GLOBAL ANALYTIC SOLVABILITY OF INVOLUTIVE SYSTEMS ON COMPACT MANIFOLDS

G. ARAÚJO, P. L. DATTORI DA SILVA, AND B. DE LESSA VICTOR

Abstract. Let M be a compact, connected, orientable and real-analytic manifold; consider closed, real-valued, real-analytic 1-forms $\omega_1, \ldots, \omega_m$ on M and the differential complex over $M \times \mathbb{T}^m$ naturally associated to the involutive system determined by them. In the real-analytic context, we completely characterize global solvability of the operators in its first (functional setting) and last (distributional setting) levels. Analogous results are obtained simultaneously in the Gevrey framework.

1. Introduction

Given Ω a closed smooth manifold and $P : C^\infty(\Omega) \to C^\infty(\Omega)$ a linear differential operator, the solvability of the equation $Pu = f$ forces certain restrictions, known as *compatibility conditions*, on the right-hand side f. Namely, if we aim to find a distribution solution $u \in \mathcal{D}'(\Omega)$, then

$$\langle f, \phi \rangle = 0, \quad \forall \phi \in C^\infty(\Omega) \cap \ker {}^{\mathrm{t}}P, \tag{1.1}$$

where ${}^{\mathrm{t}}P$ is the transpose of P. In the absence of this property the existence of a solution u is untenable, hence it only makes sense to try to solve $Pu = f$ for those f satisfying (1.1). These considerations are appropriate to f either in $C^\infty(\Omega)$ or in $\mathcal{D}'(\Omega)$, but in the former case one could want, alternatively, to solve $Pu = f$ with u smooth; in this context, we would necessarily have

$$\langle v, f \rangle = 0, \quad \forall v \in \mathcal{D}'(\Omega) \cap \ker {}^{\mathrm{t}}P, \tag{1.2}$$

a condition, in principle, more stringent than (1.1), leading to a different problem.

This digression sets the stage for different notions of *global solvability* of P, paramount in the theory of linear PDEs. Variations can be obtained by allowing P to act on other function spaces [1, 2, 16], and by considering more general vector-valued operators (making in each case the necessary adaptations). Of great importance among such equations are the so-called *tube structures* [7, 10, 11, 13, 14, 19], one of the main models of interest in the study of systems of linear PDEs, which we briefly describe.

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Given a compact n-dimensional manifold M and closed 1-forms $\omega_1, \dots, \omega_m$ over M, consider the product manifold $\Omega \doteq M \times \mathbb{T}^m$ (where $\mathbb{T}^m$ is the m-dimensional torus) and the subbundle $\mathcal{V} \subset \mathbb{C}T\Omega$ annihilated by all the forms

$$\zeta_k \doteq \mathrm{d}x_k - \omega_k, \quad k \in \{1, \dots, m\},$$

where $(x_1, \dots, x_m)$ are angular coordinates on $\mathbb{T}^m$. Such a bundle is involutive, and gives rise to a complex of vector bundles over Ω and first-order differential operators between them (see for instance [12, 24]), here concretely realized as follows.

Let Λ^q be the bundle of complex q-forms over M, and $\Lambda^{0,q}$ be its pullback to Ω via the natural projection $\Omega \to M$; we denote their spaces of smooth sections by $\Lambda^q C^\infty(M)$ and $\Lambda^{0,q} C^\infty(\Omega)$, respectively. We define $\mathbb{L}^q : \Lambda^{0,q} C^\infty(\Omega) \to \Lambda^{0,q+1} C^\infty(\Omega)$ by the formula

$$\mathbb{L}^q f \doteq \mathrm{d}_t f + \sum_{k=1}^{m} \omega_k \wedge \partial_{x_k} f, \tag{1.3}$$

where $t \in M$ and d_t is the exterior derivative on M; hence, $\mathbb{L}^q$ is a first-order differential operator, and a simple computation shows that $\mathbb{L}^{q+1} \circ \mathbb{L}^q = 0$.

Global solvability of $\mathbb{L}^q u = f$ in the smooth setup, subject to its underlying compatibility conditions, is then of major interest. Usually, it is investigated in degrees $q \in \{0, n-1\}$ (an assumption that will be kept in this work) and for corank $m = 1$ (which will be dropped).

So far we have described the environment. The problem we intend to deal with is global solvability of $\mathbb{L}^q$ in spaces of *real-analytic* sections; or, more generally, in spaces of *Gevrey* sections of order $s \geq 1$ (among which $s = 1$ comprises the real-analytic case) as well as their generalized function counterparts, the spaces of *ultradistributions* of Gevrey order s (of which the case $s = 1$ must be interpreted as the space of *hyperfunctions*). See [8, 9, 15] for some previous investigations of global solvability in the Gevrey setup for tube structures defined on tori.

Appropriate regularity of our data must then be assumed. In the present work, we will consider

$$M \text{ a real-analytic manifold and } \omega_1, \dots, \omega_m \text{ of Gevrey order } s,$$

in which case the real-analytic vector bundles $\Lambda^{0,q}$ admit spaces of G^s sections, here denoted by $\Lambda^{0,q} G^s(\Omega)$, and carry a natural locally convex topology described in Subsection 2.1; moreover, $\mathbb{L}^q : \Lambda^{0,q} G^s(\Omega) \to \Lambda^{0,q+1} G^s(\Omega)$ continuously as a differential operator with G^s coefficients. We are in position to describe our notions of solvability.

Definition 1.1. We say that $\mathbb{L}^q$ is:

(1) *globally* $(\mathcal{D}'_s, \mathcal{D}'_s)$*-solvable* if for every $f \in \Lambda^{0,q+1} \mathcal{D}'_s(\Omega)$ satisfying

$$\langle f, \phi \rangle = 0, \quad \forall \phi \in \Lambda^{0,n-q-1} G^s(\Omega) \cap \ker \mathbb{L}^{n-q-1}, \tag{1.4}$$

there exists $u \in \Lambda^{0,q} \mathcal{D}'_s(\Omega)$ solving $\mathbb{L}^q u = f$;

(2) *globally* (G^s, G^s)*-solvable* if for every $f \in \Lambda^{0,q+1} G^s(\Omega)$ satisfying

$$\langle v, f \rangle = 0, \quad \forall v \in \Lambda^{0,n-q-1} \mathcal{D}'_s(\Omega) \cap \ker \mathbb{L}^{n-q-1}, \tag{1.5}$$

there exists $u \in \Lambda^{0,q} G^s(\Omega)$ solving $\mathbb{L}^q u = f$;

(3) *globally* $(\mathcal{D}'_s, G^s)$*-solvable* if for every $f \in \Lambda^{0,q+1} G^s(\Omega)$ satisfying (1.4) there exists $u \in \Lambda^{0,q} \mathcal{D}'_s(\Omega)$ solving $\mathbb{L}^q u = f$.

Notice that the space of ultradistribution sections $\Lambda^{0,q}\mathcal{D}'_s(\Omega)$ is in natural duality with $\Lambda^{0,n-q-1}G^s(\Omega)$, even when $s = 1$; with respect to this duality, the transpose of $\mathbb{L}^q$ is essentially $\mathbb{L}^{n-q-1}$ (see (3.2)).

In the present work, our main interest will be in the case where

$$\omega_1, \dots, \omega_m \text{ are real-valued}$$

(it is worth mentioning, however, that this latter hypothesis will only kick in halfway through Section 4 and onward). It can, therefore, be regarded as a natural continuation of [4], where we characterized global Gevrey hypoellipticity of $\mathbb{L}^0$: such phenomenon, as we will see, is directly connected with our current problem. In fact, condition (d) in our main result (Theorem 1.3) can be regarded as a condition of global s-hypoellipticity in a certain closed subspace of distributions (Section 4).

Before we state our main result, we recall a definition [6, Definition 2.1]:

Definition 1.2. A real, closed $\alpha \in \Lambda^1 C^\infty(M)$ is *integral* if and only if $\int_\sigma \alpha \in 2\pi\mathbb{Z}$ for every 1-cycle σ in M; it is otherwise *rational* if $q\alpha$ is integral for some $q \in \mathbb{Z}\setminus\{0\}$.

Below we denote $\boldsymbol{\omega} \doteq (\omega_1, \dots, \omega_m)$ and

$$\Gamma_1 \doteq \left\{ \xi = (\xi_1, \dots, \xi_m) \in \mathbb{Z}^m \ ; \ \xi \cdot \boldsymbol{\omega} \doteq \sum_{k=1}^m \xi_k \omega_k \quad \text{is } \textbf{not} \text{ integral} \right\}.$$

Theorem 1.3. *Suppose that $\omega_1, \dots, \omega_m$ are real. The following are equivalent:*

(a) $\mathbb{L}^{n-1}$ is globally $(\mathcal{D}'_s, \mathcal{D}'_s)$-solvable.
(b) $\mathbb{L}^{n-1}$ is globally $(\mathcal{D}'_s, G^s)$-solvable.
(c) $\mathbb{L}^0$ is globally (G^s, G^s)-solvable.
(d) The matrix of periods $\mathsf{A}(\boldsymbol{\omega}) \in \mathsf{M}_{d\times m}(\mathbb{R})$ (see (4.4) for its definition) satisfies the following: for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|\kappa + \mathsf{A}(\boldsymbol{\omega})\xi| \geq C_\epsilon e^{-\epsilon(|\kappa|+|\xi|)^{\frac{1}{s}}}, \quad \forall (\kappa, \xi) \in \mathbb{Z}^d \times \Gamma_1.$$

(e) $\boldsymbol{\omega}$ is not (s, Γ_1)-exponential Liouville (see Definition 4.4). That is, the following property **does not** *hold: there exist $\epsilon > 0$, a sequence of integral forms $\{\theta_\nu\}_{\nu\in\mathbb{N}} \subset \Lambda^1 G^s(M;\mathbb{R})$ and $\{\xi_\nu\}_{\nu\in\mathbb{N}} \subset \Gamma_1$ such that $|\xi_\nu| \to \infty$ and*

$$\{e^{\epsilon|\xi_\nu|^{\frac{1}{s}}}(\xi_\nu \cdot \boldsymbol{\omega} - \theta_\nu)\}_{\nu\in\mathbb{N}} \text{ is bounded in } \Lambda^1 G^s(M).$$

Our proof is done is several steps along the sections of this paper, which are summarized in Section 6. The organization is as follows. In Section 2 we introduce the theoretical tools that will be necessary; mainly, the spaces of forms in which we operate, their locally convex topologies and their partial Fourier series.

In Section 3 we start our analysis of tube structures and the notions of global solvability of interest; in Subsection 3.1 we connect the contents of Theorem 1.3 with the notion of global s-hypoellipticity of $\mathbb{L}^0$ (actually, the weaker property (3.5)), for which we introduce heavier functional analytic machinery. The notion of global $(\mathcal{D}'_s, G^s)$-solvability is discussed in Subsection 3.2, where an important *a priori* inequality for it is derived.

The partial Fourier series is employed, in Section 4, to define suitable decompositions of the spaces of forms related to property *(e)* above, while the Diophantine condition in *(d)* is discussed in Subsection 4.1. A normal form for $\mathbb{L}^0$ in the set of frequencies complementary to Γ_1 is then obtained in Section 5.

Finally, after some concluding remarks in Section 6, we draw some interesting consequences and applications of Theorem 1.3 in Subsection 6.1. We explore certain substructures of $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)$ obtained by putting away some of the 1-forms ω_k, yielding new tube structures (such a procedure changes both the base space and the corank of the structure), and investigate the connection between the Gevrey solvability of the former and the latter systems. An in-depth analysis of the corank 1 case is provided (Theorem 6.4), which parallels [7, Theorem 1.9] for real forms. Even in that case, our Theorem 1.3 has conceptual advantages over theirs: for our notion of solvability *(a)* is by design stronger[1] than *(b)*, which is the de facto analogue of their solvability in the smooth setup [7, Definition 1.2]; and everything is further connected with solvability *(c)* in degree $q = 0$.

2. Preliminaries

In this section we introduce the theoretical tools that we will need along the paper.

2.1. Spaces of Gevrey sections of vector bundles. Let $s \geq 1$. Given $K \subset \mathbb{R}^n$ a regular compact set and $h > 0$ we let

$$G^{s,h}(K) \doteq \left\{ f \in C^\infty(K) \ ; \ \|f\|_{s,h,K} \doteq \sup_{\alpha \in \mathbb{Z}_+^n} h^{-|\alpha|} \alpha!^{-s} \sup_K |\partial^\alpha f| < \infty \right\}.$$

The space of Gevrey functions of order s on an open set $U \subset \mathbb{R}^n$, which we denote by $G^s(U)$, is the space of all $f \in C^\infty(U)$ such that for every regular compact set $K \subset U$ there exists $h > 0$ such that $f \in G^{s,h}(K)$. For more details about Gevrey functions see, for instance, [23].

Now let Ω be a compact real-analytic manifold and E be a real-analytic vector bundle of rank r over Ω. As in [4], given $s \geq 1$ we can endow $G^s(\Omega; E)$ – the space of sections of E of Gevrey order s – with a locally convex topology as follows. Pick

- a finite analytic atlas $\{(U_i, \chi_i)\}_{i \in I}$ of Ω,
- over which we can further find analytic local trivializations of E i.e. local frames of analytic sections $e_{i1}, \ldots, e_{ir} : U_i \to E$ and
- regular compact sets $K_i \subset U_i$ whose interiors still cover Ω.

Then every $f \in G^s(\Omega; E)$ can be written in U_i as

$$f = \sum_{j=1}^{r} f_{ij} e_{ij}, \quad f_{ij} \in G^s(U_i),$$

and we endow $G^s(\Omega; E)$ with the coarsest topology that makes all the linear maps

$$f \in G^s(\Omega; E) \longmapsto f_i \circ \chi_i^{-1} \in G^s(\chi_i(K_i))^r, \quad i \in I,$$

continuous, where $f_i \doteq (f_{i1}, \ldots, f_{ir}) : U_i \to \mathbb{C}^r$.

