RT-MAT 93-08

The Equivalence Problem in Sub-Riemannian Geometry

E. Falbel, J.A.Verderesi
and
J.M. Veloso

maio 1993

The Equivalence Problem in Sub-Riemannian Geometry

E. Falbel, J. A. Verderesi
Instituto de Matemática e Estatística
Universidade de São Paulo
CP20570-São Paulo-Brasil
and
J. M. Veloso
Universidade Federal do Pará
66000 - Belem - Pará- Brasil

1 Introduction

A sub-Riemannian manifold is a differential manifold together with a smooth distribution which carries a metric. See [S] for an introduction and references on the subject. In this work we consider the problem of equivalence in sub-Riemannian geometry. We will define a canonical connection on a non-degenerate sub-Riemannian manifold analogous to the Levi-Civita connection for Riemannian manifolds. This will allow us to study the equivalence problem using Cartan's method of moving frames. The main difference between Riemannian geometry is the essential appearence of torsion. We prove a classification theorem of sub-riemannian manifolds of constant sectional curvature and vanishing torsion.

A related structure studied in [CH] and [T] is of certain Riemannian metrics adapted to a contact form. In this work we consider a fixed metric in the distribution. This will define a canonical contact form. We would like to thank Prof. A. A. M. Rodrigues for fruitfull discussions. The first author was partially supported by CNPq.

2 Reduction of the G-structure

Definition 2.1 *A Sub-Riemann manifold is a triple (M, D, g) where M is a manifold, D is a smooth distribution on M and g is a smoothly varying quadratic form defined on D. We will say in this case that M is a sub-riemannian manifold of codimension k if D is of codimension k.*

We will concentrate in this work in the case of sub-riemannian manifolds of codimension 1. Let M be of dimension m+1.

The G-structure associated to (M, D, g) is given by the set of 1-forms

$$\left\{ \begin{array}{rcll} \theta' & = & \lambda\theta & \text{with } \lambda \neq 0 \text{ real} \\ \theta^{i'} & = & a^i_j\theta^j + v^i\theta & \text{where } (a^i_j) \in O(m) \end{array} \right. \tag{1}$$

Geometrically θ, θ^i is a basis of coframes satisfying $\theta(X_i) = 0$, $\theta^j(X_i) = \delta^j_i$ with $1 \leq i, j \leq m$ where X_i is an orthonormal basis of D.

Observe that, in general, there exists an antissymetric matrix (h_{ij}) such that

$$d\theta = h_{ij}\theta^i \wedge \theta^j + h_i\theta^i \wedge \theta$$

Although we could carry on with the theory without restrictive hypothesis, we will further restrict to the simplest case.

Definition 2.2 *(M, D, g) is said to be non-degenerate if $det(h_{ij}) \neq 0$.*

As (h_{ij}) is antissymetric, we have that in the non-degenerate case m = 2n is even. Furthermore, to show that the definition does not depend on the section of the G-structure, we choose another one as in 1. Then we see that

$$\begin{array}{rcl} h_{kl} & = & \frac{1}{\lambda}h_{ij}{}'a^i_k a^j_l \\ h_k & = & \frac{1}{\lambda}(2h_{ij}{}'v^j + \lambda h_i{}')a^i_k - \frac{\lambda_k}{\lambda} \end{array} \tag{2}$$

where $\sum \lambda_i\theta^i = d\lambda$.

It is clear now that the condition $det(h_{ij}) \neq 0$ is invariant. In fact it is equivalent to the condition that $\theta \wedge (d\theta)^n \neq 0$ for a section θ. In this case the G-structure can be reduced to a remarkably simple one.

Proposition 2.1 *The G-structure associated to a Sub-Riemannian manifold can be reduced to a $\mathbf{Z}_2 \times O(2n)$-structure in the non-degenerate case.*

Proof: We will impose the condition $det(h_{ij}) = 1$. Using the transformations 2 we see that this fixes the section θ modulo a sign. With this particular choice of the section, the second equation in 2 becomes $h_k = (2h_{ij}{}'v^j + h_i{}')a^i_k$. Again, in the non-degenerate case we can choose v^j uniquely such that $h_k = 0$, reducing the G-structure (1) to, ignoring the $\mathbf{Z}_2$ term

$$\left\{ \begin{array}{rcll} \theta' & = & \theta & \\ \theta^{i'} & = & a^i_j\theta^j & \text{where } (a^i_j) \in O(2n) \\ d\theta & = & h_{ij}\theta^i \wedge \theta^j & \text{with } \quad (h_{ij} = -h_{ji} \text{ and } det(h_{ij}) = 1) \end{array} \right. \tag{3}$$

In particular, if the distribution is orientable then it has an O(2n)-structure.

3 Adapted Connection forms

Let the Sub-Riemannian manifold (M, D, g) be given, and consider the associated O(2n)-structure (3). We will construct connection forms and torsion forms which solve the equivalence problem. We begin by considering the intrinsically defined tautological forms over the bundle 3 which we will denote by the same letters θ, θ^i.

Theorem 3.1 *There exists unique forms ω^i_j and τ^i satisfying the equation*

$$d\theta^i = \theta^j \wedge \omega^i_j + \theta \wedge \tau^i$$

with conditions i) $\omega^i_j = -\omega^j_i$ and ii) $\sum \tau^i \wedge \theta^i = 0$.

