(x,y) + y\psi_4(x) + \psi_5(x))dx.$$

The two 1-forms are dependent on the set given by

$$xy^2\psi_3 + xy\psi_4 + x\psi_5 - \lambda y^2 - \lambda y^2\psi_1 - \lambda y\psi_2 = 0.$$

Since this defines $y=0$ locally we deduce that $\psi_5 = 0$ and $\lambda\psi_2 = x\psi_4$. Replacing β by $x\beta/(x+\psi_4(x))$ establishes the result.

Applying the theorem on the reduction of a vector to its linear part again (λ is not a resonant exponent) shows that this 1-form can be reduced, by a diffeomorphism, to its 1-jet $\beta_1 = (x+\mu y)dy - \lambda ydx$ for almost all values of λ. The claim is that this diffeomorphism must be of the form

$$\Phi(x,y) = (x + y\Phi_1(x,y), y + y^2\Phi_2(x,y)).$$

Indeed, since the two forms have the same 1-jet we can write $\Phi = (x+\phi(x,y), y+\psi(x,y))$, with ϕ and ψ of order 2. Then the 1-form $\Phi^*(\beta_1)$ can be written as $Ady + Bdx$ with

$$\begin{aligned} A = {} & (x + y + \phi + \mu\psi)(1 + \psi_y) - \lambda(y+\psi)\phi_y \\ = {} & x + y + \phi + \mu\psi + (x + \mu y + \phi + \mu\psi)\psi_y + \lambda(y+\psi)\phi_y \\ B = {} & (x + \mu y + \phi + \mu\psi)\psi_x + \lambda(y+\psi)(1+\phi_x) \\ = {} & \lambda y + (x + \mu y + \phi + \mu\psi)\psi_x + \lambda y\phi_x + \lambda\psi(1+\phi_x). \end{aligned}$$

Now $\Phi^*(\beta_1)$ is equal to β if and only if

$$\begin{aligned} \phi + \mu\psi + (x + \mu y + \phi + \mu\psi)\psi_y + \lambda(y+\psi)\phi_y = {} & y\phi_1 \\ (x + \mu y + \phi + \mu\psi)\psi_x + \lambda y\phi_x + \lambda\psi(1+\phi_x) = {} & y^2\phi_2. \end{aligned}$$

Setting $y=0$ and considering initial terms one can show that, provided $\lambda \neq 1$ or $-k$ for any natural number k, the above equations are satisfied only if $\phi = y\Phi_1$ and $\psi = y^2\Phi_2$ for some analytic functions Φ_1 and Φ_2.

We now follow the proof of Theorem A.II.3 in [22]. Let $\tilde{\Phi}(x,y,t) = (x + ty\Phi_1(x,y), y + ty^2\Phi_2(x,y))$, a 1-parameter family of germs of diffeomorphisms at the origin. We denote by ω the 1-form $\tilde{\Phi}^*(\beta_1)$ in $\mathbb{R}^2 \times \mathbb{R}$. Write $\omega = A(x,y,t)dy + B(x,y,t)dx + C(x,y,t)dt$ with

$$\begin{aligned} A = {} & (x + \mu y + ty\Phi_1 + ty^2\Phi_2)(1 + 2ty\Phi_2 + ty^2\tfrac{\partial\Phi_2}{\partial y}) + \lambda(y + ty^2\Phi_2)(t\Phi_1 + ty\tfrac{\partial\Phi_1}{\partial y}) \\ B = {} & t(x + \mu y + ty\Phi_1 + ty^2\Phi_2)(y^2\tfrac{\partial\Phi_2}{\partial x}) + \lambda(y + ty^2\Phi_2)(1 + ty\tfrac{\partial\Phi_1}{\partial x}) \\ C = {} & (x + \mu y + ty\Phi_1 + ty^2\Phi_2)(y^2\Phi_2) + \lambda(y + ty^2\Phi_2)(y\Phi_1) \\ = {} & y^2\tilde{C}(x,y,t) \end{aligned}$$

for some analytic $\tilde{C}$. We need to find a vector field of the form

$$Z = U(x,y,t)\frac{\partial}{\partial x} + V(x,y,t)\frac{\partial}{\partial y} + \frac{\partial}{\partial t}$$

which is tangent to the integral surfaces in 3-space determined by ω and $\pi^*(\alpha_1)$ where π is the projection to the parameter t. This holds when $\omega(Z) = 0$ and $\alpha_1(Z) = 0$, which is equivalent to solving the following linear system for U, V.

$$\begin{aligned} xU + \lambda yV &= 0 \\ AU + BV + C &= 0. \end{aligned}$$

The solution, if it exists, is of the form $U = -\frac{\lambda yC}{\Delta}$, $V = \frac{xC}{\Delta}$, where $\Delta = \lambda yA - xB$.