Such a topology is independent of our choices (of coverings, frames, etc.) and turns $G^s(\Omega; E)$ into a DFS space (for which we refer the reader to [20]). Actually, if for each $h > 0$ we let

$$G^{s,h}(\Omega; E) \doteq \{ f \in G^s(\Omega; E) \ ; \ f_i \circ \chi_i^{-1} \in G^{s,h}(\chi_i(K_i))^r, \ \forall i \in I \}, \tag{2.1}$$

[1] Although in the end equivalent to it by virtue of the very same Theorem 1.3.

endowed with the norm

$$\|f\|_{s,h,\Omega} \doteq \sum_{i\in I}\sum_{j=1}^{r} \|f_{ij}\circ\chi_i^{-1}\|_{s,h,\chi_i(K_i)} = \sum_{i\in I}\sum_{j=1}^{r} \sup_{\alpha\in\mathbb{Z}_+^n}\sup_{\chi_i(K_i)} \left(\frac{|\partial^\alpha(f_{ij}\circ\chi_i^{-1})|}{h^{|\alpha|}\alpha!^s}\right); \quad (2.2)$$

then

$$G^s(\Omega;E) = \varinjlim_{h>0} G^{s,h}(\Omega;E)$$

as the injective limit of a system of Banach spaces with compact inclusion maps (notice that definitions (2.1)-(2.2) are not invariant though). It follows that $G^s(\Omega;E) \subset C^\infty(\Omega;E)$ continuously for the standard Fréchet topology on the space of smooth sections, and also that whenever $s_+ > s$ we have

$$G^s(\Omega;E) \subset G^{s_+,1}(\Omega;E) \quad \text{continuously.} \tag{2.3}$$

Moreover, we may easily derive the following criteria for boundedness of subsets and convergence of sequences in $G^s(\Omega;E)$.

Proposition 2.1. *A subset $B \subset G^s(\Omega;E)$ is bounded if and only if there exist constants $C, h > 0$ such that*

$$\|f_{ij}\circ\chi_i^{-1}\|_{s,h,\chi_i(K_i)} \leq C,$$

for every $f \in B$, every $i \in I$ and $j \in \{1,\dots,r\}$. A sequence $\{f_\nu\}_{\nu\in\mathbb{N}}$ converges to zero in $G^s(\Omega;E)$ if and only if there exists $h > 0$ such that for every $i \in I$ and $j \in \{1,\dots,r\}$ we have $\{(f_\nu)_{ij}\circ\chi_i^{-1}\}_{\nu\in\mathbb{N}} \subset G^{s,h}(\chi_i(K_i))$ and $\|(f_\nu)_{ij}\circ\chi_i^{-1}\|_{s,h,\chi_i(K_i)} \to 0$.

2.2. **Partial Fourier series.** Let us recall some aspects of the partial Fourier series for functions (resp. distributions) as defined in [4]. Given $f \in C^\infty(U\times\mathbb{T}^m)$, where $U \subset \mathbb{R}^n$ is an open set and $\mathbb{T}^m$ is the m-dimensional torus, we define for each $\xi \in \mathbb{Z}^m$ a function $\hat{f}_\xi \in C^\infty(U)$ by

$$\hat{f}_\xi(t) \doteq \int_{\mathbb{T}^m} e^{-ix\cdot\xi} f(t,x)\mathrm{d}x; \tag{2.4}$$

more generally, if $f \in \mathcal{D}'(U\times\mathbb{T}^m)$ we define $\hat{f}_\xi \in \mathcal{D}'(U)$ by the rule

$$\phi \in C_c^\infty(U) \longmapsto \langle f, \phi\otimes e^{-ix\cdot\xi}\rangle \in \mathbb{C}.$$

It is easy to check that this construction is local and invariant by changes of coordinates: if $\chi : U' \to U$ is a diffeomorphism between open sets in $\mathbb{R}^n$ then

$$\widehat{(X^*f)}_\xi = \chi^*\hat{f}_\xi, \quad \forall \xi \in \mathbb{Z}^m,$$

whatever $f \in \mathcal{D}'(U\times\mathbb{T}^m)$, where by definition

$$\begin{array}{cccc} X & : & U'\times\mathbb{T}^m & \longrightarrow & U\times\mathbb{T}^m \\ & & (t',x) & \longmapsto & (\chi(t'),x) \end{array}.$$

These properties allow us to define $\hat{f}_\xi \in \mathcal{D}'(M)$ for $f \in \mathcal{D}'(M\times\mathbb{T}^m)$, where M is now a smooth manifold, in which case formula (2.4) holds for all $f \in L^1_{\mathrm{loc}}(M\times\mathbb{T}^m)$.

2.3. **Forms of type $(0,q)$.** Let M be an n-dimensional smooth manifold, which we will further assume to be compact, connected, oriented, and also real-analytic from here on.

On $\Omega \doteq M \times \mathbb{T}^m$, for each $q \in \{0, \dots, n\}$ we denote by $\Lambda^{0,q}$ the bundle of q-forms over Ω that are locally spanned by $\mathrm{d}t_J \doteq \mathrm{d}t_{j_1} \wedge \cdots \wedge \mathrm{d}t_{j_q}$, $J = (j_1 < j_2 < \cdots < j_q)$, where $(t_1, \dots, t_n)$ is an analytic coordinate system on M. Given $\mathscr{F}$ a suitable sheaf of coefficients on Ω (resp. M) – say, smooth or Gevrey functions, or even their distributional counterparts – we denote by $\Lambda^{0,q}\mathscr{F}(\Omega)$ (resp. $\Lambda^q\mathscr{F}(M)$) the associated space of sections over Ω (resp. space of q-forms or q-currents over M). If $(U; t_1, \dots, t_n)$ is an analytic coordinate chart on M then any $f \in \Lambda^{0,q}\mathscr{F}(\Omega)$ can be written on $U \times \mathbb{T}^m$ as[2]

$$f = \sum_{|J|=q}{}' f_J \, \mathrm{d}t_J \tag{2.5}$$

where $f_J \in \mathscr{F}(U \times \mathbb{T}^m)$.

An $f \in \Lambda^{0,q}C^\infty(\Omega)$ acts on a $v \in \Lambda^{0,n-q}G^s(\Omega)$ in a natural way:

$$\langle f, v \rangle \doteq \int_\Omega f \wedge v \wedge \mathrm{d}x, \tag{2.6}$$

where $\mathrm{d}x \doteq \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m$. For $s \geq 1$, this pairing permits us to identify $\Lambda^{0,q}\mathcal{D}'_s(\Omega)$ with the topological dual of $\Lambda^{0,n-q}G^s(\Omega)$: the latter space is of course endowed with the DFS topology described in Subsection 2.1, so we endow the former with the strong dual topology, turning it into a so-called FS space [20].

Remark 2.2. In the case $s = 1$ the interpretation is more delicate: $\Lambda^{0,q}\mathcal{D}'_1(\Omega)$ is the space of *hyperfunction sections* of $\Lambda^{0,q}$. If we were to write an $f \in \Lambda^{0,q}\mathcal{D}'_1(\Omega)$ in local coordinates as in (2.5) we would need be very careful and take $f_J \in \mathfrak{B}(U \times \mathbb{T}^m)$, the space of hyperfunctions on $U \times \mathbb{T}^m$. However, since Ω is compact we can, and will, identify $\Lambda^{0,q}\mathcal{D}'_1(\Omega)$ with the topological dual of $\Lambda^{0,n-q}G^1(\Omega)$ (we could even take this as the *definition* of $\Lambda^{0,q}\mathcal{D}'_s(\Omega)$ when $s = 1$).

2.4. **Partial Fourier series of $(0,q)$-forms.** Similarly to what was done in [4] for $(0,1)$-forms, we extend the definition of the partial Fourier coefficients to forms $f \in \Lambda^{0,q}C^\infty(\Omega)$. Given $U \subset M$ domain of a coordinate system $(t_1, \dots, t_n)$ we write f as in (2.5), where $f_J \in C^\infty(U \times \mathbb{T}^m)$, and then define $\hat{f}_\xi \in \Lambda^q C^\infty(U)$ by

$$\hat{f}_\xi \doteq \sum_{|J|=q}{}' \widehat{(f_J)}_\xi \, \mathrm{d}t_J. \tag{2.7}$$

One can prove that this definition is independent of the choice of coordinates, which implies that $\hat{f}_\xi \in \Lambda^q C^\infty(M)$.

Lemma 2.3. *For each $\xi \in \mathbb{Z}^m$ the assignment $f \in \Lambda^{0,q}G^s(\Omega) \mapsto \hat{f}_\xi \in \Lambda^q G^s(M)$ is continuous.*

Proof. As a consequence of the topology constructed in Subsection 2.1, in addition to (2.5)-(2.7), it suffices to prove the statement for functions (i.e. the case $q = 0$). We will take the time to derive a slightly sharper estimate than required for the moment because it will be applied later on.

[2] "Primed" sums are taken only over ordered multi-indices J, being therefore unique.

If $f \in G^s(\Omega)$ then there exists $h > 0$ such that $f \in G^{s,h}(\Omega)$. Taking $U \subset M$ a coordinate open set, for any compact subset $K \subset U$ one has

$$\sup_{K\times\mathbb{T}^m} |\partial_t^\alpha \partial_x^\beta f| \leq \|f\|_{s,h,\Omega}\; h^{|\alpha|+|\beta|}(\alpha! + \beta)!^s, \quad \forall \alpha \in \mathbb{Z}_+^n,\ \forall \beta \in \mathbb{Z}_+^m,$$

where $\|\cdot\|_{s,h,\Omega}$ represents the norm in $G^{s,h}(\Omega)$. Let $\xi \in \mathbb{Z}^m$, $k \in \mathbb{Z}_+$ and pick a $\beta \in \mathbb{Z}_+^m$ such that $|\beta| = k$ and $|\xi|^k \leq m^k|\xi^\beta|$. Then

$$\begin{aligned}
|\xi|^k|\partial_t^\alpha \hat{f}_\xi(t)| &\leq m^k|\xi^\beta \partial_t^\alpha \hat{f}_\xi(t)| \\
&= m^k \left| \int_{\mathbb{T}^m} e^{-ix\cdot\xi}\partial_t^\alpha \partial_x^\beta f(t,x)\mathrm{d}x \right| \\
&\leq (2\pi)^m m^k \|f\|_{s,h,\Omega} h^{k+|\alpha|}\alpha!^s \beta!^s 2^{s(k+|\alpha|)} \\
&\leq (2\pi)^m \|f\|_{s,h,\Omega}(2^s h)^{|\alpha|}\alpha!^s (hm2^s)^k k!^s,
\end{aligned}$$

for every $\alpha \in \mathbb{Z}_+^n$ and $t \in K$. That is

$$\frac{|\xi|^k}{(hm2^s)^k k!^s}|\partial_t^\alpha \hat{f}_\xi(t)| \leq (2\pi)^m \|f\|_{s,h,\Omega}(2^s h)^{|\alpha|}\alpha!^s, \quad \forall k \in \mathbb{Z}_+,$$

which implies that

$$\frac{|\xi|^{\frac{k}{s}}}{(4(hm)^{\frac{1}{s}})^k k!}|\partial_t^\alpha \hat{f}_\xi(t)|^{\frac{1}{s}} \leq \frac{1}{2^k}(2\pi)^{\frac{m}{s}} \|f\|_{s,h,\Omega}^{\frac{1}{s}}(2h^{\frac{1}{s}})^{|\alpha|}\alpha!, \quad \forall k \in \mathbb{Z}_+.$$

Summing both sides over $k \in \mathbb{Z}_+$ and raising them back to the power of s we obtain

$$e^{\sigma|\xi|^{\frac{1}{s}}}|\partial_t^\alpha \hat{f}_\xi(t)| \leq 2^s(2\pi)^m \|f\|_{s,h,\Omega}(2^s h)^{|\alpha|}\alpha!^s,$$

where $\sigma \doteq s/4(hm)^{\frac{1}{s}}$. Hence, by suitably rearranging the terms and maximizing in both $t \in K$ and $\alpha \in \mathbb{Z}_+^n$, and noticing that these estimates are independent of our choice of U and K, we obtain

$$\|\hat{f}_\xi\|_{s,2^s h,M} \leq 2^s(2\pi)^m \|f\|_{s,h,\Omega}\; e^{-\sigma|\xi|^{\frac{1}{s}}}, \quad \forall \xi \in \mathbb{Z}^m, \tag{2.8}$$

where $\|\cdot\|_{s,h,M}$ represents the norm in $G^{s,h}(M)$ (see Subsection 2.1). This estimate (which can also be obtained for forms by working with their local coordinate components) shows, among other things, the desired continuity for each $\xi \in \mathbb{Z}^m$ fixed. □

Lemma 2.4. *For each $f \in \Lambda^{0,q}G^s(\Omega)$ we have*

$$f = \frac{1}{(2\pi)^m}\sum_{\xi\in\mathbb{Z}^m} e^{ix\cdot\xi} \otimes \hat{f}_\xi \tag{2.9}$$

with convergence in $\Lambda^{0,q}G^s(\Omega)$.

Proof. As in Lemma 2.3, it is sufficient to prove the statement for functions. We take an open covering $\{U_i\}_{i\in I}$ of M and a family of compact sets $\{K_i\}_{i\in I}$ satisfying the properties stated in the beginning of Subsection 2.1. It follows from (2.8) that

$$\|\hat{f}_\xi\|_{L^2(M)} = \left(\int_M |\hat{f}_\xi|^2 \mathrm{d}\mu\right)^{\frac{1}{2}} \leq 2^s(2\pi)^m \mathrm{vol}(M)^{\frac{1}{2}}\|f\|_{s,h,\Omega}\; e^{-\sigma|\xi|^{\frac{1}{s}}}, \quad \forall \xi \in \mathbb{Z}^m,$$

with respect to an arbitrary Riemannian metric we endow M with ($\mathrm{d}\mu$ the underlying volume form). One easily sees that the series in (2.9) converges to a $g \in L^2(\Omega)$ in the L^2 norm (now regarding the product metric on Ω). This in turn satisfies $\hat{g}_\xi = \hat{f}_\xi$ for every $\xi \in \mathbb{Z}^m$, and it follows from [4, Proposition 5.2] that $g = f$; the only thing left to prove is that the series actually converges in $G^s(\Omega)$.