Proof: Let $\tilde{\omega}^i_j$ and $\tilde{\tau}^i$ be any forms satisfying the first equation. If ω^i_j and τ^i also satisfy the equation, then

$$\theta^j \wedge (\omega^i_j - \tilde{\omega}^i_j) + \theta \wedge (\tau^i - \tilde{\tau}^i) = 0$$

From Cartan's lemma we have

$$\omega^i_j - \tilde{\omega}^i_j = a^i_{jk}\theta^k + b^i_j\theta$$

$$\tau^i - \tilde{\tau}^i = b^i_k\theta^k$$

with $a^i_{jk} = a^i_{kj}$. We will choose a^i_{jk}, b^i_j such that the conditions in the theorem be satisfied for ω^i_j, τ^i. To verify condition ii) we must have

$$0 = \sum \tau^i \wedge \theta^i = \sum \tilde{\tau}^i \wedge \theta^i + \sum\sum b^i_k\theta^k \wedge \theta^i$$

If we write $\tilde{\tau}^i = \tilde{\tau}^i_k\theta^k$, then

$$\sum\sum(\tilde{\tau}^i_k + b^i_k)\theta^k \wedge \theta^i = 0$$

and using Cartan's lemma again $\tilde{\tau}^i_k + b^i_k = a^i_k$ with $a^i_k = a^k_i$. On the other hand if i) is satisfied, and writing $\tilde{\omega}^i_j = \tilde{\omega}^i_{jk}\theta^k + \tilde{w}^i_j\theta$

$$(\tilde{\omega}^i_{jk} + \tilde{\omega}^j_{ik} + a^i_{jk} + a^j_{ik})\theta^k + (\tilde{w}^i_j + \tilde{w}^j_i + b^i_j + b^j_i)\theta = 0$$

We get two equations

$$\tilde{w}^i_j + \tilde{w}^j_i + a^i_j + a^j_i - \tilde{\tau}^i_j - \tilde{\tau}^j_i = 0$$

$$\tilde{\omega}^i_{jk} + \tilde{\omega}^j_{ik} + a^i_{jk} + a^j_{ik} = 0$$

The first equation, recalling that a^i_j is symmetric, has solution $a^i_j = \frac{\tilde{\tau}^i_j + \tilde{\tau}^j_i}{2} - \frac{\tilde{w}^i_j + \tilde{w}^j_i}{2}$ therefore b^i_j is determined. The second equation can be solved using the permutation trick, as in riemannian geometry. □

4 Reduction of the O(2n)-structure

Consider the subbundle of all orthonormal coframes $\theta^1, \cdots, \theta^{2n}$ such that $(h_{ij}) \in u(n)$, i. e.,

$$h_{kl} = h_{k+n,l+n} \quad \text{and} \quad h_{k+n,l} = -h_{k,l+n} \quad 1 \le k,l \le n$$

If $\theta'^1, \cdots, \theta'^{2n}$ is another coframe of this subbundle, with $\theta'^i = a^i_j \theta^j$, then

$$a_{kl} = a_{k+n,l+n} \quad \text{and} \quad a_{k+n,l} = -a_{k,l+n} \quad 1 \le k,l \le n$$

That is, $(a^i_j) \in U(n)$. This reduction allows us to define an operator $J : D \to D$ with $J^2 = -I$.

Theorem 4.1 *There exists unique forms $\omega^i_{1j}, \tau^i_{1j}, \tau^i_0$ satisfying the equations*

$$d\theta^i = \theta^j \wedge \omega^i_{1j} + \theta^j \wedge \tau^i_{1j} + \theta \wedge \tau^i_0$$

and conditions

- $(h_{ij}) \in u(n)$
- $\omega^i_{1j} = -\omega^j_{1i},\ \omega^{k+n}_{1l+n} = \omega^k_{1l},\ \omega^{k+n}_{1l} = -\omega^k_{1l+n}$ *for* $1 \le i,j \le 2n,\ 1 \le k,l \le n$
- $\tau^i_{1j} = -\tau^j_{1i},\ \tau^{k+n}_{1l+n} = -\tau^k_{1l},\ \tau^{k+n}_{1l} = \tau^k_{1l+n}$
- $\tau^i_0 \wedge \theta^i = 0$

We first state the following

Lemma 4.1 *Let $< A, B >= -Tr(AB)$ the scalar product in $o(2n)$. Then*

$$u(n)^{\perp} = \{B \in o(2n) \ ; \ JB + BJ = 0\}$$

Proof: If $AJ = JA$ and $JB+BJ = 0$ then $< A, B >= Tr(AJ^2B) = -Tr(JABJ) = -< A, B >$. So $< A, B >= 0$. If $X \in o(2n)$, define $A = \frac{X-JXJ}{2}$, $B = \frac{X+JXJ}{2}$. Then $X = A + B$, $A \in u(n)$ and $B \in u(n)^{\perp}$.

Proof of theorem 4.1: Let ω^i_j, τ^i_0 as in theorem 3.1. Then we decompose $\omega^i_j = \omega^i_{1j} + \tau^i_{1j}$, with $(\omega^i_{1j}) \in u(n)$ and $\tau^i_{1j} \in u(n)^{\perp}$. The unicity of ω^i_{1j} and τ^i_{1j} follows from the unicity of ω^i_j.