Since $\Delta = y^2\tilde{\Delta}$ for some analytic $\tilde{\Delta}$, and we can write $C = y^2\tilde{C}$, the system will have a solution provided that $\tilde{\Delta}$ does not vanish on the t-axis. However since ϕ is of order 2 in x, y, we deduce that Φ_1 of order 1. A calculation shows that at $(0,0,t)$ the function $\tilde{\Delta}$ is λ, and the result follows.

We now integrate the field Z. It is easy to see that U and V vanish at each point $(0,0,t)$, so the resulting family of analytic diffeomorphisms preserves the t-axis. Since Z is also tangent to the curves formed by the intersections of the integral surfaces of ω and $\pi^*(\alpha_1)$ it will take the integral curves of β_1 to those of β while preserving those of ω. This establishes the required analytic reduction.

6 Appendix 2

We give here a proof of the topological reduction of an implicit differential equations to a normal form at a well-folded saddle and node singularities. It is stated in [13] that the normal forms are $2y = p^2 - \lambda x^2$ with λ any fixed value in $]-\infty, 0[$ for a well-folded saddle and in $]0, 1/4[$ for a well-folded node. In [13] the equation is smoothly reduced to the normal form $2y = p^2 - \lambda x^2$ under the assumption that the singularity of the lifted field is normal. It is therefore not possible to deduce the topological normal forms directly from the smooth models. (There are no problems when the singularity is a well-folded focus as a focus is always normal.)

Theorem 6.1 *An implicit differential equation $F(x,y,p) = 0$ is locally topologically equivalent to $2y = p^2 + x^2$ at a well-folded saddle and to $2y = p^2 + \frac{1}{8}x^2$ at a well-folded node.*

Proof: We consider the lifted field ξ on the surface M together with the involution σ. Following [13] the problem reduces to classifying pairs (σ, ξ) or equivalently pairs (σ, α), where α is a 1-form. By a suitable choice of coordinates we can set $\sigma(x,y) = (x,-y)$ and $\alpha = (2x + \lambda y + a(x,y))dy - (\lambda x + y + b(x,y))dx$ where a, b are smooth 1-flat functions.

We consider now the polar blow-up $x = r\cos\theta$, $y = r\sin\theta$. The new 1-form in $S^1 \times \mathbb{R}^+$ is given, after a division by r, by

$$\begin{aligned} \tilde{\alpha} = \; & (-\lambda\cos^2\theta + \cos\theta\sin\theta + \lambda\sin^2\theta + r\tilde{a}(r,\theta))dr + \\ & r(2\cos^2\theta + 2\lambda\cos\theta\sin\theta + \sin^2\theta + r\tilde{b}(r,\theta))d\theta \end{aligned}$$

where $\tilde{a}(r,\theta)$ and $\tilde{b}(r,\theta)$ are smooth functions depending on a and b. The singularities of $\tilde{\alpha}$ on S^1 are thus given by

$$-\lambda \cos^2\theta + \cos\theta\sin\theta + \lambda\sin^2\theta = 0,$$

equivalently

$$\lambda\tan^2\theta - \tan\theta - \lambda = 0.$$

The equation has 4 solutions (all saddles when the singularity of α is a saddle, and two saddles and two nodes in the node case). The involution σ lifts to the involution $\tilde{\sigma}(r,\theta) = (r,-\theta)$, and the singularities of $\tilde{\alpha}$ are not paired by $\tilde{\sigma}$.

We now identify points with their images under $\tilde{\sigma}$. This identification results in a pair of 1-forms in $[0,\pi]\times\mathbb{R}^+$ with a common integral curve along $[0,\pi]\times\{0\}$. There are three situations to consider.

At $\theta = 0, \pi$ the local model is given by Theorem 4.3 in [7]. The configuration is locally smoothly equivalent to the folding of the level sets of the function $y(1+x^3)$ along the y-axis.

At the singular points of one of the forms the other form is regular, so following Proposition 4.1 in [5] the pair is locally topologically equivalent to $(dy, xdy \pm ydx)$. We note here that the Proposition 4.1 in [5] does not assume that the singularities are normal.

Away from the above points the model is given by $(dy, xdy + ydx)$ (Lemma 3.1 in [5]).

We glue the local model together by sliding along the integral curves (see [5] for details) to obtain the required homeomorphism.

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