For any $\nu \in \mathbb{N}$ we write

$$f - \underbrace{\frac{1}{(2\pi)^m} \sum_{|\xi| \leq \nu} e^{ix\cdot\xi} \otimes \hat{f}_\xi}_{\doteq g_\nu} = \frac{1}{(2\pi)^m} \sum_{|\xi| \geq \nu+1} e^{ix\cdot\xi} \otimes \hat{f}_\xi.$$

Thus again from (2.8):

$$\sup_{K_i \times \mathbb{T}^m} |\partial_t^\alpha \partial_x^\beta (f - g_\nu)| \leq \sum_{|\xi| \geq \nu+1} \sup_{\mathbb{T}^m} |\partial_x^\beta e^{ix\cdot\xi}| 2^s (2\pi)^m \|f\|_{s,h,\Omega} (2^s h)^{|\alpha|} \alpha!^s e^{-\sigma|\xi|^{\frac{1}{s}}},$$

while by [4, Lemma 6.1] there exists $h_1 > 0$ such that

$$\sup_{\mathbb{T}^m} |\partial_x^\lambda e^{ix\cdot\xi}| \leq h_1^{|\lambda|} \lambda!^s e^{\frac{\sigma}{2}|\xi|^{\frac{1}{s}}}, \quad \forall \lambda \in \mathbb{Z}_+^m.$$

Hence

$$\sup_{K_i \times \mathbb{T}^m} |\partial_t^\alpha \partial_x^\beta (f - g_\nu)| \leq h_1^{|\beta|} \beta!^s 2^s (2\pi)^m \|f\|_{s,h,\Omega} (2^s h)^{|\alpha|} \alpha!^s \sum_{|\xi| \geq \nu+1} e^{-\frac{\sigma}{2}|\xi|^{\frac{1}{s}}}.$$

Setting $h_2 \doteq \max\{2^s h, h_1\}$ one deduces that

$$\frac{\sup_{K_i \times \mathbb{T}^m} |\partial_t^\alpha \partial_x^\beta (f - g_\nu)|}{h_2^{|\alpha|+|\beta|} (\alpha+\beta)!^s} \leq 2^s (2\pi)^m \|f\|_{s,h,\Omega} \sum_{|\xi| \geq \nu+1} e^{-\frac{\sigma}{2}|\xi|^{\frac{1}{s}}}, \quad \forall \alpha \in \mathbb{Z}_+^n,\ \forall \beta \in \mathbb{Z}_+^m,$$

which allows us to conclude that the convergence stated holds in $G^{s,h_2}(K_i \times \mathbb{T}^m)$ for each $i \in I$, thus yielding the desired result (see Proposition 2.1). ☐

3. Global solvability of tube structures

For each $q \in \{0, \dots, n-1\}$ we define the differential operator $\mathbb{L}^q : \Lambda^{0,q} \to \Lambda^{0,q+1}$ between analytic vector bundles over Ω by its action on smooth sections (1.3). Since we are assuming $\omega_1, \dots, \omega_m \in \Lambda^1 G^s(M)$, we have that $\mathbb{L}^q$ has G^s coefficients (in analytic local frames and coordinates). In particular,

$$\mathbb{L}^q : \Lambda^{0,q} G^s(\Omega) \longrightarrow \Lambda^{0,q+1} G^s(\Omega) \tag{3.1}$$

continuously. With respect to the pairing (2.6), the following formula is well-known and easy to derive from Stokes Theorem:

$$\int_\Omega (\mathbb{L}^q f) \wedge v \wedge \mathrm{d}x = (-1)^{q+1} \int_\Omega f \wedge (\mathbb{L}^{n-q-1} v) \wedge \mathrm{d}x, \tag{3.2}$$

for any $f \in \Lambda^{0,q} C^\infty(\Omega)$ and $v \in \Lambda^{0,n-q-1} C^\infty(\Omega)$. It identifies

$$\mathbb{L}^q : \Lambda^{0,q} \mathcal{D}'_s(\Omega) \longrightarrow \Lambda^{0,q+1} \mathcal{D}'_s(\Omega) \tag{3.3}$$

with the transpose of the map

$$(-1)^{q+1} \mathbb{L}^{n-q-1} : \Lambda^{0,n-q-1} G^s(\Omega) \longrightarrow \Lambda^{0,n-q} G^s(\Omega)$$

and vice versa. Standard considerations about global solvability of $\mathbb{L}^q$ lead to natural compatibility conditions (1.5)-(1.4) for it, summarized in Definition 1.1. Well-known functional analytic arguments (see e.g. [3, Lemma 2.2]) entail:

$$\mathbb{L}^q \text{ is globally } (G^s, G^s)\text{-solvable} \iff (3.1) \text{ has closed range},$$
$$\mathbb{L}^q \text{ is globally } (\mathcal{D}'_s, \mathcal{D}'_s)\text{-solvable} \iff (3.3) \text{ has closed range}.$$

Moreover, the Homomorphism Theorem for Fréchet-Montel spaces [21, p. 18] yields:

$$\mathbb{L}^q \text{ is globally } (\mathcal{D}'_s, \mathcal{D}'_s)\text{-solvable} \iff \mathbb{L}^{n-q-1} \text{ is globally } (G^s, G^s)\text{-solvable}. \tag{3.4}$$

3.1. **Connections with regularity.** We recall an abstract result [5, Corollary 5.4] that will be used a couple of times:

Theorem 3.1. *For $j \in \{1, 2\}$, let $X_j, X_j^\sharp$ be DFS spaces, $X_j \subset X_j^\sharp$ with continuous inclusion mappings, and the additional property that if we write $X_1^\sharp = \varinjlim E_k$ as the injective limit of a sequence of Banach spaces with compact inclusion mappings, then*

$$X_1 \subset E_{k_0} \quad \textit{continuously for some } k_0.$$

Let $T : X_1^\sharp \to X_2^\sharp$ be a continuous linear map such that the restriction $T : X_1 \to X_2$ is well-defined, continuous and satisfies

$$\forall u \in X_1^\sharp,\ Tu \in X_2 \implies \exists v \in X_1 \textit{ such that } Tv = Tu.$$

Then $T : X_1 \to X_2$ has closed range.

Its next consequence is straightforward (recall (2.3)).

Corollary 3.2. *Suppose that for some $s_+ > s \geq 1$ the following holds:*

$$\forall u \in \Lambda^{0,q} G^{s_+}(\Omega),\ \mathbb{L}^q u \in \Lambda^{0,q+1} G^s(\Omega) \implies \exists v \in \Lambda^{0,q} G^s(\Omega) \textit{ such that } \mathbb{L}^q v = \mathbb{L}^q u. \tag{3.5}$$

Then $\mathbb{L}^q$ is globally (G^s, G^s)-solvable.

Remark 3.3. Notice the resemblance of (3.5) with *global s-hypoellipticity*, which for $q = 0$ reads:

$$\forall u \in \mathcal{D}'(\Omega),\ \mathbb{L}^0 u \in \Lambda^{0,1} G^s(\Omega) \implies u \in G^s(\Omega). \tag{3.6}$$

Property (3.6) was completely characterized in [4] when $\omega_1, \ldots, \omega_m$ are real-valued; by Corollary 3.2, if this property holds then $\mathbb{L}^0$ is globally (G^s, G^s)-solvable.

Corollary 3.4. *The partial exterior derivative $\mathrm{d}_t : G^s(\Omega) \to \Lambda^{0,1} G^s(\Omega)$ has closed range.*

Proof. By Corollary 3.2 it is enough to prove that

$$\forall u \in G^{s_+}(\Omega),\ \mathrm{d}_t u \in \Lambda^{0,1} G^s(\Omega) \implies \exists v \in G^s(\Omega) \text{ such that } \mathrm{d}_t v = \mathrm{d}_t u,$$

for some $s_+ > s$. Indeed, pick an arbitrary $t_0 \in M$. We will prove that

$$v(t, x) \doteq u(t, x) - u(t_0, x), \quad (t, x) \in \Omega,$$

defines a function of Gevrey order s. We must check, by [4, Proposition 5.2], that for every coordinate chart $(U; t_1, \ldots, t_m)$ of M and every compact set $K \subset U$ there exist constants $C, h, \epsilon > 0$ such that

$$\sup_K |\partial_t^\alpha \hat{v}_\xi| \leq C h^\alpha \alpha!^s e^{-\epsilon |\xi|^{\frac{1}{s}}}, \quad \forall \xi \in \mathbb{Z}^m, \tag{3.7}$$

for every $\alpha \in \mathbb{Z}_+^n$. But by that very proposition, since $\mathrm{d}_t u \in \Lambda^{0,1} G^s(\Omega)$ there are constants $C_1, h_1, \epsilon_1 > 0$ such that

$$\sup_K |\partial_t^\beta (\mathrm{d}\hat{u}_\xi)| \leq C_1 h_1^\beta \beta!^s e^{-\epsilon_1 |\xi|^{\frac{1}{s}}}, \quad \forall \beta \in \mathbb{Z}_+^n, \ \forall \xi \in \mathbb{Z}^m, \tag{3.8}$$

(the usual abuse of notation of treating 1-forms locally as vectors of functions is employed) which ensures (3.7) for $|\alpha| > 0$ since $\hat{v}_\xi = \hat{u}_\xi - \hat{u}_\xi(t_0)$ for every $\xi \in \mathbb{Z}^m$. We are therefore left to check that

$$\sup |\hat{u}_\xi - \hat{u}_\xi(t_0)| \leq C' e^{-\delta |\xi|^{\frac{1}{s}}}, \quad \forall \xi \in \mathbb{Z}^m, \tag{3.9}$$

for some constants $C', \delta > 0$.

We fix a finite covering $\{U_i\}_{i \in I}$ of M by coordinate balls and connect an arbitrary $t \in M$ to t_0 by a polygonal curve $\gamma_t : [0, 1] \to M$ satisfying the following properties:

- $\gamma_t(0) = t_0$, $\gamma_t(1) = t$;
- there exists a partition $0 = \tau_0 < \tau_1 < \cdots < \tau_N = 1$ such that for each $r \in \{1, \dots, N\}$ the image of the segment $\gamma_t^r \doteq \gamma_t|_{[\tau_{r-1}, \tau_r]}$ is entirely contained into a single $U_{i(r)}$ and is a straight line segment there (in the fixed set of coordinates in $U_{i(r)}$); and moreover
- N is at most $|I|$ and no $U_{i(r)}$ is repeated i.e. $r \neq r'$ implies $i(r) \neq i(r')$.

Such choices are always possible since M is compact and connected. They ensure that γ_t is a piecewise smooth curve and

$$\hat{u}_\xi(t) - \hat{u}_\xi(t_0) = \int_{\gamma_t} \mathrm{d}\hat{u}_\xi = \sum_{r=1}^N \int_{\gamma_t^r} \mathrm{d}\hat{u}_\xi.$$

Now using that estimates like (3.8) hold on all coordinate compact subsets of M one can prove the existence of constants $C'', \delta > 0$ such that

$$\left| \int_{\gamma_t^r} \mathrm{d}\hat{u}_\xi \right| \leq C'' e^{-\delta |\xi|^{\frac{1}{s}}} \operatorname{length}(\gamma_t^r), \quad \forall r \in \{1, \dots, N\}, \ \forall \xi \in \mathbb{Z}^m,$$

where, by construction,

$$\operatorname{length}(\gamma_t^r) \leq \operatorname{diam}(U_{i(r)}) \leq \max_{i \in I} \operatorname{diam}(U_i), \quad \forall r \in \{1, \dots, N\}, \ \forall t \in M,$$

from which (3.9) follows easily for some $C' > 0$, completing the proof. □

Still regarding the case $q = 0$ we have the following converse of Corollary 3.2:

Theorem 3.5. *Suppose that $\omega_1, \dots, \omega_m$ are real-analytic 1-forms. If $\mathbb{L}^0$ is globally (G^s, G^s)-solvable then*

$$\forall u \in \mathcal{D}'(\Omega), \ \mathbb{L}^0 u \in \Lambda^{0,1} G^s(\Omega) \Longrightarrow \exists v \in G^s(\Omega) \text{ such that } \mathbb{L}^0 v = \mathbb{L}^0 u.$$

Proof. Let $u \in \mathcal{D}'(\Omega)$ be such that $f \doteq \mathbb{L}^0 u \in \Lambda^{0,1} G^s(\Omega)$. Then for each $\xi \in \mathbb{Z}^m$ we have that

$$\hat{f}_\xi = \widehat{(\mathbb{L}^0 u)}_\xi = \mathrm{d}\hat{u}_\xi + i(\xi \cdot \boldsymbol{\omega})\hat{u}_\xi \doteq \mathbb{L}_\xi^0 \hat{u}_\xi$$

belongs to $\Lambda^1 G^s(M)$, which implies, due to the ellipticity of $\mathbb{L}_\xi^0$, that $\hat{u}_\xi \in G^s(M)$ by the Gevrey version[3] of Sato's Theorem [18, Theorem 8.6.1]. Moreover, by

[3]This is the only place where we employed analyticity of $\omega_1, \dots, \omega_m$, for, in that case, $\mathbb{L}_\xi^0$ is a real-analytic operator.