□

Let's now introduce complex forms. Let be

$$\zeta^{\alpha} = \theta^{\alpha} + i\theta^{\alpha+n}$$

$$\eta^\alpha_{1\beta} = \omega^\alpha_{1\beta} + i\omega^{\alpha+n}_{1\beta}$$

$$\gamma^\alpha_{1\overline{\beta}} = \tau^\alpha_{1\beta} + i\tau^{\alpha+n}_{1\beta}$$

$$\gamma^\alpha_0 = \tau^\alpha_0 + i\tau^{\alpha+n}_0$$

for $1 \leq \alpha, \beta \leq n$. If we put also $ig_{\alpha\overline{\beta}} = h_{\alpha\beta} + ih_{\alpha\beta+n}$ we get

$$d\theta = ig_{\alpha\overline{\beta}}\zeta^\alpha \wedge \zeta^{\overline{\beta}}$$

where $\zeta^{\overline{\beta}} = \overline{\zeta^\beta}$. We have $g_{\alpha\overline{\beta}} = \overline{g_{\beta\overline{\alpha}}}$. After introducing this notation, we can write theorem 4.1 as

Theorem 4.2 *There exists unique forms $\eta^\alpha_{1\beta}$, $\gamma^\alpha_{1\overline{\beta}}$, γ^α_0 such that*

$$d\zeta^\alpha = \zeta^\beta \wedge \eta^\alpha_{1\beta} + \zeta^{\overline{\beta}} \wedge \gamma^\alpha_{1\overline{\beta}} + \theta \wedge \gamma^\alpha_0$$

satisfying

- $\eta^\alpha_{1\beta} = -\eta^{\overline{\beta}}_{1\overline{\alpha}}$
- $\gamma^\alpha_{1\overline{\beta}} = -\gamma^\beta_{1\overline{\alpha}}$
- $\gamma^\alpha_0 \wedge \zeta^{\overline{\alpha}} + \gamma^{\overline{\alpha}}_0 \wedge \zeta^\alpha = 0$ *where* $\eta^{\overline{\beta}}_{1\overline{\alpha}} = \overline{\eta^\beta_{1\alpha}}$ *and* $\gamma^{\overline{\alpha}}_0 = \overline{\gamma^\alpha_0}$

The next step on this way of reduction is to observe that being $g_{\alpha\overline{\beta}}$ a hermitian matrix, it can be diagonalized using the group $U(n)$. Those eigenvalues are invariants of the subriemannian structure and are functions on the manifold. We will suppose that the diagonalization of $g_{\alpha\overline{\beta}}$ gives precisely r eigenvalues with constant multiplicities d_i. This hypothesis will be sufficient to reduce the U(n) bundle to a $U(d_1) \times \cdots \times U(d_r)$ bundle.

Diagonalizing $g_{\alpha\overline{\beta}}$ we obtain coreferentials $\zeta^1, \cdots, \zeta^n$ such that

$$g_{\alpha\overline{\beta}} = \delta^\beta_\alpha \lambda_\beta$$

and $\lambda_{d_1+\cdots+d_{k-1}+1} = \cdots = \lambda_{d_1+\cdots+d_k} = \nu_k$ for $1 \leq k \leq r$, with $d_1 + \cdots + d_r = n$, and $\nu_1 < \nu_2 < \cdots < \nu_r$, where the ν_k are real functions on M. In these coreferencials, we get

$$d\theta = i\lambda_\alpha \zeta^\alpha \wedge \zeta^{\overline{\alpha}} \tag{4}$$

If $\zeta'^1, \cdots, \zeta'^n$ is another coframe of this subbundle, with $\zeta'^\alpha = a^\alpha_\beta \zeta^\beta$, then it is easy to see that $a^\alpha_\beta \in U(d_1) \times \cdots \times U(d_r)$. This allows us to reduce the U(n)-structure to a $U(d_1) \times \cdots \times U(d_r)$-structure.

We denote by $H = U(d_1) \times \cdots \times U(d_r)$ and its Lie algebra $h = u(d_1) \times \cdots \times u(d_r)$. The inner product in o(2n), restricted to u(n), can be written in complex form as

$$< A, B >= 2ReTr(A\overline{B}^T)$$

We denote by $h^\perp$ the perpendicular space to h with respect to this inner product. If $A \in u(n)$, we may write $A = (A_{ij})_{r\times r}$, where each A_{ij} is a $d_i \times d_j$ matrix, with $\overline{A}_{ij}^T = -A_{ji}$. Analogously we may write $(g_{\alpha\overline{\beta}}) = (G_{ij})_{r\times r}$ where $G_{ij} = 0$ if $i \neq j$, and $G_{ii} = \nu_i I_{d_i}$. With this notation $A \in h$, if and only if $A_{ij} = 0$ for $i \neq j$, and $A_{ii} \in u(d_i)$. We characterize the elements of $h^\perp$ in the following

Lemma 4.2 *$B \in h^\perp$ if and only if $B_{ii} = 0$, for $1 \leq i \leq r$.*

Proof: If $A \in h$, then

$$< A, B >= -2ReTr(A_{ii}B_{ii})$$

As A_{ii} can be any element of $u(d_i)$, and $B_{ii} \in u(d_i)$, we get $< A, B >= 0$ if and only if $B_{ii} = 0$, $1 \leq i \leq r$.

By decomposing the connection η_1 obtained in theorem 4.1 in $\eta_1 = \eta + \gamma_2$, where η take values in h and γ_2 in $h^\perp$, we get the following

Theorem 4.3 *There exists unique forms $\eta^{\alpha_i}_{\beta_i}$, $\gamma^{\alpha_i}_{2\beta_j}$, $\gamma^{\alpha}_{1\overline{\beta}}$, γ^{α}_{0} with $1 \leq i,\ j \leq r$, $1 \leq \alpha,\ \beta \leq n$, and $d_1 + \cdots + d_{i-1} + 1 \leq \alpha_i,\ \beta_i \leq d_1 + \cdots + d_i$, such that*

$$d\zeta^{\alpha_i} = \zeta^{\beta_i} \wedge \eta^{\alpha_i}_{\beta_i} + \zeta^{\beta_j} \wedge \gamma^{\alpha_i}_{2\beta_j} + \zeta^{\overline{\beta}} \wedge \gamma^{\alpha_i}_{1\overline{\beta}} + \theta \wedge \gamma^{\alpha_i}_{0}$$

satisfying

- $\eta^{\alpha_i}_{\beta_i} = -\overline{\eta^{\beta_i}_{\alpha_i}}$, $\eta^{\alpha_i}_{\beta_j} = 0$ *if* $i \neq j$
- $\gamma^{\alpha_i}_{2\beta_j} = -\overline{\gamma^{\beta_j}_{2\alpha_i}}$ *if* $i \neq j$, $\gamma^{\alpha_i}_{2\beta_i} = 0$
- $\gamma^{\alpha}_{1\overline{\beta}} = -\gamma^{\beta}_{1\overline{\alpha}}$
- $\gamma^{\alpha}_{0} \wedge \zeta^{\overline{\alpha}} + \gamma^{\overline{\alpha}}_{0} \wedge \zeta^{\alpha} = 0$

We will be particularly interested in structures which have all λ_α constants and have vanishing torsions γ_1 and γ_2. The pseudo-hermitian structure and connection in [W], corresponds precisely to the case when the full reduction group is $U(n)$.

5 Curvatures

The curvature forms are defined by

$$\Pi^i_k = d\omega^i_k + \omega^i_j \wedge \omega^j_k$$

Theorem 5.1 *The curvature forms are given by*

$$\Pi^i_k = \frac{1}{2}R^i_{krs}\theta^r \wedge \theta^s + W^i_{ks}\theta^s \wedge \theta + h_{kl}\theta^l \wedge \tau^i - h_{il}\theta^l \wedge \tau^k$$

with the conditions $R^i_{krs} = -R^k_{irs}$, $R^i_{krs} = -R^i_{ksr}$, $R^i_{ksr} + R^i_{rks} + R^i_{srk} = 0$, $W^i_{ks} = -W^k_{is}$ *and* $W^i_{ks} + W^s_{ik} + W^k_{si} = 0$.

Proof: Define

$$\Omega^i_k = d\omega^i_k - \omega^j_k \wedge \omega^i_j - h_{kl}\theta^l \wedge \tau^i + h_{il}\theta^l \wedge \tau^k$$

$$\Omega^i = d\tau^i - \tau^j \wedge \omega^i_j$$

then, differentiating the equation defining the connection forms, we obtain the Bianchi identity, that is, $\theta^k \wedge \Omega^i_k + \theta \wedge \Omega^i = 0$. Observe that $\Omega^i_k = -\Omega^k_i$. To find the symmetry conditions on those forms we see that, due to the Bianchi identity, we can write

$$\Omega^i_k = \chi^i_{kr}\theta^r + \lambda^i_k\theta \text{ for 1-forms } \chi^i_{kr} \text{ and } \lambda^i_k \text{ without terms in } \theta.$$

Writing $\chi^i_{kr} = -\frac{1}{2}R^i_{krs}\theta^s$ and $\lambda^i_k = W^i_{ks}\theta^s$ and substituting in the Bianchi identities, we get

$$\Omega^i_k = \frac{1}{2}R^i_{krs}\theta^r \wedge \theta^s + W^i_{ks}\theta^s \wedge \theta$$

with the conditions $R^i_{krs} = -R^k_{irs}$, $R^i_{krs} = -R^i_{ksr}$, $R^i_{ksr} + R^i_{rks} + R^i_{srk} = 0$, and $W^i_{ks} = -W^k_{is}$. Observe that we can write, using the Bianchi identity again

$$\Omega^i = -W^i_{rs}\theta^r \wedge \theta^s + \mu^i \wedge \theta \tag{5}$$

Finally, we differentiate the condition $\theta^i \wedge \tau^i = 0$ to obtain $W^i_{ks} + W^s_{ik} + W^k_{si} = 0$.

□

In computations, we use a section of the the G-structure, that is, a moving frame which we write by the same letters as the tautological forms $\Theta^T = (\theta^1, \ldots, \theta^{2n})$ and $\tau^T = (\tau^1, \ldots, \tau^{2n})$. The structure equation is written also as $d\Theta = -\omega \wedge \Theta - \tau \wedge \theta$. If $\Theta' = g\Theta$ is a new moving frame, then

$$\begin{aligned} \omega' &= gdg^{-1} + g\omega g^{-1} \\ \tau' &= g\tau \end{aligned}$$

6 Geometrical Interpretation

In this section we give a covariant derivation interpretation of the computations with forms above.