Lemma 2.4:

$$f = \lim_{\nu\to\infty} \frac{1}{(2\pi)^m} \sum_{|\xi|\le\nu} e^{ix\cdot\xi} \otimes (\mathbb{L}^0_\xi \hat{u}_\xi) = \lim_{\nu\to\infty} \mathbb{L}^0 \left(\frac{1}{(2\pi)^m} \sum_{|\xi|\le\nu} e^{ix\cdot\xi} \otimes \hat{u}_\xi \right)$$

with convergence in $\Lambda^{0,1}G^s(\Omega)$, showing that f belongs to the closure of the range of $\mathbb{L}^0 : G^s(\Omega) \to \Lambda^{0,1}G^s(\Omega)$. Since $\mathbb{L}^0$ is globally (G^s, G^s)-solvable by assumption, there exists $v \in G^s(\Omega)$ such that $\mathbb{L}^0 v = f = \mathbb{L}^0 u$, which finishes the proof. ☐

3.2. **Global $(\mathcal{D}'_s, G^s)$-solvability and an *a priori* inequality.** Comparing the three notions of solvability laid down in Definition 1.1, it is immediate that

$$\text{global } (\mathcal{D}'_s, \mathcal{D}'_s)\text{-solvability} \implies \text{global } (\mathcal{D}'_s, G^s)\text{-solvability.} \tag{3.10}$$

Their relationship with the notion of global (G^s, G^s)-solvability is, however, subtler: although it is clear that for a given $f \in \Lambda^{0,q+1}G^s(\Omega)$ the compatibility conditions (1.5) imply (1.4), it is not know whether the converse holds, unless $q = n - 1$ according to the following digression.

Remark 3.6. We claim that, taking for granted a version of Lemma 2.4 for ultradistributions, if $f \in \Lambda^{0,n}G^s(\Omega)$ satisfies (1.4) then (1.5) holds. Indeed, let $v \in \mathcal{D}'_s(\Omega)$ be such that $\mathbb{L}^0 v = 0$ and write

$$v = \frac{1}{(2\pi)^m} \sum_{\xi\in\mathbb{Z}^m} e^{ix\cdot\xi} \otimes \hat{v}_\xi = \lim_{\nu\to\infty} \underbrace{\frac{1}{(2\pi)^m} \sum_{|\xi|\le\nu} e^{ix\cdot\xi} \otimes \hat{v}_\xi}_{\doteq v_\nu}$$

with convergence in $\mathcal{D}'_s(\Omega)$. Notice that for each $\xi \in \mathbb{Z}^m$ we have that $\mathbb{L}^0_\xi \hat{v}_\xi = 0$, which implies that $\hat{v}_\xi \in G^s(M)$, by ellipticity (this is precisely the information lacking when $v \in \Lambda^{0,q}\mathcal{D}'_s(\Omega)$ with $q > 0$). Hence, $v_\nu \in G^s(\Omega)$ and satisfies $\mathbb{L}^0 v_\nu = 0$ for every $\nu \in \mathbb{N}$. We conclude that

$$\langle v, f\rangle = \lim_{\nu\to\infty} \langle v_\nu, f\rangle = 0$$

thanks to (1.4). This proves our claim, and entails the following consequence: if $\mathbb{L}^{n-1}$ is globally (G^s, G^s)-solvable then it is globally $(\mathcal{D}'_s, G^s)$-solvable.

As is standard in the study of global solvability of linear PDEs, the notion of global $(\mathcal{D}'_s, G^s)$-solvability implies a Hörmander inequality, here involving Gevrey norms (see e.g. [22] for a local version):

Lemma 3.7. *Let $q \in \{0, \dots, n-1\}$ and suppose $\mathbb{L}^q$ is globally $(\mathcal{D}'_s, G^s)$-solvable. Then, for each $h_1, h_2 > 0$ there exist $C, h'_2 > 0$ such that*

$$\left| \int_\Omega f \wedge v \wedge \mathrm{d}x \right| \le C \|f\|_{s,h_1,\Omega} \, \|\mathbb{L}^{n-q-1} v\|_{s,h'_2,\Omega},$$

for every $f \in \Lambda^{0,q+1}G^{s,h_1}(\Omega)$ satisfying (1.4) and every $v \in \Lambda^{0,n-q-1}G^{s,h_2}(\Omega)$.

Remark 3.8. Here $h'_2 > 0$ is taken to be such that

$$\mathbb{L}^{n-q-1}(\Lambda^{0,n-q-1}G^{s,h_2}(\Omega)) \subset \Lambda^{0,n-q}G^{s,h'_2}(\Omega),$$

which always exists by continuity of $\mathbb{L}^{n-q-1} : \Lambda^{0,n-q-1}G^s(\Omega) \to \Lambda^{0,n-q}G^s(\Omega)$.

Proof. Let $E \subset \Lambda^{0,q+1}G^{s,h_1}(\Omega)$ be the subspace of all $f \in \Lambda^{0,q+1}G^{s,h_1}(\Omega)$ satisfying (1.4); as such, it is closed and therefore a Banach space for the norm $\|\cdot\|_{s,h_1,\Omega}$. Consider also $F_0 \doteq \Lambda^{0,n-q-1}G^{s,h_2}(\Omega) \cap \ker \mathbb{L}^{n-q-1}$ as well as the quotient vector space $F \doteq \Lambda^{0,n-q-1}G^{s,h_2}(\Omega)/F_0$, which we endow with the norm

$$v + F_0 \longmapsto \|\mathbb{L}^{n-q-1}v\|_{s,h_2',\Omega},$$

for some $h_2' > 0$ as in Remark 3.8. Notice that if $\|\mathbb{L}^{n-q-1}v\|_{s,h_2',\Omega} = 0$ then the equivalence class of $v \in \Lambda^{0,n-q-1}G^{s,h_2}(\Omega)$ in F is zero.

Consider the bilinear map $\mathsf{B} : E \times F \to \mathbb{C}$ given by

$$\mathsf{B}(f, v + F_0) \doteq \int_\Omega f \wedge v \wedge \mathrm{d}x.$$

It is well-defined: indeed, if $v + F_0 = v' + F_0$ then for every $f \in E$:

$$\mathbb{L}^{n-q-1}(v - v') = 0 \Longrightarrow \langle f, v - v'\rangle = 0 \Longrightarrow \mathsf{B}(f, v + F_0) = \mathsf{B}(f, v' + F_0).$$

Our thesis is equivalent to continuity of B; by the Banach-Steinhaus Theorem it suffices to check its separate continuity.

It is clear that for every v fixed the map $f \in E \mapsto \mathsf{B}(f, v + F_0) \in \mathbb{C}$ is continuous. If, on the other hand, we fix $f \in E$ then by our assumption of global $(\mathcal{D}'_s, G^s)$-solvability there exists $u \in \Lambda^{0,q}\mathcal{D}'_s(\Omega)$ such that $\mathbb{L}^q u = f$. Therefore,

$$|\mathsf{B}(f, v + F_0)| = |\langle f, v\rangle| = |\langle \mathbb{L}^q u, v\rangle| = |\langle u, \mathbb{L}^{n-q-1}v\rangle| \leq C'\|\mathbb{L}^{n-q-1}v\|_{s,h_2',\Omega},$$

for a constant $C' > 0$, by the continuity of u as a linear functional on $\Lambda^{0,n-q}G^{s,h_2'}(\Omega)$. This is continuity of $\mathsf{B}(f, \cdot)$, finishing the proof. ☐

4. Solvability and regularity in sets of frequencies

In connection with conditions (d) and (e) from Theorem 1.3, in the next sections we will see that the information relevant to the phenomenon of global solvability can be fully encoded in a special subset of frequencies (regarding the partial Fourier series) associated with our tube structure, whereas in the remaining frequencies solvability always holds, so to speak. Below we introduce such formalism and give precise meaning to this notion, by using the partial Fourier series to suitably decompose our spaces of forms and currents.

Let $\Gamma \subset \mathbb{Z}^m$ be any set. For $q \in \{0, \dots, n\}$ we let

$$\Lambda^{0,q}\mathcal{D}'_\Gamma(\Omega) \doteq \{f \in \Lambda^{0,q}\mathcal{D}'(\Omega) \ ; \ \hat{f}_\xi = 0, \ \forall \xi \in \mathbb{Z}^m \setminus \Gamma\}.$$

We define $\Lambda^{0,q}G^s_\Gamma(\Omega)$ accordingly, which is a closed subspace of $\Lambda^{0,q}G^s(\Omega)$ thanks to Lemma 2.3, hence itself a DFS space with respect to the subspace topology. It also carries a continuous projection

$$f \in \Lambda^{0,q}G^s(\Omega) \longmapsto f_\Gamma \doteq \frac{1}{(2\pi)^m} \sum_{\xi \in \Gamma} e^{ix\cdot\xi} \otimes \hat{f}_\xi \in \Lambda^{0,q}G^s_\Gamma(\Omega),$$

thanks to Lemma 2.4. Moreover

$$\Lambda^{0,q}G^s(\Omega) = \Lambda^{0,q}G^s_\Gamma(\Omega) \oplus \Lambda^{0,q}G^s_{\mathbb{Z}^m\setminus\Gamma}(\Omega), \tag{4.1}$$

the sum being direct as a consequence of [4, Lemma 5.1].

It is clear that

$$\mathbb{L}^q(\Lambda^{0,q}\mathcal{D}'_\Gamma(\Omega)) \subset \Lambda^{0,q+1}\mathcal{D}'_\Gamma(\Omega), \quad \forall q \in \{0, \dots, n-1\},$$

thus inducing linear maps

$$\mathbb{L}^q_\Gamma : \Lambda^{0,q}\mathcal{D}'_\Gamma(\Omega) \longrightarrow \Lambda^{0,q+1}\mathcal{D}'_\Gamma(\Omega),$$

and by restriction

$$\mathbb{L}^q_\Gamma : \Lambda^{0,q}G^s_\Gamma(\Omega) \longrightarrow \Lambda^{0,q+1}G^s_\Gamma(\Omega), \tag{4.2}$$

which is continuous. In what follows, we relate the closedness of the range of the map (3.1) with the ones in (4.2); the proof is elementary.

Proposition 4.1. *Let $\Gamma \subset \mathbb{Z}^m$. Then $\mathbb{L}^q : \Lambda^{0,q}G^s(\Omega) \to \Lambda^{0,q+1}G^s(\Omega)$ has closed range if and only if the same property holds for both $\mathbb{L}^q_\Gamma : \Lambda^{0,q}G^s_\Gamma(\Omega) \to \Lambda^{0,q+1}G^s_\Gamma(\Omega)$ and $\mathbb{L}^q_{\mathbb{Z}^m\setminus\Gamma} : \Lambda^{0,q}G^s_{\mathbb{Z}^m\setminus\Gamma}(\Omega) \to \Lambda^{0,q+1}G^s_{\mathbb{Z}^m\setminus\Gamma}(\Omega)$.*

In this section we will be interested in the following property.

Definition 4.2. We say that $\mathbb{L}^0$ is *globally (s,Γ)-hypoelliptic* provided that

$$\forall u \in \mathcal{D}'_\Gamma(\Omega),\ \mathbb{L}^0 u \in \Lambda^{0,1}G^s(\Omega) \Longrightarrow u \in G^s(\Omega).$$

Remark 4.3. By Theorem 3.1, if $\mathbb{L}^0$ is globally (s,Γ)-hypoelliptic then $\mathbb{L}^0_\Gamma : G^s_\Gamma(\Omega) \to \Lambda^{0,1}G^s_\Gamma(\Omega)$ has closed range.

Up to this point, our closed 1-forms $\omega_1,\dots,\omega_m$ could be, in principle, complex-valued. From here on, however, we will further assume them to be real. In what follows we lay down a generalization of the main results and definitions in [4].

Definition 4.4. Assume each ω_k real. We say that the system $\boldsymbol{\omega} = (\omega_1,\dots,\omega_m)$ is:

(1) Γ-*rational* if there exists $\xi \in \Gamma$ such that $\xi \cdot \boldsymbol{\omega}$ is an integral 1-form.
(2) (s,Γ)-*exponential Liouville* if
  (a) $\boldsymbol{\omega}$ is not Γ-rational and
  (b) there exist $\epsilon > 0$, a sequence of integral forms $\{\theta_\nu\}_{\nu\in\mathbb{N}} \subset \Lambda^1 G^s(M;\mathbb{R})$ and $\{\xi_\nu\}_{\nu\in\mathbb{N}} \subset \Gamma$ such that $|\xi_\nu| \to \infty$ and

$$\{e^{\epsilon|\xi_\nu|^{\frac{1}{s}}}(\xi_\nu \cdot \boldsymbol{\omega} - \theta_\nu)\}_{\nu\in\mathbb{N}} \text{ is bounded in } \Lambda^1 G^s(M). \tag{4.3}$$

Remark 4.5. It is necessary for $\Gamma \subset \mathbb{Z}^m$ to be an infinite set in order for any system $\boldsymbol{\omega} = (\omega_1,\dots,\omega_m)$ to be (s,Γ)-exponential Liouville.

These definitions recover their standard counterparts when $\Gamma \doteq \mathbb{Z}^m \setminus \{0\}$, see [4, Definition 3.2]. Notice that, with respect to Γ, property (2b) is obviously preserved by inclusion, whilst property (2a) is preserved by reverse inclusion; hence, if $\Gamma \subset \Gamma'$ there is no direct relationship between $\boldsymbol{\omega}$ being (s,Γ)-exponential Liouville and being (s,Γ')-exponential Liouville.