Let (M,D,g) be an orientable non-degenerate metric distribution of codimension 1. Let dV be the volume form on D. The contact form is the unique form such that

$$\begin{aligned} Ker(\theta) &= D \\ d\theta^n_{|D} &= 2^n n! dV \end{aligned}$$

We denote by ξ the unique field such that

$$\begin{aligned} \theta(\xi) &= 1 \\ \iota_\xi d\theta &= 0 \end{aligned}$$

We define the antissimetric form $h : D \to D$ by $d\omega(X,Y) = < h(X), Y >$. The connection defined above is translated into the following proposition. Let U be any vector and X, Y be in the distribution.

Proposition 6.1 *There exists a unique connection* $\nabla : TM \to TM^* \otimes TM$ *with the following properties, where* T *is the torsion tensor of the connection.*

- $\nabla_U : D \to D$
- $\nabla \xi = 0$
- $\nabla g = 0$
- $\begin{aligned} T(X,Y) &= d\theta(X,Y)\xi \\ T(\xi, X) &= \tau(X) \end{aligned}$ *where* τ *is a symmetric tensor with* $\iota_\xi \tau = 0$.

Observe that the torsion tensor is given by $T = \theta \wedge \tau + d\theta \otimes \xi$. If (e_i) is a local frame and (θ^i) is its dual coframe, we can write in those coordinates $\nabla e_i = \omega_i^j e_j$ and $\tau = \tau^j \otimes e_j$. The structure equation

$$d\omega = \hat{\nabla}\omega + \omega \circ T$$

where, $\hat{\nabla}\omega(X,Y) = (\nabla_X \omega)(Y) - (\nabla_Y \omega)(X)$, translates into the structure equations of theorem 3.1.

We collect in the following proposition some straightforward, though usefull, properties of the connection.

Proposition 6.2 *The connection* ∇ *has the following properties*

- $L_\xi : \underline{D} \to \underline{D}$

- $d\theta(X,Y) = \theta(T(X,Y))$
- $< \tau(X), Y >= \frac{1}{2} L_\xi g(X,Y)$
- $\nabla_\xi h = -(h \circ \tau + \tau \circ h)$

Definition 6.1 *The metric distribution is called h-compatible if its adapted connection satisfies* $\nabla h = 0$.

Observe also that, using coframes, this condition is

$$dh_{ij} - h_{lj}\omega_i^l - h_{li}\omega_j^l = 0$$

It is easy to see that in the reduction of theorem 4.3, if we suppose λ_α constants, an h-compatible connection has λ_α constants, $\gamma_1 = \gamma_2 = 0$. In this case the condition $\gamma_0^\alpha \wedge \zeta^{\overline{\alpha}} + \gamma_0^{\overline{\alpha}} \wedge \zeta^\alpha = 0$ in theorem 4.3 can be substituted for $\gamma_0^\alpha \wedge \zeta^{\overline{\alpha}} = 0$ as can be seen differentiating equation 4. We see than, that the pseudo-hermitian structure and connection defined by Webster [W] is equivalent to an h-compatible sub-Riemannian structure which has the group $U(n)$ as structure group after reducing completely as in theorem 4.3.

The curvature of this connection is given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

From general theory of connections we have the Bianchi identity

$$PR(X,Y)Z = PT(T(X,Y),Z) + P(\nabla_X T)(Y,Z)$$

where P denotes the cyclic summation. In the case of the adapted connection we get the following identities

- $PR(X,Y)Z = P < h(X), Y > \tau(Z)$
- $P < (\nabla_X h)(Y), Z >= 0$
- $R(\xi,Y)Z - R(\xi,Z)Y = (\nabla_Z \tau)(Y) - (\nabla_Y \tau)(Z)$

where we supposed $X, Y, Z \in D$ and ξ the transverse field.

A geometrical interpretation of the vanishing torsion condition is obtained in the following

Proposition 6.3 *The transversal vector field is an infinitesimal equivalence if and only if* $\tau^i = 0$.

In this case we see that $\Omega^i = d\tau^i - \tau^j \wedge \omega_j^i = 0$, and therefore from the symmetries of W^i_{rs} we get also $W^i_{rs} = 0$. We then write the curvature

$$\Pi^i_j = \frac{1}{2} R^i_{jrs} \theta^r \wedge \theta^s$$

The adapted Riemannian metric is defined by the reduction (3), where $d\theta = h_{ij}\theta^i \wedge \theta^j$. The Levi-Civita connection $\tilde{\omega}^i_j, \tilde{\omega}^i, \tilde{\omega}_i$ satisfies the structure equation

$$\begin{aligned} d\theta^i &= \theta^j \wedge \tilde{\omega}^i_j + \theta \wedge \tilde{\omega}^i \\ d\theta &= \theta^j \wedge \tilde{\omega}_j \end{aligned}$$

with conditions $\tilde{\omega}^i_j = -\tilde{\omega}^j_i$ and $\tilde{\omega}^i = -\tilde{\omega}_i$. Writing $\tilde{\omega}_i = b_{ij}\theta^j + b_i\theta$ and substituting in the second equation above follows that $b_i = 0$ and $\tilde{\omega}_i = (h_{ij} + a_{ij})\theta^j$ with $a_{ij} = a_{ji}$. Using the convention $h^i_j = -h_{ij}, a^i_j = -a_{ij}$ we prove the following

Proposition 6.4 *The adapted connection and torsion forms are given by*

$$\begin{aligned} \omega^i_j &= \tilde{\omega}^i_j - h^i_j\theta \\ \tau^i &= a^i_j\theta^j \end{aligned}$$

In analogy to the second fundamental form for submanifolds in a riemannian manifold, we define

$$II = a_{ij}\theta^i \cdot \theta^j = \sum \tau^i \cdot \theta^i$$

7 Constant Curvature Models

Definition 7.1 *Two metric distributions M and M' are equivalent, if there exists a diffeomorphism $f : M \to M'$ such that $f_*(D) = (D')$ and it preserves the metric in the distribution.*

To establish the equivalence between two metric distributions, we will consider the $U(d_1) \times \cdots \times U(d_r)$-reduction as in theorem 4.3. We will establish the models for h-compatible sub-riemannian metrics with vanishing torsion. In the notation of theorem 4.3 we write the structure equations in this case

$$d\theta = i\lambda_{\alpha_i}\zeta^{\alpha_i} \wedge \zeta^{\overline{\alpha_i}}$$

$$d\zeta^{\alpha_i} = \zeta^{\beta_i} \wedge \eta^{\alpha_i}_{\beta_i}$$

with $1 \le i \le r$, and $d_1 + \cdots + d_{i-1} + 1 \le \alpha_i,\ \beta_i \le d_1 + \cdots + d_i$, satisfying and $\eta^{\alpha_i}_{\beta_i} = -\overline{\eta^{\beta_i}_{\alpha_i}}$

The curvature forms may be written as

$$\Psi^{\alpha_i}_{\beta_i} = d\eta^{\alpha_i}_{\beta_i} + \eta^{\alpha_i}_{\gamma_i} \wedge \eta^{\gamma_i}_{\beta_i}$$

Definition 7.2 *The h-compatible subriemannian manifold has constant h-sectional curvature if*

$$\Psi^{\alpha_i}_{\beta_i} = C_i(\zeta^{\alpha_i} \wedge \overline{\zeta}^{\beta_i} + \delta^{\alpha_i}_{\beta_i} \sum \zeta^{\gamma_i} \wedge \overline{\zeta}^{\gamma_i})$$

where C_i is a constant for each $1 \leq i \leq r$.

We are now ready to state the local classification theorem

Theorem 7.1 *Suppose M and M' are two h-compatible sub-riemannian manifolds satisfying the conditions*

i) $r = r'$, $d_i = d_i'$, and $\lambda_{\alpha_i} = \lambda'_{\alpha_i}$

ii) They have null torsion and the same constant h-sectional curvatures, $C_i = C_i'$

Then they are locally equivalent.

Proof. We follow the proof of the local equivalence between constant curvature riemannian manifolds. We construct the manifold $P \times P'$ the product of the H-structures (reduced as above) Consider over this manifold the pull-back of the tautological forms on P and P' by the projections and denote them by the same letters as in the G-structures. We form then the system

$$\begin{aligned} \omega^{\alpha_i} &= \zeta^{\alpha_i} - \zeta'^{\alpha_i} \\ \omega &= \theta - \theta' \\ \omega^{\alpha_i}_{\beta_i} &= \eta^{\alpha_i}_{\beta_i} - \eta'^{\alpha_i}_{\beta_i} \end{aligned}$$

Using conditions i) and ii), we prove that it is involutive. Fixing coframes Θ and Θ' in P and P' there exists an integral submanifold passing through $\Theta \times \Theta'$. This integral submanifold is given by the graph of a function $F : P \to P'$ which gives rise to an equivalence $f : M \to M'$.

□

We now obtain the models realizing each situation arising in theorem 7. We start with the building blocks of the theorem, that is, $r = 1$, $d_1 = n$ and $\lambda_{\alpha_1} = \lambda$ constant. We will give a construction in the most appropriate form for the further construction of the composite models.

1) Consider in $\mathbf{C}^{n+1}$ the hermitian product $< z, w >= \sum_{j=0}^{n} z^j \overline{w}^j$. We define

$$S^{2n+1}(r) = \{ \ z \in \mathbf{C}^{n+1} \mid \ < z, z >= r^2 \ \}$$

The Hopf fibration is given by $S^{2n+1}(r) \to \mathbf{CP}^n$. Observe that the group $U(n+1)$ acts on the sphere preserving the fibration.

Let $(b_0, b_1, \cdots, b_n)$ be a unitary frame in $\mathbf{C}^{n+1}$. We consider $z = rb_0 \in S^{2n+1}(r)$ and define the coframe θ, ζ^α by the formula

$$dz = \frac{1}{r}\theta(ib_0) + \zeta^\alpha b_\alpha$$

where we observe that ib_0 is tangent to the sphere. A calculation, using $db_\alpha = \omega_\alpha^\beta b_\beta$ and observing that $\zeta^\alpha = r\omega_0^\alpha$, shows that

$$d\theta = i\sum_{\alpha=1}^{n} \zeta^\alpha \wedge \overline{\zeta}^\alpha$$

$$\eta_\beta^\alpha = \omega_\alpha^\beta - \delta_\beta^\alpha \frac{i}{r^2}\theta$$

$$\Psi_\beta^\alpha = \frac{1}{r^2}(\zeta^\alpha \wedge \overline{\zeta}^\beta + \delta_\beta^\alpha \sum \zeta^\gamma \wedge \overline{\zeta}^\gamma)$$

2) Consider in $\mathbf{C}^{n+1}$ the hermitian product $< z, w >= -z_0\overline{w}_0 + \sum_{j=1}^{n} z^j\overline{w}^j$. We define

$$Q^{2n+1}(r) = \{\ z \in \mathbf{C}^{n+1} \mid\ < z,z >= -r \text{ and } z_0 > 0\ \}$$