Recall [4, 14] that a matrix of periods $\mathsf{A}(\boldsymbol{\omega}) \in \mathsf{M}_{d\times m}(\mathbb{R})$ is defined by the rule

$$\mathsf{A}(\boldsymbol{\omega})\xi \doteq \frac{1}{2\pi}\left(\int_{\sigma_1} \xi\cdot\boldsymbol{\omega},\dots,\int_{\sigma_d}\xi\cdot\boldsymbol{\omega}\right), \quad \xi \in \mathbb{Z}^m, \tag{4.4}$$

where $\sigma_1,\dots,\sigma_d$ are 1-cycles in M whose classes $[\sigma_1],\dots,[\sigma_d]$ in $H_1(M;\mathbb{Z})$ form a basis of its free part – and therefore also an $\mathbb{R}$-basis for $H_1(M;\mathbb{R})$. In order to characterize the properties introduced in Definition 4.4 by means of $\mathsf{A}(\boldsymbol{\omega})$ (Proposition 4.11 below), we must first bring in their corresponding Diophantine conditions.

4.1. **Diophantine conditions for matrices in sets of frequencies.** Let $\Gamma \subset \mathbb{Z}^m$ be any set and $\mathbf{A} \in \mathsf{M}_{d\times m}(\mathbb{R})$ be a matrix.

Definition 4.6. We say that $\mathbf{A}$ satisfies *condition* $(\mathrm{DC})_{s,\Gamma}$ if for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|\kappa + \mathbf{A}\xi| \geq C_\epsilon e^{-\epsilon(|\kappa|+|\xi|)^{\frac{1}{s}}}, \quad \forall (\kappa,\xi) \in \mathbb{Z}^d \times \Gamma.$$

Example 4.7. Recall from [15, Section 3] that $\mathbf{A}$ is said to satisfy condition $(\mathrm{DC})^1_s$ if for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|\kappa + \mathbf{A}\xi| \geq C_\epsilon e^{-\epsilon(|\kappa|+|\xi|)^{\frac{1}{s}}}, \quad \forall (\kappa,\xi) \in \mathbb{Z}^d \times \mathbb{Z}^m \text{ with } \kappa + \mathbf{A}\xi \neq 0.$$

This is nothing but $(\mathrm{DC})_{s,\Gamma_1}$ for $\Gamma_1 \doteq \{\xi \in \mathbb{Z}^m \ ; \ \mathbf{A}\xi \notin \mathbb{Z}^d\}$. Indeed, on the one hand

$$\mathbb{Z}^d \times \Gamma_1 \subset \{(\kappa,\xi) \in \mathbb{Z}^d \times \mathbb{Z}^m \ ; \ \kappa + \mathbf{A}\xi \neq 0\},$$

hence $(\mathrm{DC})^1_s$ implies $(\mathrm{DC})_{s,\Gamma_1}$; assuming, on the other hand, condition $(\mathrm{DC})_{s,\Gamma_1}$, given $(\kappa,\xi) \in \mathbb{Z}^d \times \mathbb{Z}^m$ such that $\kappa + \mathbf{A}\xi \neq 0$, either $\xi \in \Gamma_1$ – so the desired inequality is ensured by $(\mathrm{DC})_{s,\Gamma_1}$ – or $\kappa + \mathbf{A}\xi$ is an integer, ensuring that

$$|\kappa + \mathbf{A}\xi| \geq 1 \geq e^{-\epsilon(|\kappa|+|\xi|)^{\frac{1}{s}}}.$$

This easily implies condition $(\mathrm{DC})^1_s$.

Example 4.8. By the same token, condition $(\mathrm{DC})^2_s$ from [15, Section 3] – which says that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|\kappa + \mathbf{A}\xi| \geq C_\epsilon e^{-\epsilon(|\kappa|+|\xi|)^{\frac{1}{s}}}, \quad \forall (\kappa,\xi) \in (\mathbb{Z}^d \times \mathbb{Z}^m) \setminus \{(0,0)\},$$

and is related to the phenomenon of global s-hypoellipticity (3.6) (see [4]) – is precisely $(\mathrm{DC})_{s,\Gamma_2}$ with $\Gamma_2 \doteq \mathbb{Z}^m \setminus \{0\}$. In fact, since $\mathbb{Z}^d \times \Gamma_2 \subset (\mathbb{Z}^d \times \mathbb{Z}^m) \setminus \{(0,0)\}$ it is immediate that, on the one hand, $(\mathrm{DC})^2_s$ trivially implies $(\mathrm{DC})_{s,\Gamma_2}$; whilst, on the other hand, if $\xi = 0$ and $\kappa \in \mathbb{Z}^d \setminus \{0\}$ then $|\kappa + \mathbf{A}\xi| = |\kappa| \geq 1$, and, by the previous argument, $(\mathrm{DC})_{s,\Gamma_2}$ implies $(\mathrm{DC})^2_s$.

The characterizations below are easy to prove (see [15]) and will be useful later.

Lemma 4.9. *Condition* $(\mathrm{DC})_{s,\Gamma}$ *is equivalent to the following: for every* $\epsilon > 0$ *there exists* $C_\epsilon > 0$ *such that*

$$|\kappa + \mathbf{A}\xi| \geq C_\epsilon e^{-\epsilon|\xi|^{\frac{1}{s}}}, \quad \forall (\kappa,\xi) \in \mathbb{Z}^d \times \Gamma.$$

Lemma 4.10. *If* $(\mathrm{DC})_{s,\Gamma}$ *holds then for every* $\epsilon > 0$ *there exists* $C_\epsilon > 0$ *such that*

$$\max_{\ell} |e^{2\pi i a_\ell\cdot\xi} - 1| \geq C_\epsilon e^{-\epsilon|\xi|^{\frac{1}{s}}}, \quad \forall \xi \in \Gamma,$$

where $a_\ell \in \mathbb{R}^m$ *denotes the* ℓ*-th row of* $\mathbf{A}$.

The next result follows from the same proof as that of [4, Proposition 4.2].

Proposition 4.11. *Let* $\Gamma \subset \mathbb{Z}^m$ *be any set. The system* $\boldsymbol{\omega}$ *is* Γ*-rational if and only if* $\mathsf{A}(\boldsymbol{\omega})(\Gamma) \cap \mathbb{Z}^d \neq \emptyset$. *It is an* (s,Γ)*-exponential Liouville system if and only if it is not* Γ*-rational and* $\mathsf{A}(\boldsymbol{\omega})$ *does not satisfy* $(\mathrm{DC})_{s,\Gamma}$.

Using the notion introduced above (as well as Lemmas 4.9 and 4.10, and Proposition 4.11), a detailed inspection in the arguments in [4] further shows that:

Theorem 4.12. *Let $\omega_1, \ldots, \omega_m \in \Lambda^1 G^s(M)$ be real and closed and $\Gamma \subset \mathbb{Z}^m$. If $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)$ is neither Γ-rational nor (s, Γ)-exponential Liouville then $\mathbb{L}^0$ is globally (s, Γ)-hypoelliptic.*

Remark 4.13. The converse of Theorem 4.12 holds under some mild extra assumptions on Γ e.g. that it is *conic*, in the sense that $\mathbb{N}\Gamma \subset \Gamma$ (required for the rational case). We will not use this fact in our arguments though.

Corollary 4.14. *If $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)$ is neither Γ-rational nor (s, Γ)-exponential Liouville then $\mathbb{L}^0_\Gamma : G^s_\Gamma(\Omega) \to \Lambda^{0,1} G^s_\Gamma(\Omega)$ has closed range.*

The proof of our next result is inspired by that of [7, Lemma 3.1]. However, in order to estimate the corresponding Gevrey norms, we employ a much more refined version of an argument appearing the proof of [4, Theorem 3.4].

Theorem 4.15. *If $\boldsymbol{\omega}$ is (s, Γ)-exponential Liouville for some $\Gamma \subset \mathbb{Z}^m$ then $\mathbb{L}^{n-1}$ is not globally $(\mathcal{D}'_s, G^s)$-solvable.*

Proof. Suppose that $\boldsymbol{\omega}$ is not Γ-rational and there exist $\epsilon > 0$, a sequence of integral forms $\{\theta_\nu\}_{\nu \in \mathbb{N}} \subset \Lambda^1 G^s(M; \mathbb{R})$ and $\{\xi_\nu\}_{\nu \in \mathbb{N}} \subset \Gamma$ such that $|\xi_\nu| \to \infty$ and (4.3) holds. Denoting by $\Pi : \widetilde{M} \to M$ the universal covering of M, we take for each $\nu \in \mathbb{N}$ a $\psi_\nu \in G^s(\widetilde{M}; \mathbb{R})$ such that $\mathrm{d}\psi_\nu = \Pi^* \theta_\nu$. It is known that, since θ_ν is integral,

$$\text{for every } p, q \in \widetilde{M} \text{ such that } \Pi(p) = \Pi(q) \text{ we have } \psi_\nu(p) - \psi_\nu(q) \in 2\pi\mathbb{Z},$$

which is the condition one needs to descend $e^{i\psi_\nu}$ to M via Π to a function $g_\nu \in G^s(M)$ i.e. such that $\Pi^* g_\nu = e^{i\psi_\nu}$ on $\widetilde{M}$. Therefore,

$$\mathrm{d}g_\nu = i g_\nu \theta_\nu \quad \text{on } M.$$

Moreover, g_ν never vanishes. Set

$$f_\nu \doteq e^{-ix\cdot\xi_\nu} \otimes g_\nu \mathrm{d}\mu \in \Lambda^{0,n} G^s(\Omega), \tag{4.5}$$

$$v_\nu \doteq e^{ix\cdot\xi_\nu} \otimes g_\nu^{-1} \in G^s(\Omega), \tag{4.6}$$

where $\mathrm{d}\mu$ stands for the volume form associated to a real-analytic Riemannian metric on M. Below, we will show how the sequences $\{f_\nu\}_{\nu \in \mathbb{N}}$ and $\{v_\nu\}_{\nu \in \mathbb{N}}$ violate the inequality presented in Lemma 3.7.

Let $\phi \in G^s(\Omega)$ be such that $\mathbb{L}^0 \phi = 0$: we prove next that

$$\int_\Omega f_\nu \wedge \phi \wedge \mathrm{d}x = 0,$$

that is, each f_ν satisfies the correct compatibility conditions (1.4). In fact, notice that for every $\xi \in \mathbb{Z}^m$ we have that

$$0 = \widehat{(\mathbb{L}^0 \phi)}_\xi = \mathrm{d}\hat{\phi}_\xi + i(\xi \cdot \boldsymbol{\omega})\hat{\phi}_\xi;$$

hence, $\hat{\phi}_\xi = 0$ whenever $\xi \cdot \boldsymbol{\omega}$ is non-integral by [7, Lemma 2.1]. Therefore,

$$\int_\Omega f_\nu \wedge \phi \wedge \mathrm{d}x \sim \int_M g_\nu \left(\int_{\mathbb{T}^m} e^{-ix\cdot\xi_\nu} \phi \mathrm{d}x \right) \mathrm{d}\mu = \int_M g_\nu \hat{\phi}_{\xi_\nu} \mathrm{d}\mu = 0,$$

since $\xi_\nu \cdot \boldsymbol{\omega}$ is non-integral for every $\nu \in \mathbb{N}$ by hypothesis. On the other hand:

$$\begin{aligned}\mathbb{L}^0 v_\nu &= -e^{ix\cdot\xi_\nu} \otimes g_\nu^{-2} \mathrm{d} g_\nu + i(e^{ix\cdot\xi_\nu} \otimes g_\nu^{-1})\xi_\nu \cdot \boldsymbol{\omega} \\ &= -i(e^{ix\cdot\xi_\nu} \otimes g_\nu^{-1})\theta_\nu + i(e^{ix\cdot\xi_\nu} \otimes g_\nu^{-1})\xi_\nu \cdot \boldsymbol{\omega} \\ &= i v_\nu(\xi_\nu \cdot \boldsymbol{\omega} - \theta_\nu), \quad \forall \nu \in \mathbb{N}, \end{aligned} \tag{4.7}$$

by (4.5)-(4.6). We proceed to compute the G^s norms of f_ν and $\mathbb{L}^0 v_\nu$.

Take a coordinate domain $(V; t_1, \dots, t_n)$ in M, so small that $\Pi : \widetilde{V} \to V$ is a real-analytic diffeomorphism for some open set $\widetilde{V} \subset \widetilde{M}$, and $\phi_\nu \in G^s(V; \mathbb{R})$ given by $\phi_\nu \circ \Pi = \psi_\nu|_{\widetilde{V}}$; hence $g_\nu = e^{i\phi_\nu}$ on V. In the topology of $\Lambda^1 G^s(M)$,

$$e^{\epsilon|\xi_\nu|^{\frac{1}{s}}}(\xi_\nu \cdot \boldsymbol{\omega} - \theta_\nu) \text{ bounded} \Longrightarrow |\xi_\nu|^{-1}(\xi_\nu \cdot \boldsymbol{\omega} - \theta_\nu) \text{ bounded} \Longrightarrow |\xi_\nu|^{-1}\theta_\nu \text{ bounded},$$

where we have used that $|\xi_\nu| \to \infty$. We conclude that given a compact set $K \subset V$ there exist $C_1, h_1 > 0$ such that

$$\sup_K \left|\partial_t^\alpha(|\xi_\nu|^{-1}\theta_\nu)\right| \le C_1 h_1^{|\alpha|}\alpha!^s, \quad \sup_K |\partial_t^\alpha \rho_\nu| \le C_1 h_1^{|\alpha|}\alpha!^s, \tag{4.8}$$

for every $\alpha \in \mathbb{Z}_+^n$ and $\nu \in \mathbb{N}$, where

$$\rho_\nu \doteq e^{\epsilon|\xi_\nu|^{\frac{1}{s}}}(\xi_\nu \cdot \boldsymbol{\omega} - \theta_\nu). \tag{4.9}$$