The ball in $\mathbf{C}P^n$ is given by $B^n = \{\ [z] \mid\ < z,z >< 0\ \}$. So we have the fibration $Q^{2n+1}(r) \to B^n$ as a S^1 bundle. Observe that the group $U(n,1)$ acts on the quadric preserving the fibration. Let $(b_0, b_1, \cdots, b_n)$ be a frame in $\mathbf{C}^{n+1}$ with respect to the above hermitian product. We consider $z = rb_0 \in Q^{2n+1}(r)$ and define the coframe θ, ζ^α by the formula

$$dz = \frac{1}{r}\theta(ib_0) + \zeta^\alpha b_\alpha$$

where we observe that ib_0 is tangent to the quadric. A calculation, using $db_\alpha = \omega_\alpha^\beta b_\beta$, as above, shows that

$$d\theta = i\sum_{\alpha=1}^{n} \zeta^\alpha \wedge \overline{\zeta}^\alpha$$

$$\eta_\beta^\alpha = \omega_\alpha^\beta - \delta_\beta^\alpha \frac{i}{r^2}\theta$$

$$\Psi_\beta^\alpha = -\frac{1}{r^2}(\zeta^\alpha \wedge \overline{\zeta}^\beta + \delta_\beta^\alpha \sum \zeta^\gamma \wedge \overline{\zeta}^\gamma)$$

3) Let the Heisenberg group $H^{2n+1} \in \mathbf{C}^{n+1}$ be defined by $\sum_{j=1}^{n} z^j\overline{w}^j + i(z^{n+1} - \overline{z}^{n+1}) = 0$. Then $H^{2n+1} = \{\ (z, x+iy) \mid z \in \mathbf{C}^n \text{ and } y = \frac{1}{2}|z|^2\ \}$. So we have the $\mathbf{R}$-fibration $H^{2n+1} \to \mathbf{C}^n$. The group acting on H^{2n+1} is represented by

$$HU(n+1) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ z & A & 0 \\ x + \frac{i}{2}|z|^2 & i\overline{z}^t A & 1 \end{bmatrix} \quad A \in U(n), z \in \mathbf{C}^n, x \in \mathbf{R} \right\}$$

and analogously to the previous cases, it preserves the fibration. Let $(b_0, b_1, \cdots, b_{n+1})$ be a frame in HU(n+1). From the Lie algebra we obtain

$$\begin{cases} db_0 & = \zeta^\alpha b_\alpha + \theta b_{n+1} \\ db_\alpha & = \omega_\alpha^\beta b_\alpha + i\overline{\zeta}^\alpha b_{n+1} \\ db_{n+1} & = 0 \end{cases}$$

We consider $z = b_0 \in H^{2n+1}$. then define the coframes using the formula

$$dz = \theta b_{n+1} + \zeta^\alpha b_\alpha$$

We get

$$d\theta = i \sum_{\alpha=1}^{n} \zeta^\alpha \wedge \overline{\zeta}^\alpha$$

$$\Psi_\beta^\alpha = 0$$

To construct the composite models, we will define a certain product of fiber bundles. Let $E_1 \to M_1$ and $E_2 \to M_2$ be S^1 bundles. Consider the product bundle $E_1 \otimes E_2 \to M_1 \times M_2$ with fiber S^1, obtained considering the tensor product of the $\mathbf{C}$ bundles which are extensions of the S^1 bundles and taking the image of $E_1 \times E_2$. We denote this bundle by $E_1 \bullet E_2$.

The models are sub-Riemannian structures over an S^1 bundle obtained from the models defined above. Let $N = (n_1, n_2, \cdots, n_m)$ a sequence of positive integers and $R = (r_1, r_2, \cdots, r_m)$ a sequence of real numbers $r_1 \leq r_2 \leq \cdots \leq r_m$. In the following definition we will denote by $H_\Gamma^{2n+1}(r)$ the quotient of $H^{2n+1}(r)$ by the discrete group $\mathbf{Z}$ acting on the fibers. The resulting bundle is a S^1 bundle. Let

$$M^{2n+1}(r) = \begin{cases} S^{2n+1}(r) & r > 0 \\ H_\Gamma^{2n+1}(r) & r = 0 \\ Q^{2n+1}(r) & r < 0 \end{cases}$$

Let

$$M^N(R) = M^{2n+1}(r_1) \bullet \cdots \bullet M^{2n+1}(r_m)$$

Let $(f_{j0}, f_{j1}, \cdots, (f_{jn_j})$ be a unitary base for each $\mathbf{C}^{n_j+1}$ with respect to the corresponding hermitian forms above. Then define the forms η_{jk}^l by the formula $df_{jk} = \omega_{jk}^l f_{jl}$.