Observe in addition that $\mathrm{d} g_\nu = i g_\nu \theta_\nu$ on M implies $\mathrm{d}\phi_\nu = \theta_\nu$ on V. From the first inequality in (4.8) we conclude the existence of $C_2, h_2 > 0$ such that

$$\sup_K \left|\partial_t^\alpha(|\xi_\nu|^{-1}\phi_\nu)\right| \le C_2 h_2^{|\alpha|}\alpha!^s, \quad \forall \alpha \in \mathbb{Z}_+^n, \ \forall \nu \in \mathbb{N};$$

hence by [4, Lemma 6.1] one can find $h_3 > 0$ (depending only on C_2, h_2, ϵ) such that

$$\sup_K |\partial_t^\alpha(g_\nu^{\pm 1})| = \sup_K \left|\partial_t^\alpha e^{\pm i\phi_\nu}\right| = \sup_K \left|\partial_t^\alpha e^{i|\xi_\nu|(\pm|\xi_\nu|^{-1}\phi_\nu)}\right| \le h_3^{|\alpha|}\alpha!^s e^{\frac{\epsilon}{8}|\xi_\nu|^{\frac{1}{s}}}, \tag{4.10}$$

for every $\alpha \in \mathbb{Z}_+^n$ and $\nu \in \mathbb{N}$. Also in V we have

$$f_\nu = e^{-ix\cdot\xi_\nu} g_\nu(t) D(t) \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n, \tag{4.11}$$

where $D \in C^\omega(V; \mathbb{R})$. Hence, there exist $C_4, h_4 > 0$ depending on D, and by [4, Lemma 6.1] an $h_5 > 0$, such that

$$\begin{aligned}|\partial_t^\alpha \partial_x^\beta e^{-ix\cdot\xi_\nu} g_\nu D| &\le |\partial_x^\beta e^{-ix\cdot\xi_\nu}| \sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}|\partial_t^\gamma g_\nu||\partial_t^{\alpha-\gamma} D| \\ &\le h_5^{|\beta|} e^{\frac{\epsilon}{4}|\xi_\nu|^{\frac{1}{s}}}\beta!^s \sum_{\gamma\le\alpha}\binom{\alpha}{\gamma} h_3^{|\gamma|}\gamma!^s e^{\frac{\epsilon}{8}|\xi_\nu|^{\frac{1}{s}}} C_4 h_4^{|\alpha-\gamma|}(\alpha-\gamma)! \\ &\le C_6 h_6^{|\alpha|+|\beta|} e^{\frac{3\epsilon}{8}|\xi_\nu|^{\frac{1}{s}}}(\alpha+\beta)!^s \end{aligned} \tag{4.12}$$

on $K \times \mathbb{T}^m$, for some constants $C_6, h_6 > 0$ independent of $\alpha \in \mathbb{Z}_+^n$, $\beta \in \mathbb{Z}_+^m$ and $\nu \in \mathbb{N}$. Note that, by estimate (4.10), it is not difficult, with an analogous computation, to deduce that

$$|\partial_t^\alpha \partial_x^\beta v_\nu| \le C_6 h_6^{|\alpha|+|\beta|} e^{\frac{3\epsilon}{8}|\xi_\nu|^{\frac{1}{s}}}(\alpha+\beta)!^s \tag{4.13}$$

on $K \times \mathbb{T}^m$, for every $\alpha \in \mathbb{Z}_+^n$, $\beta \in \mathbb{Z}_+^m$ and $\nu \in \mathbb{N}$.

Moving to another direction, we estimate (4.7). One deduces from (4.8), (4.9) and (4.10) that

$$\begin{aligned} |\partial_t^\alpha \partial_x^\beta (\mathbb{L}^0 v_\nu)| &\leq e^{-\epsilon|\xi_\nu|^{\frac{1}{s}}} |\partial_x^\beta e^{ix\cdot\xi_\nu}| \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} |\partial_t^\gamma (g_\nu^{-1})| |\partial_t^{\alpha-\gamma} \rho_\nu| \\ &\leq e^{-\epsilon|\xi_\nu|^{\frac{1}{s}}} h_5^{|\beta|} e^{\frac{\epsilon}{4}|\xi_\nu|^{\frac{1}{s}}} \beta!^s \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} h_3^{|\gamma|} \gamma!^s e^{\frac{\epsilon}{8}|\xi_\nu|^{\frac{1}{s}}} C_1 h_1^{|\alpha-\gamma|} (\alpha-\gamma)!^s \\ &\leq C_7 h_7^{|\alpha|+|\beta|} (\alpha+\beta)!^s e^{-\frac{5\epsilon}{8}|\xi_\nu|^{\frac{1}{s}}}, \end{aligned} \tag{4.14}$$

on $K \times \mathbb{T}^m$, for constants $C_7, h_7 > 0$ independent of $\alpha \in \mathbb{Z}_+^n$, $\beta \in \mathbb{Z}_+^m$ and $\nu \in \mathbb{N}$.

Finally, it follows from (4.11), (4.12), (4.13) and (4.14) that

$$f_\nu \in \Lambda^{0,n} G^{s,h_6}(K), \quad v_\nu \in G^{s,h_6}(K), \quad \mathbb{L}^0 v_\nu \in \Lambda^{0,1} G^{s,h_7}(K),$$

with

$$\|f_\nu\|_{s,h_6,K} \leq C_6 e^{\frac{3\epsilon}{8}|\xi_\nu|^{\frac{1}{s}}}, \quad \|\mathbb{L}^0 v_\nu\|_{s,h_7,K} \leq C_7 e^{-\frac{5\epsilon}{8}|\xi_\nu|^{\frac{1}{s}}},$$

for every $\nu \in \mathbb{N}$. Since these estimates can be obtained for each K satisfying the hypotheses previously imposed, we conclude the existence of $C', C'', h', h'' > 0$ for which

$$f_\nu \in \Lambda^{0,n} G^{s,h'}(\Omega), \quad v_\nu \in G^{s,h'}(\Omega), \quad \mathbb{L}^0 v_\nu \in \Lambda^{0,1} G^{s,h''}(\Omega),$$

with

$$\|f_\nu\|_{s,h',\Omega} \leq C' e^{\frac{3\epsilon}{8}|\xi_\nu|^{\frac{1}{s}}}, \quad \|\mathbb{L}^0 v_\nu\|_{s,h'',\Omega} \leq C'' e^{-\frac{5\epsilon}{8}|\xi_\nu|^{\frac{1}{s}}},$$

for every $\nu \in \mathbb{N}$. In particular,

$$\|f_\nu\|_{s,h',\Omega} \|\mathbb{L}^0 v_\nu\|_{s,h'',\Omega} \longrightarrow 0 \quad \text{as } \nu \to \infty.$$

On the other hand, by construction,

$$\int_\Omega f_\nu \wedge v_\nu \wedge \mathrm{d}x = (2\pi)^m \int_M \mathrm{d}\mu \neq 0, \quad \forall \nu \in \mathbb{N},$$

which contradicts the thesis of Lemma 3.7 and allows us to conclude that $\mathbb{L}^{n-1}$ is not globally $(\mathcal{D}'_s, G^s)$-solvable. ☐

5. The normal form over integral frequencies

From here on, given a system $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)$ of real closed 1-forms on M, we will be especially interested in the following set of frequencies:

$$\Gamma_0 \doteq \{\xi \in \mathbb{Z}^m \ ; \ \xi \cdot \boldsymbol{\omega} \text{ is an integral 1-form}\}.$$

In this section we investigate closedness of the range of $\mathbb{L}^0_{\Gamma_0} : G^s_{\Gamma_0}(\Omega) \to \Lambda^{0,1} G^s_{\Gamma_0}(\Omega)$; as we will see below, such a property always holds.

Notice that Γ_0 is a subgroup of $\mathbb{Z}^m$, hence we have

$$\Gamma_0 \cong \mathbb{Z}^r \quad \text{for some } r \leq m.$$

We then take $\eta_1, \ldots, \eta_r$ a basis of Γ_0 as a $\mathbb{Z}$-module: for each $\xi \in \Gamma_0$ there exist unique $\alpha_1, \ldots, \alpha_r \in \mathbb{Z}$ such that

$$\xi = \alpha_1 \eta_1 + \cdots + \alpha_r \eta_r \doteq \alpha_\xi \cdot \eta, \tag{5.1}$$

where $\alpha_\xi \doteq (\alpha_1, \dots, \alpha_r) \in \mathbb{Z}^r$ and $\eta \doteq (\eta_1, \dots, \eta_r)$. For each $j \in \{1, \dots, r\}$ we let $\psi_j \in G^s(\widetilde{M}; \mathbb{R})$ be such that $\mathrm{d}\psi_j = \Pi^*(\eta_j \cdot \boldsymbol{\omega})$; recall that since $\eta_j \cdot \boldsymbol{\omega}$ is integral there exists $g_j \in G^s(M)$ such that $\Pi^* g_j = e^{i\psi_j}$. In particular,

$$\mathrm{d}g_j = i g_j (\eta_j \cdot \boldsymbol{\omega}), \quad j \in \{1, \dots, r\}. \tag{5.2}$$

Let $g^{-\alpha_\xi} \doteq g_1^{-\alpha_1} \cdots g_r^{-\alpha_r} \in G^s(M)$ for each $\xi \in \Gamma_0$, and consider the map

$$\mathscr{T} : \Lambda^{0,q} G^s_{\Gamma_0}(\Omega) \longrightarrow \Lambda^{0,q} G^s_{\Gamma_0}(\Omega)$$

defined by

$$\mathscr{T}(f) \doteq \frac{1}{(2\pi)^m} \sum_{\xi \in \Gamma_0} e^{ix\cdot\xi} \otimes g^{-\alpha_\xi} \hat{f}_\xi.$$

Proposition 5.1. *$\mathscr{T}$ is well-defined and a topological isomorphism.*

Proof. Let $U \subset M$ be a coordinate domain so small that $\Pi : \widetilde{U} \to U$ is a real-analytic diffeomorphism for some open set $\widetilde{U} \subset \widetilde{M}$. Let $\phi_j \in G^s(U; \mathbb{R})$ be given by

$$\phi_j \circ \Pi = \psi_j|_{\widetilde{U}}, \quad j \in \{1, \dots, r\};$$

hence, on U,

$$g_j^{-\alpha_j} = e^{-i\alpha_j\phi_j} \Longrightarrow g^{-\alpha_\xi} = e^{-i\alpha_\xi\cdot\boldsymbol{\phi}},$$

where $\boldsymbol{\phi} \doteq (\phi_1, \dots, \phi_r)$. Now take $K \subset U$ a compact set. Since each $\phi_j \in G^s(U; \mathbb{R})$ it follows from [4, Lemma 6.1] that for each $\delta > 0$ we can find $h_1 > 0$ such that

$$\sup_K |\partial_t^\gamma g^{-\alpha_\xi}| \leq h_1^{|\gamma|} \gamma!^s e^{\delta|\alpha_\xi|^{\frac{1}{s}}}, \quad \forall \gamma \in \mathbb{Z}_+^n, \ \forall \xi \in \Gamma_0. \tag{5.3}$$

We claim that, due to the fact that $\eta_1, \dots, \eta_r$ form a basis of the $\mathbb{Z}$-module Γ_0, there exists a constant $C > 0$ such that

$$|\alpha_\xi| \leq C|\xi|, \quad \forall \xi \in \Gamma_0.$$

Indeed, since $\eta_1, \dots, \eta_r$ are linearly independent over $\mathbb{Z}$ it is well-known that they are also linearly independent over $\mathbb{R}$, forming thus a basis of an r-dimensional linear subspace of $\mathbb{R}^m$, namely $\mathrm{span}_\mathbb{R} \Gamma_0$. We extend (5.1) to all $\xi \in \mathrm{span}_\mathbb{R} \Gamma_0$, now with $\alpha_\xi \in \mathbb{R}^r$, and $\xi \mapsto |\alpha_\xi|$ defines a norm over $\mathrm{span}_\mathbb{R} \Gamma_0$, which is therefore equivalent to the Euclidean norm inherited from $\mathbb{R}^m$ (or any other, for that matter). This proves the existence of such a $C > 0$.

Therefore, by (5.3), for each $\delta > 0$ we can find $h_1 > 0$ such that

$$\sup_K |\partial_t^\gamma g^{-\alpha_\xi}| \leq h_1^{|\gamma|} \gamma!^s e^{C^{\frac{1}{s}}\delta|\xi|^{\frac{1}{s}}}, \quad \forall \gamma \in \mathbb{Z}_+^n, \ \forall \xi \in \Gamma_0, \tag{5.4}$$

and similarly, by [4, Lemma 6.1], for each $\delta > 0$ there exists $h_2 > 0$ such that

$$\sup_{\mathbb{T}^m} |\partial_x^\lambda e^{ix\cdot\xi}| \leq h_2^{|\lambda|} \lambda!^s e^{\delta|\xi|^{\frac{1}{s}}}, \quad \forall \lambda \in \mathbb{Z}_+^m, \ \forall \xi \in \mathbb{R}^m. \tag{5.5}$$

On the other hand, since f is a G^s form, it is a consequence of (2.8) that

$$\sup_K |\partial_t^\beta \hat{f}_\xi| \leq 2^s (2\pi)^m \|f\|_{s,h,\Omega} (2^s h)^{|\beta|} \beta!^s e^{-\sigma|\xi|^{\frac{1}{s}}}, \quad \forall \beta \in \mathbb{Z}_+^n, \ \forall \xi \in \mathbb{Z}^m,$$

for some $h > 0$, and a $\sigma > 0$ that depends on h, m and s. Hence, if we choose $\delta \doteq \sigma/(4C^{\frac{1}{s}})$ in (5.4) and $\delta \doteq \sigma/4$ in (5.5) we obtain $h_1, h_2 > 0$ such that

$$\begin{aligned}\sup_{K\times\mathbb{T}^m} |\partial_x^\lambda \partial_t^\beta \mathscr{T}(f)| &\leq \frac{1}{(2\pi)^m} \sum_{\xi\in\Gamma_0} \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \sup_K |\partial_t^\gamma \hat{f}_\xi| \sup_K |\partial_t^{\beta-\gamma} g^{-\alpha_\xi}| \sup_{\mathbb{T}^m} |\partial_x^\lambda e^{ix\cdot\xi}| \\ &\leq 2^s \sum_{\xi\in\Gamma_0} \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \|f\|_{s,h,\Omega} (2^s h)^{|\gamma|} \gamma!^s e^{-\sigma|\xi|^{\frac{1}{s}}} h_1^{|\beta-\gamma|} (\beta-\gamma)!^s e^{\frac{\sigma}{4}|\xi|^{\frac{1}{s}}} h_2^{|\lambda|} \lambda!^s e^{\frac{\sigma}{4}|\xi|^{\frac{1}{s}}}.\end{aligned}$$