Consider $Z = (r_1 f_{10}) \bullet (r_2 f_{20}) \bullet \cdots \bullet (r_m f_{m0}) \in M^N(R)$. Then $T_Z M^N(R)$ is generated by the vectors

$$\{i f_{10} \bullet \cdots \bullet f_{m0} \ , \ (f_{10} \bullet \cdots \bullet f_{jk} \bullet \cdots \bullet f_{m0})\}$$

where $1 \leq j \leq m$ and $1 \leq k \leq n_j$. The dual basis is identified with

$$\{\eta_{10}^0 = \eta_{20}^0 = \cdots = \eta_{m0}^0 \ , \ \zeta^{k_j} = r_j \omega_{j0}^k \ \}$$

We define the contact form to be

$$\theta = i \sum_{j=1}^{n} \frac{\lambda_j}{r_j} \eta_{j0}^{0}$$

Observe that $\theta(i f_{10} \bullet \cdots \bullet f_{m0}) = \sum \frac{\lambda_j}{r_j}$ and $\theta(f_{10} \bullet \cdots \bullet f_{jk} \bullet \cdots \bullet f_{m0}) = 0$. Then

$$d\theta = i\lambda_j \zeta^{\alpha_j} \wedge \zeta^{\overline{\alpha_j}}$$

where $1 \leq \alpha_j \leq n_j$.

It is easy to see that the constructed models have h-sectional curvature constant and vanishing torsion.

The complete simply connected sub-Riemannian manifolds with constant h-sectional curvature, vanishing torsion are precisely the universal coverings of the models obtained above. By completeness we mean the standard definition on a structure which carries a connection.

8 References

CH Chern, S. S. , Hamilton, R. S.: On Riemannian metrics adapted to three-dimensional contact-manifolds. Lect. Not. Math. 1111, 279-305.

S Strichartz, R. S. : Sub-Riemannian Geometry. Journal Diff. Geom. 24, (1986), 221-263.

T Tanno, S. : Variational Problems on Contact Riemannian Manifolds. Trans. Am. Math. Soc. 314, n1,(1989), 349-379.

W Webster, S. M. : Pseudo-Hermitian structures on a real hypersurface. Journal Diff. Geom. 13,(1978), n1, 25-41.

TRABALHOS DO DEPARTAMENTO DE MATEMÁTICA

TÍTULOS PUBLICADOS

92-01 COELHO, S.P. The automorphism group of a structural matrix algebra. 33p.

92-02 COELHO, S.P. & POLCINO MILIES , C. Group rings whose torsion units form a subgroup. 7p.

92-03 ARAGONA, J. Some results for the operator on generalized differential forms. 9p.

92-04 JESPERS, E. & POLCINO MILIES, F.C. Group rings of some p-groups. 17p.

92-05 JESPERS, E., LEAL G. & POLCINO MILIES, C. Units of Integral Group Rings of Some Metacyclic Groups. 11p.

92-06 COELHO, S.P., Automorphism Groups of Certain Algebras of Triangular Matrices. 9p.

92-07 SCHUCHMAN, V., Abnormal solutions of the Evolution Equations, I. 16p.

92-08 SCHUCHMAN, V., Abnormal solutions of the Evolution Equations, II. 13p.

92-09 COELHO, S.P., Automorphism Groups of Certain Structural Matrix Rings. 23p.

92-10 BAUTISTA, R. & COELHO, F.U. On the existence of modules which are neither preprojective nor preinjectives. 14p.

92-11 MERKLEN, H.A., Equivalence modulo preprojectives for algebras which are a quotient of a hereditary. 11p.

92-12 BARROS, L.G.X. de, Isomorphisms of Rational Loop Algebras. 18p.

92-13 BARROS, L.G.X. de, On semisimple Alternative Loop Algebras. 21p.

92-14 MERKLEN, H.A., Equivalências Estáveis e Aplicações 17 p.

92-15 LINTZ , R.G., The theory of -generators and some questions in analysis. 26p.

92-16 CARRARA ZANETIC, V.L. Submersions Maps of Constant Rank Submersions with Folds and Immersions. 6p.

92-17 BRITO, F.G.B. & EARP, R.S. On the Structure of certain Weingarten Surfaces with Boundary a Circle. 8p.

92-18 COSTA, R. & GUZZO JR.,H. Indecomposable baric algebras, II. 10p.

92-19 GUZZO JR., H. A generalization of Abraham's example 7p.

92-20 JURIAANS, O.S. Torsion Units in Integral Group Rings of Metabelian Groups. 6p.

92-21 COSTA, R. Shape identities in genetic algebras. 12p.

92-22 COSTA, R. & VEGA R.B. Shape identities in genetic algebras II. 11p.

92-23 FALBEL, E. A Note on Conformal Geometry. 6p.

93-01 COELHO, F.U. A note on preinjective partial tilting modules. 7p.

93-02 ASSEM, I. & COELHO, F.U. Complete slices and homological properties of tilted algebras. 11p.

93-03 ASSEM, I. & COELHO, F.U. Glueings of tilted algebras 20p.

93-04 COELHO, F.U. Postprojective partitions and Auslander-Reiten quivers. 26p.

93-05 MERKLEN, H.A. Web modules and applications. 14p.

93-06 GUZZO JR., H. The Peirce decomposition for some commutative train algebras of rank n. 12p.

93-07 PERESI, L.A. Minimal Polynomial Identities of Baric Algebras. 11p.

93-08 FALBEL E., VERDERESI J.A. & VELOSO J.M. The Equivalence Problem in Sub-Riemannian Geometry. 14p.

NOTA: Os títulos publicados dos Relatórios Técnicos dos anos de 1980 a 1991 estão à disposição no Departamento de Matemática do IME-USP.

Cidade Universitária "Armando de Salles Oliveira"
Rua do Matão, 1010 - Butantã
Caixa Postal - 20.570 (Ag. Iguatemi)
CEP: 01498 - São Paulo - Brasil