If we take $h_3 \doteq \max\{h_1, h_2, 2^s h\}$ it follows that

$$\begin{aligned}\sup_{K\times\mathbb{T}^m} |\partial_x^\lambda \partial_t^\beta \mathscr{T}(f)| &\leq 2^s \|f\|_{s,h,\Omega} (2h_3)^{|\beta+\lambda|} (\beta+\lambda)!^s \sum_{\xi\in\Gamma_0} e^{-\frac{\sigma}{2}|\xi|^{\frac{1}{s}}} \\ &\leq C_3 \|f\|_{s,h,\Omega} (2h_3)^{|\beta+\lambda|} (\beta+\lambda)!^s, \quad \forall \lambda \in \mathbb{Z}_+^m,\ \forall \beta \in \mathbb{Z}_+^n,\end{aligned}$$

for some $C_3 > 0$. Observe that the estimate above holds for any K with the properties previously described. Hence, it not only implies that $\mathscr{T}$ is well-defined, but also that it is continuous. Finally, note that the same arguments apply to its inverse:

$$\mathscr{T}^{-1}(f) = \frac{1}{(2\pi)^m} \sum_{\xi\in\Gamma_0} e^{ix\cdot\xi} \otimes g^{\alpha_\xi} \hat{f}_\xi,$$

which completes the proof. □

A simple computation using (5.2) shows that for every $\xi = \alpha_\xi \cdot \eta \in \Gamma_0$ we have

$$\mathrm{d} g^{-\alpha_\xi} = -i(\xi \cdot \boldsymbol{\omega}) g^{-\alpha_\xi};$$

hence, for any $f \in \Lambda^{0,q} G^s_{\Gamma_0}(\Omega)$ we have

$$\begin{aligned}(\widehat{\mathbb{L}^q \mathscr{T}(f)})_\xi &= \mathrm{d} \widehat{\mathscr{T}(f)}_\xi + i(\xi\cdot\boldsymbol{\omega}) \wedge \widehat{\mathscr{T}(f)}_\xi \\ &= \mathrm{d}(g^{-\alpha_\xi} \hat{f}_\xi) + i(\xi\cdot\boldsymbol{\omega}) \wedge (g^{-\alpha_\xi} \hat{f}_\xi) \\ &= g^{-\alpha_\xi} \mathrm{d} \hat{f}_\xi \\ &= \widehat{\mathscr{T}(\mathrm{d}_t f)}_\xi, \quad \forall \xi \in \Gamma_0,\end{aligned}$$

which implies that

$$\mathbb{L}^q \circ \mathscr{T} = \mathscr{T} \circ \mathrm{d}_t \quad \text{on } \Lambda^{0,q} G^s_{\Gamma_0}(\Omega). \tag{5.6}$$

Proposition 5.2. *The map* $\mathbb{L}^0_{\Gamma_0} : G^s_{\Gamma_0}(\Omega) \to \Lambda^{0,1} G^s_{\Gamma_0}(\Omega)$ *always has closed range.*

Proof. From Corollary 3.4, $\mathrm{d}_t : G^s(\Omega) \to \Lambda^{0,1} G^s(\Omega)$ has closed range; thus

$$X \doteq \operatorname{ran}\{\mathrm{d}_t : G^s_{\Gamma_0}(\Omega) \longrightarrow \Lambda^{0,1} G^s_{\Gamma_0}(\Omega)\}$$

is a closed subspace of $\Lambda^{0,1} G^s_{\Gamma_0}(\Omega)$, by Proposition 4.1. Thanks to (5.6) we have

$$\mathscr{T}(X) = \operatorname{ran}\{\mathbb{L}^0 : G^s_{\Gamma_0}(\Omega) \longrightarrow \Lambda^{0,1} G^s_{\Gamma_0}(\Omega)\},$$

which by Proposition 5.1 is closed in $\Lambda^{0,1} G^s_{\Gamma_0}(\Omega)$. □

6. Conclusion and further results

It follows from Propositions 4.1 and 5.2 that $\mathbb{L}^0 : G^s(\Omega) \to \Lambda^{0,1}G^s(\Omega)$ has closed range if and only if the same holds for $\mathbb{L}^0_{\Gamma_1} : G^s_{\Gamma_1}(\Omega) \to \Lambda^{0,1}G^s_{\Gamma_1}(\Omega)$, where

$$\Gamma_1 \doteq \mathbb{Z}^m \setminus \Gamma_0 = \{\xi \in \mathbb{Z}^m \ ; \ \xi \cdot \boldsymbol{\omega} \text{ is } \textit{not} \text{ an integral 1-form}\}.$$

By Definition 4.4, the system $\boldsymbol{\omega}$ is not Γ_1-rational; we conclude from Corollary 4.14:

Corollary 6.1. *If $\boldsymbol{\omega}$ is not (s, Γ_1)-exponential Liouville – which is equivalent to $\mathsf{A}(\boldsymbol{\omega})$ satisfying condition $(\mathrm{DC})_{s,\Gamma_1}$ – then $\mathbb{L}^0_{\Gamma_1} : G^s_{\Gamma_1}(\Omega) \to \Lambda^{0,1}G^s_{\Gamma_1}(\Omega)$ has closed range.*

The proof of Theorem 1.3 is now essentially complete; we summarize it below, referring to its statement in the Introduction.

Proof of Theorem 1.3. Equivalence *(a)* $\Leftrightarrow$ *(c)* is a particular instance of the duality (3.4). Implication *(a)* $\Rightarrow$ *(b)* is general, as pointed out in (3.10). The equivalence *(d)* $\Leftrightarrow$ *(e)* follows from Proposition 4.11, while Theorem 4.15 yields *(b)* $\Rightarrow$ *(e)* as a special case. A conjunction of Proposition 5.2 and Corollary 6.1 then entails *(e)* $\Rightarrow$ *(c)*, thus closing the chain of equivalences. □

6.1. **Solvability of associated substructures.** Theorem 1.3 gives a numerical condition in terms of the matrix of periods $\mathsf{A}(\boldsymbol{\omega})$ that completely characterizes global (G^s, G^s)-solvability of $\mathbb{L}^0$. In this section, we use this result in order to relate the aforementioned solvability to that of certain substructures associated to $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)$. The reader must keep in mind that a consequence of Theorem 1.3 is that all the properties of interest of $\boldsymbol{\omega}$ (as far as our discussion is concerned) are independent of the order of $\omega_1, \ldots, \omega_m$. Also, for convenience we will use, in our arguments below, the alternative characterization of condition $(\mathrm{DC})_{s,\Gamma}$ presented in Lemma 4.9.

Let $\dot{m} \in \{1, \ldots, m-1\}$ and consider the system

$$\dot{\boldsymbol{\omega}} \doteq (\omega_1, \ldots, \omega_{\dot{m}}),$$

which generates a corank $\dot{m}$ tube structure on $\dot{\Omega} \doteq M \times \mathbb{T}^{\dot{m}}$: we denote, for $q \in \{0, \ldots, n\}$,

$$\dot{\mathbb{L}}^q \doteq \mathrm{d}_t + \sum_{k=1}^{\dot{m}} \omega_k \wedge \partial_{x_k} \quad \text{acting on } (0,q)\text{-forms on } \dot{\Omega}. \tag{6.1}$$

In relation to it, below we decompose a vector $\xi \in \mathbb{Z}^m$ as $\xi = (\dot{\xi}, \ddot{\xi}) \in \mathbb{Z}^{\dot{m}} \times \mathbb{Z}^{m-\dot{m}}$.

Theorem 6.2. *If $\mathbb{L}^0$ is globally (G^s, G^s)-solvable then the same holds for $\dot{\mathbb{L}}^0$.*

Proof. Suppose that $\dot{\mathbb{L}}^0$ is not globally (G^s, G^s)-solvable, and let

$$\dot{\Gamma}_0 \doteq \{\dot{\xi} \in \mathbb{Z}^{\dot{m}} \ ; \ \dot{\xi} \cdot \dot{\boldsymbol{\omega}} \text{ is an integral 1-form}\},$$
$$\dot{\Gamma}_1 \doteq \mathbb{Z}^{\dot{m}} \setminus \dot{\Gamma}_0.$$

Then there exist $\epsilon_0 > 0$ and sequences $\{\kappa_\nu\}_{\nu \in \mathbb{N}} \subset \mathbb{Z}^d$, $\{\dot{\xi}_\nu\}_{\nu \in \mathbb{N}} \subset \dot{\Gamma}_1$ such that

$$|\kappa_\nu + \mathsf{A}(\dot{\boldsymbol{\omega}})\dot{\xi}_\nu| < \nu^{-1} e^{-\epsilon_0 |\dot{\xi}_\nu|^{\frac{1}{s}}}, \quad \forall \nu \in \mathbb{N},$$

where, as we recall, $\mathsf{A}(\dot{\boldsymbol{\omega}}) \in \mathsf{M}_{d \times \dot{m}}(\mathbb{R})$ is computed by

$$\mathsf{A}(\dot{\boldsymbol{\omega}})\dot{\xi} = \frac{1}{2\pi} \left(\int_{\sigma_1} \dot{\xi} \cdot \dot{\boldsymbol{\omega}}, \ldots, \int_{\sigma_d} \dot{\xi} \cdot \dot{\boldsymbol{\omega}} \right), \quad \forall \dot{\xi} \in \mathbb{Z}^{\dot{m}}.$$

Now consider the sequence $\{\xi_\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{Z}^m$ defined by $\xi_\nu \doteq (\dot{\xi}_\nu, 0) \in \mathbb{Z}^{\dot{m}} \times \mathbb{Z}^{m-\dot{m}}$. Then $\xi_\nu \cdot \boldsymbol{\omega} = \dot{\xi}_\nu \cdot \dot{\boldsymbol{\omega}}$ is not an integral form for every $\nu \in \mathbb{N}$; hence $\{\xi_\nu\}_{\nu\in\mathbb{N}} \subset \Gamma_1$. Moreover, $\mathsf{A}(\boldsymbol{\omega})\xi_\nu = \mathsf{A}(\dot{\boldsymbol{\omega}})\dot{\xi}_\nu$ and $|\xi_\nu| = |\dot{\xi}_\nu|$. Therefore,

$$|\kappa_\nu + \mathsf{A}(\boldsymbol{\omega})\xi_\nu| < \nu^{-1} e^{-\epsilon_0 |\xi_\nu|^{\frac{1}{s}}}, \quad \forall \nu \in \mathbb{N},$$

proving that $\mathbb{L}^0$ is not globally (G^s, G^s)-solvable either. □

Corollary 6.3. *If $\mathbb{L}^0$ is globally (G^s, G^s)-solvable then the same holds for each operator*

$$\mathcal{L}_k \doteq \mathrm{d}_t + \omega_k \wedge \partial_x : G^s(M \times \mathbb{T}^1) \longrightarrow \Lambda^{0,1} G^s(M \times \mathbb{T}^1) \tag{6.2}$$

associated with a corank 1 tube structure on $M \times \mathbb{T}^1$.

Based on this result, we focus next on the particular case of corank 1.

Theorem 6.4. *Let $\vartheta \in \Lambda^1 G^s(M)$ be real and closed. Then*

$$\mathcal{L} \doteq \mathrm{d}_t + \vartheta \wedge \partial_x : G^s(M \times \mathbb{T}^1) \longrightarrow \Lambda^{0,1} G^s(M \times \mathbb{T}^1)$$

is globally (G^s, G^s)-solvable if and only if either ϑ is rational or non s-exponential Liouville[4]*. The latter is equivalent to $\mathcal{L}$ being globally s-hypoelliptic.*

Proof. If ϑ is not rational then

$$\Gamma_0 \doteq \{\xi \in \mathbb{Z} \ ; \ \xi\vartheta \text{ is an integral 1-form}\}$$

equals $\{0\}$; hence, by Theorem 1.3, it follows that $\mathcal{L}$ is globally (G^s, G^s)-solvable if and only if ϑ is non s-exponential Liouville (equivalent to $\mathcal{L}$ being globally s-hypoelliptic by virtue of [4, Theorem 3.4]).

It remains to prove that when ϑ is rational (in which case $\Gamma_0 \neq \{0\}$) then $\mathcal{L}$ is in fact globally (G^s, G^s)-solvable. Due to the fact that $\mathbb{Z}$ is a principal ideal domain, we have that $\Gamma_0 = q\mathbb{Z}$ for some $q \in \mathbb{N}$. This implies that

$$\mathsf{A}(\vartheta) = \frac{p}{q} \quad \text{for some } p \in \mathbb{Z}^d.$$

Therefore, given $\kappa \in \mathbb{Z}^d$ and $\xi \in \Gamma_1 \doteq \mathbb{Z} \setminus \Gamma_0$, we have

$$\xi\mathsf{A}(\vartheta) \notin \mathbb{Z}^d \Longrightarrow \kappa + \xi\mathsf{A}(\vartheta) \neq 0.$$

Since $\Gamma_0 = q\mathbb{Z}$, we have that $q(\kappa + \xi\mathsf{A}(\vartheta)) \in \mathbb{Z}^d$, which allows us to deduce that

$$|q(\kappa + \xi\mathsf{A}(\vartheta))| \geq 1 \Longrightarrow |\kappa + \xi\mathsf{A}(\vartheta)| \geq \frac{1}{q}, \quad \forall (\kappa, \xi) \in \mathbb{Z}^d \times \Gamma_1,$$

an inequality that clearly implies that $\mathsf{A}(\vartheta)$ satisfies condition $(\mathrm{DC})_{s,\Gamma_1}$ (in a trivial way when $q = 1$). The conclusion follows from Theorem 1.3. □

Corollary 6.5. *Suppose that $\mathbb{L}^0$ is globally (G^s, G^s)-solvable. Then, for each $k \in \{1, \ldots, m\}$ the 1-form ω_k is either rational or non s-exponential Liouville.*

Once we have found necessary conditions for global (G^s, G^s)-solvability of $\mathbb{L}^0$ in terms of properties of each ω_k, we now look for conditions that turn out to be sufficient. First, we prove that if each ω_k is rational then the system is globally (G^s, G^s)-solvable.

Proposition 6.6. *If every component of $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)$ is a rational 1-form then $\mathbb{L}^0$ is globally (G^s, G^s)-solvable.*

[4] That is: non $(s, \mathbb{Z} \setminus \{0\})$-exponential Liouville, in the terminology of Definition 4.4.

Proof. The argument is quite similar to the one applied in the proof of Theorem 6.4. We once again let

$$\Gamma_0 \doteq \{\xi \in \mathbb{Z}^m \ ; \ \xi \cdot \boldsymbol{\omega} \text{ is an integral 1-form}\},$$
$$\Gamma_1 \doteq \mathbb{Z}^m \setminus \Gamma_0.$$

By hypothesis, each ω_k is rational, which means that

$$\mathsf{A}(\omega_k) = \frac{p_k}{q_k} \quad \text{for some } p_k \in \mathbb{Z}^d \text{ and } q_k \in \mathbb{N}.$$

Furthermore, if $\xi \in \Gamma_1$ and $\kappa \in \mathbb{Z}^d$, it follows that

$$0 \neq \kappa + \mathsf{A}(\boldsymbol{\omega})\xi = \kappa + \sum_{k=1}^{m} \xi_k \mathsf{A}(\omega_k);$$

hence, for $q \doteq \prod_{k=1}^m q_k$ we have $q(\kappa + \mathsf{A}(\boldsymbol{\omega})\xi) \in \mathbb{Z}^d \setminus \{0\}$, which implies that

$$q|\kappa + \mathsf{A}(\boldsymbol{\omega})\xi| \geq 1 \Longrightarrow |\kappa + \mathsf{A}(\boldsymbol{\omega})\xi| \geq \frac{1}{q}, \quad \forall (\kappa, \xi) \in \mathbb{Z}^d \times \Gamma_1.$$

Since this estimate clearly implies condition $(\mathrm{DC})_{s,\Gamma_1}$ for $\mathsf{A}(\boldsymbol{\omega})$, our result follows. □

Next we show that, generally speaking, the rational components of $\boldsymbol{\omega}$ can be neglected regarding the global (G^s, G^s)-solvability of $\mathbb{L}^0$.

Theorem 6.7. *Suppose that at least one component of* $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)$ *is* not *rational. Actually, by rearranging them, without loss of generality we assume that*

$$\omega_k \text{ is a rational 1-form} \Longleftrightarrow k > \dot{m},$$

for a certain $\dot{m} \in \{1, \ldots, m-1\}$. *Let* $\dot{\boldsymbol{\omega}} \doteq (\omega_1, \ldots, \omega_{\dot{m}})$. *Then,* $\mathbb{L}^0$ *is globally* (G^s, G^s)*-solvable if and only if the same holds for* $\dot{\mathbb{L}}^0$ *as in* (6.1).

Proof. Note that the direct implication is a consequence of Theorem 6.2; below, we borrow the notation established in its proof. Suppose conversely that $\dot{\mathbb{L}}^0$ is globally (G^s, G^s)-solvable. Then, for every $\delta > 0$ there exists $C_\delta > 0$ such that

$$|\kappa + \mathsf{A}(\dot{\boldsymbol{\omega}})\dot{\xi}| \geq C_\delta e^{-\delta|\dot{\xi}|^{\frac{1}{s}}}, \quad \forall (\kappa, \dot{\xi}) \in \mathbb{Z}^d \times \dot{\Gamma}_1. \tag{6.3}$$

Now let $\ddot{\boldsymbol{\omega}} \doteq (\omega_{\dot{m}+1}, \ldots, \omega_m)$. Then, for any $\kappa \in \mathbb{Z}^d$ and $\xi \in \Gamma_1$ we have

$$\kappa + \mathsf{A}(\boldsymbol{\omega})\xi = \kappa + \mathsf{A}(\dot{\boldsymbol{\omega}})\dot{\xi} + \mathsf{A}(\ddot{\boldsymbol{\omega}})\ddot{\xi} \neq 0.$$

In this context, there are two possibilities: either $\dot{\xi} \in \dot{\Gamma}_0$ or $\dot{\xi} \in \dot{\Gamma}_1$. In the former case, $\mathsf{A}(\dot{\boldsymbol{\omega}})\dot{\xi} \in \mathbb{Z}^d$ which implies that

$$\kappa + \mathsf{A}(\dot{\boldsymbol{\omega}})\dot{\xi} \in \mathbb{Z}^d.$$

Since $\mathsf{A}(\ddot{\boldsymbol{\omega}})$ has rational coefficients, there exists a non-zero $q \in \mathbb{N}$ such that

$$q(\kappa + \mathsf{A}(\dot{\boldsymbol{\omega}})\dot{\xi} + \mathsf{A}(\ddot{\boldsymbol{\omega}})\ddot{\xi}) \in \mathbb{Z}^d \setminus \{0\},$$

which again entails

$$|\kappa + \mathsf{A}(\boldsymbol{\omega})\xi| \geq \frac{1}{q}, \quad \forall (\kappa, \xi) \in \mathbb{Z}^d \times \Gamma_1, \ \dot{\xi} \in \dot{\Gamma}_0. \tag{6.4}$$

In the latter case, for $\dot{\xi} \in \dot{\Gamma}_1$ we write

$$q(\kappa + \mathsf{A}(\boldsymbol{\omega})\xi) = \underbrace{q\kappa + q\mathsf{A}(\ddot{\boldsymbol{\omega}})\ddot{\xi}}_{\in \mathbb{Z}^d} + \mathsf{A}(\dot{\boldsymbol{\omega}})(q\dot{\xi}).$$

We may assume without loss of generality that $q\dot{\xi} \in \dot{\Gamma}_1$ (otherwise $\mathsf{A}(\dot{\boldsymbol{\omega}})(q\dot{\xi}) \in \mathbb{Z}^d$ and we fall back into the previous reasoning); fixing an $\epsilon > 0$ and applying (6.3) with $\delta \doteq \epsilon q^{-\frac{1}{s}}$ we conclude the existence of $C_\epsilon > 0$ such that

$$q|\kappa + \mathsf{A}(\dot{\boldsymbol{\omega}})\dot{\xi} + \mathsf{A}(\ddot{\boldsymbol{\omega}})\ddot{\xi}| \geq C_\epsilon e^{-\epsilon q^{-\frac{1}{s}}|q\dot{\xi}|^{\frac{1}{s}}} \geq C_\epsilon e^{-\epsilon|\xi|^{\frac{1}{s}}},$$

which allows us to deduce that for any $\epsilon > 0$ there exists $C'_\epsilon > 0$ such that

$$|\kappa + \mathsf{A}(\boldsymbol{\omega})\xi| \geq C'_\epsilon e^{-\epsilon|\xi|^{\frac{1}{s}}}, \quad \forall(\kappa, \xi) \in \mathbb{Z}^d \times \Gamma_1, \ \dot{\xi} \in \dot{\Gamma}_1. \tag{6.5}$$

By associating (6.4) and (6.5) we conclude that $\mathsf{A}(\boldsymbol{\omega})$ satisfies condition $(\mathrm{DC})_{s,\Gamma_1}$, from which the theorem follows. □

Corollary 6.8. *Suppose that $\boldsymbol{\omega}$ has a single non-rational component ω_k. Then $\mathbb{L}^0$ is globally (G^s, G^s)-solvable if and only if ω_k is* not *s-exponential Liouville.*

We close this section showing that the solvability of $\boldsymbol{\omega}$ cannot be always retrieved from that of its proper substructures. Concretely, we intend to show that the converse of Corollary 6.5 does not hold. Before proceeding to the proof, we need a couple of auxiliary results. Recall that an $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is *not* an s-exponential Liouville number if and only if for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|\kappa - \alpha\xi| \geq \ C_\epsilon e^{-\epsilon|\xi|^{-\frac{1}{s}}}, \quad \forall(\kappa, \xi) \in \mathbb{Z}^2 \setminus \{(0,0)\};$$

that is, α satisfies condition $(\mathrm{DC})_{s,\mathbb{Z}\setminus\{0\}}$ (regarded as a 1×1 matrix). We denote

$$\mathscr{N\!L}_s \doteq \{\alpha \in \mathbb{R} \setminus \mathbb{Q} \ ; \ \alpha \text{ is not an } s\text{-exponential Liouville number}\},$$
$$\mathscr{L}_s \doteq \{\alpha \in \mathbb{R} \setminus \mathbb{Q} \ ; \ \alpha \text{ is an } s\text{-exponential Liouville number}\}.$$

Both sets are non-empty: for instance, every non-Liouville number belongs to $\mathscr{N\!L}_s$ (see also [17, Theorem 5.1]). The next lemma follows easily from the definition.

Lemma 6.9. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $b \in \mathbb{Q}$ and $c \in \mathbb{Q} \setminus \{0\}$. Then, $\alpha \in \mathscr{L}_s$ if and only if $c\alpha + b \in \mathscr{L}_s$.*

Incidentally, this entails that both $\mathscr{L}_s$ and $\mathscr{N\!L}_s$ are dense in $\mathbb{R}$, since given $\alpha \in \mathscr{L}_s$ (resp. $\alpha \in \mathscr{N\!L}_s$) the homeomorphism $\beta \in \mathbb{R} \mapsto \alpha + \beta \in \mathbb{R}$ maps $\mathbb{Q}$ into $\mathscr{L}_s$ (resp. $\mathscr{N\!L}_s$).

Proposition 6.10. $\mathscr{N\!L}_s + \mathscr{N\!L}_s = \mathbb{R}$.

Proof. The set of Liouville numbers has Lebesgue measure zero; hence, the same holds for its subset $\mathscr{L}_s$. Therefore, $\mathscr{N\!L}_s$ has positive measure, and by Steinhaus Theorem the difference set

$$\mathscr{N\!L}_s - \mathscr{N\!L}_s = \mathscr{N\!L}_s + \mathscr{N\!L}_s$$

contains a neighborhood of the origin (notice that $\mathscr{N\!L}_s$ is symmetric, by Lemma 6.9). This ensures that for any $r \in \mathbb{R}$ there exists a $\nu \in \mathbb{N}$ such that $r/\nu \in \mathscr{N\!L}_s + \mathscr{N\!L}_s$; hence, $r \in \mathscr{N\!L}_s + \mathscr{N\!L}_s$, as a consequence of Lemma 6.9. □

Next we exhibit a corank 2 system $\boldsymbol{\omega} = (\omega_1, \omega_2)$ whose corank 1 substructures (namely, the ones determined by each ω_k) are all individually globally (G^s, G^s)-solvable on their own, but $\boldsymbol{\omega}$ itself is not. On $M \doteq \mathbb{T}^1$ we take

$$\omega_k \doteq c_k \ \mathrm{d}t \in \Lambda^1 C^\omega(M; \mathbb{R}), \quad k \in \{1, 2\},$$

for a convenient choice of $c_1, c_2 \in \mathbb{R}$. Then

$$\mathbb{L}^0 f = (\partial_t f + c_1 \partial_{x_1} f + c_2 \partial_{x_2} f)\ \mathrm{d}t.$$

In this context, $d = 1$ and

$$\kappa + \mathsf{A}(\boldsymbol{\omega})\xi = \kappa + c_1\xi_1 + c_2\xi_2, \quad \kappa \in \mathbb{Z},\ \xi = (\xi_1, \xi_2) \in \mathbb{Z}^2. \tag{6.6}$$

If $c_1, c_2 \in \mathscr{NL}_s$ then the system of corank 1 associated to each ω_k is globally (G^s, G^s)-solvable (Theorem 6.4). If, however, $\lambda \doteq c_1 + c_2 \in \mathscr{L}_s$ (which is possible by Proposition 6.10) then there exist $\epsilon > 0$ and a sequence $\{(\kappa_\nu, \xi_\nu)\}_{\nu\in\mathbb{N}} \subset \mathbb{Z}^2 \setminus \{(0,0)\}$ such that

$$|\kappa_\nu + \lambda\xi_\nu| < \nu^{-1} e^{-\epsilon|\xi_\nu|^{\frac{1}{s}}}, \quad \forall \nu \in \mathbb{N}.$$

Consider now $\vartheta_\nu \doteq (\xi_\nu, \xi_\nu)$; it follows from (6.6) that

$$|\kappa_\nu + \mathsf{A}(\boldsymbol{\omega})\vartheta_\nu| = |\kappa_\nu + \lambda\xi_\nu| < \nu^{-1} e^{-\epsilon|\xi_\nu|^{\frac{1}{s}}} \le \nu^{-1} e^{-\epsilon 2^{-\frac{1}{s}}|\vartheta_\nu|^{\frac{1}{s}}},$$

which proves that $\mathbb{L}^0$ is not globally (G^s, G^s)-solvable.

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Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, Caixa Postal 668, São Carlos, SP 13560-970, Brazil

Email address: gccsa@icmc.usp.br

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, Caixa Postal 668, São Carlos, SP 13560-970, Brazil

Email address: dattori@icmc.usp.br

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, Caixa Postal 668, São Carlos, SP 13560-970, Brazil

Email address: brunodelessa@icmc.usp.br